TECHNICAL TRANSACTIONS 1/2017
CZASOPISMO TECHNICZNE 1/2017
ELECTRICAL ENGINEERING
DOI: 10.4467/2353737XCT.17.008.6105

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Digital filters: hermitian, antihermitian, unitary and their applications

Filtry cyfrowe: hermitowski, antyhermitowski, unitarny i ich zastosowania

Abstract

The paper presents the relation between the three types of noncausal digital filters: unitary, Hermitian and antihermitian filters. A decomposition has been made of a causal filter into a Hermitian and unitary filter cascade, and the use of this decomposition to calculate the electric power quality of receivers has been highlighted. Reference has also been made to the analogy between the set of unitary filters and the unit circle in the Gaussian plane.

Keywords: unitary and Hermitian digital filters, operators, convolution

Streszczenie

W artykule podano związek między trzema typami nieprzyczynowych filtrów cyfrowych: filtrami unitarnymi, hermitowskimi i antyhermitowskimi. Dokonano też rozkładu filtru przyczynowego na kaskadę filtru hermitowskiego i unitarnego, oraz zwrócono uwagę na zastosowanie tego rozkładu do jakościowo-energetycznej oceny odbiorników energii elektrycznej. Nawiązano też do analogii zbioru filtrów unitarnych i okręgu jednostkowego na płaszczyźnie Gaussa.

Słowa kluczowe: unitarne i hermitowskie filtry cyfrowe, operatory, splot

1. Introduction

A digital filter is a transducer of a signal $\{x_n\}_{n=-\infty}^{\infty}$ that operates according to the convolution operator rules

$$(Ax)_n = \sum_m A_m x_{n-m} \qquad n,m \in (-\infty,\infty)$$

The digital filter identifier is a string of weights $\{A_n\}_{n=-\infty}^{\infty}$ or its 'z' form

$$A(z) = \sum_n A_n z^n$$

where:

$$A_n = \frac{1}{n!}\left[\frac{d^n A(z)}{dz^n}\right]_{z=0} \qquad \text{for n} \geq 0$$

$$A_n = \frac{1}{2\pi j}\oint A(z)\, d(\ln z) \qquad \text{for n} < 0$$

the integral is taken over the unit circle. In this way, the digital filter has two equivalent identifiers

$$A(z) \leftrightarrow \{A_n\}$$

For the two filters $A(z)$ and $B(z)$ the Borel theorem of convolution takes place:

$$A(z)B(z) \leftrightarrow \sum_m A_m x_{n-m} \doteqdot \{A_n\} * (B_n)$$

The filter is called causal (past) if

$$A_n = 0 \qquad \text{for } n < 0$$

The filter A^* is coupled to A if for any two signals x and y it meets the equation

$$(Ax,y) = (x,A^* y)$$

where the dot product of signals or filters is defined as

$$(A,B) = \sum_{n=-\infty}^{\infty} A_n B_n \text{ or } (A,B) = \sum_{n=0}^{N-1} A_n B_n$$

for signals or filters respectively and infinite or finite time support (in particular N-periodic).

For a digital filter realizing convolution-type operator it holds that

$$A^*_n = A_{-n} \leftrightarrow A^*(z) = A\left(z^{-1}\right)$$

The filter is called Hermitian if $A^* = A$, i.e. when

$$A_{-n} = A_n \leftrightarrow A\left(z^{-1}\right) = A(z)$$

and Antihermitian if $A^* = -A$, i.e. when

$$A_{-n} = -A_n \leftrightarrow A\left(z^{-1}\right) = -A(z)$$

The filter is stable if and only if

$$\sum_{n=-\infty}^{\infty} |A_n| < \infty \leftrightarrow |A(z)| < \infty \text{ for } z\text{: } |z| \le 1$$

1.1. N-periodic Filter

An N-periodic filter is defined by the formula:

$$\tilde{A}_n = A_n + \sum_{p=1}^{\infty}\left(A_{n+pN} + A_{n-pN}\right) \qquad n \in \{0,1,\ldots,N-1\}$$

It holds that $\tilde{A}^*_n = \tilde{A}_{N-n}$ for $n \in \{0,1,\ldots,N-1\}$

In particular

$$\tilde{A}_n = A_n + \sum_{p=1}^{\infty}\left(A_{n+pN}\right)$$

for the causal filter

$$\tilde{A}_n = A_n + \sum_{p=1}^{\infty}\left(A_{pN+n} + A_{pN-n}\right)$$

for the Hermitian filter

$$\tilde{A}_n = A_n + \sum_{p=1}^{\infty}\left(A_{pN+n} - A_{pN-n}\right) \text{ for } n \in \{0,1,\ldots,N-1\}$$

for the antihermitian filter.

A periodic filter operates on the N-periodic input signal $\{x_n\}_{n=0}^{N-1}$ according to the cyclic convolution operator

$$\left(\tilde{A}x\right)_n = \sum_{m=0}^{n} \tilde{A}_{n-m} x_m + \sum_{m=n+1}^{N-1} \tilde{A}_{n-m+N} x_m \quad \text{for } n \in \{0,1,\dots,N-1\}$$

2. Functional filters

The functional filter is defined as a digital filter formed from the digital filter A using some transforming function

$$f : A \to f(A)$$

Commonly it is assumed that both A and $f(A)$ are causal filters. The following notation is also assumed:

$$\left\{(f(A))_n\right\} \leftrightarrow (f(A)(z)) = \sum_n (f(A))_n z^n = f\left(\sum_n A_n z^n\right)$$

Thus the first two weights of the filter can easily be determined

$$(f(A))_0 = f(A_0) \qquad (f(A))_1 = \left[\frac{df(A)}{dA}\right]_{z=0} A_1$$

Further, the weight of the function filter can be determined directly according to the formula

$$(f(A))_n = \frac{1}{n!}\left[\frac{d^n f(A)(z)}{dz^n}\right]_{z=0}$$

However, it seems scarcely useful due to the presence of the composite function. Often it is better to use a recursive method, which is more effective for some function $f(A)$. More about the functional filters can be found in [6]. In this paper two important functional filters are used: $f(A) = \sqrt{A}$ (the square root filter) and $f(A) = A^{-1}$ (the inverse filter).

The square root filter meets the convolution equation:

$$\sum_{m=0}^{n} \left(\sqrt{A}\right)_{n-m} \left(\sqrt{A}\right)_m = A_n$$

Thus a recursive formula can be derived

$$\left(\sqrt{A}\right)_n = \frac{A_n}{2\sqrt{A_0}} - \frac{1}{2\sqrt{A_0}} \sum_{m=0}^{n-1} \left(\sqrt{A}\right)_{n-m} \left(\sqrt{A}\right)_m \tag{1}$$

for $n = 2, 3, \dots$

$$\left(\sqrt{A}\right)_0 = \sqrt{A_0}, \; \left(\sqrt{A}\right)_1 = \frac{A_1}{2\sqrt{A_0}}$$

The inverse-filter weights can also be determined by the convolution equation

$$\sum_{m=0}^{n}\left(A^{-1}\right)_{n-m}A_m = I_n \doteqdot \begin{cases}1 & n=0\\ 0 & n\neq 0\end{cases}$$

its solution is also a recursive formula

$$\left(A^{-1}\right)_n = -\left(A_0\right)^{-1}\sum_{m=1}^{n}\left(A^{-1}\right)_{n-m}A_m \quad \text{for } n=2,3,\ldots \tag{2}$$
$$\left(A^{-1}\right)_0 = -\left(A_0\right)^{-1}$$

For example, the weight of the digital filter of impedance Z, which is the inverse operator of an electrical two-terminal circuit Y admittance, is determined by the recursive formula

$$z_n = -z_0\sum_{m=1}^{n} z_{n-m}y_m\,,\; z_0 = \left(y_0\right)^{-1}$$

The function filter is analytical over the conjugation operation only if $\left[f(A)\right]^* = f\left(A^*\right)$. It is not difficult to prove that this condition is met for any function having the Taylor expansion form with respect to A, and that the inverse function to the analytic one over the inverse operation also fulfils this condition. Thus, in particular

$$\left(\sqrt{A}\right)^* = \sqrt{A^*} \;\text{ and }\; \left(A^{-1}\right)^* = \left(A^*\right)^{-1}$$

2.1. Hermitian, antihermitian and unitary filters

The weights representing the convolution-type filters are listed below:

$$\begin{array}{llll} H^* = H & H_{-n} = H_n & \text{Hermitian} & \text{(equal)} \\ A^* = -A & A_{-n} = -A_n & \text{antihermitian} & \text{(reverse)} \\ U^* = U^{-1} & U_{-n} = (U^{-1})_n & \text{unitary} & \text{(inverse)} \end{array} \tag{3}$$

It is worth noting that the unitary filter is a generalization of a complex number over the unit circle. Just as with any digital filter, the unitary filter can be decomposed into the sum of Hermitian and antihermitian filters

$$U = H + A \text{ ie. } U^* = H - A \tag{4}$$

thus

$$H = \frac{1}{2}\left(U + U^*\right) = \frac{1}{2}\left(U + U^{-1}\right) \tag{5}$$

$$A = \frac{1}{2}\left(U - U^*\right) = \frac{1}{2}\left(U - U^{-1}\right)$$

Hence (convolution filters commute!)

$$(H+A)(H-A)=H^2-A^2=1 \tag{6}$$

using the general composition formula of filters *A* with conjugated B

$$\left(AB^*\right)_n=\sum_m A_{n+m}B_m \tag{7}$$

it is obtained successively:

$$\left(H^2\right)_n=\sum_m H_{n+m}H_m$$

$$\left(A^2\right)_n=-\sum_m A_{n+m}A_m$$

$$\left(UU^*\right)_n=\sum_m U_{n+m}U_m=I_n \qquad m\in(-\infty,\infty)$$

It is also easy to obtain the following expression for the dot products and filter norms:

$$(H,A)=\left(A^*H,I\right)=\left(A^*,H^*\right)=-(H,A)=0$$

Thus *H* and *A* are orthogonal filters.
In addition:

$$\|U\|^2=(U,U)=\left(U^*U,I\right)=(I,I)=I$$

$$\|H\|^2=(H,\mathrm{H})=\frac{1}{4}\left(U+U^*,\mathrm{U}+\mathrm{U}^*\right)=\frac{1}{2}\left(1+\left(U^2,I\right)\right) \tag{8}$$

$$\|A\|^2=(A,\mathrm{A})=\frac{1}{4}\left(U-U^*,\mathrm{U}-\mathrm{U}^*\right)=\frac{1}{2}\left(1-\left(U^2,I\right)\right) \tag{9}$$

therefore

$$\|H\|^2+\|A\|^2=1 \tag{10}$$

3. 'Pole' decomposition of causal filter Y

The logic diagram can be written as follows:

$$
\begin{array}{lll}
Y^{*}=HU^{-1} & H^{2}=YY^{*} & \left(H^{2}\right)_{n}=\sum_{m=0}^{\infty} Y_{n+m} Y_{m} \\
\uparrow & & \\
\boxed{Y=HU} \longrightarrow & U^{2}=YZ^{*} & \left(U^{2}\right)_{n}=\sum_{m=0}^{\infty} Y_{n+m} Z_{m} \\
\downarrow & & \\
Y^{-1}=Z=H^{-1}U^{-1} & U^{-2}=ZY & \left(U^{-2}\right)_{n}=\sum_{m=0}^{\infty} Z_{n+m} Y_{m} \\
Z^{*}=H^{-1}U & &
\end{array}
$$

thus

$$
\begin{array}{ll}
H=\sqrt{Y}\left(\sqrt{Y}\right)^{*} & H_{n}=\sum_{m=0}^{\infty}\left(\sqrt{Y}\right)_{n+m}\left(\sqrt{Y}\right)_{m} \\
U=\sqrt{Y}\left(\sqrt{Z}\right)^{*} \longrightarrow & U_{n}=\sum_{m=0}^{\infty}\left(\sqrt{Y}\right)_{n+m}\left(\sqrt{Z}\right)_{m} \\
U^{-1}=\sqrt{Z}\left(\sqrt{Y}\right)^{*} & U_{n}^{-1}=\sum_{m=0}^{\infty}\left(\sqrt{Z}\right)_{n+m}\left(\sqrt{Y}\right)_{m}
\end{array}
\tag{11}
$$

A square-root-causal filter is used in the above formulas.
Combining equation (8), (9) (11) we obtain

$$
\left(U^{2},I\right)=\left(Z^{*}Y,I\right)=\left(Y,Z\right) \tag{12}
$$

and then

$$
\|H\|^{2}=1+\frac{1}{2}\sum_{n=1}^{\infty} Y_{n} Z_{n} \tag{13}
$$

$$
\|A\|^{2}=-\frac{1}{2}\sum_{n=1}^{\infty} Y_{n} Z_{n} \tag{14}
$$

The (6) is a hyperbolic relation. If *H, A* operators have the meaning of real numbers and the identity *I* operator the meaning of the ordinary number equals one then in the *H, A* coordinates this relation will be a hyperbole equation with asymptotes of $A=\pm H$ (Fig.1).

A comparison between the operator relation (6) and the hyperbole equation is presented below:

$$
\sum_{m} H_{n+m} H_{m}-\left(-\sum_{m} A_{n+m} A_{m}\right)=\begin{cases}1 & n=0 \\ 0 & n\neq 0\end{cases} \tag{15}
$$

$$
H^{2}-A^{2}=1 \tag{16}
$$

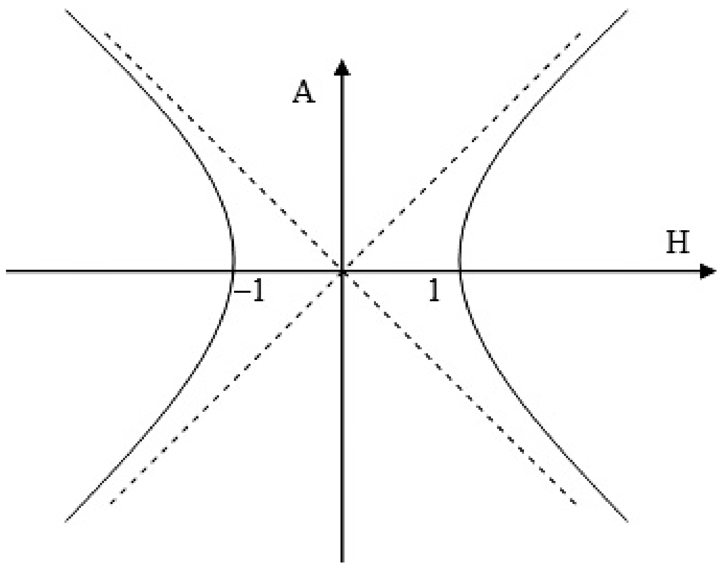

Fig. 1. A hyperbole in real-number coordinates of the equation $H^2 - A^2 = 1$ that corresponds to the operator relation (6)

It is also worth noting that for $n = 0$ the relation (15) becomes

$$\sum_m H_m^2 + \sum_m A_m^2 = 1$$

which coincides with the Pythagorean condition (10).

The Hermitian, antihermitian and unitary filters operate on signal $\{x_n\}_{n=-\infty}^{\infty}$ according to the convolution formulas

$$(Hx)_n = H_0 x_n + \sum_{m=1}^{n} H_m \left(x_{n-m} + x_{n+m}\right) \tag{17}$$

$$(Ax)_n = \sum_{m=1}^{n} A_m \left(x_{n-m} - x_{n+m}\right) \tag{18}$$

$$(Ux)_n = U_0 x_n + \sum_{m=1}^{n} \left(U_m x_{n-m} + \left(U^{-1}\right)_m x_{n+m}\right) \tag{19}$$

The weights of the periodic filters have been calculated using the following formulas (the first two were presented in the introduction):

$$\tilde{H}_n = H_n + \sum_{p=1}^{\infty} \left(H_{pN+n} + H_{pN-n}\right) \tag{20}$$

$$\tilde{A}_n = A_n + \sum_{p=1}^{\infty} \left(A_{pN+n} - A_{pN-n}\right) \tag{21}$$

$$\tilde{U}_n = U_n + \sum_{p=1}^{\infty}\left(U_{pN+n} + U_{pN-n}\right) \text{ for } n \in \{0,1,\dots,N-1\} \tag{22}$$

The periodic filter operates on a periodic input signal $\{x_n\}_{n=0}^{N-1}$ according to (common to *H, A and U* operators) the periodic convolution formula:

$$\left(\tilde{G}_x\right)_n = \sum_{m=0}^{n} x_m \tilde{G}_{n-m} + \sum_{m=n+1}^{N-1} x_m \tilde{G}_{n-m+N} \tag{23}$$

Unitary filter can also have the exponential form

$$U = e^{\phi} = \sum_{n=0}^{n} \frac{\phi^n}{n!} \quad : \quad \phi^* = -\phi \tag{24}$$

Any power of the ϕ operation, used in the series, can be determined with the recursive convolution formula:

$$\left(\phi^{p+1}\right)_n = \phi^p \phi$$

which has the following implementation

$$\left(\phi^{p+1}\right)_n = \sum_m \left(\phi^p\right)_{n-m} \phi_m = \sum_{m=1}^{\infty}\left[\left(\phi^p\right)_{n-m} + \left(\phi^p\right)_{n+m}\right]\phi_m \tag{25}$$

for $n = 2, 3, \dots$

Then the components of the decomposition (4) can be interpreted as

$$H = ch(\phi) = \sum_{n=0}^{\infty} \frac{\phi^{2n}}{(2n)!}; \qquad A = sh(\phi) = \sum_{n=0}^{\infty} \frac{\phi^{2n+1}}{(2n+1)!} \tag{26}$$

thus

$$e^{\phi} = ch(\phi) + sh(\phi) \rightarrow ch(\phi) = \frac{1}{2}\left(e^{\phi} + e^{-\phi}\right);\ sh(\phi) = \frac{1}{2}\left(e^{\phi} - e^{-\phi}\right) \tag{27}$$

Formulas (6) and (10) take then the form in which the first formula is analogous to the well-known hyperbolic function in the common "numerical" theory:

$$ch(\phi)^2 - sh(\phi)^2 = I \tag{28}$$

$$\left\|ch(\phi)\right\|^2 + \left\|sh(\phi)\right\|^2 = 1 \tag{29}$$

From (24) it also follows that the relation between the *U* operator and the Euler's number 'e' is

$$e^I = e^I \rightarrow \left(e^I\right)_n = e\begin{cases}1 \\ 0\end{cases} = \begin{cases} e & n = 0 \\ 0 & n \neq 0\end{cases} \tag{30}$$

4. Conclusion

The decomposition of a unitary convolution operator into the Hermitian and antihermitian operators presented in the article (4) is a significant generalization in digital filter theory of Euler's famous formula:

$$e^{j\phi} = ch(j\phi) + sh(j\phi)$$

where ϕ– real number, $j = \sqrt{-1}$
commonly used for complex numbers.

The unitary digital filter-operator itself corresponds to a complex number in the Gauss plane over the unit circle. On the other hand the decomposition (11), i.e. $Y = HU$ is an extension on digital filters of the ordinary admittance of the complex two-terminal circuit decomposition into a module and an argument, hence the name "pole decomposition", while the decomposition

$$Y = H\left(ch(\phi) + sh(\phi)\right)$$

is an operational generalization of the decomposition form of that complex admittance into a conductance and a susceptance:

$$Y = |Y|\left(ch(j\phi) + sh(j\phi)\right),\ |Y| > 0,\ \phi - \text{real numbers}$$

The formula (14) in the form of a negative half of the incomplete scalar product of the two causal mutually-inverse digital filters of an impedance and an admittance may be of practical significance. It is a counterpart of the susceptance measure of a digital receiver and thus it is an indicator of harmful reactive power of that receiver.

This allows, in particular, to make a comparative evaluation of the i-th receiver in relation to the others connected in parallel by simply calculating the fraction

$$\frac{\|sh\phi\|_i^2}{\sum_n \|sh\phi\|_n^2}$$

It is obvious that the "pure resistance" receiver impedance corresponds to the digital filter:

$$z = z_0 I \rightarrow y = y_0 I \ \ : \ \ z_0 y_0 = 1$$

and finally combining expressions the following beautiful theorem can be formulated

$$\|sh(\phi)\| = 0 \leftrightarrow \begin{matrix}(Y,Z)=1 \\ YZ = I\end{matrix}$$

which we can rewrite using digital filters weights in the following form

$$\sum_{n=-\infty}^{\infty}\left[(sh\phi)_n\right]^2=1\leftrightarrow \begin{array}{l}\displaystyle\sum_{n=0}^{\infty}Y_nZ_n=1\\ \displaystyle\sum_{m=0}^{n}Y_{n-m}Z_m=I_n=\begin{cases}1 & n=0\\ 0 & n\neq 0\end{cases}\end{array}$$

5. Calculation example

The evaluation of the reactive power factor $\|sh\phi\|^2$ of the wave-impedance operator:

$$\hat{z}=\sqrt{\frac{a-z}{b-z}}; \qquad a>1, \qquad b>1$$

Since it is an integral-derivative operator of order ½ its inversion has the form:

$$y=\sqrt{\frac{b-z}{a-z}}$$

The time samples of the above operators are given by a power series expansion with respect to the variable 'z' [1, 2, 4, 5]

$$\hat{z}_n=\sqrt{\frac{a}{b}}a^{-n}\alpha_n; \qquad y_n=\sqrt{\frac{b}{a}}a^{-n}\beta_n; \quad n=0,1,2,...$$

$$\alpha_n=\sum_{m=0}^{n}\left(\frac{a}{b}\right)^m k_{n-m}h_m; \qquad \beta_n=\sum_{m=0}^{n}\left(\frac{a}{b}\right)^m h_{n-m}k_m$$

Where $\{k_n\}$, $\{h_n\}$ – the universal sequences determined by the formulas:

$$k_m=-\frac{1}{2}\frac{1}{4}\frac{3}{6}\frac{5}{8}...\frac{2m-3}{2m} \qquad \text{derivative}$$

$$h_m=-\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{7}{8}...\frac{2m-1}{2m} \qquad \text{integral}$$

while $k_0=h_0=1$

Therefore, the reactive power factor is obtained from:

$$\|\sinh\phi\|^2=-\frac{1}{2}\sum_{m=1}^{\infty}y_m\hat{z}_m=-\frac{1}{2}\sum_{m=1}^{\infty}a^{-2m}\alpha_m\beta_m$$

Let us take into account the first two terms of this expansion and set $x = a/b$, the first component is positive

$$\frac{1}{8}a^{-2}(x-1)^2=\frac{1}{8}\left(\frac{a-b}{ab}\right)^2$$

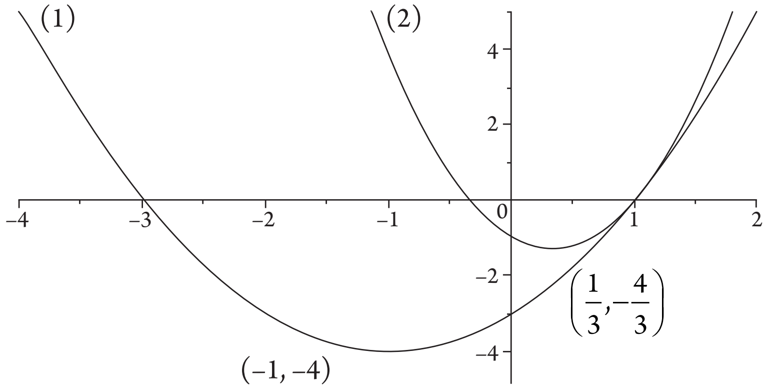

Fig. 2. Graph of the two factors of the second-order series expansion term of the reactive power factor $||\mathrm{sh}\phi||^2$

and the second is:

$$\frac{1}{2}a^{-4}\frac{1}{64}\left(\overset{(1)}{3x^2-2x-1}\right)\left(\overset{(2)}{x^2+2x-3}\right)$$

Figure 2 shows that the second term of the series expansion (as a product of 1 and 2) is also positive, therefore the reactive power factor of the wave impedance is a positive function.

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