TECHNICAL TRANSACTIONS 1/2017
CZASOPISMO TECHNICZNE 1/2017
MATHEMATICS
DOI: 10.4467/2353737XCT.17.013.6110

Ludwik Byszewski (lbyszews@pk.edu.pl)
Institute of Mathematics, Faculty of Physics, Mathematics and Computer Science, Cracow University of Technology

Tadeusz Wacławski
Institute of Electrical Engineering and Computer Science, Faculty of Electrical and Computer Engineering, Cracow University of Technology

On the uniqueness of solutions to parabolic semilinear problems under nonlocal conditions with integrals

O jednoznaczności rozwiązań parabolicznych zagadnień z nielokalnymi warunkami z całkami

Abstract

The uniqueness of classical solutions to parabolic semilinear problems together with nonlocal initial conditions with integrals, for the operator $\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial}{\partial x_j}\right)+c(x,t)-\frac{\partial}{\partial t},\ x=(x_1,...,x_n)$, in the cylindrical domain $D:=D_0\times(t_0,t_0+T)\subset\Re^{n+1}$, where $t_0\in\Re$, $0<T<\infty$, are studied. The result requires that the nonlocal conditions with integrals be introduced.

Keywords: parabolic problems, semilinear equation, nonlocal initial condition with integral, cylindrical domain, uniqueness of solutions

Streszczenie

W artykule omówiono jednoznaczność klasycznych rozwiązań parabolicznych semiliniowych zagadnień z nielokalnymi początkowymi warunkami z całkami dla operatora $\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial}{\partial x_j}\right)+c(x,t)-\frac{\partial}{\partial t},\ x=(x_1,...,x_n)$, w walcowym obszarze $D:=D_0\times(t_0,t_0+T)\subset\Re^{n+1}$, gdzie $t_0\in\Re$, $0<T<\infty$. Wynik polega na tym, że zostały wprowadzone warunki nielokalne z całkami.

Słowa kluczowe: zagadnienia paraboliczne, równanie semiliniowe, nielokalny warunek początkowy z całką, obszar walcowy, jednoznaczność rozwiązań

1. Introduction

In this paper we prove two theorems on the uniqueness of classical solutions to parabolic semilinear problems, for the equation

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u(x,t)}{\partial x_j}\right) + c(x,t)u(x,t) - \frac{\partial u(x,t)}{\partial t} = f(x,t,u(x,t)), \qquad (1)$$

$$= f(x,t,u(x,t)),$$

$$(x,t) \in D := D_0 \times (t_0, t_0+T) \subset \Re^{n+1},$$

where $t_0 \in \Re$, $0 < T < \infty$. The coefficients $a_{ij}\,(i,j=1,...n)$, c and the function f are given. The nonlocal initial condition considered in the paper is of the form

$$u(x,t_0) + \frac{h(x)}{T}\int_{t_0}^{t_0+T} u(x,\tau)d\tau = f_0(x), \;\; x \in D_0,$$

where $|h(x)| \le 1$ for $x \in D_0$.

The result obtained is a continuation of the results given by Rabczuk in [5], by Chabrowski in [3], by Brandys in [1] and by the first author in [1] and [2].

In monograph [5], Rabczuk gives two uniqueness criteria for classical solutions for initial – boundary problems to the equation

$$\sum_{i=1}^{n} \frac{\partial^2 u(x,t)}{\partial x_i^2} - \frac{\partial u(x,t)}{\partial t} = f(x,t,u(x,t)), \;\; x \in D_0 \subset \Re^n, \;\; t > 0.$$

In paper [3], Chabrowski studies nonlocal problems for the equation

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u(x,t)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u(x,t)}{\partial x_i} + c(x,t)u(x,t) - \frac{\partial u(x,t)}{\partial t} =$$

$$= f(x,t), \;\; x \in D_0 \subset \Re^n, \;\; t \in (0,T).$$

The nonlocal initial condition, considered in [3], is of the form

$$u(x,0) + \sum_i \beta_i(x)u(x,T_i) = \psi(x), \;\; x \in D_0,$$

where $t_0 \in (0, T)$ and was introduced in this form as the first by Chabrowski.

In publication [2], two uniqueness criteria for classical solutions for equation (1) together with the nonlocal condition $u(x,t_0) + h(x)u(x,t_0+T) = f_0(x), \;\; x \in D_0$, are studied.

2. Preliminaries

The notation, definitions and assumptions from this section are valid throughout this paper.

We will need the set $\Re_- := (-\infty, 0]$.

Let t_0 be a real number, $0 < T < \infty$ and $x = (x_1, ..., x_n) \in \Re^n$.

Define the domain (see [1] or [2])

$$D := D_0 \times (t_0, t_0 + T),$$

where D_0 is an open and bounded domain in $\Re^n$ such that the boundary ∂D_0 satisfies the following conditions:

(i) If $n \geq 2$ then ∂D_0 is a union of a finite number of surface patches of class C^1 which have no common interior points but have common boundary points.

(ii) If $n \geq 3$ then all the edges of ∂D_0 are sums of finite numbers of $(n-2)$ – dimensional surface patches of class C^1.

Assumption (A_1). $a_{ij}, \dfrac{\partial a_{ij}}{\partial x_s} \in C(\bar{D}, \Re)$ $(i, j, s = 1, \ldots, n)$, where $a_{ij} = a_{ij}(x,t)$ for $(x,t) \in \bar{D}$ $(i, j = 1, \ldots, n)$; $a_{ij}(x,t) = a_{ji}(x,t)$ for $(x,t) \in D$ $(i, j = 1, \ldots, n)$ and $\sum_{i,j=1}^{n} a_{ij}(x,t)\lambda_i \lambda_j \geq 0$ for arbitrary $(x,t) \in$ D and $(\lambda_1, \ldots, \lambda_n) \in \Re^n$; $c \in C(\bar{D}, \Re_-)$.

Assumption (A_2).

(i) $f : \bar{D} \times \Re \ni (x,t,z) \rightarrow f(x,t,z) \in \Re$, $f \in C(\bar{D} \times \Re, \Re)$, $\dfrac{\partial f}{\partial z} \in C(\bar{D} \times \Re, \Re)$ and $\dfrac{\partial f(x,t,z)}{\partial z} > 0$ for $(x,t) \in \bar{D}$, $z \in \Re$;

(ii) $f_1 : \partial D_0 \times [0,T] \rightarrow \Re$;

(ii′) $k \in C(\partial D_0 \times [0,T], \Re_-)$;

(iii) $f_0 : D_0 \rightarrow \Re$.

Assumption (A_3). $h \in C(\bar{D}_0, \Re)$ and $|h(x)| \leq 1$ for $x \in D_0$.

Let $C^{2,1}(\bar{D}, \Re)$ be the space of all $w \in C(\bar{D}, \Re)$ such that $\dfrac{\partial w}{\partial x_i}, \dfrac{\partial^2 w}{\partial x_i \partial x_j} \in C(\bar{D}, \Re)$ for $i, j = 1, \ldots, n$ and $\dfrac{\partial w}{\partial t} \in C(\bar{D}, \Re)$.

The symbols L and P are reserved for two operators given by the formulas

$$(Lw)(x,t) := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t) \frac{\partial w(x,t)}{\partial x_j} \right) \tag{2}$$

and

$$(Pw)(x,t) := (Lw)(x,t) + c(x,t)w(x,t) - \frac{\partial w(x,t)}{\partial t} \tag{3}$$

for $w \in C^{2,1}(\bar{D}, \Re)$, $(x,t) \in \bar{D}$.

By n_x where $x \in \partial D_0$, we denote the interior normal to ∂D_0 at x. In short, we denote, also, n_x by n.

Let $C^{2,1}(\bar{D},\mathfrak{R})$, $x \in \partial D_0$ and $t \in [t_0, t_0 + T]$. The expression

$$\frac{du(x,t)}{d\upsilon(x_0,t)} := \sum_{i=1}^{n} \frac{\partial u(x_0,t)}{\partial x_i} \sum_{j=1}^{n} a_{ij}(x_0,t)\cos(n_{x_0},x_j) \tag{4}$$

is called the transversal derivative of the function u at the point (x_0, t). If it does not lead to misunderstanding the transversal derivative $\frac{du(x,t)}{d\upsilon(x_0,t)}$ will be denoted by $\frac{d}{d\upsilon}u(x_0,t)$ or by $\frac{du}{d\upsilon_{x_0}}$.

For the given functions $a_{ij}(i,j=1,...,n)$ and c satisfying Assumption (A_1) and for the given functions f, f_1, f_0 and h satisfying Assumptions (A_2) (i)–(iii) and (A_3) the first Fourier's semilinear nonlocal problem in D consists in finding a function $u \in C^{2,1}(\bar{D},\mathfrak{R})$ satisfying the equation

$$(Pu)(x,t) = f(x,t,u(x,t)) \quad \text{for} \quad (x,t) \in D, \tag{5}$$

the nonlocal initial condition

$$u(x,t_0) + \frac{h(x)}{T}\int_{t_0}^{t_0+T} u(x,\tau)d\tau = f_0(x) \quad \text{for} \quad x \in D_0 \tag{6}$$

and the boundary condition

$$u(x,t) = f_1(x,t) \quad \text{for} \quad x \in \partial D_0 \times [t_0, t_0 + T]. \tag{7}$$

A function u possessing the above properties is called a solution of the first Fourier semilinear nonlocal problem (5)–(7) in D.

If condition (7) from the first Fourier semilinear nonlocal problem (5)–(7) is replaced by the condition

$$\frac{d}{d\upsilon_x}u(x,t) + k(x,t)u(x,t) = f_1(x,t) \quad \text{for} \quad x \in \partial D_0 \times [t_0, t_0 + T], \tag{8}$$

where k is the given function satisfying Assumption (A_2)(ii′) then problem (5), (6) and (8) is said to be the mixed semilinear nonlocal problem in D. A function $u \in C^{2,1}(\bar{D},\mathfrak{R})$ satisfying equation (5) and conditions (6), (8) is called a solution of the mixed semilinear nonlocal problem (5), (6) and (8) in D.

Assumption (A_4). For each two solutions w_1 and w_2 of problem (5)–(7) or of problem (5), (6) and (8) the following inequality

$$\left[\frac{1}{T}\int_{t_0}^{t_0+T}(w_1(x,\tau) - w_2(x,\tau))d\tau\right]^2 \le \left[w_1(x,t_0+T) - w_2(x,t_0+T)\right]^2 \quad \text{for } x \in D_0$$

is satisfied.

Remark 2.1. The reason for which Assumption (A_4) is introduced is that the problems considered are nonlocal.

3. Theorems about uniqueness

In this section we shall prove two theorems about the uniqueness of solutions of parabolic semilinear problems together with nonlocal initial conditions with integrals.

Theorem 3.1. *Suppose that the coefficients a_{ij} $(i,j=1,...,n)$ and c of the differential equation (5) satisfy Assumption (A_1) and the functions f, f_1, f_0 and h satisfy Assumptions (A_2)(i)–(iii) and (A_3). Then the first Fourier semilinear nonlocal problem (5)–(7) admits at most one solution in D in the class of the solutions satisfying Assumption (A_4).*

Proof. Suppose that u_1 and u_2 are two solutions of problem (5)–(7) in D and let

$$v := u_1 - u_2 \quad \text{in } \bar{D}. \tag{9}$$

Then the following formulas hold:

$$(Pv)(x,t) = f(x,t,u_1(x,t)) - f(x,t,u_2(x,t)) \text{ for } (x,t)\in\bar{D}, \tag{10}$$

$$v(x,t_0) + \frac{h(x)}{T}\int_{t_0}^{t_0+T} v(x,\tau)d\tau = 0 \text{ for } x\in\bar{D}_0, \tag{11}$$

$$v(x,t) = 0 \text{ for } (x,t)\in\partial D_0\times[t_0,t_0+T]. \tag{12}$$

From the assumption that $u_1, u_2 \in C^{2,1}(\bar{D},\Re)$, from the second and third part of Assumption (A_2)(i) and from the mean value theorem, there exists $\theta\in(0,1)$ such that

$$\begin{aligned} & f(x,t,u_1(x,t)) - f(x,t,u_2(x,t)) \\ & = v(x,t)\frac{\partial f(x,t,u_2(x,t)+\theta v(x,t))}{\partial z} \text{ for } (x,t)\in\bar{D}. \end{aligned} \tag{13}$$

By (13), (10), by Assumption (A_1) by (2) and (3) and by [4] (Section 17.11),

$$\begin{aligned} & \int_{t_0}^{t_0+T}\left[\int_{D_0} v^2\frac{\partial f(x,t,u_2+\theta v)}{\partial z}dx\right]dt \\ & = \int_{t_0}^{t_0+T}\left[\int_{D_0} vPv dx\right]dt \\ & = \int_{t_0}^{t_0+T}\left[\int_{D_0} vLv dx\right]dt + \int_{t_0}^{t_0+T}\left[\int_{D_0} cv^2 dx\right]dt \end{aligned} \tag{14}$$

$$-\int_{t_0}^{t_0+T}\left[\int_{D_0}\frac{\partial v}{\partial t}v\,dx\right]dt$$

$$=-\int_{t_0}^{t_0+T}\left[\int_{\partial D_0} v\sum_{i=1}^{n}\cos(n,x_i)\sum_{j=1}^{n}a_{ij}\frac{\partial v}{\partial x_j}d\sigma_x\right]dt$$

$$-\int_{t_0}^{t_0+T}\left[\int_{D_0}\sum_{i,j=1}^{n}a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}dx\right]dt$$

$$+\int_{t_0}^{t_0+T}\left[\int_{D_0}cv^2dx\right]dt-\int_{t_0}^{t_0+T}\left[\int_{D_0}\frac{\partial v}{\partial t}v\,dx\right]dt,$$

where $d\sigma_x$ is a surface element in $\Re^n$.

From (14), (12) and from Assumption (A_1),

$$\int_{t_0}^{t_0+T}\left[\int_{D_0}v^2\frac{\partial f(x,t,u_2+\theta v)}{\partial z}dx\right]dt\le-\int_{t_0}^{t_0+T}\left[\int_{D_0}\frac{\partial v}{\partial t}v\,dx\right]dt. \tag{15}$$

Using integration by parts, it is easy to see that

$$\int_{t_0}^{t_0+T}\left[\int_{D_0}\frac{\partial v}{\partial t}v\,dx\right]dt=\frac{1}{2}\int_{D_0}v^2(x,t_0+T)dx-\frac{1}{2}\int_{D_0}v^2(x,t_0)dx. \tag{16}$$

Formulae (15) and (16) imply the inequality

$$\int_{t_0}^{t_0+T}\left[\int_{D_0}v^2\frac{\partial f(x,t,u_2+\theta v)}{\partial z}dx\right]dt \tag{17}$$

$$\le-\frac{1}{2}\int_{D_0}v^2(x,t_0+T)dx+\frac{1}{2}\int_{D_0}v^2(x,t_0)dx.$$

From (17) and (11), we have

$$\int_{t_0}^{t_0+T}\left[\int_{D_0}v^2\frac{\partial f(x,t,u_2+\theta v)}{\partial z}dx\right]dt \tag{18}$$

$$\le-\frac{1}{2}\int_{D_0}v^2(x,t_0+T)dx+\frac{1}{2}\int_{D_0}\left[\frac{h(x)}{T}\int_{t_0}^{t_0+T}v(x,\tau)d\tau\right]^2dx.$$

By (18) and Assumption (A_4),

$$\int_{t_0}^{t_0+T}\left[\int_{D_0}v^2\frac{\partial f(x,t,u_2+\theta v)}{\partial z}dx\right]dt \tag{19}$$

$$\leq -\frac{1}{2}\int_{D_0} v^2(x,t_0+T)dx + \frac{1}{2}\int_{D_0} h^2(x)\left[\frac{1}{T}\int_{t_0}^{t_0+T} v(x,\tau)d\tau\right]^2 dx$$

$$\leq -\frac{1}{2}\int_{D_0} v^2(x,t_0+T)dx + \frac{1}{2}\int_{D_0} h^2(x)v^2(x,t_0+T)dx$$

$$= -\frac{1}{2}\int_{D_0} v^2(x,t_0+T)[1-h^2(x)]dx.$$

From (19) and from Assumption (A_3) we obtain

$$\int_{t_0}^{t_0+T}\left[\int_{D_0} v^2 \frac{\partial f(x,t,u_2+\theta v)}{\partial z}dx\right]dt \leq 0.$$

By the above inequality and by Assumption (A_2)(i), we obtain

$$v^2(x,t)\leq 0 \quad \text{for} \quad (x,t)\in D$$

and therefore

$$v(x,t)=0 \quad \text{for} \quad (x,t)\in D.$$

The proof of Theorem 3.1 is thereby complete.

Theorem 3.2. *Suppose that the assumptions of Theorem 3.1, concerning to the coefficients* a_{ij} $(i,j=1,\ldots,n)$, *c and the functions f,* f_1, f_0 *and h, are satisfied and that the function k satisfies Assumption* (A_2)(ii′). *Then the mixed semilinear nonlocal problem* (5), (6) *and* (8) *admits at most one solution in D in the class of the solutions satisfying Assumption* (A_4).

Proof. Suppose that u_1 and u_2 are two solutions of problem (5), (6) and (8) in *D*, and let

$$v := u_1 - u_2 \quad \text{in} \quad \bar{D}. \tag{20}$$

Then the following formulas hold:

$$(Pv)(x,t) = f(x,t,u_1(x,t)) - f(x,t,u_2(x,t)) \quad \text{for} \quad (x,t)\in \bar{D}, \tag{21}$$

$$v(x,t_0) + \frac{h(x)}{T}\int_{t_0}^{t_0+T} v(x,\tau)d\tau = 0 \quad \text{for} \quad x\in \bar{D}_0, \tag{22}$$

$$\frac{d}{d\upsilon_x}v(x,t) + k(x,t)v(x,t) = 0 \quad \text{for} \quad (x,t)\in \partial D_0\times[t_0,t_0,T]. \tag{23}$$

Applying a similar argument as in the proof of Theorem 3.1 and using the definition of $\frac{du}{d\upsilon_x}$ (see (4)), we have

$$\int_{t_0}^{t_0+T}\left[\int_{D_0} v^2 \frac{\partial f(x,t,u_2+\theta v)}{\partial z}dx\right]dt \tag{24}$$

$$=-\int_{t_0}^{t_0+T}\left[\int_{\partial D_0} v\frac{dv}{d\upsilon}d\sigma_x\right]dt$$

$$-\int_{t_0}^{t_0+T}\left[\int_{D_0}\sum_{i,j=1}^{n} a_{i,j}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}dx\right]dt$$

$$+\int_{t_0}^{t_0+T}\left[\int_{D_0} cv^2 dx\right]dt-\int_{t_0}^{t_0+T}\left[\int_{D_0}\frac{dv}{dt}v dx\right]dt.$$

From (24), (23), and as in the proof of Theorem 3.1

$$\int_{t_0}^{t_0+T}\left[\int_{D_0} v^2\frac{\partial f(x,t,u_2+\theta v)}{\partial z}dx\right]dt \tag{25}$$

$$\leq\int_{t_0}^{t_0+T}\left[\int_{\partial D_0} kv^2 d\sigma_x\right]dt-\frac{1}{2}\int_{D_0} v^2(x,t_0+T)[1-h^2(x)]dx.$$

By (25) and Assumptions $(A_2)(\mathrm{ii}')$ and (A_3) we obtain the inequality

$$\int_{t_0}^{t_0+T}\left[\int_{D_0} v^2\frac{\partial f(x,t,u_2+\theta v)}{\partial z}dx\right]dt\leq 0.$$

Consequently, as in the proof of Theorem 3.1,

$$v(x,t)=0 \quad \text{for} \quad (x,t)\in D$$

and the proof of Theorem 3.2 is complete.

4. Physical interpretation of the nonlocal condition (6)

Theorems 3.1 and 3.2 can be applied to descriptions of physical problems in heat conduction theory for which we cannot measure the temperature at the initial instant but we can measure the temperature in the form of the nonlocal condition (6).

Observe, also, that in Theorem 3.1 and 3.2, the nonlocal condition (6) considered is more general than the classical initial condition and the integral periodic condition and the integral anti-periodic condition. Namely, if the function h from condition (6) satisfies the relation

$h(x)=0$ for $x\in D_0$ then condition (6) is reduced to

the initial condition

$$u(x,t_0) = f_0(x) \quad \text{for} \quad x \in D_0.$$

Instead if the function h and f in (6) satisfy the conditions

$$h(x) = -1[h(x) = 1] \quad \text{for} \quad x \in D_0,$$

$$f_0(x) = 0 \quad \text{for} \quad x \in D_0,$$

then condition (6) is reduced, respectively, to the integral periodic [antiperiodic] initial condition:

$$u(x,t_0) = \frac{1}{T}\int_{t_0}^{t_0+T} u(x,\tau)d\tau \quad \left[\, u(x,t_0) = -\frac{1}{T}\int_{t_0}^{t_0+T} u(x,\tau)d\tau \,\right] \text{ for } \; x \in D_0.$$

References

[1] Brandys J., Byszewski L., *Uniqueness of solutions to inverse parabolic problems,* Comment. Math. Prace Matem. 42.1, 2002, 17–30.

[2] Byszewski L., *Uniqueness of solutions of parabolic semilinear nonlocal-boundary problems,* J. Math. Anal. Appl. 165.2, 1992, 472–478.

[3] Chabrowski J., *On nonlocal problems for parabolic equations,* Nagoya Math. J. 93, 1984, 109–131.

[4] Krzyżański M., *Partial Differential Equations of Second Order*, Vol. 1, PWN, Warszawa 1971.

[5] Rabczuk R., *Elements of Differential Inequalities,* PWN, Warszawa 1976 (in Polish).