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ON THE BIVARIATE BASKAKOV-DURRMEYER TYPE OPERATORS

O OPERATORACH DWÓCH ZMIENNYCH
TYPU BASKAKOWA-DURRMAYERA

A b s t r a c t

In this paper we introduce some linear positive operators of the Baskakov-Durrmeyer type in the space of continuous functions of two variables. The theorems on convergence and the degree of approximation are established.

Keywords: *Baskakov-Durrmeyer type operators, linear operators, approximation order*

S t r e s z c z e n i e

W artykule definiuje się dodatnie operatory liniowe typu Baskakowa-Durrmeyeera w przestrzeni ciągłych funkcji dwóch zmiennych. Formuluje się i dowodzi twierdzenia dotyczące zbieżności oraz rzędu zbieżności.

Słowa kluczowe: *operatory typu Baskakowa-Durrmeyeera, operatory liniowe, rzяд aproksymacji*

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1. Introduction

In recent years, several researchers have studied various modifications of the Baskakov-Durrmeyer operators. The approximation properties of these operators in many different spaces were considered, for example, in [4, 8, 10, 11, 18, 19].

A large amount of literature is available on approximation of function of one variable, but the corresponding problem for bivariate functions has received less attention. The bivariate Bernstein operator was first introduced by Dhingas [3] and it was also considered by Lorentz [9] and Stancu [14]. Recently, some positive linear operators for function of two variables and their approximation properties were investigated in a series of research articles (e.g. [2, 5, 6, 7, 12, 13, 15, 17, 20, 21]).

In this paper, we will introduce the Baskakov-Durrmeyer type operators in the space of continuous functions of two variables. This is an extension of the paper [10] for a bivariate case.

Let $\mathbb{R}_0^+ = [0, \infty)$ and $\mathbb{R}_+^2 = \mathbb{R}_0^+ \times \mathbb{R}_0^+$. We denote by $C(\mathbb{R}_+^2)$ the space of all real-valued functions continuous on \mathbb{R}_+^2 and by $C_B(\mathbb{R}_+^2)$ – the space of functions continuous and bounded on \mathbb{R}_+^2 . The norm on $C_B(\mathbb{R}_+^2)$ is defined by

$$\|f\|_{C_B(\mathbb{R}_+^2)} = \sup_{(x,y) \in \mathbb{R}_+^2} |f(x,y)|.$$

Let

$$W_{n,k}^a(x) = e^{-\frac{ax}{1+x}} \sum_{i=0}^k \binom{k}{i} (n)_i a^{k-i} \frac{x^k}{k!(1+x)^{n+k}},$$

where $a \in \mathbb{R}_0^+$, $(n)_0 = 1$, $(n)_i = n(n+1)\dots(n+i-1)$, $i \geq 1$.

We consider the class of operators $M_{n,m}^{\alpha,\beta,a,b}$ given by the formula

$$\begin{aligned} M_{n,m}^{\alpha,\beta,a,b}(f; x, y) &= mn \sum_{k,l=0}^{\infty} W_{n,k}^a(x) W_{m,l}^b(y) \frac{1}{\Gamma(\alpha+k+1)} \frac{1}{\Gamma(\beta+l+1)} \\ &\quad \times \int_0^\infty \int_0^\infty e^{-ns} (ns)^{\alpha+k} e^{-mz} (mz)^{\beta+l} f(s, z) ds dz \end{aligned}$$

for $(x, y) \in \mathbb{R}_+^2$, where $m, n \in \mathbb{N}$, $a, b \in \mathbb{R}_0^+$, $\alpha, \beta > -1$. It is clear that the operator $M_{n,m}^{\alpha,\beta,a,b}$ is linear and positive on \mathbb{R}_+^2 . In this paper we study some approximation properties of $M_{n,m}^{\alpha,\beta,a,b}$ in the space of continuous functions of two variables on a compact set. We find the order of this approximation using full and partial modulus of continuity.

Observe that if $f(s, z) = f_1(s)f_2(z)$, then

$$M_{n,m}^{\alpha,\beta,a,b}(f; x, y) = M_n^{\alpha,a}(f_1; x) M_m^{\beta,b}(f_2; y), \quad (1.1)$$

where

$$M_n^{\alpha,a}(f_1; x) = n \sum_{k=0}^{\infty} W_{n,k}^a(x) \frac{1}{\Gamma(\alpha+k+1)} \int_0^{\infty} e^{-ns} (ns)^{\alpha+k} f_1(s) ds.$$

Some properties of the operator $M_n^{\alpha,a}$, in particular, an estimation of the rate of convergence, were studied in [10].

Let $(x, y) \in \mathbb{R}_+^2$ and

$$e^{i,j}(s, z) = s^i z^j, \quad \phi_{x,y}^{i,j}(s, z) = (s-x)^i (z-y)^j, \quad i, j = 0, 1, 2, 4, \quad (s, z) \in \mathbb{R}_+^2.$$

Now, we give some lemmas which will be useful in the future proofs of the main results. The following lemmas are simple consequences of the above definitions and the results obtained in [10, Lemma 2.2, Lemma 2.3].

Lemma 1. Let $m, n \in \mathbb{N}$, $a, b \in \mathbb{R}_0^+$, $\alpha, \beta > -1$. For $(x, y) \in \mathbb{R}_+^2$ we get

$$M_{n,m}^{\alpha,\beta,a,b}(e^{0,0}; x, y) = 1, \quad (1.2)$$

$$M_{n,m}^{\alpha,\beta,a,b}(e^{1,0}; x, y) = \frac{\alpha+1}{n} + x + \frac{ax}{n(1+x)}, \quad (1.3)$$

$$M_{n,m}^{\alpha,\beta,a,b}(e^{0,1}; x, y) = \frac{\beta+1}{m} + y + \frac{by}{m(1+y)}, \quad (1.4)$$

$$\begin{aligned} M_{n,m}^{\alpha,\beta,a,b}(e^{2,0}; x, y) &= \frac{(\alpha+1)(\alpha+2)}{n^2} + \frac{2(\alpha+2)x+x^2}{n} + x^2 + \frac{a^2x^2}{n^2(1+x)^2} \\ &\quad + \frac{2ax^2}{n(1+x)} + \frac{2(\alpha+2)ax}{n^2(1+x)}, \end{aligned} \quad (1.5)$$

$$\begin{aligned} M_{n,m}^{\alpha,\beta,a,b}(e^{0,2}; x, y) &= \frac{(\beta+1)(\beta+2)}{m^2} + \frac{2(\beta+2)y+y^2}{m} + y^2 + \frac{b^2y^2}{m^2(1+y)^2} \\ &\quad + \frac{2by^2}{m(1+y)} + \frac{2(\beta+2)by}{m^2(1+y)}. \end{aligned} \quad (1.6)$$

Lemma 2. Let $m, n \in \mathbb{N}$, $a, b \in \mathbb{R}_0^+$, $\alpha, \beta > -1$. For $(x, y) \in \mathbb{R}_+^2$ we get

$$M_{n,m}^{\alpha,\beta,a,b}(\phi_{x,y}^{1,0}; x, y) = \frac{\alpha+1}{n} + \frac{ax}{n(1+x)},$$

$$M_{n,m}^{\alpha,\beta,a,b}(\phi_{x,y}^{0,1}; x, y) = \frac{\beta+1}{m} + \frac{by}{m(1+y)},$$

$$M_{n,m}^{\alpha,\beta,a,b}(\phi_{x,y}^{1,1}; x, y) = \frac{(\alpha+1)(\beta+1)}{nm} + \frac{(\alpha+1)by}{nm(1+y)} + \frac{(\beta+1)ax}{nm(1+x)} + \frac{abxy}{nm(1+x)(1+y)},$$

$$M_{n,m}^{\alpha,\beta,a,b}(\phi_{x,y}^{2,0};x,y) = \frac{(\alpha+1)(\alpha+2)}{n^2} + \frac{2x+x^2}{n} + \frac{a^2x^2}{n^2(1+x)^2} + \frac{2(\alpha+2)\alpha x}{n^2(1+x)},$$

$$M_{n,m}^{\alpha,\beta,a,b}(\phi_{x,y}^{0,2};x,y) = \frac{(\beta+1)(\beta+2)}{m^2} + \frac{2y+y^2}{m} + \frac{b^2y^2}{m^2(1+y)^2} + \frac{2(\beta+2)\beta y}{m^2(1+y)}.$$

Theorem 1. For each $f \in C_B(\mathbb{R}_+^2)$, we have

$$\|M_{n,m}^{\alpha,\beta,a,b}(f)\|_{C_B(\mathbb{R}_+^2)} \leq \|f\|_{C_B(\mathbb{R}_+^2)}$$

for all $n, m \in \mathbb{N}$.

Proof. Using the definition $M_{n,m}^{\alpha,\beta,a,b}$, we obtain

$$\begin{aligned} |M_{n,m}^{\alpha,\beta,a,b}(f; x, y)| &\leq mn \sum_{k,l=0}^{\infty} W_{n,k}^a(x) W_{m,l}^b(y) \frac{1}{\Gamma(\alpha+k+1)} \frac{1}{\Gamma(\beta+l+1)} \\ &\quad \times \int_0^{\infty} \int_0^{\infty} e^{-ns} (ns)^{\alpha+k} e^{-mz} (mz)^{\beta+l} |f(s, z)| ds dz \\ &\leq \sup_{(s,z) \in \mathbb{R}_+^2} |f(s, z)| mn \sum_{k,l=0}^{\infty} W_{n,k}^a(x) W_{m,l}^b(y) \frac{1}{\Gamma(\alpha+k+1)} \frac{1}{\Gamma(\beta+l+1)} \\ &\quad \times \int_0^{\infty} \int_0^{\infty} e^{-ns} (ns)^{\alpha+k} e^{-mz} (mz)^{\beta+l} ds dz \\ &= \sup_{(s,z) \in \mathbb{R}_+^2} |f(s, z)| M_{n,m}^{\alpha,\beta,a,b}(e^{0,0}; x, y) = \sup_{(s,z) \in \mathbb{R}_+^2} |f(s, z)| = \|f\|, \end{aligned}$$

which gives the result. \square

Theorem 2 [22]. Let I_1 and I_2 be compact intervals of the real line. Let $n, m \in \mathbb{N}$ and $T_{n,m} : C(I_1 \times I_2) \rightarrow C(I_1 \times I_2)$ be linear positive operators. If

$$\lim_{n,m \rightarrow \infty} T_{n,m}(e^{i,j}) = e^{i,j}, \quad (i, j) \in \{(0,0), (1,0), (0,1)\}$$

and

$$\lim_{n,m \rightarrow \infty} T_{n,m}(e^{2,0} + e^{0,2}) = e^{2,0} + e^{0,2},$$

uniformly on $I_1 \times I_2$, then the sequence $(T_{n,m} f)$ converges to f uniformly on $I_1 \times I_2$, for any $f \in C(I_1 \times I_2)$.

Let $A, B > 0$. Throughout the rest of this paper we will denote $\mathbb{R}_{AB}^2 = [0, A] \times [0, B]$.

Theorem 3. Let $(x, y) \in \mathbb{R}_{AB}^2$ are fixed. If $f \in C(\mathbb{R}_{AB}^2)$, then

$$\lim_{n,m \rightarrow \infty} M_{n,m}^{\alpha,\beta,a,b}(f; x, y) = f(x, y).$$

Moreover, this convergence is uniform on \mathbb{R}_{AB}^2 .

Proof. Using (1.2)–(1.6), we have

$$\lim_{n,m \rightarrow \infty} M_{n,m}^{\alpha,\beta,a,b}(e^{i,j};x,y) = e^{i,j}(x,y), \quad (i,j) \in \{(0,0),(1,0),(0,1)\}$$

and

$$\lim_{n,m \rightarrow \infty} M_{n,m}^{\alpha,\beta,a,b}(e^{2,0} + e^{0,2};x,y) = e^{2,0}(x,y) + e^{0,2}(x,y)$$

uniformly on \mathbb{R}_{AB}^2 . Applying Theorem 2, the proof of the theorem is completed. \square

2. Local approximation results

In this section we will investigate the degree of approximation for functions of two variables by operators $M_{n,m}^{\alpha,\beta,a,b}$ in terms of the modulus of continuity on a compact set.

Let $f \in C(\mathbb{R}_{AB}^2)$ and $\delta > 0$. The full continuity modulus of the function f is defined as (see [1], [16])

$$\omega(f;\delta) = \sup_{\substack{(s,z),(x,y) \in \mathbb{R}_{AB}^2 \\ (s-x)^2 + (z-y)^2 \leq \delta^2}} |f(s,z) - f(x,y)|$$

and its partial continuity moduli are given by

$$\omega^{(1)}(f;\delta) = \sup_{\substack{0 \leq z \leq B \\ |s-x| \leq \delta}} |f(s,z) - f(x,z)|,$$

$$\omega^{(2)}(f;\delta) = \sup_{\substack{0 \leq s \leq A \\ |z-y| \leq \delta}} |f(s,z) - f(s,y)|.$$

It is known that $\lim_{\delta \rightarrow 0} \omega(f;\delta) = 0$, $\omega(f;\delta_1) \leq \omega(f;\delta_2)$ for $0 < \delta_1 \leq \delta_2$ and for any $\lambda > 0$, $\omega(f;\lambda\delta) \leq (1+\lambda)\omega(f;\delta)$. The same properties are satisfied by partial continuity moduli. The details of the modulus of continuity for the bivariate case can be found in [1].

Theorem 4. Let $f \in C(\mathbb{R}_{AB}^2)$. For $(x,y) \in \mathbb{R}_{AB}^2$, we have

$$|M_{n,m}^{\alpha,\beta,a,b}(f;x,y) - f(x,y)| \leq 2\omega(f;\delta),$$

where

$$\begin{aligned} \delta = & \left(\frac{(\alpha+1)(\alpha+2)}{n^2} + \frac{2x+x^2}{n} + \frac{a^2x^2}{n^2(1+x)^2} + \frac{2(\alpha+2)ax}{n^2(1+x)} \right. \\ & \left. + \frac{(\beta+1)(\beta+2)}{m^2} + \frac{2y+y^2}{m} + \frac{b^2y^2}{m^2(1+y)^2} + \frac{2(\beta+2)by}{m^2(1+y)} \right)^{1/2}. \end{aligned}$$

Proof. Let $\delta > 0$. If $\sqrt{(s-x)^2 + (z-y)^2} \leq \delta$, then $|f(s, z) - f(x, y)| \leq \omega(f; \delta)$. If $\sqrt{(s-x)^2 + (z-y)^2} > \delta$, then

$$\frac{(s-x)^2 + (z-y)^2}{\delta^2} > \frac{\sqrt{(s-x)^2 + (z-y)^2}}{\delta} > 1.$$

Therefore, we obtain

$$\begin{aligned} |f(s, z) - f(x, y)| &\leq \omega\left(f; \sqrt{(s-x)^2 + (z-y)^2}\right) \\ &\leq \left(1 + \frac{\sqrt{(s-x)^2 + (z-y)^2}}{\delta}\right) \omega(f; \delta) \leq \left(1 + \frac{(s-x)^2 + (z-y)^2}{\delta^2}\right) \omega(f; \delta). \end{aligned}$$

The operator $M_{n,m}^{\alpha,\beta,a,b}$ is positive and linear, so

$$\begin{aligned} |M_{n,m}^{\alpha,\beta,a,b}(f; x, y) - f(x, y)| &\leq M_{n,m}^{\alpha,\beta,a,b}(|f - f(x, y)|; x, y) \\ &\leq \omega(f; \delta) \left(M_{n,m}^{\alpha,\beta,a,b}(e^{0,0}; x, y) + \frac{1}{\delta^2} M_{n,m}^{\alpha,\beta,a,b}(\phi_{x,y}^{2,0} + \phi_{x,y}^{0,2}; x, y) \right). \end{aligned}$$

From Lemma 2 we obtain

$$\begin{aligned} |M_{n,m}^{\alpha,\beta,a,b}(f; x, y) - f(x, y)| &\leq M_{n,m}^{\alpha,\beta,a,b}(|f - f(x, y)|; x, y) \\ &\leq \omega(f; \delta) \left\{ 1 + \frac{1}{\delta^2} \left(\frac{(\alpha+1)(\alpha+2)}{n^2} + \frac{2x+x^2}{n} + \frac{a^2x^2}{n^2(1+x)^2} + \frac{2(\alpha+2)ax}{n^2(1+x)} \right. \right. \\ &\quad \left. \left. + \frac{(\beta+1)(\beta+2)}{m^2} + \frac{2y+y^2}{m} + \frac{b^2y^2}{m^2(1+y)^2} + \frac{2(\beta+2)by}{m^2(1+y)} \right) \right\}, \end{aligned}$$

which ends the proof. \square

Theorem 5. If $f \in C(\mathbb{R}_{AB}^2)$, then for all $(x, y) \in \mathbb{R}_{AB}^2$, we have

$$\begin{aligned} |M_{n,m}^{\alpha,\beta,a,b}(f; x, y) - f(x, y)| &\leq \left(1 + \frac{(\alpha+1)(\alpha+2)}{n} + 2x+x^2 + \frac{a^2x^2}{n(1+x)^2} + \frac{2(\alpha+2)ax}{n(1+x)} \right) \omega^{(1)}\left(f; \frac{1}{\sqrt{n}}\right) \\ &\quad + \left(1 + \frac{(\beta+1)(\beta+2)}{m} + 2y+y^2 + \frac{b^2y^2}{m(1+y)^2} + \frac{2(\beta+2)by}{m(1+y)} \right) \omega^{(2)}\left(f; \frac{1}{\sqrt{m}}\right). \end{aligned}$$

Proof. Let $f \in C(\mathbb{R}_{AB}^2)$. Observe that

$$\begin{aligned} \left| M_{n,m}^{\alpha,\beta,a,b}(f; x, y) - f(x, y) \right| &\leq mn \sum_{k,l=0}^{\infty} W_{n,k}^a(x) W_{m,l}^b(y) \frac{1}{\Gamma(\alpha+k+1)} \frac{1}{\Gamma(\beta+l+1)} \\ &\quad \times \int_0^{\infty} \int_0^{\infty} e^{-ns} (ns)^{\alpha+k} e^{-mz} (mz)^{\beta+l} |f(s, z) - f(x, z)| ds dz \\ &\quad + mn \sum_{k,l=0}^{\infty} W_{n,k}^a(x) W_{m,l}^b(y) \frac{1}{\Gamma(\alpha+k+1)} \frac{1}{\Gamma(\beta+l+1)} \\ &\quad \times \int_0^{\infty} \int_0^{\infty} e^{-ns} (ns)^{\alpha+k} e^{-mz} (mz)^{\beta+l} |f(x, z) - f(x, y)| ds dz \\ &= J_1 + J_2. \end{aligned}$$

Using the properties of the modulus of continuity and (1.5), we have

$$\begin{aligned} J_1 &= mn \sum_{k,l=0}^{\infty} W_{n,k}^a(x) W_{m,l}^b(y) \frac{1}{\Gamma(\alpha+k+1)} \frac{1}{\Gamma(\beta+l+1)} \\ &\quad \times \int_0^{\infty} \int_0^{\infty} e^{-ns} (ns)^{\alpha+k} e^{-mz} (mz)^{\beta+l} |f(s, z) - f(x, z)| ds dz \\ &\leq \omega^{(1)}(f; \delta_n) \left\{ 1 + \frac{1}{\delta_n^2} M_{n,m}^{\alpha,\beta,a,b}(\phi_{x,y}^{2,0}; x, y) \right\} \\ &\leq \left(1 + \frac{(\alpha+1)(\alpha+2)}{n} + 2x + x^2 + \frac{a^2 x^2}{n(1+x)^2} + \frac{2(\alpha+2)ax}{n(1+x)} \right) \omega^{(1)}\left(f; \frac{1}{\sqrt{n}}\right), \end{aligned}$$

where $\delta_n = \frac{1}{\sqrt{n}}$. Similarly, we obtain

$$J_2 \leq \left(1 + \frac{(\beta+1)(\beta+2)}{m} + 2y + y^2 + \frac{b^2 y^2}{m(1+y)^2} + \frac{2(\beta+2)by}{m(1+y)} \right) \omega^{(2)}\left(f; \frac{1}{\sqrt{m}}\right).$$

Hence, the proof is completed. \square

Now, we consider the mixed modulus of smoothness and the modulus of smoothness (see [16]). Let $\delta_j > 0, j = 1, 2$.

The mixed modulus of smoothness is defined as

$$\omega_{\text{mix}}(f; \delta_1, \delta_2) = \sup_{\substack{|s-x| \leq \delta_1, |z-y| \leq \delta_2 \\ (x,y), (s,z) \in \mathbb{R}_{AB}^2}} |f(s, z) - f(x, z) - f(s, y) + f(x, y)|$$

and the modulus of smoothness of the first and the second order are given by

$$\omega_1(f; \delta_1, \delta_2) = \sup_{\substack{0 \leq h \leq \delta_1, 0 \leq k \leq \delta_2 \\ (x,y), (x+h, y+k) \in \mathbb{R}_{AB}^2}} |f(x+h, y+k) - f(x, y)|,$$

$$\omega_2(f; \delta_1, \delta_2) = \sup_{\substack{0 \leq h \leq \delta_1, 0 \leq k \leq \delta_2 \\ (x,y), (x+2h, y+2k) \in \mathbb{R}_{AB}^2}} |f(x+2h, y+2k) - 2f(x+h, y+k) + f(x, y)|,$$

respectively.

Theorem 6. Let $f \in C(\mathbb{R}_{AB}^2)$ and

$$H_{n,m}^{\alpha,\beta,a,b}(f; x, y) = M_{n,m}^{\alpha,\beta,a,b}(f; x, y) - f\left(\frac{\alpha+1}{n}x + \frac{ax}{n(1+x)}, \frac{\beta+1}{m}y + \frac{by}{m(1+y)}\right) + f(x, y).$$

There exists a positive constant C such that, for all $(x, y) \in \mathbb{R}_{AB}^2$, we have

$$|H_{n,m}^{\alpha,\beta,a,b}(g; x, y) - g(x, y)| \leq C \left\{ \frac{1}{n} \left\| \frac{\partial^2 g}{\partial u^2} \right\|_{C(\mathbb{R}_{AB}^2)} + \frac{1}{m} \left\| \frac{\partial^2 g}{\partial v^2} \right\|_{C(\mathbb{R}_{AB}^2)} + \frac{1}{nm} \left\| \frac{\partial^2 g}{\partial u \partial v} \right\|_{C(\mathbb{R}_{AB}^2)} \right\}$$

for any function g , such that $g, \frac{\partial^i g}{\partial x^i}, \frac{\partial^i g}{\partial y^i}, \frac{\partial^2 g}{\partial x \partial y}$ ($i = 1, 2$) belong to $C(\mathbb{R}_{AB}^2)$.

Proof. Let $(x, y) \in \mathbb{R}_{AB}^2$. Observe that

$$\begin{aligned} g(s, z) - g(x, y) &= (s-x) \frac{\partial g(x, y)}{\partial x} + (z-y) \frac{\partial g(x, y)}{\partial y} \\ &\quad + \int_x^s (s-u) \frac{\partial^2 g(u, y)}{\partial u^2} du + \int_y^z (z-v) \frac{\partial^2 g(x, v)}{\partial v^2} dv + \int_x^s \int_y^z \frac{\partial^2 g(u, v)}{\partial u \partial v} dv du. \end{aligned}$$

We have

$$H_{n,m}^{\alpha,\beta,a,b}(e^{0,0}; x, y) = 1, \quad H_{n,m}^{\alpha,\beta,a,b}(\phi_{x,y}^{1,0}; x, y) = H_{n,m}^{\alpha,\beta,a,b}(\phi_{x,y}^{0,1}; x, y) = 0.$$

Let

$$\xi_g^{i,j}(s, z) = \left(\int_x^s (s-u) \frac{\partial^2 g(u, y)}{\partial u^2} du \right)^i \left(\int_y^z (z-v) \frac{\partial^2 g(x, v)}{\partial v^2} dv \right)^j, \quad i, j = 0, 1$$

and

$$\xi_g(s, z) = \int_x^s \int_y^z \frac{\partial^2 g(u, v)}{\partial u \partial v} du dv.$$

Hence

$$H_{n,m}^{\alpha,\beta,a,b}(g; x, y) - g(x, y) = H_{n,m}^{\alpha,\beta,a,b}(\xi_g^{1,0}; x, y) + H_{n,m}^{\alpha,\beta,a,b}(\xi_g^{0,1}; x, y) + H_{n,m}^{\alpha,\beta,a,b}(\xi_g; x, y).$$

Using the definition of $H_{n,m}^{\alpha,\beta,a,b}$, we can write

$$\begin{aligned}
|H_{n,m}^{\alpha,\beta,a,b}(\xi_g^{1,0};x,y)| &= \left| M_{n,m}^{\alpha,\beta,a,b}(\xi_g^{1,0};x,y) \right. \\
&\quad \left. - \int_x^{\frac{\alpha+1}{n}+x+\frac{ax}{n(1+x)}} \left(\frac{\alpha+1}{n} + x + \frac{ax}{n(1+x)} - u \right) \frac{\partial^2 g(u,y)}{\partial u^2} du \right| \\
&\leq \left| M_{n,m}^{\alpha,\beta,a,b}(\xi_g^{1,0};x,y) \right| \\
&\quad + \left| \int_x^{\frac{\alpha+1}{n}+x+\frac{ax}{n(1+x)}} \left(\frac{\alpha+1}{n} + x + \frac{ax}{n(1+x)} - u \right) \frac{\partial^2 g(u,y)}{\partial u^2} du \right| \\
&\leq \frac{1}{2} \sup_{(u,v) \in \mathbb{R}_{AB}^2} \left| \frac{\partial^2 g(u,y)}{\partial u^2} \right| M_{n,m}^{\alpha,\beta,a,b}(\phi_{x,y}^{2,0};x,y) \\
&\quad + \frac{1}{2} \sup_{(u,v) \in \mathbb{R}_{AB}^2} \left| \frac{\partial^2 g(u,y)}{\partial u^2} \right| \left(\frac{\alpha+1}{n} + \frac{ax}{n(1+x)} \right)^2 \\
&\leq C_1 \frac{1}{n} \left\| \frac{\partial^2 g}{\partial u^2} \right\|_{C(\mathbb{R}_{AB}^2)}
\end{aligned}$$

and similarly, we get

$$\begin{aligned}
|H_{n,m}^{\alpha,\beta,a,b}(\xi_g^{0,1};x,y)| &\leq C_2 \frac{1}{m} \left\| \frac{\partial^2 g}{\partial v^2} \right\|_{C(\mathbb{R}_{AB}^2)}, \\
|H_{n,m}^{\alpha,\beta,a,b}(\xi_g; x, y)| &\leq C_3 \frac{1}{nm} \left\| \frac{\partial^2 g}{\partial u \partial v} \right\|_{C(\mathbb{R}_{AB}^2)},
\end{aligned}$$

where C_1, C_2, C_3 are positive constants. Hence

$$|H_{n,m}^{\alpha,\beta,a,b}(g;x,y) - g(x,y)| \leq C \left\{ \frac{1}{n} \left\| \frac{\partial^2 g}{\partial x^2} \right\|_{C(\mathbb{R}_{AB}^2)} + \frac{1}{m} \left\| \frac{\partial^2 g}{\partial v^2} \right\|_{C(\mathbb{R}_{AB}^2)} + \frac{1}{nm} \left\| \frac{\partial^2 g}{\partial u \partial v} \right\|_{C(\mathbb{R}_{AB}^2)} \right\}$$

for some $C > 0$ and the theorem is proved. \square

Theorem 7. If $f \in C(\mathbb{R}_{AB}^2)$, then

$$\begin{aligned}
|M_{n,m}^{\alpha,\beta,a,b}(f;x,y) - f(x,y)| &\leq C \left\{ \omega_2 \left(f; \sqrt{\frac{1}{n}}, \sqrt{\frac{1}{m}} \right) + \omega_{mix} \left(f; \sqrt{\frac{1}{n}}, \sqrt{\frac{1}{m}} \right) \right. \\
&\quad \left. + \omega_1 \left(f; \frac{1}{n} \left(\alpha + 1 + \frac{ax}{1+x} \right), \frac{1}{m} \left(\beta + 1 + \frac{by}{1+y} \right) \right) \right\},
\end{aligned}$$

where $C > 0$, $(x,y) \in \mathbb{R}_{AB}^2$.

Proof. Let $f \in C(\mathbb{R}_{AB}^2)$ and $\delta_j > 0, j = 1, 2$. We shall use the Steklov function of second order defined by

$$\begin{aligned} f_{\delta_1 \delta_2}(x, y) &= \frac{16}{\delta_1^2 \delta_2^2} \int_0^{\frac{\delta_2}{2}} \int_0^{\frac{\delta_2}{2}} \int_0^{\frac{\delta_1}{2}} \int_0^{\frac{\delta_1}{2}} 2f(x + s_1 + s_2, y + z_1 + z_2) \\ &\quad - f(x + 2(s_1 + s_2), y + 2(z_1 + z_2)) ds_1 ds_2 dz_1 dz_2. \end{aligned}$$

Observe that

$$|f_{\delta_1 \delta_2}(x, y) - f(x, y)| \leq \omega_2(f; \delta_1, \delta_2)$$

and

$$\begin{aligned} f_{\delta_1 \delta_2}(x, y) &= \frac{32}{\delta_1^2 \delta_2^2} \int_0^{\frac{\delta_2}{2}} \int_0^{\frac{\delta_2}{2}} \int_x^{x+\frac{\delta_1}{2}} \int_u^{u+\frac{\delta_1}{2}} f(s, y + z_1 + z_2) ds du dz_1 dz_2 \\ &\quad - \frac{4}{\delta_1^2 \delta_2^2} \int_0^{\frac{\delta_2}{2}} \int_0^{\frac{\delta_2}{2}} \int_x^{x+\delta_1} \int_u^{u+\delta_1} f(s, y + 2(z_1 + z_2)) ds du dz_1 dz_2 \\ &= \frac{32}{\delta_1^2 \delta_2^2} \int_y^{y+\frac{\delta_2}{2}} \int_v^{v+\frac{\delta_2}{2}} \int_0^{\frac{\delta_1}{2}} \int_0^{\frac{\delta_1}{2}} f(x + s_1 + s_2, w) ds_1 ds_2 dw dv \\ &\quad - \frac{4}{\delta_1^2 \delta_2^2} \int_y^{y+\delta_2} \int_v^{v+\delta_2} \int_0^{\frac{\delta_1}{2}} \int_0^{\frac{\delta_1}{2}} f(x + 2s_1 + 2s_2, w) ds_1 ds_2 dw dv \\ &= \frac{32}{\delta_1^2 \delta_2^2} \int_0^{\frac{\delta_2}{2}} \int_y^{y+\frac{\delta_2}{2}} \int_0^{\frac{\delta_1}{2}} \int_x^{x+\frac{\delta_1}{2}} f(u + s_2, v + z_2) du ds_2 dv dz_2 \\ &\quad - \frac{4}{\delta_1^2 \delta_2^2} \int_0^{\frac{\delta_2}{2}} \int_y^{y+\delta_2} \int_0^{\frac{\delta_1}{2}} \int_x^{x+\delta_1} f(u + 2s_2, v + 2z_2) du ds_2 dv dz_2. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\partial^2}{\partial x^2} f_{\delta_1 \delta_2}(x, y) &= \frac{32}{\delta_1^2 \delta_2^2} \int_0^{\frac{\delta_2}{2}} \int_0^{\frac{\delta_2}{2}} \left[f(x + \delta_1, y + z_1 + z_2) \right. \\ &\quad \left. - 2f\left(x + \frac{\delta_1}{2}, y + z_1 + z_2\right) + f(x, y + z_1 + z_2) \right] dz_1 dz_2 \\ &\quad - \frac{4}{\delta_1^2 \delta_2^2} \int_0^{\frac{\delta_2}{2}} \int_0^{\frac{\delta_2}{2}} \left[f(x + 2\delta_1, y + 2(z_1 + z_2)) \right. \\ &\quad \left. - 2f(x + \delta_1, y + 2(z_1 + z_2)) + f(x, y + 2(z_1 + z_2)) \right] dz_1 dz_2 \end{aligned}$$

and $\left| \frac{\partial^2}{\partial x^2} f_{\delta_1 \delta_2}(x, y) \right| \leq \frac{8}{\delta_1^2} \omega_2\left(f; \frac{\delta_1}{2}, \frac{\delta_2}{2}\right) + \frac{1}{\delta_1^2} \omega_2(f; \delta_1, \delta_2) \leq \frac{9}{\delta_1^2} \omega_2(f; \delta_1, \delta_2).$

Similarly, we get

$$\begin{aligned} \left| \frac{\partial^2}{\partial y^2} f_{\delta_1 \delta_2}(x, y) \right| &\leq \frac{9}{\delta_2^2} \omega_2(f; \delta_1, \delta_2), \\ \left| \frac{\partial^2}{\partial x \partial y} f_{\delta_1 \delta_2}(x, y) \right| &\leq \frac{9}{\delta_1 \delta_2} \omega_{\text{mix}}(f; \delta_1, \delta_2), \quad (x, y) \in \mathbb{R}_{AB}^2. \end{aligned}$$

From the above and by Theorem 6, we obtain

$$\begin{aligned} &|M_{n,m}^{\alpha,\beta,a,b}(f; x, y) - f(x, y)| \\ &\leq H_{n,m}^{\alpha,\beta,a,b}(|f - f_{\delta_1 \delta_2}|; x, y) + |H_{n,m}^{\alpha,\beta,a,b}(f_{\delta_1 \delta_2}; x, y) - f_{\delta_1 \delta_2}(x, y)| + |f_{\delta_1 \delta_2}(x, y) - f(x, y)| \\ &\quad + \left| f\left(\frac{\alpha+1}{n} + x + \frac{ax}{n(1+x)}, \frac{\beta+1}{m} + y + \frac{by}{m(1+y)}\right) - f(x, y) \right| \\ &\leq C \left\{ \omega_2(f; \delta_1, \delta_2) + \omega_{\text{mix}}(f; \delta_1, \delta_2) + \omega_1\left(f; \frac{\alpha+1}{n} + \frac{ax}{n(1+x)}, \frac{\beta+1}{m} + \frac{by}{m(1+y)}\right) \right\}, \end{aligned}$$

where C is a positive constant. This completes the proof. \square

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