

Krzysztof Tomczyk

Marek Sieja (msieja@pk.edu.pl)

Department of Automatic Control and Information Technology, Faculty of Electrical and Computer Engineering, Cracow University of Technology

RELATIONSHIP BETWEEN THE ABSOLUTE ERROR AND PARAMETER VALUES OF VOLTAGE OUTPUT ACCELEROMETER

ZALEŻNOŚĆ BŁĘDU BEZWZGLĘDNEGO OD WARTOŚCI PARAMETRÓW AKCELEROMETRU Z WYJŚCIEM NAPIĘCIOWYM

Abstract

The functional relationship between the absolute error and the parameter values of a voltage output accelerometer is presented in this paper. The theoretical basis for the mathematical model of the accelerometer and its standard, which is a reference for the error determination, is discussed. The polynomial function is applied for error approximation. Optimal orders of the polynomial along with their parameters and associated uncertainties are determined. The results presented in this paper concern the maximum errors obtained by exciting the accelerometer by the signal of constrained magnitude.

Keywords: absolute error, approximation function, voltage output accelerometer

Streszczenie

W artykule przedstawiono funkcyjną zależność błędu bezwzględnego od wartości parametrów akcelerometru z wyjściem napięciowym. Omówiono podstawy teoretyczne dotyczące matematycznych modeli akcelerometru i jego wzorca, który stanowi odniesienie do wyznaczenia błędu. Do aproksymacji błędów zastosowano funkcję wielomianową. Wyznaczono optymalne rzędy wielomianów wraz z ich parametrami i towarzyszącymi niepewnościami. Przedstawione w artykule wyniki dotyczą maksymalnych błędów uzyskanych w wyniku pobudzania akcelerometru sygnałem ograniczonym w amplitudzie.

Słowa kluczowe: błąd bezwzględny, funkcja aproksymująca, akcelerometr z wyjściem napięciowym

1. Introduction

The theoretical basis of calibrating instruments intended for dynamic measurements based on the criterion of absolute error is discussed in the papers [1–3]. The procedures for determining such errors are also presented here. These procedures are designed to determine the input signals that maximise the error at the output of the instrument. Only one constraint, referring to the magnitude, or two constraints relating simultaneously to the magnitude and the rate of change, are imposed on these signals.

In the above procedures, the basic task is to determine the relationship between the error and the time of instrument testing. This error has such a property that, for a specified time, its value stops increasing. This time depends on the operating bandwidth of the instrument and corresponds to the steady state of its impulse response. Examples of the examination of selected vibration sensors based on the solutions above are presented in [4, 5]. The maximising signals are determined along with the absolute error values. The study was conducted for chosen values of time with the assumed calculation step. However, these results do not provide information about error for other testing times and are only valid for the specific sensors.

Taking into account the above limitations, this paper proposes solutions based on polynomial functions that approximate the relationship between error and time and also the relationship between error and selected parameters of the instrument model. This approach is based on theoretical solutions presented in [6, 7]. The calculations refer to the voltage output accelerometer [8, 9], whose mathematical model contains three main parameters: the voltage sensitivity, damping ratio, and non-damped natural frequency. The studies of the relationship between error and model parameters are presented for the times corresponding to the steady state of error.

2. Model of Accelerometer and Its Standard

The mechanical construction of the voltage output accelerometer is shown in Fig. 1.

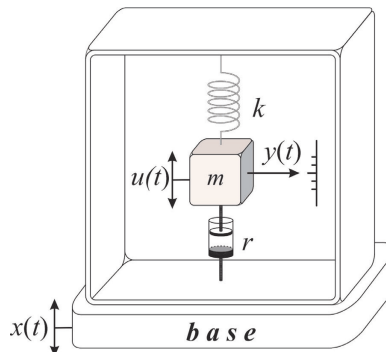


Fig. 1. Mechanical construction of the voltage output accelerometer

The construction of the voltage output accelerometer is represented by a differential equation, as follows:

$$m\ddot{u}(t) + r\dot{y}(t) + ky(t) = 0, \quad (1)$$

where $u(t)$, $y(t)$, $x(t)$, $m[\text{kg}]$, $r[\text{kg/s}]$, $k[\text{N/m}]$, $m\dot{y}(t)$, $ru(t)$ and $ku(t)$ are the absolute mass displacement, relative mass displacement, vibration (excitation), seismic mass, dumping coefficient, spring constant, moment of inertia, moment of dumping, and moment of elasticity, respectively.

Considering the absolute mass displacement in (1) as follows:

$$u(t) = x(t) + y(t), \quad (2)$$

we finally obtain

$$m\ddot{y}(t) + r\dot{y}(t) + ky(t) = -m\ddot{x}(t). \quad (3)$$

Presenting (3) in the domain s , we have:

$$K_a(s) = \frac{-S\omega_0^2}{s^2 + 2\beta\omega_0 s + \omega_0^2}, \quad (4)$$

where S is the voltage sensitivity, while:

$$\omega_0 = 2\pi f_0 = \sqrt{\frac{k}{m}} \quad (5)$$

and:

$$\beta = \frac{r}{2\sqrt{km}} \quad (6)$$

are the non-damped natural frequency and the damping ratio.

The state-space representation of (4) is:

$$K_a(s) = \mathbf{C}_a (s\mathbf{I} - \mathbf{A}_a)^{-1} \mathbf{B}_a, \quad (7)$$

where:

$$\mathbf{A}_a = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\beta\omega_0 \end{bmatrix}, \mathbf{B}_a = [0 \quad -S\omega_0^2]^T, \mathbf{C}_a = [1 \quad 0], \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (8)$$

are the state matrix, input vector, output vector, and identity matrix, respectively.

The model of the standard should meet the assumptions of non-distortion transformation. This means that in the range of the accelerometer work, the amplitude characteristic should be constant, while the phase characteristic should decrease linearly. These conditions fulfill the model of the Butterworth filter described by:

$$K_s(s) = \frac{n_M}{s^M + d_1 s^{M-1} + d_2 s^{M-2} + \dots + d_{M-1} s + d_M} = \frac{S}{\prod_{m=1}^M \left(\frac{s}{2\pi f_c} - p_m \right)}, \quad (9)$$

where the m -th pole is calculated by:

$$p_m = e^{\frac{j(2m+M-1)\pi}{2M}} \quad (10)$$

while f_c is the filter's cut-off frequency, which is equal to the accelerometer bandwidth, and M , n , and d_1, d_1, \dots, d_M are the order of the standard model, coefficient of the numerator, and coefficients of the denominator, respectively.

The transfer function (9) can be presented by:

$$K_s(s) = \mathbf{C}_s (s\mathbf{I} - \mathbf{A}_s)^{-1} \mathbf{B}_s, \quad (11)$$

where:

$$\mathbf{A}_s = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -d_M & -d_{M-1} & -d_{M-2} & \dots & -d_2 & -d_1 \end{bmatrix}, \quad (12)$$

$$\mathbf{B}_s = [0 \ 0 \ \dots \ 0 \ 0 \ n_M]^T, \mathbf{C}_s = [1 \ 0 \ \dots \ 0 \ 0 \ 0].$$

Let us introduce a new state-space representation based on (7)–(12), as below [2]:

$$K(s) = K_a(s) - K_s(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}, \quad (13)$$

where:

$$\mathbf{A} = \begin{bmatrix} A_a & 0 \\ 0 & A_s \end{bmatrix}, \mathbf{B} = [B_a \ B_s]^T, \mathbf{C} = [C_a \ -C_s]. \quad (14)$$

This space is a combination of the accelerometer and standard models.

The cut-off frequency f_c is calculated based on the parameters β , S and f_0 , by solving the equation which results from the amplitude characteristic of the accelerometer, as follows:

$$\lambda = \frac{S}{\sqrt{[1 - (f_c / f_0)^2]^2 + 4\beta^2 (f_c / f_0)^2}} \quad (15)$$

where:

$$\lambda = S + \frac{S}{100\%} \cdot \delta \quad (16)$$

and δ is the tolerance of the amplitude characteristic.

3. Determining the Absolute Error

If the input signal is constrained in magnitude,

$$|x(t)| \leq a \quad (17)$$

the absolute error can be calculated using a continuous formula:

$$D_t = a \int_0^T |k(t)| dt \quad (18)$$

or a discrete one:

$$D_n = a \Delta \sum_{n=0}^{N-1} |k[n]| \quad (19)$$

where:

$$k(t) = \mathbf{C} e^{\mathbf{A}t} \mathbf{B} \quad (20)$$

and:

$$k[n] = \mathbf{C} e^{\mathbf{A}\Delta n} \mathbf{B} \quad (21)$$

are the continuous and discrete impulse responses and Δ denotes the sampling interval of the state-space equations over the interval $[0, T]$.

The maximising signal with one constraint is determined by:

$$x_0(t) = a \operatorname{sign}\{k(T-t)\} \quad (22)$$

and:

$$x_0[n] = a \operatorname{sign}\{k[N-n]\}, \quad n=0, 1, \dots, N-1 \quad (23)$$

based on (20) and (21), where N denotes the number of samples $[1-3]$.

4. Determining the Approximate Polynomial

Let the vector:

$$\mathbf{Z} = [z_0, z_1, \dots, z_{J-1}]^T \quad (24)$$

be represented below by the assumed values of T , β , S , or f_0 . The vector of absolute errors that corresponds to \mathbf{Z} is:

$$\mathbf{D} = [D(x_0)_0, D(x_0)_1, \dots, D(x_0)_{J-1}]^T, \quad (25)$$

The polynomial of order α that approximates the error has the form:

$$d_i(z) = g_0 + g_1 z_i + g_2 z_i^2 + \dots + g_\alpha z_i^\alpha + \varepsilon_{i1}, \quad i=0, 1, \dots, J-1, \quad (26)$$

where g_0, g_1, \dots, g_{J-1} denote the coefficients of the polynomial and ε_1 is the approximation error [6].

In matrix form, we have:

$$\mathbf{D} = \Phi \mathbf{G} + \mathbf{E}_1 \quad (27)$$

where:

$$\Phi = \begin{bmatrix} 1 & z_0 & z_0^2 & \dots & z_0^\alpha \\ 1 & z_1 & z_1^2 & \dots & z_1^\alpha \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & z_{J-1} & z_{J-1}^2 & \dots & z_{J-1}^\alpha \end{bmatrix}. \quad (28)$$

and:

$$\mathbf{G} = [g_0, g_1, \dots, g_{J-1}]^T \quad (29)$$

$$\mathbf{E}_1 = [\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{J-1}]^T \quad (30)$$

We want to estimate the value \mathbf{G} that minimise $\mathbf{E}_1^T \mathbf{E}_1$. Hence:

$$\min \mathbf{E}_1^T \mathbf{E}_1 = (\mathbf{D} - \Phi \mathbf{G})^T (\mathbf{D} - \Phi \mathbf{G}) \quad (31)$$

Simplification of the matrices gives:

$$\mathbf{E}_1^T \mathbf{E}_1 = (\mathbf{D}^T - \mathbf{G}^T \Phi^T)(\mathbf{D} - \Phi \mathbf{G}) = \mathbf{D}^T \mathbf{D} - 2\mathbf{G}^T \Phi^T \mathbf{D} + \mathbf{G}^T \Phi^T \Phi \mathbf{G} \quad (32)$$

Taking the first derivative with respect to \mathbf{G} and equating to zero, we have:

$$-2\Phi^T \mathbf{D} + 2\Phi^T \Phi \mathbf{G} = 0 \quad (33)$$

Based on the first-order condition and after simplification, we have:

$$\Phi^T \Phi \tilde{\mathbf{G}} = \Phi^T \mathbf{D} \quad (34)$$

where $\tilde{\mathbf{G}}$ denotes the estimate of \mathbf{G} that minimises the error \mathbf{E}_1 .

Multiplying both sides of (34) by $(\Phi^T \Phi)^{-1}$, we finally have:

$$\tilde{\mathbf{G}} = (\Phi^T \Phi)^{-1} \Phi^T \mathbf{D} \quad (35)$$

which is the least-squares estimator for the polynomial approximation in matrix form.

The uncertainty of approximation [7] is given by:

$$u_A[d(z)] = \sqrt{\frac{(\Phi \tilde{\mathbf{G}} - \mathbf{D})^T (\Phi \tilde{\mathbf{G}} - \mathbf{D})}{J - \alpha}} = \sqrt{\frac{\sum_{j=0}^{J-1} (d[g_0, g_1, \dots, g_\alpha, z_j] - D(x_0)_j)^2}{J - \alpha}}. \quad (36)$$

The standard uncertainty for particular coefficients $g_0, g_1, \dots, g_\alpha$ is:

$$u_A(g_i) = u_A[d(z)] \cdot \sqrt{\Theta_{i,i}}, \quad i = 0, 1, \dots, \alpha, \quad (37)$$

where:

$$\Theta = (\Phi^T \Phi)^{-1}. \quad (38)$$

The relative uncertainty for the coefficients $g_0, g_1, \dots, g_\alpha$ is:

$$\delta(g_i) = \frac{u_A(g_i)}{|g_i|}, \quad i = 0, 1, \dots, \alpha. \quad (39)$$

The order of the polynomial (26) is determined by checking the chi-square test:

$$\chi = \frac{\sum_{j=0}^{J-1} (\Xi_j)^2}{\sigma^2(\Xi)}, \quad (40)$$

where:

$$\Xi = \sum_{j=0}^{J-1} (d(g_0, g_1, \dots, g_\alpha, z_j) - D(x_0)_j). \quad (41)$$

5. Example of Calculation

Below, based on Eqs. (1)–(23) and for the values of parameters β , S , and f_0 assumed in advance, the characteristic $d(T)$ was determined. An approximation of this characteristic was made based on Eqs. (24)–(41). For the assumed ranges of β and S with the steps Δ_β and Δ_S , the values of error D for constant values of $T=100\text{ms}$ and $f_0=1\text{kHz}$ were determined. The value of the amplitude characteristic tolerance was assumed to be equal to 10%.

Then, based on Eqs. (24)–(41), the functions $d(\beta)$ and $d(S)$, which approximate the error D , were determined. The orders of approximating polynomials were determined by checking the chi-square test.

5.1. Determining the Error D

The values of error D for $T=0, 0.01, \dots, 0.1\text{s}$ as well as $\beta=0.015$, $S=0.15\text{V}/(\text{ms}^{-2})$, and $f_0=1\text{kHz}$ are reported in Table 1. For such a coefficient of the accelerometer model, the value of the cut-off frequency f_c determined by (15) is equal to 931Hz.

Table 1. Values of error D

$T[\text{ms}]$	0	10	20	30	40	50
$D[\text{Vs}]$	0	0.580	0.808	0.896	0.931	0.944
$T[\text{ms}]$	60	70	80	90	100	
$D[\text{Vs}]$	0.949	0.951	0.952	0.953	0.953	

Figure 1 shows the approximation of the values of error D , while the coefficients and associated uncertainties for the sixth-order polynomial are reported in Table 2. This order is optimal because it ensures that the result of the chi-squares test will be less than 1.

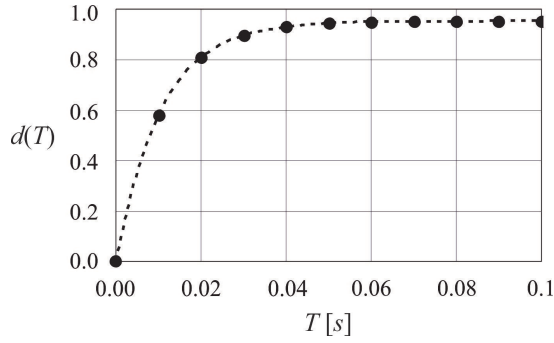


Fig. 2. Approximation of the error by the sixth-order polynomial

Table 2. Determined parameters and associated uncertainties for polynomial $d(T)$

g_0	g_1	g_2	g_3	g_4
$3.51 \cdot 10^{-4}$	85.5	$-3.44 \cdot 10^3$	$7.65 \cdot 10^4$	$-9.59 \cdot 10^5$
g_5	g_6	$u_A[d(T)][Vs]$	$u_A(g_0)$	$u_A(g_1)$
$6.32 \cdot 10^6$	$-1.70 \cdot 10^7$	$2.36 \cdot 10^{-3}$	$2.35 \cdot 10^{-3}$	0.879
$u_A(g_2)$	$u_A(g_3)$	$u_A(g_4)$	$u_A(g_5)$	$u_A(g_6)$
91.7	$3.71 \cdot 10^3$	$6.98 \cdot 10^4$	$6.13 \cdot 10^5$	$2.04 \cdot 10^6$
$\delta(g_0)[\%]$	$\delta(g_1)[\%]$	$\delta(g_2)[\%]$	$\delta(g_3)[\%]$	$\delta(g_4)[\%]$
$2.75 \cdot 10^{-3}$	0.0260	2.66	4.85	7.27
$\delta(g_5)[\%]$	$\delta(g_6)[\%]$	χ		
9.70	12.0	0.184		

From Figure 1, it follows that with the increase of time T , the error becomes constant. This corresponds to the steady state of the accelerometer impulse response, determined as the inverse Laplace transform of the transfer function (4). This impulse is presented in Fig. 2 and was obtained for $t = 0, \Delta_t, \dots, T$, where $\Delta_t = 10^{-5}$ s and $T = 100$ ms.

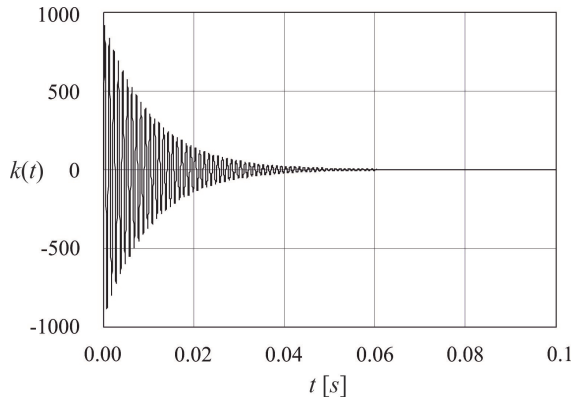


Fig. 3. Accelerometer impulse response for $\beta=0.015$, $S=0.15V/(ms^{-2})$, and $f_0=1kHz$

The values of error D determined for $\beta=0.015, \Delta_\beta, \dots, 0.015$, and $S=0.1, \Delta_S, \dots, 0.15$, where $\Delta_\beta=0.0005$ and $\Delta_S=0.005$, are reported in Table 3. The error was calculated for $T=100$ ms.

Table 3. Matrix (44) of the errors D

		β										
		0.0100	0.0105	0.0110	0.0115	0.0120	0.0125	0.0130	0.0135	0.0140	0.0145	0.0150
S	0.100	0.636	0.606	0.579	0.554	0.531	0.510	0.490	0.472	0.455	0.450	0.425
	0.105	0.701	0.668	0.638	0.610	0.585	0.562	0.540	0.520	0.502	0.484	0.468
	0.110	0.769	0.733	0.700	0.669	0.642	0.616	0.592	0.571	0.550	0.531	0.514
	0.115	0.840	0.800	0.764	0.731	0.701	0.673	0.647	0.623	0.601	0.580	0.561
	0.120	0.915	0.971	0.832	0.796	0.763	0.733	0.704	0.679	0.654	0.632	0.611
	0.125	0.992	0.945	0.903	0.864	0.828	0.795	0.764	0.736	0.710	0.685	0.662
	0.130	1.07	1.02	0.976	0.934	0.895	0.859	0.826	0.796	0.767	0.741	0.716
	0.135	1.16	1.10	1.05	1.01	0.965	0.926	0.891	0.858	0.827	0.799	0.772
	0.140	1.24	1.19	1.13	1.08	1.04	0.996	0.958	0.922	0.889	0.859	0.830
	0.145	1.33	1.27	1.21	1.16	1.11	1.07	1.03	0.989	0.954	0.921	0.890
	0.15	1.43	1.36	1.30	1.24	1.19	1.14	1.10	1.06	1.02	0.986	0.953

5.2. Determining the Approximate Functions

Below, we determine the functions $d(\beta)$ and $d(S)$ approximating the error based on Table 3 as well as for $S=0.15\text{V}/(\text{ms}^{-2})$ and $\beta=0.015$ respectively.

Figure 4 shows the approximation functions $d(\beta)$ and $d(S)$ while Tables 4 and 5 report the polynomial coefficients and associated uncertainties. The first function was approximated by a third-order polynomial while the second one was approximated by a second-order one.

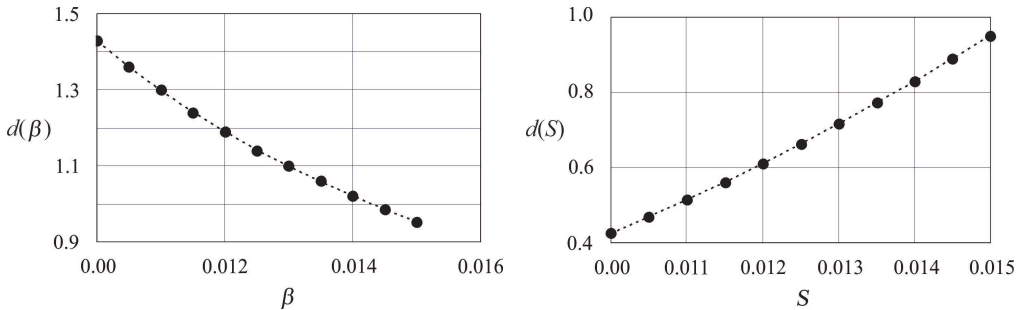


Fig. 4. Approximation functions $d(\beta)$ and $d(S)$

Table 4. Determined parameters and associated uncertainties for the polynomial $d(\beta)$

g_0	g_1	g_2	g_3	$u_A[d(T)][Vs]$
85.5	-656	$3.74 \cdot 10^4$	$-7.86 \cdot 10^5$	$1.50 \cdot 10^{-3}$
$u_A(g_0)$	$u_A(g_1)$	$u_A(g_2)$	$u_A(g_3)$	$\delta(g_0)[\%]$
0.292	71.1	$5.73 \cdot 10^3$	$1.53 \cdot 10^5$	0.0440
$\delta(g_1)[\%]$	$\delta(g_2)[\%]$	$\delta(g_3)[\%]$	χ	
0.190	15.3	19.4	0.0420	

Table 5. Determined parameters and associated uncertainties for the polynomial $d(S)$

g_0	g_1	g_2	$u_A[d(T)][Vs]$	$u_A(g_0)$
$7.47 \cdot 10^{-3}$	-0.0901	42.6	$3.07 \cdot 10^{-4}$	$6.50 \cdot 10^{-3}$
$u_A(g_1)$	$u_A(g_2)$	$\delta(g_0)[\%]$	$\delta(g_1)[\%]$	$\delta(g_2)[\%]$
0.105	0.420	7.21	0.247	0.986
χ				
0.449				

From Fig. 4, it follows that as β increases, the error decreases, whereas when S increases, the error increases. The approximation polynomials were determined by successively increasing their order from the first order and simultaneously checking the result of the χ test. This order was assumed to be an optimal solution for which the result of this test is less than one for the first time.

It should be added that with the increase of the non-damped natural frequency, the time of steady of the absolute error decreases. The steady values of error have the same value for the different frequencies, and therefore the determination of the relationship between the error and this parameter has been omitted from this paper.

6. Conclusions

The solutions obtained in this paper can be successfully extended to a wider group of measuring instruments with more parameters and for a wider range of their variability. Based on the solutions presented in Section 4, the functions approximating the error were obtained. Such functions will allow easy determination of the maximum dynamic errors based on the results of parametric identification of measuring instruments. Analogous functions can also be successfully determined for other error criteria, for example, for the integral square criterion.

References

- [1] Layer E., Gawedzki W., *Dynamika Aparatury Pomiarowej. Badania i Ocena*, PWN, Warszawa 1991.
- [2] Layer E., Tomczyk K., *Determination of Non-Standard Input Signal Maximizing the Absolute Error*, *Metrology and Measurement Systems*, Vol. 17, 2009, pp. 199–208.
- [3] Tomczyk K., *Applied Measurement Systems. Calibration of measuring systems based on maximum dynamic error*, Published by InTech, Rijeka 2012, pp. 189–210.
- [4] Tomczyk K., *Impact of uncertainties in accelerometer modeling on the maximum values of absolute dynamic error*, *Measurement*, Vol. 80, 2016, pp. 71–78.
- [5] Tomczyk K., Layer E., *Energy density for signals maximizing the integral-square error*, *Measurement*, Vol. 90, 2016, pp. 224–232.
- [6] Foszcz D., *Estymacja Parametrów Funkcji Regresji Metodą Klasyczną oraz Metodami Bootstrapowymi*, *Górnictwo i Geoinżynieria*, Vol. 3(1), 2016, pp. 67–78.
- [7] Dorozhovets M., *Ocena wpływu oddziaływan systematycznych na parametry niepewności podczas aproksymacji metoda najmniejszych kwadratów*, *PAK*. Vol. 12, 2006, pp. 22–25.
- [8] Sun X.T., Jing X.J., Xu J., Cheng L., *A Quasi-Zero-Stiffness-Based Sensor System in Vibration Measurement*, *IEEE Transactions on Industrial Electronics*, Vol. 61, no. 10, 2014, pp. 5606–6114.
- [9] Di Natale C., Ferrari V., Ponzoni A., Sberveglieri G., Ferrari M., *Sensors and Microsystems*, Springer, 2013.