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The evaluation of control quality in automatic systems based on maximum error

Ocena jakości sterowania w układach automatyki na podstawie błędu maksymalnego

Abstract
The article presents an approach that uses the value of the maximum error to assess the quality of control in automatic control systems. The integral-square-error criterion is analysed together with the signals that enable its derivation. Signals with two constraints are considered.
Keywords: Control error, integral-square criterion, maximising signals, constraints of signals

Streszczenie
W artykule przedstawiono zastosowanie błędu maksymalnego do oceny jakości sterowania w układach automatyki. Analizowane jest kryterium całki z kwadratu błędu oraz sygnały umożliwiające jego wyznaczenie. Rozpatrywane są sygnały z jednym oraz dwoma ograniczeniami.
Słowa kluczowe: Błąd sterowania, kryterium całkowo-kwadratowe, sygnały maksymalizujące, ograniczenia sygnałów

1. Introduction

In order to evaluate the quality of control in many automatic systems, we need to determine the control error. Unless it is equal to zero, the value of the error does not provide an unequivocal answer to the question of whether the control system works well or whether it behaves poorly. This is because there is no reference to which one could compare the error. This paper proposes that a solution to this problem is to refer to the actual error as a percentage of the maximum error value for the chosen error criterion. This percentage provides a clear measure of the quality or effectiveness of our control. Many different criteria are available for measuring error, but the most commonly used is the integral-square-error criterion. The method of determining the shape of signals maximising the value of the integral-square-error criterion is derived in this paper for linear, time invariant control systems. The paper presents solutions referring to the existence and attainability of signals with two constraints imposed on them. These constraints relate to the amplitude and to the maximum rate of signal change. The last constraint is due to the need to match the dynamic properties of the signal to the dynamic properties of the control system under testing. It has been proved that signals maximising the integral-square criterion always reach one of the constraints imposed on them. For this reason, a signal with two constraints always takes the form of triangles or trapezoids while a signal with a amplitude constraint only corresponds to a signal of the 'bang-bang' type.

2. Errors in Automatic Control Systems (ACS)

In automatic control systems (Fig. 1) where P is a controlled plant and C its controller, the control error expressed in "s" domain has the form

$$E(s) = D(s) \cdot U(s) \tag{2.1}$$

where

$$D(s) = \frac{1}{1 + P(s) \cdot C(s)} \tag{2.2}$$

and can easily be transformed in "t" domain by the inverse Laplace transform

$$e(t) = L^{-1} D(s) U(s) \tag{2.3}$$

The quality of control in the automatic control system (ACS) is verified through the measurement of a chosen error criterion. The most commonly used criteria here are the integral criteria and maximum of over-overshoot criterion. The criterion which presents the value of the error surface integral is very popular for monotonic signals. Its value can be easy obtained for the error given in the form of $E(s)$. Then we have

$$I_1 = \int_0^\infty e(t)dt = E(s)\Big|_{s=0} \tag{2.4}$$

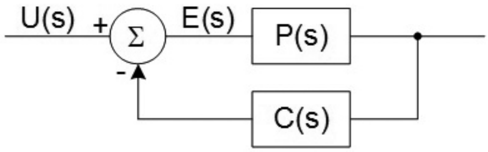

Fig. 1. Automatic control system

Criteria with time as a weight function are used in these cases where, for the initial period of control, substantial error values are acceptable and in the longer intervals, they should decrease

$$I_{t,k} = \int_0^{\infty} t^k e(t)dt = (-1)^k \left. \frac{d^k E(s)}{ds^k} \right|_{s=0} \tag{2.5}$$

For non-monotonic signals, the error $e(t)$ in integrals (2.4)–(2.5) is replaced with its module value $|e(t)|$.

The energy criterion presents the integral square error, which we can obtain in a simple way having spectral form $E(j\omega)$ of the transform $E(s)$

$$I_2 = \int_0^{\infty} [e(t)]^2 dt = \frac{1}{2\pi j} \int_{-j\omega}^{+j\omega} E(s) \cdot E(-s) ds = \frac{1}{\pi} \int_0^{\infty} |e(j\omega)|^2 d\omega \tag{2.6}$$

In [1], the method of calculating the integral (2.6) is presented based on the values of the coefficients of the numerator and the denominator of the error transfer function. For $E(s)$ given in the form (2.7)

$$E(s) = \frac{\sum a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_0}{\sum b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0} \quad m > n, \tag{2.7}$$

the first three values of I_2 are as follows

$$m = 1 \quad I_2 = \frac{a_0}{2b_1 b_0} \tag{2.8}$$

$$m = 2 \quad I_2 = \frac{a_1^2 + \frac{b_2}{b_0} a_0^2}{2b_2 b_1} \tag{2.9}$$

$$m = 3 \quad I_2 = \frac{b_1 a_2^2 + b_3(a_1^2 - 2a_0 a_2) + a_1^2 + \frac{b_1 b_2}{b_0} a_0^2}{2b_3(b_1 b_2 - b_0 b_3)} \tag{2.10}$$

From the non-integral error criteria, it is worth mentioning the overshoot error $e_{ovsh}(t)$ determining in % the ratio of the maximum value of the error $e_{\max}(t)$ to its steady value $e_{st}(t)$.

3. Controllers

In ACS three basic types of controllers are used. Proportional controller P, presented by means of constant gain K_p usually used at ACS in order to reduce the steady state error; integral controller $I = \frac{1}{T_i s}$ where T_i is integral constant, used at ASC in the case of astatic control and derivative controller $D = T_d s$, where T_d is derivative constant, which is not used separately because it amplifies the noise signals. These controllers in ACS, depending on the needs of control, are applied in different combination as a sum of *PI*, *PD* or *PID*.

Due to the minimisation of the error, for a given model of plant $P(s)$ and error criterion, the optimum values of controller parameters can be determined by calculating their derivatives relative to individual components

$$\frac{\partial I}{\partial K_p} = 0, \quad \frac{\partial I}{\partial T_d} = 0 \quad \frac{\partial I}{\partial T_i} = 0 \tag{3.1}$$

However, it is worth noting here that in some cases, the solution of the set of equation (3.1) can be difficult to solve and a correct result is not always easy to obtain.

4. Models of ACS

Since more and more physical experiments have been replaced with computer simulations, the synthesis of mathematical models has become an important problem in many types of ACS. The success of such a simulation is mainly conditioned by the correctness of the model. One could think therefore that the best solution would be to create higher order models of high accuracy. In practice, however, an advantage coming from the use of models of high and very high order is often illusory, since the analysis of their properties is more complicated, labour-consuming and costly. This fact leads to the replacement of the high order model by simplified models described by differential equations of a lower order [2, pp. 792–800; 3; 4, pp. 19–30]. In general, methods of model simplification can be divided into two groups. The first group includes methods based on the minimisation of a chosen form of error between the responses of the models. The second group is based on neglecting those poles which are furthest from the origin and retain only dominant poles. The retention of the dominant poles makes the response of the reduced model approximate that of the original, since the neglected poles make a highly insignificant contribution to the total response except at the beginning.

Papers [2, pp. 792–800; 3; 4, pp. 19–30] show also the synthesis of the methods of simplified models enabling the creation of a lower order model, which near the beginning of the time interval maps the model of the higher order with an error approaching zero. A great advantage of this method is that it does not require the computation of poles as in the case of other methods where it is often needed. The accuracy of models near the beginning of the time interval is especially important with reference to the systems working in a dynamic mode far from a steady state.

5. Maximum errors

It is easy to observe that the control errors (2.1), (2.3) can be determined if, and only if, the mathematical model of the system and the model of the controller are provided in advance and the control signal is known. Traditionally, these errors are determined for a standard input signal, most often in the form of a unit step function, Dirac's impulse or, less often, in the form of ramp or sinusoidal inputs. As a result, different error values are obtained, since they essentially depend on the input signal for which they are computed. This is a significant limitation of their usefulness because, in practice, real ACS are not excited by standard signals but usually by signals which are decidedly different from the standard signals. In such a situation, the received error values will be different from each other even for this same criterion and the lack of reference for them makes it impossible to assess the quality of ACS.

Our proposal is to apply the maximum error as a reference to the actual error in the estimation of control quality. It is worth noting that such an estimation is universal for any input signal in such a sense that the maximum error ensures that its value will always be greater or, at least, equal to the value resulting from a signal of any shape which could appear at the input of the ACS. Effectively, all the possible input signals to a real system are taken into consideration at the same time. Therefore, the value of maximum errors can create a reference valid for the chosen error criterion. However, the procedure of the determination of maximum errors requires special input signals to be used which warrant that the error values determined with them will always be higher than, or at least equal to, the value generated by any other signal. Below, we present an analytical method for determining the shapes of signals that maximise the integral-square-error as an example [3; 5, pp. 179–186].

Let us express for this purpose the error (2.1) by inner product

$$I_2(u) = \int_0^T [e(t)]^2\, dt = \langle Du, Du \rangle \quad u \in U \tag{5.1}$$

where in (5.1), the error $e(t)$ equals

$$e(t) = Du = \int_0^t k(t-\tau)u(\tau)d\tau \tag{5.2}$$

Let us assume that U is the set of input signals u piecewise C^1 over the interval $[0.T]$ and k is the impulse response of $D(s)$.

Let us additionally assume that

$$\forall 0 < b < c < T \;\; \exists x \in U : supp\, x \subset [b,c] \tag{5.3}$$

such that

$$I_2(x) > 0 \tag{5.4}$$

In order to match the dynamic properties of the signal to the dynamic properties of the ACS, let us define a set A of signals with imposed constraints on amplitude a and a rate of change ϑ[3], [6, pp. 550–553; 7, pp. 147–175]

$$A:\left\{u(t)\in U:\ u(t)\le a,\ \left|u'_{+}(t)\right|\le\vartheta,\ \left|u'_{-}(t)\right|\le\vartheta,\ t\in[0,T]\right\} \tag{5.5}$$

where $u'_{+}(t)$ and $u'_{-}(t)$ are increasing and decreasing derivative of $u(t)$, respectively.

Let us assume that $u_0(t)\in A$ fulfils the condition

$$I_2(u_0)=\sup\left\{I_2(u):\ u\in A\right\} \tag{5.6}$$

Then, we put the following theorem:

$$\forall t\in[0,T]\ \left|u_0(t)\right|=a\ \ or\ \left|u'_{0+}(t)\right|=\vartheta\ \ or\ \left|u'_{0-}(t)\right|=\vartheta \tag{5.7}$$

The proof (not direct) is as follows:

Suppose that (5.7) is not true. Then

$$\exists\varepsilon>0,\ \ \exists\, 0<b<c<T \tag{5.8}$$

such that

$$\left|u_0(t)\right|\le a-\varepsilon,\ \left|u'_{+}(t)\right|\le\vartheta-\varepsilon,\ \left|u'_{-}(t)\right|\le\vartheta-\varepsilon,\ t\in(b,c) \tag{5.9}$$

and for a small $r\in\Re$

$$u_0+rx\in A\ \ \forall r\in(-\delta,\delta) \tag{5.10}$$

From the optimum condition in u_0, it results that

$$I_2<u_0>\ \ge\ I_2<u_0+rx> \tag{5.11}$$

hence

$$<Du_0,Du_0>\ \ge\ <Du_0,Du_0>+<Drx,Du_0>+ \tag{5.12}$$

$$+<Du_0,Drx>+<Drx,Drx>$$

and

$$0\ \ge\ 2r<Du_0,Dx>+r^2<Dx,Dx> \tag{5.13}$$

Coming back to the record as in (5.11), we have

$$0\ \ge\ 2rI_2<u_0,x>+r^2I_2<x> \tag{5.14}$$

However, it can be easily seen that solution (5.14) represents a parabola crossing zero and directed upwards, so that the last inequality will never be fulfilled for $I_2(x)>0$, $r\in(-\delta,\delta)$.

As a result, we get a contradiction to the assumption that $I_2(x)>0$. We can therefore infer that $I(u_0)$ can fulfil condition (5.6) only if the input signal $u_0(t)$reaches one of the constraints given in (5.7). This means that the space of the solution of the signals $u_0(t)$ maximising functional (5.6) is therefore limited to the form of triangles with the slope inclination $|u'_{0+}(t)|=\vartheta$ or $|u'_{0-}(t)|=\vartheta$ or of trapezoids with slopes $|u'_{0+}(t)|=\vartheta$ and $|u'_{0-}(t)|=\vartheta$ and an amplitude of a.

Carrying out the proof in an identical manner to that above, it can be shown that if only one of the constraints is applied to the signal, either of amplitude a or of the rate of change ϑ, then the functional $I_2(u_0)$ reaches maximum if the signal reaches this constraint over the interval [0, T]. It means that if only amplitude constraint is imposed on signal $u(t)$ it will take the shape of 'bang-bang' signal with the amplitude $a = 1$ The analytical solution referring to the switching moments of the 'bang-bang' signal maximising integral-square-criterion is considered in detail in [2, pp. 792–800; 3].

6. Conclusions

The important achievement of this paper is the presentation of the possibility of an unequivocal evaluation of the control quality in automatic systems by means of signals maximising control errors. The paper also provides mathematical proof that such signals exist and shows the space in which they are available. Knowing the shapes of maximising signals in advance is especially useful if we waive the requirement of an exact calculation in favour of the approximate programs of computation as this narrows down the domain of search for the correct signal. This significantly increases the likelihood of obtaining the proper solution and considerably reduces the computation time. Good results among others give here, for example, the genetic algorithms program. Verification of the results can be easily obtained based on analysis of the signal energy density. It transpires that the maximum energy of the correctly determined signal is accumulated near the beginning and the end of the time scale. For non-maximising signals, energy density is completely asymmetric with respect to the centre of the time-scale plane [8, p. 224–232].

References

[1] James H.M., Nichols N.B., Phillips R.S., *Theory of Servomechanisms*, McGraw-Hill, New York, l947.

[2] Layer E., *Mapping Error of Linear Dynamic Systems Caused by Reduced-Order Model*, IEEE Trans. Inst. Meas., 2001,Vol. 50.

[3] Layer E., *Modelling of Simplified Dynamical Systems*, Springer-Verlag, Berlin Heidelberg, New York, 2002.

[4] Layer E., Piwowarczyk T., *Application of the generalized Fibinacci sequences to the simplification of mathematical models of linear dynamic systems*, Archives of Electrical Engineering, PAN, Warszawa 1999, vol. XLVIII.

[5] Layer E., *Non-standard input signals for the calibration and optimization of the measuring systems*, Measurement, Elsevier Science Ltd, London 2003, Vol. 34, Issue 2.
[6] Rutland N.K., *The Principle of Matching: Practical Conditions for Systems with Inputs Restricted in Amplitude and Rate of Change*, IEEE Trans. Autom. Control, 1994, Vol. 39.
[7] Zakian V., *Perspectives of the principle of matching and the method of inequalities,* Int. J. Control, 1996, Vol. 65.
[8] Tomczyk K., Layer E., *Energy density for signals maximizing the integral-square error*, Measurement, Elsevier Science Ltd, London 2016, Vol. 90.

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