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AN INVESTIGATION OF THE APPROXIMATION OF
FUNCTIONS OF TWO VARIABLES BY THE POISSON
INTEGRAL FOR HERMITE EXPANSIONS

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POISSONA ZWIĄZANĄ Z WIELOMIANAMI HERMITE'A

Abstract

This paper presents a study of the approximation properties of the Poisson integral for Hermite expansions in the space L^p . The rate of convergence of functions of two variables by these integrals is established.

Keywords: rate of convergence; Poisson integral; Hermite expansions, positive linear operators

Streszczenie

Artykuł poświęcony jest własnościom aproksymacyjnym całek Poissona związanych z wielomianami Hermite'a. Udowodniono twierdzenie o rzędzie zbieżności funkcji dwóch zmiennych w przestrzeni L^p tymi operatorami.

Słowa kluczowe: promień zbieżności, całka Poissona, wielomiany Hermite'a, dodatnie operatory liniowe

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1. Introduction

Let $1 \leq p \leq \infty$, we denote by $L^p(\mathbf{R}^2)$ the set of all the Lebesgue measurable functions f defined on \mathbf{R}^2 such that $\int_{-\infty-\infty}^{\infty\infty} |f(t_1, t_2)|^p dt_1 dt_2 < \infty$ if $1 \leq p < \infty$, and if $p = \infty$ we require f to be bounded almost everywhere on \mathbf{R}^2 .

In this paper, we present approximation properties of the Poisson integral \bar{A} in the space $L^p(\mathbf{R}^2)$, $1 \leq p \leq \infty$ defined by:

$$\bar{A}(f; r, y_1, y_2) = \int_{-\infty-\infty}^{\infty\infty} \int_{-\infty-\infty}^{\infty\infty} r K(r, y_1, z_1) K(r, y_2, z_2) f(z_1, z_2) dz_1 dz_2, \quad 0 < r < 1,$$

where:

$$K(r, y, z) = \sum_{n=0}^{\infty} r^n h_n(y) h_n(z) = \frac{1}{(\pi(1-r^2))^{\frac{1}{2}}} \exp\left(-\frac{1}{2} \cdot \frac{1+r^2}{1-r^2} (y^2 + z^2) + \frac{2ryz}{1-r^2}\right),$$

$$h_n(x) = \left(2^n n! \sqrt{\pi}\right)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{2}\right) H_n(x)$$

and H_n is the n th Hermite polynomial (see [11]). The norm in $L^p(\mathbf{R}^2)$ is given by:

$$\|f\|_p = \begin{cases} \left(\int_{-\infty-\infty}^{\infty\infty} \int_{-\infty-\infty}^{\infty\infty} |f(t_1, t_2)|^p dt_1 dt_2 \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \sup_{(t_1, t_2) \in \mathbf{R}^2} \text{ess } |f(t_1, t_2)|, & p = \infty. \end{cases}$$

Some convergence theorems, the Voronovskaya formula, and a boundary value problem for the integral \bar{A} were presented in [5]. The following result was proved (see [5]):

Theorem 1 Let $\bar{y} = (\bar{y}_1, \bar{y}_2) \in \mathbf{R}^2$ and $f = f_1 + f_2$, where $f_1 \in L^1(\mathbf{R}^2)$, $f_2 \in L^\infty(\mathbf{R}^2)$. If f is continuous at \bar{y} , then:

$$\lim_{(r, y) \rightarrow (1^-, \bar{y})} \bar{A}(f; r, y) = f(\bar{y}), \quad y = (y_1, y_2).$$

In this paper we shall give an order of approximation of functions belonging to $L^p(\mathbf{R}^2)$ by the operator \bar{A} . It is worth mentioning that approximation properties of Poisson integrals for orthogonal expansions and their various modifications were also studied in [4, 12, 6–10], in one and two dimensions.

Some auxiliary results, which will be needed in the next part of this paper, are now presented. It is clear that $\bar{A}(f; r, y_1, y_2) = rA(f_1; r, y_1)A(f_2; r, y_2)$ for $f_1, f_2 \in L^p(\mathbf{R})$ and such that $f(z_1, z_2) = f_1(z_1)f_2(z_2)$, where $A(f)(r, y) = A(f; r, y) = \int_{-\infty}^{\infty} K(r, y, z) f(z) dz$, $0 < r < 1$.

The operator \bar{A} is linear and positive. Basic facts on positive linear operators and its applications can be found in [2, 3].

In paper [7], we can find the following equalities:

$$A(1; r, y) = \left(\frac{2}{1+r^2} \right)^{1/2} \exp\left(-\frac{1}{2} \cdot \frac{1-r^2}{1+r^2} y^2 \right),$$

$$A(\phi_{m,y}; r, y) = \left(\frac{2}{1+r^2} \right)^{1/2} \exp\left(-\frac{1}{2} \cdot \frac{1-r^2}{1+r^2} y^2 \right)$$

$$\times \sum_{p=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{p} \frac{(m-p)!}{(m-2p)! 2^p} \cdot \left(\frac{1-r^2}{1+r^2} \right)^p \left(-\frac{(1-r)^2}{1+r^2} y \right)^{m-2p}$$

for $0 < r < 1$, $y \in \mathbb{R}$, where $[a]$ denotes the integral part of $a \in \mathbb{R}$ and $\phi_{m,y}(z) = (z-y)^m$.

From the above, we have the following result in the bivariate case.

Lemma 1 Let $\phi_{n,y_i}(z_1, z_2) = (z_i - y_i)^n$, $y_i, z_i \in \mathbb{R}$, $i = 1, 2$, $n \in \mathbb{N}$. It holds

$$\bar{A}\left(|\phi_{1,y_i}|; r, y_1, y_2\right) \leq \frac{2r}{1+r^2} \cdot \left(\frac{1-r^2}{1+r^2} + \frac{(r-1)^4}{(1+r^2)^2} y_i^2 \right)^{\frac{1}{2}} \exp\left(-\frac{1}{2} \cdot \frac{1-r^2}{1+r^2} (y_1^2 + y_2^2) \right) \quad (1)$$

for $0 < r < 1$.

Proof. Using Hölder's inequality, we get:

$$\bar{A}\left(|\phi_{1,y_1}|; r, y_1, y_2\right) \leq \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r K(r, y_1, z_1) K(r, y_2, z_2) dz_1 dz_2 \right)^{\frac{1}{2}}$$

$$\times \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r K(r, y_1, z_1) K(r, y_2, z_2) |z_1 - y_1|^2 dz_1 dz_2 \right)^{\frac{1}{2}} = \left(\bar{A}(1; r, y_1, y_2) \right)^{\frac{1}{2}} \cdot \left(\bar{A}(\phi_{2,y_1}; r, y_1, y_2) \right)^{\frac{1}{2}} \quad (2)$$

for $(y_1, y_2) \in \mathbb{R}^2$ and $0 < r < 1$. We have (see [5]):

$$\bar{A}(1; r, y_1, y_2) = \frac{2r}{1+r^2} \exp\left(-\frac{1}{2} \cdot \frac{1-r^2}{1+r^2} (y_1^2 + y_2^2) \right),$$

$$\bar{A}(\phi_{2,y_i}; r, y_1, y_2) = \frac{2r}{1+r^2} \left(\frac{1-r^2}{1+r^2} + \frac{(r-1)^4}{(1+r^2)^2} y_i^2 \right) \exp\left(-\frac{1}{2} \cdot \frac{1-r^2}{1+r^2} (y_1^2 + y_2^2) \right), i = 1, 2.$$

From this and (2) we obtain (1) for $i = 1$. Analogously, we calculate (1) for $i = 2$.

2. Rate of convergence

In this section, we give an order of approximation of function of two variables in the space L^p .

We achieve this using the modulus of continuity $\omega(f; \delta_1, \delta_2)$, $\delta_1, \delta_2 > 0$ of $f \in L^p(\mathbb{R}^2)$ defined as follows:

$$\omega(f; \delta_1, \delta_2) = \sup_{\substack{0 < h_1 \leq \delta_1 \\ 0 < h_2 \leq \delta_2}} \left\{ \sup_{(y_1, y_2) \in \mathbb{R}^2} |f(y_1 + h_1, y_2 + h_2) - f(y_1, y_2)| \right\}.$$

First, we prove the following lemma, which we will use in the proof of the approximation theorem.

We shall apply the method used in [12].

Lemma 2 *Let $f \in C^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$, $1 \leq p \leq \infty$. Therefore*

$$\begin{aligned} & \left| \bar{A}(f; r, y_1, y_2) - f(y_1, y_2) \bar{A}(1; r, y_1, y_2) \right| \leq \frac{2r}{1+r^2} \exp\left(-\frac{1}{2} \cdot \frac{1-r^2}{1+r^2} (y_1^2 + y_2^2)\right) \\ & \times \left\{ \left(\frac{1-r^2}{1+r^2} + \frac{(r-1)^4 y_1^2}{(1+r^2)^2} \right)^{\frac{1}{2}} \sup_{(y_1, y_2) \in \mathbb{R}^2} \left| \frac{\partial f(y_1, y_2)}{\partial y_1} \right| + \left(\frac{1-r^2}{1+r^2} + \frac{(r-1)^4 y_2^2}{(1+r^2)^2} \right)^{\frac{1}{2}} \sup_{(y_1, y_2) \in \mathbb{R}^2} \left| \frac{\partial f(y_1, y_2)}{\partial y_2} \right| \right\}. \end{aligned}$$

for $0 < r < 1$ and all $(y_1, y_2) \in \mathbb{R}^2$.

Proof. Let $(y_1, y_2) \in \mathbb{R}^2$. be a fixed point and $f \in C^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$. We have:

$$f(z_1, z_2) - f(y_1, y_2) = \int_{y_1}^{z_1} \frac{\partial}{\partial u} f(u, z_2) du + \int_{y_2}^{z_2} \frac{\partial}{\partial v} f(y_1, v) dv$$

for all $(z_1, z_2) \in \mathbb{R}^2$. Let us denote:

$$\lambda_{y_1}(z_1, z_2) = \int_{y_1}^{z_1} \frac{\partial}{\partial u} f(u, z_2) du, \quad \tau_{y_2}(z_1, z_2) = \int_{y_2}^{z_2} \frac{\partial}{\partial v} f(y_1, v) dv.$$

Observe that:

$$\left| \lambda_{y_1}(z_1, z_2) \right| \leq |z_1 - y_1| \sup_{(y_1, y_2) \in \mathbb{R}^2} \left| \frac{\partial f(y_1, y_2)}{\partial y_1} \right|, \quad \left| \tau_{y_2}(z_1, z_2) \right| \leq |z_2 - y_2| \sup_{(y_1, y_2) \in \mathbb{R}^2} \left| \frac{\partial f(y_1, y_2)}{\partial y_2} \right|. \quad (3)$$

From (3) and Lemma 1, we obtain:

$$\bar{A}\left(\left|\lambda_{y_1}\right|; r, y_1, y_2\right) \leq \bar{A}\left(\left|\varphi_{1, y_1}\right|; r, y_1, y_2\right) \sup_{(y_1, y_2) \in \mathbb{R}^2} \left| \frac{\partial f(y_1, y_2)}{\partial y_1} \right|$$

$$\leq \frac{2r}{1+r^2} \left(\frac{1-r^2}{1+r^2} + \frac{(r-1)^4 y_1^2}{(1+r^2)^2} \right)^{\frac{1}{2}} \exp \left(-\frac{1}{2} \cdot \frac{1-r^2}{1+r^2} (y_1^2 + y_2^2) \right) \sup_{(y_1, y_2) \in \mathbb{R}^2} \left| \frac{\partial f(y_1, y_2)}{\partial y_1} \right|,$$

$$\bar{A} \left(|\tau_{y_2}|; r, y_1, y_2 \right) \leq \frac{2r}{1+r^2} \left(\frac{1-r^2}{1+r^2} + \frac{(r-1)^4 y_2^2}{(1+r^2)^2} \right)^{\frac{1}{2}} \exp \left(-\frac{1}{2} \cdot \frac{1-r^2}{1+r^2} (y_1^2 + y_2^2) \right)$$

$$\sup_{(y_1, y_2) \in \mathbb{R}^2} \left| \frac{\partial f(y_1, y_2)}{\partial y_2} \right|.$$

Hence:

$$\left| \bar{A}(f; r, y_1, y_2) - f(y_1, y_2) \bar{A}(1; r, y_1, y_2) \right| \leq \frac{2r}{1+r^2} \exp \left(-\frac{1}{2} \cdot \frac{1-r^2}{1+r^2} (y_1^2 + y_2^2) \right)$$

$$\times \left\{ \left(\frac{1-r^2}{1+r^2} + \frac{(r-1)^4 y_1^2}{(1+r^2)^2} \right)^{\frac{1}{2}} \sup_{(y_1, y_2) \in \mathbb{R}^2} \left| \frac{\partial f(y_1, y_2)}{\partial y_1} \right| + \left(\frac{1-r^2}{1+r^2} + \frac{(r-1)^4 y_2^2}{(1+r^2)^2} \right)^{\frac{1}{2}} \sup_{(y_1, y_2) \in \mathbb{R}^2} \left| \frac{\partial f(y_1, y_2)}{\partial y_2} \right| \right\}$$

and the proof of the lemma is completed.

We are now in a position to prove the approximation theorem.

Theorem 2 Let $f \in C(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$, $1 \leq p \leq \infty$. Therefore

$$\left| \bar{A}(f; r, y_1, y_2) - f(y_1, y_2) \bar{A}(1; r, y_1, y_2) \right|$$

$$\leq 6\omega \left(f; \sqrt{\frac{1-r^2}{1+r^2} + \frac{(r-1)^4}{(1+r^2)^2}} y_1^2, \sqrt{\frac{1-r^2}{1+r^2} + \frac{(r-1)^4}{(1+r^2)^2}} y_2^2 \right)$$

for $0 < r < 1$ and all $(y_1, y_2) \in \mathbb{R}^2$.

Proof. Let $(y_1, y_2) \in \mathbb{R}^2$ be a fixed point and f_{δ_1, δ_2} be the Steklov mean defined by:

$$f_{\delta_1, \delta_2}(y_1, y_2) = \frac{1}{\delta_1 \delta_2} \int_0^{\delta_1} \int_0^{\delta_2} f(y_1 + u, y_2 + v) du dv \quad \text{for } (y_1, y_2) \in \mathbb{R}^2, \delta_1, \delta_2 > 0.$$

From this definition, we conclude that:

$$f_{\delta_1, \delta_2}(y_1, y_2) - f(y_1, y_2) = \frac{1}{\delta_1 \delta_2} \int_0^{\delta_1} \int_0^{\delta_2} \Delta_{u,v} f(y_1, y_2) du dv,$$

$$\frac{\partial}{\partial y_1} f_{\delta_1, \delta_2}(y_1, y_2) = \frac{1}{\delta_1 \delta_2} \int_0^{\delta_2} \left(\Delta_{\delta_1, v} f(y_1, y_2) - \Delta_{0, v} f(y_1, y_2) \right) dv,$$

$$\frac{\partial}{\partial y_2} f_{\delta_1, \delta_2}(y_1, y_2) = \frac{1}{\delta_1 \delta_2} \int_0^{\delta_1} (\Delta_{u, \delta_2} f(y_1, y_2) - \Delta_{u, 0} f(y_1, y_2)) du,$$

where

$$\Delta_{u, v} f(y_1, y_2) = f(y_1 + u, y_2 + v) - f(y_1, y_2).$$

Hence, if $f \in C(R^2) \cap L^p(R^2)$, then $f_{\delta_1, \delta_2} \in C^1(R^2) \cap L^p(R^2)$. Moreover

$$\begin{aligned} \sup_{(y_1, y_2) \in R^2} |f_{\delta_1, \delta_2}(y_1, y_2) - f(y_1, y_2)| &\leq \omega(f; \delta_1, \delta_2), \\ \sup_{(y_1, y_2) \in R^2} \left| \frac{\partial}{\partial y_1} f_{\delta_1, \delta_2}(y_1, y_2) \right| &\leq 2\delta_1^{-1} \omega(f; \delta_1, \delta_2), \end{aligned} \quad (4)$$

for all $\delta_1, \delta_2 > 0$. Observe that

$$\begin{aligned} & \left| \bar{A}(f; r, y_1, y_2) - f(y_1, y_2) \bar{A}(1; r, y_1, y_2) \right| \\ & \leq \left| \bar{A}(f - f_{\delta_1, \delta_2}; r, y_1, y_2) \right| + \left| \bar{A}(f_{\delta_1, \delta_2}; r, y_1, y_2) - f_{\delta_1, \delta_2}(y_1, y_2) \bar{A}(1; r, y_1, y_2) \right| \\ & \quad + \left| f_{\delta_1, \delta_2}(y_1, y_2) - f(y_1, y_2) \right| \cdot \bar{A}(1; r, y_1, y_2), \quad (y_1, y_2) \in R^2, \delta_1, \delta_2 > 0. \end{aligned}$$

From Lemma 2 and (4) we obtain

$$\begin{aligned} & \left| \bar{A}(f_{\delta_1, \delta_2}; r, y_1, y_2) - f_{\delta_1, \delta_2}(y_1, y_2) \bar{A}(1; r, y_1, y_2) \right| \\ & \leq \frac{2r}{1+r^2} \exp\left(-\frac{1}{2} \cdot \frac{1-r^2}{1+r^2} (y_1^2 + y_2^2)\right) \left\{ 2\delta_1^{-1} \omega(f; \delta_1, \delta_2) \left(\frac{1-r^2}{1+r^2} + \frac{(r-1)^4 y_1^2}{(1+r^2)^2} \right)^{\frac{1}{2}} \right. \\ & \quad \left. + 2\delta_2^{-1} \omega(f; \delta_1, \delta_2) \left(\frac{1-r^2}{1+r^2} + \frac{(r-1)^4 y_2^2}{(1+r^2)^2} \right)^{\frac{1}{2}} \right\} \\ & \leq 2\omega(f; \delta_1, \delta_2) \left\{ \delta_1^{-1} \left(\frac{1-r^2}{1+r^2} + \frac{(r-1)^4 y_1^2}{(1+r^2)^2} \right)^{\frac{1}{2}} + \delta_2^{-1} \left(\frac{1-r^2}{1+r^2} + \frac{(r-1)^4 y_2^2}{(1+r^2)^2} \right)^{\frac{1}{2}} \right\}. \end{aligned}$$

Using (4) we have:

$$\left| f_{\delta_1, \delta_2}(y_1, y_2) - f(y_1, y_2) \right| \cdot \bar{A}(1; r, y_1, y_2) \leq \bar{A}(1; r, y_1, y_2) \omega(f; \delta_1, \delta_2) \leq \omega(f; \delta_1, \delta_2)$$

and

$$\begin{aligned} \left| \bar{A}(f - f_{\delta_1, \delta_2}; r, y_1, y_2) \right| &\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r K(r, y_1, z_1) K(r, y_2, z_2) |f(z_1, z_2) - f_{\delta_1, \delta_2}(z_1, z_2)| dz_1 dz_2 \\ &\leq \sup_{(y_1, y_2) \in \mathbb{R}^2} \left| f_{\delta_1, \delta_2}(y_1, y_2) - f(y_1, y_2) \right| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r K(r, y_1, z_1) K(r, y_2, z_2) dz_1 dz_2 \\ &\leq \bar{A}(1; r, y_1, y_2) \omega(f; \delta_1, \delta_2) \leq \omega(f; \delta_1, \delta_2). \end{aligned}$$

Finally, we get:

$$\begin{aligned} &\left| \bar{A}(f; r, y_1, y_2) - f(y_1, y_2) \bar{A}(1; r, y_1, y_2) \right| \\ &\leq 2\omega(f; \delta_1, \delta_2) \left\{ 1 + \delta_1^{-1} \left(\frac{1-r^2}{1+r^2} + \frac{(r-1)^4}{(1+r^2)^2} y_1^2 \right)^{\frac{1}{2}} + \delta_2^{-1} \left(\frac{1-r^2}{1+r^2} + \frac{(r-1)^4}{(1+r^2)^2} y_2^2 \right)^{\frac{1}{2}} \right\} \end{aligned}$$

for all $(y_1, y_2) \in \mathbb{R}^2, \delta_1, \delta_2 > 0$. Choosing:

$$\delta_1 = \left(\frac{1-r^2}{1+r^2} + \frac{(r-1)^4}{(1+r^2)^2} y_1^2 \right)^{\frac{1}{2}}, \quad \delta_2 = \left(\frac{1-r^2}{1+r^2} + \frac{(r-1)^4}{(1+r^2)^2} y_2^2 \right)^{\frac{1}{2}},$$

we obtain the desired estimation for \bar{A} .

From Theorem 2, we can derive the following result.

Corollary 1 *Let $f \in C(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$, $1 \leq p \leq \infty$. Then it holds*

$$\begin{aligned} \left| \bar{A}(f; r, y_1, y_2) - f(y_1, y_2) \right| &\leq 6\omega \left(f; \sqrt{\frac{1-r^2}{1+r^2} + \frac{(r-1)^4}{(1+r^2)^2} y_1^2}, \sqrt{\frac{1-r^2}{1+r^2} + \frac{(r-1)^4}{(1+r^2)^2} y_2^2} \right) \\ &\quad + |f(y_1, y_2)| \cdot \left| \bar{A}(1; r, y_1, y_2) - 1 \right| \end{aligned}$$

for $0 < r < 1$ and all $(y_1, y_2) \in \mathbb{R}^2$.

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