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An abstract nonlocal functional-differential second order evolution Cauchy problem

Abstrakcyjne nielokalne funkcjonalno-różniczkowe ewolucyjne zagadnienie Cauchy'ego rzędu drugiego

Abstract

The aim of the paper is to prove two theorems on the existence and uniqueness of mild and classical solutions of a semilinear functional-differential evolution second order equation together with nonlocal initial conditions. The theory of strongly continuous cosine families of linear operators in a Banach space is applied. The paper is based on publications [1–12] and is a generalization of paper [7].

Keywords: nonlocal, second order, functional-differential, evolution Cauchy problem, Banach spaces

Streszczenie

W artykule udowodniono dwa twierdzenia o istnieniu i jednoznaczności całkowych i klasycznych rozwiązań semiliniowego funkcjonalno-różniczkowego zagadnienia ewolucyjnego Cauchy'ego rzędu drugiego z nielokalnymi warunkami początkowymi. W tym celu zastosowano teorię rodziny cosinus liniowych operatorów w przestrzeni Banacha. Artykuł bazuje na publikacjach [1–12] i jest uogólnieniem publikacji [7].

Słowa kluczowe: nielokalne, rzędu drugiego, funkcjonalno-różniczkowe, zagadnienie ewolucyjne Cauchy'ego, przestrzenie Banacha

1. Introduction

In this paper, we consider the abstract nonlocal semilinear functional-differential second order evolution Cauchy problem

$$u''(t)=Au(t)+f\big(t,u(t),u(a_1(t)),\ldots,u(a_m(t)),u'(t)\big),\ t\in(0,T], \tag{1.1}$$

$$u(0)=x_0, \tag{1.2}$$

$$u'(0)+\sum_{i=1}^{p}h_i u_i(t_i)=x_i, \tag{1.3}$$

where A is a linear operator from a real Banach space X into itself, $u:[0,T]\to X$, $f:[0,T]\times X^{m+2}\to X$, $a_i:[0,T]\to[0,T]$ $(i=1,2,\ldots,m)$, x_0, $x_1\in X$, $h_i\in\mathbb{R}$ $(i=1,2,\ldots,p)$ and

$$0<t_1<t_2<\ldots<t_p\le T.$$

We prove two theorems on the existence and uniqueness of mild and classical solutions of the problem (1.1) - (1.3). For this purpose, we apply the theory of strongly continuous cosine families of linear operators in a Banach space. We also apply the Banach contraction theorem and the Bochenek theorem (see Theorem 1.1 in this paper).

Let A be the same linear operator as in (1.1). We will need the following assumption:

Assumption (A_1). Operator A is the infinitesimal generator of a strongly continuous cosine family $\{C(t): t\in\mathbb{R}\}$ of bounded linear operators from X into itself.

Recall that the infinitesimal generator of a strongly continuous cosine family $C(t)$ is the operator $A: X\supset D(A)\to X$ defined by

$$Ax:=\frac{d^2}{dt^2}C(t)x\Big|_{t=0},\ x\in D(A),$$

where

$$D(A):=\{x\in X: C(t)x \text{ is of class } C^2 \text{ with respect to } t\}.$$

Let

$$E:=\{x\in X: C(t)x \text{ is of class } C^1 \text{ with respect to } t\}.$$

The associated sine family $\{S(t): t\in\mathbb{R}\}$ is defined by

$$S(t)x:=\int_0^t C(s)x\,ds,\ x\in X,\ t\in\mathbb{R}.$$

From Assumption (A_1), it follows (see [12]) that there are constants $M\ge 1$ and $\omega\ge 0$ such that

$$\|C(t)\|\le Me^{\omega|t|} \text{ and } \|S(t)\|\le Me^{\omega|t|} \text{ for } t\in\mathbb{R}.$$

We also will use the following assumption:

Assumption (A_2). The adjoint operator A^* is densely defined in X^*, that is, $\overline{D(A^*)}=X^*$.

The paper is based on publications [1 - 12] and is a generalization of paper [7] in this sense that, now, a more general functional – differential problem is considered than in [7].

For convenience of the reader, a result obtained by J. Bochenek (see [3]) will be presented here.

Let us consider the Cauchy problem

$$u''(t)=Au(t)+h(t),\ t\in(0,T], \tag{1.4}$$

$$u(0)=x_0, \tag{1.5}$$

$$u'(0)=x_1. \tag{1.6}$$

A function $u:[0,T]\to X$ is said to be a classical solution of the problem (1.4) – (1.6) if

$$u\in C^1([0,T],X)\cap C^2((0,T],X), \tag{a}$$

$$u(0)=x_0 \text{ and } u'(0)=x_1, \tag{b}$$

$$u''(t)=Au(t)+h(t) \text{ for } t\in(0,T]. \tag{c}$$

Theorem 1.1. *Suppose that:*

(i) Assumptions (A_1) and (A_2) *are satisfied,*

(ii) $h:[0,T]\to X$ *is Lipschitz continuous,*

(iii) $x_0\in D(A)$ *and* $x_1\in E$.

Then u given by the formula

$$u(t)=C(t)x_0+S(t)x_1+\int_0^t S(t-s)h(s)ds,\ t\in[0,T],$$

is the unique classical solution of the problem (1.4) – (1.6).

2. Theorem on mild solutions

A function u belonging to $C^1([0,T],X)$ and satisfying the integral equation

$$u(t)=C(t)x_0+S(t)x_1-S(t)\left(\sum_{i=1}^{p}h_iu(t_i)\right)$$

$$+\int_0^t S(t-s)f\big(s,u(s),u(a_1(s)),\ldots,u(a_m(s)),u'(s)\big)ds,\ t\in[0,T],$$

is said to be a mild solution of the nonlocal Cauchy problem (1.1) – (1.3).

Theorem 2.1. *Suppose that:*

(i) *Assumption* (A_1) *is satisfied,*

(ii) $a_i:[0,T]\to[0,T]$ $(i=1,2,\dots,m)$ *are continuous on* $[0,T]$, $f:[0,T]\times X^{m+2}\to X$ *is continuous with respect to the first variable* $t\in[0,T]$ *and there exists a positive constant* L_1 *such that*

$$\left\|f\left(s,z_1,z_2,\dots,z_{m+2}\right)-f\left(s,\tilde{z}_1,\tilde{z}_2,\dots,\tilde{z}_{m+2}\right)\right\|\le L_1\sum_{i=1}^{m+2}\left\|z_i-\tilde{z}_i\right\| \text{ for } s\in[0,T],\ \tilde{z}_i,\tilde{z}_i\in X$$

$$(i=1,2,\dots,m+2),$$

(iii) $2C\left((m+1)TL_1+\sum_{i=1}^{p}|h_i|\right)<1,$

where $C:=\sup\left\{\|C(t)\|+\|S(t)\|+\|S'(t)\|:t\in[0,T]\right\},$

(iv) $x_0\in E$ *and* $x_1\in X$.

Then, the nonlocal Cauchy problem $(1.1)-(1.3)$ *has a unique mild solution.*

Proof. Let the operator $F:C^1([0,T],X)\to C^1([0,T],X)$ be given by

$$(Fu)(t)=C(t)x_0+S(t)x_1-S(t)\left(\sum_{i=1}^{p}h_iu(t_i)\right)$$

$$+\int_0^t S(t-s)f\left(s,u(s),u(a_1(s)),\dots,u(a_m(s)),u'(s)\right)ds,\ t\in[0,T].$$

Now, we shall show that F is a contraction on the Banach space $C^1([0,T],X)$ equipped with the norm

$$\|w\|_1:=\sup\left\{\|w(t)\|+\|w'(t)\|:t\in[0,T]\right\}.$$

To do this, observe that

$$\left\|(Fw)(t)-(F\tilde{w})(t)\right\|=\left\|S(t)\left(\sum_{i=1}^{p}h_i\left(\tilde{w}(t_i)-w(t_i)\right)\right)\right.$$

$$+\int_0^t S(t-s)\Big(f\left(s,w(s),w(a_1(s)),\dots,w(a_m(s)),w'(s)\right)$$

$$\left.-f\left(s,\tilde{w}(s),\tilde{w}(a_1(s)),\dots,\tilde{w}(a_m(s)),\tilde{w}'(s)\right)\Big)ds\right\|\le C\left(\sum_{i=1}^{p}|h_i|\right)\|w-\tilde{w}\|_1$$

$$+\int_0^t\|S(t-s)\|L_1\Big(\|w(s)-\tilde{w}(s)\|+\|w(a_1(s))-\tilde{w}(a_1(s))\|+$$

$$\dots+\|w(a_m(s))-\tilde{w}(a_m(s))\|+\|w'(s)-\tilde{w}'(s)\|\Big)ds$$

$$\le C\left((m+1)TL_1 + \sum_{i=1}^{p} |h_i| \right) \| w - \tilde{w} \|_1$$

and

$$\|(Fw)'(t) - (F\tilde{w})'(t)\| = \left\| S'(t) \left(\sum_{i=1}^{p} h_i \left(\tilde{w}(t_i) - w(t_i) \right) \right) \right.$$

$$+ \int_0^t C(t-s) \Big(f\big(s, w(s), w(a_1(s)), \ldots, w(a_m(s)), w'(s)\big)$$

$$\left. - f\big(s, \tilde{w}(s), \tilde{w}(a_1(s)), \ldots, \tilde{w}(a_m(s)), \tilde{w}'(s)\big) \Big) ds \right\| \le C \left(\sum_{i=1}^{p} |h_i| \right) \| w - \tilde{w} \|_1$$

$$+ \int_0^t \|C(t-s)\| L_1 \Big(\|w(s) - \tilde{w}(s)\| + \|w(a_1(s)) - \tilde{w}(a_1(s))\| + \ldots + \|w(a_m(s)) - \tilde{w}(a_m(s))\|$$

$$+ \|w'(s) - \tilde{w}'(s)\| \Big) ds \le C\left((m+1)TL_1 + \sum_{i=1}^{p} |h_i| \right) \| w - \tilde{w} \|_1, \; t \in [0,T].$$

Consequently,

$$\|Fw - F\tilde{w}\|_1 \le 2C\left((m+1)TL_1 + \sum_{i=1}^{p} |h_i| \right) \| w - \tilde{w} \|_1 \text{ for } w, \tilde{w} \in C^1([0,T], X).$$

Therefore, in space $C^1([0,T],X)$, there is the only one fixed point of F and this point is the mild solution of the nonlocal Cauchy problem (1.1) – (1.3). So, the proof of Theorem 2.1 is complete.

Remark 2.1. The application of a Bielecki norm in the proof of Theorem 2.1 does not give any benefit.

3. Theorem about classical solutions

A function $u:[0,T] \to X$ is said to be a classical solution of the problem (1.1) – (1.3) if

$$u \in C^1([0,T], X) \cap C^2((0,T], X), \tag{a}$$

$$u(0) = x_0 \text{ and } u'(0) + \sum_{i=1}^{p} h_i u(t_i) = x_1, \tag{b}$$

$$u''(t) = Au(t) + f\big(t, u(t), u(a_1(t)), \ldots, u(a_m(t)), u'(t)\big) \text{ for } t \in [0,T]. \tag{c}$$

Theorem 3.1. *Suppose that:*

(i) *Assumptions* (A_1) *and* (A_2) *are satisfied, and* $a_i:[0,T] \to [0,T]$ $(i=1,2,\ldots,m)$ are *of class* C^1 *on* $[0,T]$.

(ii) *There exists a positive constant L_2 such that*

$$\left\|f\left(s,z_1,z_2,\dots,z_{m+2}\right)-f\left(\tilde{s},\tilde{z}_1,\tilde{z}_2,\dots,\tilde{z}_{m+2}\right)\right\|\le L_2\left(\left|s-\tilde{s}\right|+\sum_{i=1}^{m+2}\left\|z_i-\tilde{z}_i\right\|\right)$$

for $s,\tilde{s}\in[0,T],\ z_i,\tilde{z}_i\in X\ (i=1,2,\dots,m+2).$

(iii) $2C\left((m+1)TL_2+\sum_{i=1}^{p}\left|h_i\right|\right)<1.$

(iv) $x_0\in E$ *and* $x_1\in X$.

Then, the nonlocal Cauchy problem (1.1) - (1.3) has a unique mild solution u. Moreover, if

$$x_0\in D(A),\ \ x_1\in E\ \ \text{and}\ \ u(t_i)\in E\ \ (i=1,2,\dots,p),$$

and if there exists a positive constant κ such that

$$\left\|u\left(a_i(s)\right)-u\left(a_i(\tilde{s})\right)\right\|\le\kappa\left\|u(s)-u(\tilde{s})\right\|\ \text{for}\ s,\tilde{s}\in[0,T]\ (i=1,2,\dots,m)$$

then u is the unique classical solution of nonlocal problem (1.1) – (1.3).

Proof. Since the assumptions of Theorem 2.1 are satisfied, the nonlocal Cauchy problem (1.1) – (1.3) possesses a unique mild solution, which is denoted by u.

Now, we shall show that u is the classical solution of problem (1.1) – (1.3).

Firstly, we shall prove that u, $u(a_i(\cdot))$ $(i=1,2,\dots,m)$ and u' satisfy the Lipschitz condition on $[0,T]$. Let t and $t+h$ be any two points belonging to $[0,T]$. Observe that

$$\begin{aligned}u(t+h)-u(t)&=C(t+h)x_0+S(t+h)x_1-S(t+h)\left(\sum_{i=1}^{p}h_iu(t_i)\right)\\&+\int_0^{t+h}S(t+h-s)f\left(s,u(s),u\left(a_1(s)\right),\dots,u\left(a_m(s)\right),u'(s)\right)ds\\&-C(t)x_0-S(t)x_1+S(t)\left(\sum_{i=1}^{p}h_iu(t_i)\right)\\&-\int_0^{t}S(t-s)f\left(s,u(s),u\left(a_1(s)\right),\dots,u\left(a_m(s)\right),u'(s)\right)ds.\end{aligned}$$

Since

$$C(t)x_0+S(t)\left(x_1-\sum_{i=1}^{p}h_iu(t_i)\right)$$

is of class C_2 in $[0,T]$, there are $C_1>0$ and $C_2>0$ such that

$$\left\| \left(C(t+h)-C(t)\right)x_0+\left(S(t+h)-S(t)\right)\left(x_1-\sum_{i=1}^{p}h_iu(t_i)\right) \right\| \le C_1|h|$$

and

$$\left\| \left(\left(C(t+h)-C(t)\right)x_0\right)'+\left(\left(S(t+h)-S(t)\right)\left(x_1-\sum_{i=1}^{p}h_iu(t_i)\right)\right)' \right\| \le C_2|h|.$$

Hence

$$\|u(t+h)-u(t)\|\le C_1|h|$$

$$+\left\|\int_0^t S(s)\Big(f\big(t+h-s,u(t+h-s),u(a_1(t+h-s)),\ldots,u(a_m(t+h-s)),u'(t+h-s)\big)\right.$$

$$\left. -f\big(t-s,u(t-s),u(a_1(t-s)),\ldots,u(a_m(t-s)),u'(t-s)\big)\Big)ds\right\|$$

$$+\left\|\int_t^{t+h} S(s)\Big(f\big(t+h-s,u(t+h-s),u(a_1(t+h-s)),\ldots,u(a_m(t+h-s)),u'(t+h-s)\big)ds\right\|$$

$$\le C_1|h|+\int_0^t Me^{\omega T}L_2\Big(|h|+\|u(t+h-s)-u(t-s)\|+\|u(a_1(t+h-s))-u(a_1(t-s))\|+\ldots$$

$$\ldots+\|u(a_m(t+h-s))-u(a_m(t-s))\|+\|u'(t+h-s)-u'(t-s)\|\Big)ds+Me^{\omega T}N|h|,$$

where

$$N:=\sup\left\{\left\|f\big(s,u(s),u(a_1(s)),\ldots,u(a_m(s)),u'(s)\big)\right\|:s\in[0,T]\right\}.$$

From this, we obtain

$$\|u(t+h)-u(t)\|\le C_3|h|+C_4\int_0^t\Big(\|u(s+h)-u(s)\|+\|u'(s+h)-u'(s)\|\Big)ds. \tag{3.1}$$

Moreover, we have

$$u'(t)=\left(C(t)x_0+S(t)\left(x_1-\sum_{i=1}^{p}h_iu(t_i)\right)\right)'+\int_0^t C(t-s)f\big(s,u(s),u(a_1(s)),\ldots,u(a_m(s)),u'(s)\big)ds.$$

From the above formula, we obtain, analogously,

$$\|u'(t+h)-u'(t)\|\le C_5|h|+C_6\int_0^t\Big(\|u(s+h)-u(s)\|+\|u'(s+h)-u'(s)\|\Big)ds. \tag{3.2}$$

By inequalities (3.1) and (3.2), we get

$$\|u(t+h)-u(t)\|+\|u'(t+h)-u'(t)\|\le C_*|h|+C_{**}\int_0^t\left(\|u(s+h)-u(s)\|+\|u'(s+h)-u'(s)\|\right)ds.$$

From Gronwall's inequality, we have

$$\|u(t+h)-u(t)\|+\|u'(t+h)-u'(t)\|\le \tilde{C}|h|, \tag{3.3}$$

where $\tilde{C}$ is a positive constant.

By (3.3), it follows that u, $u(a_i(\cdot))$ $(i=1,2,\ldots,m)$ and u' satisfy the Lipschitz conditions on $[0,T]$. This implies that the mapping

$$[0,T]\ni t\to f\left(t,u(t),u(a_1(t)),\ldots,u(a_m(t)),u'(t)\right)\in X$$

also satisfies the Lipschitz condition.

The above property of f together with the assumptions of Theorem 3.1 imply, by Theorem 1.1 and by Theorem 2.1, that the linear Cauchy problem

$$v''(t)=Av(t)+f\left(t,u(t),u(a_1(t)),\ldots,u(a_m(t)),u'(t)\right),\ t\in[0,T],$$

$$v(0)=x_0,$$

$$v'(0)=x_1-\sum_{i=1}^{p}h_iu(t_i)$$

has a unique classical solution v such that

$$v(t)=C(t)x_0+S(t)\left(x_1-\sum_{i=1}^{p}h_iu(t_i)\right)$$

$$+\int_0^t S(t-s)f\left(s,u(s),u(a_1(s)),\ldots,u(a_m(s)),u'(s)\right)ds=u(t),\ t\in[0,T].$$

Consequently, u is the unique classical solution of the semilinear Cauchy problem (1.1) – (1.3) and, therefore, the proof of Theorem 3.1 is complete.

References

[1] Balachandran K., Ilamaran S., *Existence and uniqueness of mild and strong solutions of a semilinear evolution equation with nonlocal conditions,* Indian J. Pure Appl. Math. 25.4, 1994, 411–418.

[2] Bednarz A., Byszewski L., *On abstract nonlocal Cauchy problem*, Technical Transactions, 1-NP/2013, 11–17.

[3] Bochenek J., *An abstract nonlinear second order differential equation,* Ann. Polon. Math. 54.2, 1991, 155–166.

[4] Bochenek J., *Cosine operator functions and condition (F),* Selected Problems of Mathematics, Cracow University of Technology Monographs, Anniversary Issue 6, 1995, 13–24.

[5] Bochenek J., Winiarska T., *Second order evolution equations with parameter,* Ann. Polon. Math. 59.1, 1994, 41–52.

[6] Byszewski L., *On some application of the Bochenek theorem,* Univ. Iagel. Acta. Math. 45, 2007, 147–153.

[7] Byszewski L., *An abstract nonlocal functional-differential second order evolution problem,* Technical Transactions 1/2017, 147–154.

[8] Byszewski L., Winiarska T., *An abstract nonlocal second order evolution problem,* Opuscula Mathematica 32.1, 2012, 75–82.

[9] Cholewa J.W., Dłotko T., *Global Attractors in Abstract Parabolic Problems,* London Mathematical Society Lecture Notes, Series 278, Cambridge University Press, Cambridge 2000.

[10] Fattorini H. O., *Ordinary differential equation in linear topological spaces,* J. Differential Equations 5, 1968, 72–105.

[11] Nagy B., *Cosine operator functions and the abstract Cauchy problem,* Periodica Math. Hungary 7, 1976, 213–217.

[12] Travis C.C., Webb G.F., *Cosine family and abstract nonlinear second order differential equations,* Acta Math. Hungar. 32, 1978, 75–96.

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