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MODELING OF SPACE HARMONIC INTERACTIONS IN AXIAL FLUX PERMANENT MAGNET GENERATORS

MODELOWANIE ODDZIAŁYWAŃ HARMONICZNYCH PRZESTRZENNYCH W GENERATORACH Z MAGNESAMI TRWAŁYMI O STRUMIENIU OSIOWYM

A b s t r a c t

The paper presents the methodology of determining the distribution of flux density in the magnetic circuit of a 3-phase axial flux machine with permanent magnets located on the rotor. It allowed to create an analytical dependences of the inductances and flux linkages. Particular attention was paid to such processing of mathematical model equations, so as to create a possibility to determine the machine properties in steady state on the basis of equations, without even having to use quantitative solutions. The theory of linear differential equations used here, with periodically variable coefficients, allowed to determine the Fourier spectra with respect to winding currents and electromagnetic torque in steady state operation.

Keywords: permanent magnets, axial flux, generator, space harmonic interaction

S t r e s z c z e n i e

W artykule przedstawiono metodykę wyznaczania rozkładu indukcji w obwodzie magnetycznym 3-fazowej maszyny z osiowym strumieniem magnesów trwałych umieszczonych na wirniku. Dzięki temu sformułowano analityczne zależności dające możliwość określenia indukcyjności oraz strumieni skojarzonych. Szczególną uwagę zwrócono na takie przetworzenie równań modelu matematycznego, aby było możliwe jakościowe określenie własności maszyny w stanach ustalonych już na podstawie równań, bez konieczności ich ilościowego rozwiązywania. Zastosowana tutaj teoria równań różniczkowych liniowych o okresowo zmiennych współczynnikach pozwoliła na określenie widm Fouriera prądów uzwojeń i momentu elektromagnetycznego w ustalonym stanie pracy.

Słowa kluczowe: magnesy trwałe, generator, strumień osiowy, oddziaływanie harmonicznych

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1. Introduction

The interest in new a construction of generators is associated with a global trend to support the local energy supply with small renewable energy sources. Typical water and wind turbines are characterized by relatively low rotational speeds [1, 2], which, in classic solutions, usually require the use of a mechanical gear. The elimination of a gear, which is expensive and uncomfortable in maintenance and operation, requires the use of low-speed generators excited by Permanent Magnets (PM), driven directly from the turbine shaft. Slow-speed generators with a traditional design are usually relatively heavy and have significant radial dimensions, which often makes them difficult to use in small scale applications. Currently, an optimal design is being intensively sought, and literature mentions a lot of interesting solutions [3]. However, it seems that the traditional layout of the PM synchronous machine with a radial flux is still an optimal construction. It is believed that PM disc-type solutions with axial flux also enable the creation of simple structures with a very large number of poles.

The Axial Flux Permanent Magnet Generators (AFPMG) typically have a disc construction with a coreless stator. Design and mathematical modelling techniques for these class of machines are currently developed and improved [3–6]. For this reason, the purpose of this paper is to show the methodology of mathematical modelling for this specific class of machines. The main task of the presented study is the use of the harmonic balance method in the analyses of steady-states in AFPMG. The problem was presented in order to make attempts to determine whether, by means of mathematical modelling, it is possible to distinguish and quantify the interaction of space harmonics for winding currents and electromagnetic torque.

2. Distribution of magnetic field in coreless AFPMG

For the simplest models of classic machines, inductances are calculated based on the distribution of the radial component of the field in the air gap, because this is possible thanks to a specific structure of the magnetic circuit. The geometry of the magnetic circuit in the Axial Flux Permanent Magnet (AFPM) machine is different, and simple relationships, sufficiently accurate for conventional machines, should be modified. Thus, in many cases, the calculation of the parameters of equations for PM machines requires a numerical method of field distribution analysis (Finite Elements Analysis). Over the years, AFPM machines have been developed, and in a number of different topologies, a single-sided machine and double-sided machine with internal stator and double external rotors can be distinguished [3]. This work focuses primarily on the AFPM double-sided machine, with coreless stator (Fig. 1) because such solutions exist for popular low power generators. The winding topologies of the non-overlapping and overlapping windings as well as PM shapes [3–6] for the AFPM machines are illustrated in Fig. 2 and 3 respectively, and the main features of the AFPM machines are presented in Table 1. For these examples, it is possible to create analytical formulas describing the parameters of the mathematical model. The basis of this analysis is the magnetic field distribution in the machine air gap.

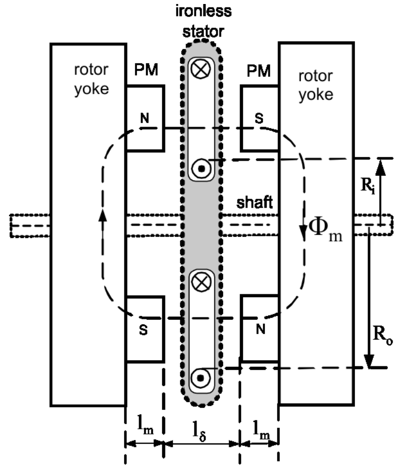

Fig. 1. Construction of dual-rotor AFPMG

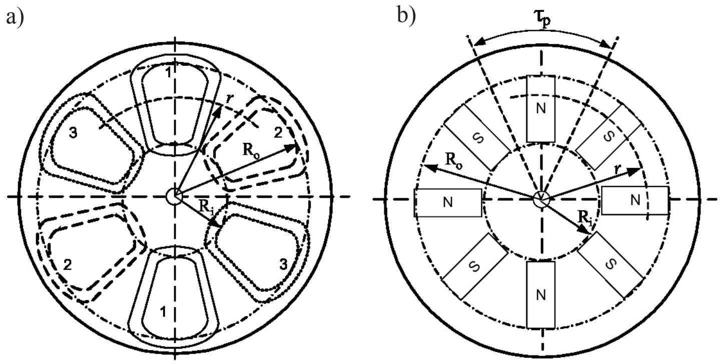

Fig. 2. Cross-section of AFPMG: a) stator (3-phsae non-overlapping winding $p_s = 2$); b) rotor ($p = 4$)

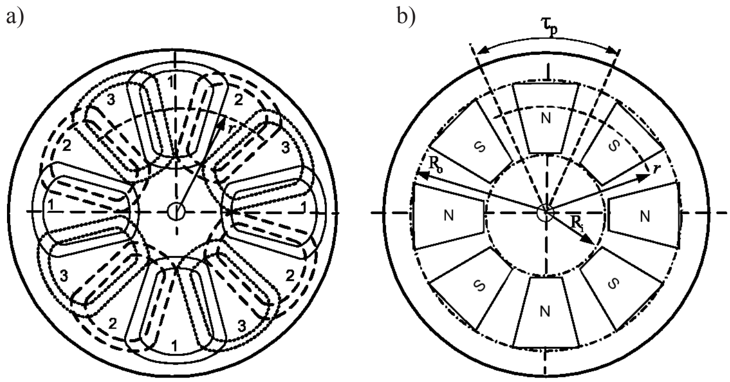

Fig. 3. Construction of AFPMG: a) stator (3-phsae overlapping winding $p_s = 4$); b) rotor ($p = 4$)

Table 1

The main differences between machine construction with various windings

–	**Non-overlapping windings**	**Overlapping windings**
Number of phase coils	p_s	p_s
Total number of stator coils	3 p_s	3 p_s
Number of magnets (one rotor side)	4 p_s = 2p	2 p_s = 2p
p/p_s	**2**	**1**
Maximal angle of coil pitch	$\varepsilon_{max} = \frac{2\pi}{3p_s} = \frac{4\pi}{3p}$	$\varepsilon_{max} = \frac{\pi}{p_s} = \frac{\pi}{p}$
where: p_s – **number of stator pole pairs,** p – **number of rotor pole pairs**		

In order to illustrate the methodology of field distribution in the air gap, a model of coordinates was used, depicting the AFPM machine, as shown in Fig. 4.

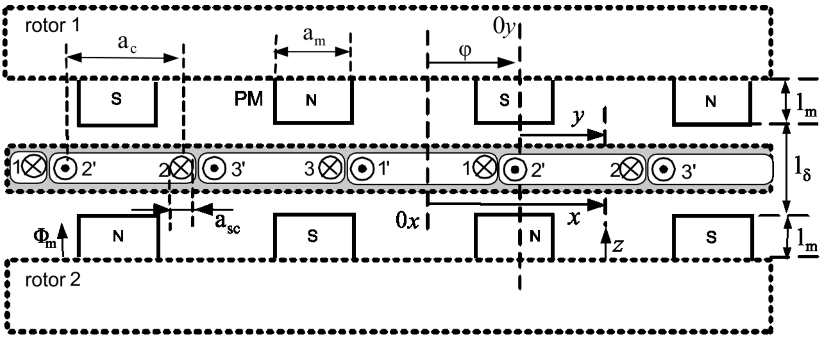

Fig. 4. Model of coordinate system for flux density distribution

For the AFPM machine, an assumption was made concerning a linear approximation for the characteristics of permanent magnet demagnetization $B_m = B_r + \mu_0 \cdot \mu_{rm} \cdot H_m$ (where: B_r – magnet remanence, μ_{rm} – relative magnet permeability), which is a good approximation for modern rare-earth PM, and magnetic drops in the machine iron were neglected. For a one-dimensional (1-D) model of magnetic field distribution in the machine, which assumes the occurrence of only the axial component (independent on the axial coordinate *z*), magnetic flux density in the air gap and magnet is generally a function of three variables (dependent on the location according to stator *x*, dependent on the angle of rotor position φ and radial location $R_i \leq r \leq R_o$):

$$B_\delta(z,x,\varphi,r) = B_\mathrm{m}(z,x,\varphi,r) = B(x,\varphi,r) \tag{1}$$

where:

$B_\delta(z,x,\varphi,r)$, $B_\mathrm{m}(z,x,\varphi,r)$ – magnetic flux density in the air gap and in PM, respectively.

The general formula representing the one-dimensional magnetic field distribution in the AFPM machine is as follows:

$$B(x,\varphi,r) = B_\Theta(x,r) + B_\mathrm{m}(x-\varphi,r) \tag{2}$$

where:

$B_\Theta(x,r)$ – axial component of flux density induced by winding MMF's,
$B_m(x-\varphi,r)$ – axial component of flux density induced by PM.

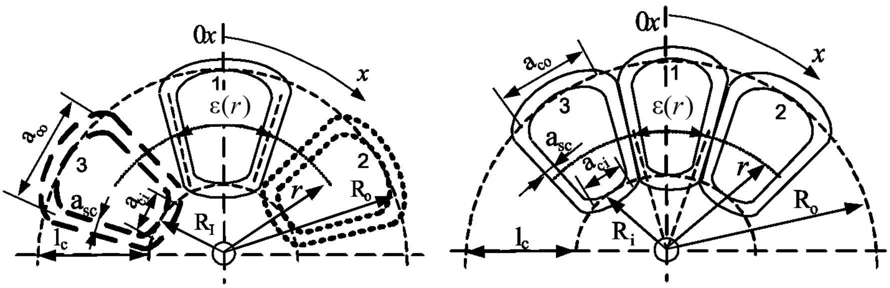

Fig. 5. Layouts of different non-overlapping winding types

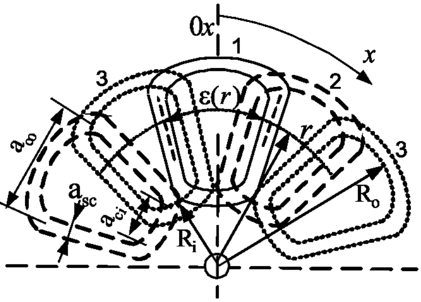

Fig. 6. Layouts of overlapping winding type

The MMF component describing the field distribution dependent on the winding currents (2) and for 3-phase AFPM machines with symmetrical design (layouts of stator windings are shown at Fig. 5, 6 can be presented in a form [7–9]:

$$B_\Theta(x,r) = \sum_{a=1}^{3} B_{\Theta a}(x,r) \tag{3}$$

where:

$$B_{\Theta a}(x,r) = \lambda_0 \cdot \Theta_a(x,r) \tag{4}$$

$B_{\Theta}(x,r)$ – a function of winding “a” MMF,
λ_0 – unit permeance function.

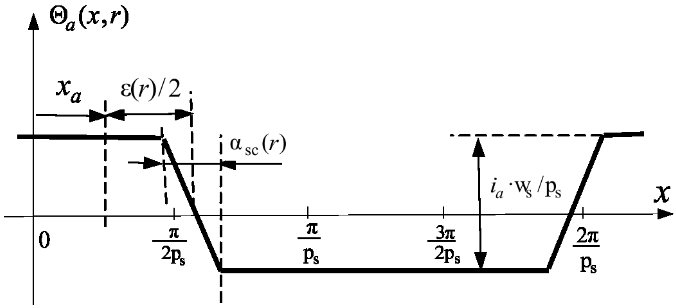

Fig. 7. Winding “a” MMF distribution

The Fourier distribution of winding “a” MMF (Fig. 7) is represented by following formula:

$$\Theta_a(x,r) = \sum_{\nu \in P} \Theta_\nu^a(r) \cdot e^{j\nu(x-x_a)} \tag{5}$$

where Fourier spectra of the MMF’s contain harmonics of the ν^{th} order belongs to set P:

$$P = \{-\nu_{max} \ldots -5p_s, -4p_s, -3p_s, -2p_s, -p_s, p_s, 2p_s, 3p_s, 4p_s, 5p_s \ldots \nu_{max}\} \tag{6}$$

$$\Theta_\nu^a(r) = i_a \frac{1}{\pi} W_\nu^s(r) \qquad W_\nu^s(r) = \frac{w_s \cdot k_s^{|\nu|}(r)}{|\nu|} \tag{7}$$

w_s – total number of phase winding turns,
$k_s^{|\nu|}(r)$ – winding factor for ν^{th} harmonic,

$$x_a = (a-1)\frac{2\pi}{3p_s} \quad \text{for } a = 1, 2, 3.$$

For concentrated coils, winding factor contains only pitch factor and can be written as [3]:

$$k_s^{|\nu|}(r) = \sin\left(|\nu| \frac{\varepsilon(r)}{2}\right) \cdot \frac{\sin\left(|\nu| \frac{\alpha_{sc}(r)}{2}\right)}{|\nu| \frac{\alpha_{sc}(r)}{2}} \tag{8}$$

where:

$\varepsilon(r) = \frac{a_c}{r}$ – angle of coil pitch or coil span at coordinate r $\left(a_c \approx \frac{a_{co} + a_{ci}}{2}\right)$,

$\alpha_{sc}(r) = \frac{a_{sc}}{r}$ – angle of coil side width at coordinate r.

The unit permeance function, due to the properties of the magnet being similar to the air, $(\mu_{rm} \cong 1.01 \cdots 1.1)$, becomes virtually constant.

$$\lambda_0 = \frac{\mu_0}{l_\delta + 2l'_m} \tag{9}$$

where:
$l'_m = \frac{l_m}{\mu_{rm}}$,
l_m – magnet height
l_δ – length of air gap.

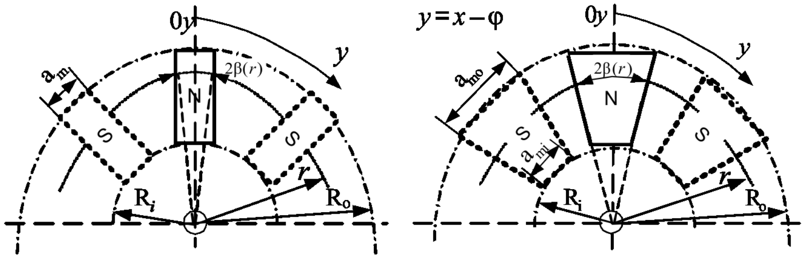

Fig. 8. Layouts of exemplary PM shapes located on the rotor

The axial component of PM flux density (2) for different layouts of rotor construction (Fig. 8.) can be presented in form [8] as shown in Fig. 9.

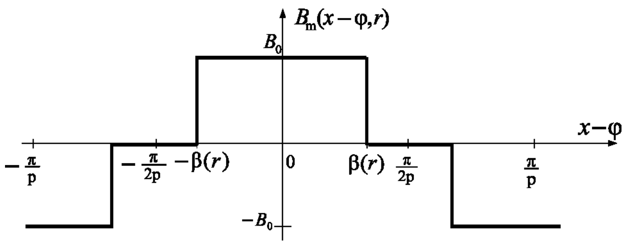

Fig. 9. PM flux density distribution

Fourier series coefficients, of the PM flux density distribution (Fig. 9) can be determined from the following formulas [7, 8]:

$$B_m(x - \varphi, r) = \sum_{\varsigma \in Q} B_\varsigma^{PM}(r) \cdot e^{j\varsigma(x-\varphi)} \tag{10}$$

where:

$$Q = \{-\varsigma_{max} \ldots -5p, -3p, -p, p, 3p, 5p \ldots \varsigma_{max}\} \tag{11}$$

$$B_{\varsigma}^{\rm PM}(r)=\frac{2}{\pi}\frac{B_0}{\varsigma}\,{\rm p}\cdot\sin(\varsigma\cdot\beta(r))\qquad B_0={\rm B_r}\,\frac{2{\rm l}'_{\rm m}}{2{\rm l}'_{\rm m}+{\rm l}_{\delta}} \tag{12}$$

$\beta_{\rm m}(r)=\dfrac{{\rm a_m}}{2r}$ – half angular pitch of magnet pole at coordinate r $\left({\rm a_m}\approx\dfrac{{\rm a_{mo}}+{\rm a_{mi}}}{2}\right)$,
B_0 – air gap flux density for PM area in currentless state.

3. Mathematical model of 3-phase coreless AFPMG

Using Lagrange's formalism [8, 9] to create a description of AFPM machine, mathematical model can be written in a standard matrix form:

$$\begin{aligned}&\frac{\rm d}{{\rm d}t}\{\mathbf{L}\cdot\mathbf{i}+\mathbf{\Psi_{PM}}(\varphi)\}+\mathbf{R}_{\rm s}\cdot\mathbf{i}=\mathbf{u}\\&J\frac{{\rm d}^2\varphi}{{\rm d}t^2}=T_{\rm em}(\varphi,i_1,i_2,i_3)+T_{\rm L}-D\frac{{\rm d}\varphi}{{\rm d}t}\end{aligned} \tag{13}$$

wherein electromagnetic torque:

$$T_{\rm em}(i_1,i_2,i_3,\varphi)=\mathbf{i}^T\cdot\frac{\partial}{\partial\varphi}\mathbf{\Psi_{PM}}(\varphi) \tag{14}$$

where:

$$\mathbf{L}=\mathbf{L_{\sigma s}}+\mathbf{L_s}=\begin{bmatrix}L_{\sigma s}&&\\&L_{\sigma s}&\\&&L_{\sigma s}\end{bmatrix}+\begin{bmatrix}L_{11}&L_{12}&L_{13}\\L_{21}&L_{22}&L_{23}\\L_{31}&L_{32}&L_{33}\end{bmatrix}$$

$$\mathbf{i}=\begin{bmatrix}i_1\\i_2\\i_3\end{bmatrix}\quad\mathbf{u}=\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\qquad\mathbf{\Psi_{PM}}(\varphi)=\begin{bmatrix}\psi_{\rm PM1}(\varphi)\\\psi_{\rm PM2}(\varphi)\\\psi_{\rm PM3}(\varphi)\end{bmatrix}\qquad\mathbf{R}_{\rm s}=\begin{bmatrix}R_{\rm s}&&\\&R_{\rm s}&\\&&R_{\rm s}\end{bmatrix}$$

For mathematical models of electric machines used to assess the impact of space harmonic interactions, it is very important to correctly identify the qualitative characteristics of flux linkage, as a function of the angle of rotor position. Flux ψ_a, linked with „a" winding can be expressed as:

$$\psi_a(\varphi)=\int_{\rm R_i}^{\rm R_o}{\rm w_s}\left\{\int_{\frac{-\varepsilon(r)+\alpha_{\rm sc}(r)}{2}+x_a}^{\frac{\varepsilon(r)-\alpha_{\rm sc}(r)}{2}+x_a}B(x,\varphi,r){\rm d}x\right\}r{\rm d}r \tag{15}$$

For the assumed flux density distribution $B(x,\varphi,r)$, integrations in expression (15) are tedious and difficult. However, we can get a solution under some simplifying assumptions averaged angles $\varepsilon(r),\alpha_{\rm sc}(r),\beta_{\rm m}(r)$ by introducing average value for r coordinate:

$$r \approx \mathrm{r_s} = \frac{\mathrm{R_o} + \mathrm{R_i}}{2} \tag{16}$$

The averaged angles are as follows:

$$\varepsilon(r) \approx \frac{\mathrm{a_c}}{\mathrm{r_s}} = \varepsilon(\mathrm{r_s}), \qquad \alpha_{\mathrm{sc}}(r) \approx \frac{\mathrm{a_{sc}}}{\mathrm{r_s}} = \alpha_{\mathrm{sc}}(\mathrm{r_s}), \qquad \beta_{\mathrm{m}}(r) \approx \frac{\mathrm{a_m}}{2\mathrm{r_s}} = \beta_{\mathrm{m}}(\mathrm{r_s}) \tag{17}$$

For this assumption, generally, the flux density (2) is function of two variables [8, 9]:

$$B(x, \varphi, \mathrm{r_s}) = \sum_{\zeta} B_{\zeta}(\varphi) \cdot \mathrm{e}^{\mathrm{j}\zeta(x - x_0)} \tag{18}$$

and we obtain the dependence:

$$\psi_a(\varphi) = \sum_{\nu \in P} 2W_{\nu}^{\mathrm{s}}(\mathrm{r_s}) \cdot B_{-\nu}(\varphi) \cdot \frac{\mathrm{R_o^2} - \mathrm{R_i^2}}{2} \cdot \mathrm{e}^{-\mathrm{j}\nu(x_a - x_0)} \tag{19}$$

from which it can be concluded that the magnetic flux linkage of winding contains only the space harmonics related to the distribution of the magnetic flux density in air gap, belonging to set P, and corresponding to MMF harmonics of the winding „*a*" (6).
After the introduction into relation (18), the function of the flux density distribution produced by coil „*b*":

$$B_b(x, \mathrm{r_s}) = \sum_{\nu \in P} i_b \frac{1}{\pi} W_{\nu}(\mathrm{r_s}) \cdot \mathrm{e}^{\mathrm{j}\nu(x - x_b)} \tag{20}$$

then, flux linkage $\psi_a(\varphi)$ can be determined and inductance of windings „*a*" and „*b*":

$$L_{ab}(\varphi) = \frac{\psi_a(\varphi)}{i_b} \tag{21}$$

The form of the expression describing inductance (21) after completion of the formal mathematical operations is represented by a general dependence [8, 9]:

$$L_{ab}(\varphi) = \sum_{\nu \in P} L_{\nu}^{\mathrm{ss}} \cdot \mathrm{e}^{\mathrm{j}\nu(a-b)\frac{2\pi}{3\mathrm{p_s}}} \quad \text{for} \quad a, b = 1, 2, 3 \tag{22}$$

where:

$$L_{\nu}^{\mathrm{ss}} = \frac{2}{\pi} \cdot [W_{\nu}^{\mathrm{s}}(\mathrm{r_s})]^2 \cdot \mathrm{r_s} \cdot \mathrm{l_c} \cdot \lambda_0 \tag{23}$$

$\mathrm{l_c} = \mathrm{R_o} - \mathrm{R_i}$ is length of the coil side.

A flux linkage component for winding „*a*", in a currentless state, can be derived using the substitution in formula (18), function of the flux density distribution produced by PM:

$$B_{\mathrm{m}}(x - \varphi, \mathrm{r_s}) = \sum_{\varsigma \in Q} B_{\varsigma}^{\mathrm{PM}}(\mathrm{r_s}) \cdot \mathrm{e}^{\mathrm{j}\varsigma(x - \varphi)} \tag{24}$$

After formal mathematical transformations, the "a" winding flux linkage ψ_{PMa} produced by magnets in a currentless state is represented by the relationship [8]:

$$\psi_{PMa}(\varphi) = \sum_{\varsigma \in Q} \psi_{\varsigma}^{PM} \cdot e^{j\varsigma\{(a-1)\frac{2\pi}{3p} - \varphi\}} \text{ for } a = 1, 2, 3 \tag{25}$$

where for stators with non-overlapping windings:

$$\psi_{\varsigma}^{PM} = 2 \cdot B_{\varsigma}^{PM}(r_s) \cdot W_{2\varsigma}^{s}(r_s) \cdot r_s \cdot l_c \cdot \lambda_0 \tag{26}$$

and for stator with overlapping windings:

$$\psi_{\varsigma}^{PM} = 2 \cdot B_{\varsigma}^{PM}(r_s) \cdot W_{\varsigma}^{s}(r_s) \cdot r_s \cdot l_c \cdot \lambda_0 \tag{27}$$

Generally, the matrix of main stator inductances and vector of PM flux linkages take the following forms [8, 9]:

$$\mathbf{L_s} = \sum_{\nu \in P} L_{\nu}^{ss} \cdot \begin{bmatrix} 1 & \underline{b}^{-\nu} & \underline{b}^{-2\nu} \\ \underline{b}^{\nu} & 1 & \underline{b}^{-\nu} \\ \underline{b}^{2\nu} & \underline{b}^{\nu} & 1 \end{bmatrix} \text{ where: } \underline{b} = e^{j\frac{2\pi}{3p_s}} \tag{28}$$

$$\mathbf{\Psi_{PM}}(\varphi) = \sum_{\varsigma \in Q} \psi_{\varsigma}^{PMs} \cdot e^{-j\varsigma\varphi} \cdot \begin{bmatrix} 1 \\ \underline{d}^{\varsigma} \\ \underline{d}^{2\varsigma} \end{bmatrix} \text{ where: } \underline{d} = e^{j\frac{2\pi}{3p}} \tag{29}$$

The leakage inductances are expressed analytically as a sum of two components [3]. One of them is connected with the leakage flux around radial portions of conductors (corresponding to slot leakage in classical machines) and second is dependent on the leakage flux around the coil end connections. The leakage inductance coefficient can be determined from the formula:

$$L_{\sigma s} \approx 2\mu_0 \cdot (w_s)^2 (l_c + a_c) \cdot 0.3 \tag{30}$$

4. Mathematical model of 3-phase AFPMG for steady state operation

In steady-state operation, the mathematical model of a three-phase AFPM machine is reduced to a system of linear differential equations with periodically variable coefficients. A detailed analysis and solution of this system of equations using the harmonic balance method enables qualitative (frequency determination) and quantitative (amplitude determination) assessment of Fourier spectrum of currents and electromagnetic torque [8–10]. The main problem here is to define the parameters of the model, which should highlight the impact of all relevant harmonics of a spatial field distribution in the machine, and the conversion of the mathematical model in order to be able to track the interactions of these harmonics. Very useful is the transformation of symmetrical components, which describes the machine in orthogonal bases, and puts, in order, the inductance matrices and vectors of PM flux linkages, so that even on the basis of equations, we can analyse how space harmonics affect currents and electromagnetic torque [8–10].

If mathematical model equations (13) and (14) are transformed into symmetrical components, using a matrix transformation:

$$\mathbf{T} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \underline{a} & \underline{a}^2 \\ 1 & \underline{a}^2 & \underline{a} \end{bmatrix}_{(3\times3)} \quad \text{where: } \underline{a} = e^{\mathrm{j}\frac{2\pi}{3}} \tag{31}$$

then, the machine voltage equations take the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\{[\mathbf{L}^{\mathbf{s}}_{\boldsymbol{\sigma}\mathbf{s}} + \mathbf{L}^{\mathbf{s}}_{\mathbf{s}}]\cdot \mathbf{i}^{\mathbf{s}}\} + \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Psi}^{\mathbf{s}}_{\mathbf{PM}}(\varphi) + \mathbf{R}_{\mathrm{s}}\cdot \mathbf{i}^{\mathbf{s}} = \mathbf{u}^{\mathbf{s}} \tag{32}$$

and the formula for the electromagnetic torque may be expressed as:

$$T_{\mathrm{em}}(i^{\mathrm{s}0}\ i^{\mathrm{s}1}\ i^{\mathrm{s}2}, \varphi) = (\check{\mathbf{i}}^{\mathbf{s}})^{\mathrm{T}} \cdot \frac{\partial}{\partial\varphi}\boldsymbol{\Psi}^{\mathbf{s}}_{\mathbf{PM}}(\varphi) \tag{33}$$

where:

$$\mathbf{u}^{\mathbf{s}} = \mathbf{T}\cdot\mathbf{u}; \mathbf{u}^{\mathbf{s}} = [u^{\mathrm{s}0}u^{\mathrm{s}1}u^{\mathrm{s}2}]^{\mathrm{T}} \qquad \mathbf{i}^{\mathbf{s}} = \mathbf{T}\cdot\mathbf{i}; \mathbf{i}^{\mathbf{s}} = [i^{\mathrm{s}0}i^{\mathrm{s}1}i^{\mathrm{s}2}]^{\mathrm{T}}$$

$$\mathbf{L}^{\mathbf{s}}_{\boldsymbol{\sigma}\mathbf{s}} = \mathbf{T}\cdot\mathbf{L}_{\boldsymbol{\sigma}\mathbf{s}}\cdot\mathbf{T}^{-1} = \mathbf{L}_{\boldsymbol{\sigma}\mathbf{s}} \qquad \mathbf{L}^{\mathbf{s}}_{\mathbf{s}} = \mathbf{T}\cdot\mathbf{L}_{\mathbf{s}}\cdot\mathbf{T}^{-1}$$

$$\boldsymbol{\Psi}^{\mathbf{s}}_{\mathbf{PM}}(\varphi) = \mathbf{T}\cdot\boldsymbol{\Psi}_{\mathbf{PM}}(\varphi)\ ;\ \boldsymbol{\Psi}^{\mathbf{s}}_{\mathbf{PM}}(\varphi) = [\psi^{\mathrm{s}0}(\varphi)\ \ \psi^{\mathrm{s}1}(\varphi)\ \ \psi^{\mathrm{s}2}(\varphi)]^{\mathrm{T}}$$

Winding inductance matrix after the symmetrical components transformation is dependent on the distribution of MMF harmonics and can be written in the following form:

$$\mathbf{L}^{\mathbf{s}}_{\mathbf{s}} = \begin{bmatrix} L^{\mathrm{ss}0} & 0 & 0 \\ 0 & L^{\mathrm{ss}1} & 0 \\ 0 & 0 & L^{\mathrm{ss}2} \end{bmatrix} \tag{34}$$

where:

$$L^{\mathrm{ss}0} = 3\sum_{k=0,\pm1,\pm2...} L^{\mathrm{ss}}_{(6k-3)\mathrm{p_s}} \qquad L^{\mathrm{ss}1} = 3\sum_{k=0,\pm1,\pm2...} L^{\mathrm{ss}}_{(3k+1)\mathrm{p_s}} \qquad L^{\mathrm{ss}2} = 3\sum_{k=0,\pm1,\pm2...} L^{\mathrm{ss}}_{(3k-1)\mathrm{p_s}} \tag{35}$$

Vector of PM flux linkages, after the symmetrical component transformation can be written as follows:

$$\boldsymbol{\Psi}^{\mathbf{s}}_{\mathbf{PM}}(\varphi) = \sum_{\varsigma\in Q}\boldsymbol{\Psi}^{\mathbf{PMs}}_{\varsigma}\cdot \mathrm{e}^{\mathrm{j}\varsigma\varphi} \tag{36}$$

where:

$$\mathbf{\Psi}_{\varsigma}^{\mathbf{PMs}} = \sqrt{3}\begin{bmatrix} \psi_{\varsigma}^{\mathrm{PMs}} \\ 0 \\ 0 \end{bmatrix} \quad \text{for} \quad \varsigma = \pm 3\mathrm{p}, \pm 9\mathrm{p}... \tag{37}$$

$$\mathbf{\Psi}_{\varsigma}^{\mathbf{PMs}} = \sqrt{3}\begin{bmatrix} 0 \\ \psi_{\varsigma}^{\mathrm{PMs}} \\ 0 \end{bmatrix} \quad \text{for} \quad \varsigma = ... -5\mathrm{p}, \ \mathrm{p}, 7\mathrm{p}... \tag{38}$$

$$\mathbf{\Psi}_{\varsigma}^{\mathbf{PMs}} = \sqrt{3}\begin{bmatrix} 0 \\ 0 \\ \psi_{\varsigma}^{\mathrm{PMs}} \end{bmatrix} \quad \text{for} \quad \varsigma = ... -7\mathrm{p}, \ -\mathrm{p}, 5\mathrm{p}... \tag{39}$$

If we consider an AFPM machine operating in generator mode, then winding currents' arrows must be drawn to indicate proper energy flow direction.

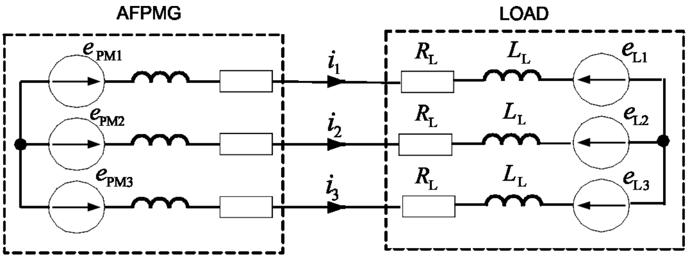

Fig. 10. Diagram of AFPMG connected to the load

Circuit equation for generator shown in Fig. 10 may be expressed in accordance with equation (32) as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{\Psi}_{\mathbf{PM}}^{\mathbf{s}}(\varphi) - \mathbf{R}_{\mathrm{s}} \cdot \mathbf{i}^{\mathrm{s}} - \frac{\mathrm{d}}{\mathrm{d}t}\{[\mathbf{L}_{\boldsymbol{\sigma}\mathbf{s}}^{\mathbf{s}} + \mathbf{L}_{\mathbf{s}}^{\mathbf{s}}] \cdot \mathbf{i}^{\mathbf{s}}\} = \mathbf{u}^{\mathbf{s}} \tag{40}$$

where:

$$\mathbf{u}^{\mathbf{s}} = \mathbf{e}_{\mathrm{L}}^{\mathrm{s}} + \mathbf{R}_{\mathrm{L}} \cdot \mathbf{i}^{\mathbf{s}} + \frac{\mathrm{d}}{\mathrm{d}t}\{\mathbf{L}_{\mathrm{L}} \cdot \mathbf{i}^{\mathbf{s}}\} \tag{41}$$

$$\mathbf{R}_{\mathrm{L}} = \begin{bmatrix} R_{\mathrm{L}} & & \\ & R_{\mathrm{L}} & \\ & & R_{\mathrm{L}} \end{bmatrix} \qquad \mathbf{L}_{\mathrm{L}} = \begin{bmatrix} L_{\mathrm{L}} & & \\ & L_{\mathrm{L}} & \\ & & L_{\mathrm{L}} \end{bmatrix}$$

Dependences (40) and (41) can be grouped and written in a simpler form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\{[\mathbf{L}_{\boldsymbol{\sigma}\mathbf{s}}^{\mathbf{s}} + \mathbf{L}_{\mathbf{s}}^{\mathbf{s}} + \mathbf{L}_{\mathrm{L}}] \cdot \mathbf{i}^{\mathbf{s}}\} + (\mathbf{R}_{\mathrm{s}} + \mathbf{R}_{\mathrm{L}}) \cdot \mathbf{i}^{\mathbf{s}} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{\Psi}_{\mathbf{PM}}^{\mathbf{s}}(\varphi) - \mathbf{e}_{\mathrm{L}}^{\mathrm{s}} \tag{42}$$

If we assume that the EMFs on load side form a balanced three-phase voltage system, then it will correspond to the generator working on a grid. Expressions of these voltages in symmetrical components are expressed as:

$$\mathbf{e}_{\mathrm{L}}^{\mathrm{s}} = \sum_{\eta=\pm 1} \mathbf{E}_{\boldsymbol{\eta}}^{\mathbf{s}} \cdot \mathrm{e}^{\mathrm{j}\eta\omega_0 t} = \begin{bmatrix} 0 \\ E \\ 0 \end{bmatrix} \cdot \mathrm{e}^{\mathrm{j}\omega_0 t} + \begin{bmatrix} 0 \\ 0 \\ E \end{bmatrix} \cdot \mathrm{e}^{-\mathrm{j}\omega_0 t} \tag{43}$$

$E = \sqrt{\frac{3}{2}}\, E_{\mathrm{ph}}$, E_{ph} – is the RMS value of grid phase voltage (line-neutral).

The steady state is considered when angular velocity of the rotor $\dot{\varphi} = \Omega$ is constant then $\varphi = \Omega \cdot t + \varphi_0$. If steady-state dependence $\omega_0 = \mathrm{p}\Omega$ is fulfilled, then the inductance matrix and vector of PM flux linkages become periodic, and we can assume solutions for set of equations (42), as:

$$\mathbf{i}^{\mathbf{s}} = \sum_{\nu} \mathbf{I}_{\boldsymbol{\nu}}^{\mathbf{s}} \cdot \mathrm{e}^{\mathrm{j}\nu\Omega t} \qquad \text{where:} \quad \mathbf{I}_{\boldsymbol{\nu}}^{\mathbf{s}} = [\underline{I}_{\nu}^{\mathrm{s}0} \;\; \underline{I}_{\nu}^{\mathrm{s}1} \;\; \underline{I}_{\nu}^{\mathrm{s}2}]^{\mathrm{T}} \tag{44}$$

Solutions (44) satisfy, according to the harmonic balance method [8–10], an infinite dimensional system of algebraic equations:

$$\mathrm{diag}\begin{bmatrix} \vdots \\ \mathrm{j}3\mathrm{p}\Omega \\ \mathrm{j}\mathrm{p}\Omega \\ -\mathrm{j}\mathrm{p}\Omega \\ -\mathrm{j}3\mathrm{p}\Omega \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} \ddots & & \vdots & \vdots & \vdots & \\ & L^{\mathrm{s}0} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & L^{\mathrm{s}1} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & L^{\mathrm{s}2} & 0 & \cdots \\ \cdots & 0 & 0 & 0 & L^{\mathrm{s}0} & \\ & \vdots & \vdots & \vdots & & \ddots \end{bmatrix} \cdot \begin{bmatrix} \vdots \\ \underline{I}_{3\mathrm{p}}^{\mathrm{s}0} \\ \underline{I}_{\mathrm{p}}^{\mathrm{s}1} \\ \underline{I}_{-\mathrm{p}}^{\mathrm{s}2} \\ \underline{I}_{-3\mathrm{p}}^{\mathrm{s}0} \\ \vdots \end{bmatrix} +$$

$$+\mathrm{diag}\begin{bmatrix} \vdots \\ R_{\mathrm{s}} + R_{\mathrm{L}} \\ R_{\mathrm{s}} + R_{\mathrm{L}} \\ R_{\mathrm{s}} + R_{\mathrm{L}} \\ R_{\mathrm{s}} + R_{\mathrm{L}} \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} \vdots \\ \underline{I}_{3\mathrm{p}}^{\mathrm{s}0} \\ \underline{I}_{\mathrm{p}}^{\mathrm{s}1} \\ \underline{I}_{-\mathrm{p}}^{\mathrm{s}2} \\ \underline{I}_{-3\mathrm{p}}^{\mathrm{s}0} \\ \vdots \end{bmatrix} = \mathrm{diag}\begin{bmatrix} \vdots \\ \mathrm{j}3\mathrm{p}\Omega \\ \mathrm{j}\mathrm{p}\Omega \\ -\mathrm{j}\mathrm{p}\Omega \\ -\mathrm{j}3\mathrm{p}\Omega \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} \vdots \\ \underline{\psi}_{3\mathrm{p}}^{\mathrm{s}} \\ \underline{\psi}_{\mathrm{p}}^{\mathrm{s}} \\ \underline{\psi}_{-\mathrm{p}}^{\mathrm{s}} \\ \underline{\psi}_{-3\mathrm{p}}^{\mathrm{s}} \\ \vdots \end{bmatrix} - \begin{bmatrix} \vdots \\ 0 \\ \mathrm{E} \\ \mathrm{E} \\ 0 \\ \vdots \end{bmatrix} \tag{45}$$

where:

$$L^{\mathrm{s}0} = L_{\sigma\mathrm{s}} + L^{\mathrm{ss}0} + L_{\mathrm{L}}; \quad L^{\mathrm{s}1} = L_{\sigma\mathrm{s}} + L^{\mathrm{ss}1} + L_{\mathrm{L}}; \quad L^{\mathrm{s}2} = L_{\sigma\mathrm{s}} + L^{\mathrm{ss}2} + L_{\mathrm{L}}; \quad \underline{\psi}_{\varsigma}^{\mathrm{s}} = \psi_{\varsigma}^{\mathrm{PMs}} \cdot \mathrm{e}^{\mathrm{j}\varsigma\varphi_0} \tag{46}$$

From symmetry properties of system equation (45), it can be concluded that:

$$\underline{I}_{\nu}^{\mathrm{s}0} = \check{\underline{I}}_{-\nu}^{\mathrm{s}0} \text{ and } \underline{I}_{\nu}^{\mathrm{s}1} = \check{\underline{I}}_{-\nu}^{\mathrm{s}2} \tag{47}$$

The formula showing the time form of the first phase is as follows:

$$i_1(t) = \frac{2}{\sqrt{3}} \mathrm{Re}\left\{ \sum_{n=0}^{\infty} \underline{I}_{(2n+1)\mathrm{p}}^{\mathrm{s}(2n+1)\bmod 3} \cdot \mathrm{e}^{\mathrm{j}(2n+1)\cdot \mathrm{p}\Omega t} \right\} \tag{48}$$

After formal mathematical transformations to the general formula for electromagnetic torque (33), we can derive the following equations describing electromagnetic torque:

$$T_{em} = -\mathrm{Im}\left\{ \sum_{k=0,\pm1,\pm2,\ldots} \begin{bmatrix} \cdots & \overset{\vee}{\underline{I}}{}_{3\mathrm{p}}^{\mathrm{s}0} & \overset{\vee}{\underline{I}}{}_{\mathrm{p}}^{\mathrm{s}1} & \overset{\vee}{\underline{I}}{}_{-\mathrm{p}}^{\mathrm{s}2} & \cdots \end{bmatrix} \cdot \begin{bmatrix} \vdots \\ (3\mathrm{p}+6k\cdot\mathrm{p})\,\underline{\psi}_{3\mathrm{p}+k_1 6\mathrm{p}}^{\mathrm{s}} \\ (\mathrm{p}+6k\cdot\mathrm{p})\,\underline{\psi}_{\mathrm{p}+k_1 6\mathrm{p}}^{\mathrm{s}} \\ (-\mathrm{p}+6k\cdot\mathrm{p})\,\underline{\psi}_{-\mathrm{p}+k_1 6\mathrm{p}}^{\mathrm{s}} \\ \vdots \end{bmatrix} \cdot \mathrm{e}^{\mathrm{j}6k\cdot \mathrm{p}\Omega t} \right\} \tag{49}$$

5. Conclusions

This paper presents an analytical description of the magnetic field distribution in the air gap of the AFPM machine, discusses the methodology concerning the creation of circuit models 3-phase AFPMG and deals with analyses of a steady-state operation. The harmonic balance method was used for model in symmetrical components, enabling direct tracking of interactions between space harmonics, generated in winding currents and in the electromagnetic torque. By analyzing those formulas, it can be concluded that, in the steady-state AFPMG produced by the interaction of space harmonics, currents at pulsation orders of pΩ, 3pΩ, 5pΩ, 7pΩ…, while in electromagnetic torque components, pulsations at orders of 6pΩ, 12pΩ, 18pΩ…are present.

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