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CONVERGENCE IN MEASURE THROUGH COMPACTIFICATIONS

ZBIEŻNOŚĆ WEDŁUG MIARY POPRZEZ UZWARCENIA

Abstract

For a metrizable space X, concepts of metric convergence in measure and of functional convergence in measure of sequences of measurable mappings taking their values in X are introduced and applied to a comparison of compactifications of X.

Keywords: Metrizable space, Hausdorff compactification, mimimum uniform compactification, infinite σ-finite measure, metric convergence in measure, functional convergence in measure

Streszczenie

Dla przestrzeni metryzowalnej X, wprowadza się pojęcia metrycznej zbieżności według miary i funkcyjnej zbieżności według miary ciągów odwzorowań mierzalnych, przyjmujących swe wartości w X oraz stosuje się te pojęcia do porównywania uzwarceń przestrzeni X.

Słowa kluczowe: przestrzeń metryzowalna, uzwarcenie Hausdorffa, minimalne uzwarcenie jednostajne, nieskończona miara σ-skończona, metryczna zbieżność według miary, funkcyjna zbieżność według miary

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In memory of Stanisław Łojasiewicz [1926–2002]

1. Introduction

A convenient interpretation of ZFC which agrees with that of [8] is our basic set-theoretic assumption. An evident frequent use of the axiom of countable choice (CC) makes it impossible to rewrite in ZF most of the results of this work (cf. [4] and [6]–[8]); however, some of the theorems presented below are also theorems of, for instance, ZF+UFT+CC (cf [6]).

In what follows, $X = (X,\tau)$ is a non-void metrizable space, $\mathfrak{B}(X)$ is the σ-field of all Borel sets in X, and $\mathfrak{B}_s(X)$ is the collection of all separable Borel sets in X. Moreover, $\mathfrak{M}$ is a σ-field of subsets of a set E, while μ is an infinite σ-finite measure on $\mathfrak{M}$. Let $\mathcal{M}(E,X)$ be the family of all $(\mathfrak{M},\mathfrak{B}(X))$-measurable functions $f: E \to X$ such that $\mu[f^{-1}(X \setminus B_f)] = 0$ for some $B_f \in \mathfrak{B}_s(X)$. Of course, a function $f: E \to X$ is $(\mathfrak{M},\mathfrak{B}(X))$-measurable if and only if $f^{-1}(V) \in \mathfrak{M}$ whenever $V \in \tau$. If one wants to try to work without CC, since second countability and separability are not equivalents in ZF+$\neg$CC, it might be more preferable to define $\mathfrak{B}_s(X)$ as the collection of all these Borel sets of X that are second-countable as topological subspaces of X. Clearly, the second definition of $\mathfrak{B}_s(X)$ is equivalent in ZFC to our previous definition of $\mathfrak{B}_s(X)$ but not equivalent to it in ZF.

Every compactification of X is assumed to be Hausdorff. For a compactification αX of X, the collection of all bounded continuous real functions on X that are continuously extendable over αX is denoted by $C_\alpha(X)$. As usual, βX stands for the Čech-Stone compactification of X. The collection of all bounded continuous real functions on X is $C_\beta(X)$. A great role in the theory of compactifications is played by the collection $\mathcal{E}(X)$ of all sets $F \subseteq C_\beta(X)$ such that the evaluation mapping $e_F : X \to \mathbb{R}^F$ is a homeomorphic embedding where $[e_F(x)](f) = f(x)$ for all $f \in F$ and $x \in X$ (cf. e.g.[1], [2] and [11]–[13]). If $F \in \mathcal{E}(X)$, then the closure in $\mathbb{R}^F$ of the set $e_F(X)$ is a compactification of X called generated by F and denoted by $e_F X$. In particular, every compactification αX of X is generated by $C_\alpha(X)$. Since, in ZF, the sentence that Tikhonov cubes (called Hilbert cubes in [6]) are compact is equivalent with the ultrafilter theorem UFT (cf. Theorem 4.70 of [6]), it is true in ZF+UFT that, for every $F \in \mathcal{E}(X)$, the compactification $e_F X$ of X exists. This is why some theorems on compactifications in ZFC are also theorems of ZF+UFT. It is still an open problem to investigate all significant details on compactifications in ZF+UFT and show possible differences between the theories of compactifications in ZFC and in ZF+UFT. Let us leave this problem for future considerations not described in this article and, for simplicity, let us work in ZFC to avoid troublesome disasters without AC. All topological and set-theoretic concepts that we use are standard and they can be found in [2], [3], [6]–[8] and [10]. Useful facts of measure theory are taken from [5] and [9].

The paper is mainly about the following concepts of metric and functional convergence in measure:

Definition 1. Let d be a compatible metric on X and let f_n, f be functions from $\mathcal{M}(E,X)$ where $n \in \omega$. We say that the sequence $\langle f_n \rangle$ is d-convergent in measure μ to f if, for each positive real number ε, the sequence

$$\langle \mu(\{t \in E : d(f_n(t), f(t)) \geq \varepsilon\}) \rangle$$

converges to zero in $\mathbb{R}$ with the usual topology. For every compatible metric ρ on X, the ρ-convergence in μ will be called a metric convergence in μ.

Definition 2. Suppose that $\varnothing \neq F \subseteq C_\beta(X)$ and let f_n, f be functions from $\mathcal{M}(E, X)$ where $n \in \omega$. We say that the sequence $\langle f_n \rangle$ is F-convergent in measure μ to f if, for each positive real number ε and for each $\phi \in F$, the sequence

$$\langle \mu(\{t \in E : |\phi(f_n(t)) - \phi(f(t))| \geq \varepsilon\}) \rangle$$

converges to zero, i.e. if for each $\phi \in F$, the sequence $\langle \phi \circ f_n) \rangle$ converges in μ to $\phi \circ f$. For each set $H \in \mathcal{E}(X)$, the H-convergence in μ will be called a functional convergence in μ.

Definition 3. Let d, ρ be compatible metrics on X and let F, H be non-void subsets of $C_\beta(X)$. For $i, j \in \{d, \rho, F, H\}$, we say that:

1. i-convergence in μ implies j-convergence in μ if every sequence of functions from $\mathcal{M}(E, X)$ which is i-convergent in μ to a function $f \in \mathcal{M}(E, X)$ is also j-convergent in μ to f;
2. i-convergence in μ is equivalent with j-convergence in μ if i-convergence in μ implies j-convergence in μ and j-convergence in μ implies i-convergence in μ.

In the sequel, the notions of d-convergence and F-convergence in μ are applied to a comparison of compactifications of X. Recall that, for compactifications αX and γX of X, the inequality $\alpha X \leq \gamma X$ holds if and only if $C_\alpha(X) \subseteq C_\gamma(X)$; moreover, αX and γX are equivalent if and only if $C_\alpha(X) = C_\gamma(X)$. We write $\alpha X = \gamma X$ to say that αX is identified with γX, i.e. to denote that αX and γX are equivalent. One of the most interesting theorems of this paper asserts that if there exists a metrizable compactification αX of X such that $C_\alpha(X)$-convergence in μ implies $C_\beta(X)$-convergence in μ, then the space X is compact. Moreover, among other results, it is shown that if αX and γX are metrizable compactifications of X, then $\alpha X \leq \gamma X$ if and only if $C_\gamma(X)$-convergence in μ implies $C_\alpha(X)$-convergence in μ. Ideas of simple examples relevant to convergence in μ are described.

2. Metric convergence in measure and minimum uniform compactifications

For a compatible metric d on X, R. Grant Woods investigated in [14] the compactification $u_d X$ generated by the collection $\mathcal{U}_d^*(X)$ of all these bounded real functions on X that are uniformly continuous with respect to d and the standard metric induced by the absolute value on $\mathbb{R}$. If the metric d is not totally bounded, $u_d X$ is not metrizable (cf. Theorem 3.3 (b) of [14]). If the metric d is totally bounded, then $u_d X$ is the Hausdorff metric completion of the metric space (X, d) (cf. Theorem 3.3 (a) of [14] and Problem 4.5.6 of [3]). If one wants to consider minimum uniform compactifications in ZF, one should be warned that models of ZF in which there are infinite Dedekind-finite dense subsets of $\mathbb{R}$ (cf. [6]–[8]) can be used to deduce the following:

Proposition 1. *If X is an infinite Dedekind-finite dense subset of the unit interval* $[0; 1]$ *and* $d(x, y) = |x - y|$ *for* $x, y \in X$, *then d is a totally bounded complete metric on X such that* $u_d X = [0; 1]$, *while the Hausdorff metric completion of* (X, d) *is (up to an obvious isometry)* (X, d). *Therefore, it is unprovable in ZF that, for every totally bounded metric space* (X, d), *the minimum uniform compactification $u_d X$ is the Hausdorff metric completion of* (X, d).

That $u_d X = [0; 1]$ for each dense in $[0; 1]$ infinite Dedekind-finite set X in the proposition above can be shown in ZF by using Lemma 4.3.16 of [3]. Interesting problems on Hausdorff metric completions of metric spaces in ZF are described in [4]. To avoid misunderstanding, let us recall that ZFC is our basic assumption throughout this paper.

For every metrizable compactification αX of X, there exists a totally bounded metric ρ on X such that $\alpha X = u_\rho X$. To apply metric convergence in measure to minimum uniform compactifications, the following notion is useful:

Definition 4. Let d and ρ be compatible metrics on X. We say that d is uniformly smaller than ρ if the following condition holds:

$$\forall_{\varepsilon\in(0;+\infty)}\exists_{\delta\in(0;+\infty)}\forall_{x,y\in X}[\rho(x,y)<\delta \Rightarrow d(x,y)<\varepsilon].$$

Proposition 2. *Let d and ρ be compatible metrics on X such that d is not uniformly smaller than* ρ. *Then there exist functions* $f_n, f \in \mathcal{M}(E, X)$ *where* $n \in \omega$, *such that the sequence* $\langle f_n \rangle$ *is* ρ*-convergent in* μ *to f but* $\langle f_n \rangle$ *is not d-convergent in* μ *to f.*

Proof. Let us take $\varepsilon \in (0, +\infty)$ such that, for each $\delta \in (0, +\infty)$, there are $x, y \in X$ such that $\rho(x, y) < \delta$, while $d(x, y) \geq \varepsilon$. Using CC, we find sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ of points of X such that $\lim_{n\to+\infty}\rho(x_n, y_n) = 0$, while $d(x_n, y_n) \geq \varepsilon$ for each $n \in \omega$. Let $\langle E_n \rangle$ be a sequence of sets from $\mathfrak{M}$ such that $\bigcap_{n\in\omega} E_n = \emptyset$, $\mu(E \setminus E_n) < +\infty$, $\mu(E_n) = +\infty$ and $E_{n+1} \subsetneq E_n$ for all $n \in \omega$. Such a sequence $\langle E_n \rangle$ exists because the measure μ is infinite and σ-finite. Define $f_n(t) = y_0$ for $t \in E \setminus E_0$ and, for each $t \in E_i \setminus E_{i+1,}$ let $f_n(t) = y_i$ if $i \leq n$, while $f_n(t) = x_i$ if $i > n$. Moreover, put $f(t) = y_0$ for $t \in E \setminus E_1$ and, for each $i \in \omega$, let $f(t) = y_i$ when $t \in E_i \setminus E_{i+1}$. The sequence $\langle f_n \rangle$ ρ-converges in μ to f but it does not d-converge in μ to f. □

The proof to Proposition 2 can serve as a scheme of examples of sequences ρ-convergent in μ that are not d-convergent in μ.

In much the same way as for the classical convergence in measure, one can prove Propositions 3–5.

Proposition 3. *Let d be a compatible metric on X and let* $f, g \in \mathcal{M}(E, X)$. *If a sequence of functions from* $\mathcal{M}(E, X)$ *is d-convergent in* μ *to f and to g, then* $\mu(\{t \in E : f(t) \neq g(t)\}) = 0$.

Definition 5. When d is a compatible metric on X, then we say that a sequence $\langle f_n \rangle$ of functions from $\mathcal{M}(E, X)$ converges (d, μ)-uniformly on E to a function $f \in \mathcal{M}(E, X)$ if, for each $\varepsilon \in (0, +\infty)$, there exists a set $A \in \mathfrak{M}$ such that $\mu(E \setminus A) < \varepsilon$ and the convergence of $\langle f_n \rangle$ to f is uniform with respect to d on A.

Proposition 4. *When d is a compatible metric on X, then a sequence* $\langle f_n \rangle$ *of functions from* $\mathcal{M}(E,X)$ *is d-convergent in* μ *to a function* $f \in \mathcal{M}(E,X)$ *if and only if each subsequence of* $\langle f_n \rangle$ *contains a subsequence which converges* (d,μ)*-uniformly on E to f.*

Proposition 5. *If d is a compatible metric on X, then every sequence of functions from* $\mathcal{M}(E,X)$ *which is d-convergent in* μ *to a function* $f \in \mathcal{M}(E,X)$ *contains a subsequence which pointwise converges* μ*-almost everywhere on E to f.*

In the light of Proposition 5, for every pair d, ρ of compatible metrics on X and for every pair f, g of functions from $\mathcal{M}(E,X)$, it is true that if there exists a sequence $\langle f_n \rangle$ of functions from $\mathcal{M}(E,X)$ such that $\langle f_n \rangle$ is both d-convergent in μ to f and ρ-convergent in μ to g, then $f = g$ μ-almost everywhere on E, i.e. $\mu(\{t \in E : f(t) \neq g(t)\}) = 0$. Therefore, if for compatible metrics d and ρ on X, a sequence $\langle h_n \rangle$ of functions from $\mathcal{M}(E,X)$ is d-convergent in μ to a function $h \in \mathcal{M}(E,X)$ and the same sequence $\langle h_n \rangle$ is not ρ-convergent in μ to h, then there does not exist a function in $\mathcal{M}(E,X)$ such that $\langle h_n \rangle$ is ρ-convergent in μ to it.

Theorem 1. *For every pair* d, ρ *of compatible metrics on X, the following conditions are equivalent:*

1. *d is uniformly smaller than* ρ;
2. $\mathcal{U}_d^*(X) \subseteq \mathcal{U}_\rho^*(X)$;
3. $u_d X \leq u_\rho X$;
4. *for every pair* A, B *of subsets of X such that* $d(A,B) > 0$, *the inequality* $\rho(A,B) > 0$ *holds*;
5. ρ*-convergence in* μ *implies d-convergence in* μ.

Proof. It is obvious that implications (i)⇒(ii)⇒(iii) and (i)⇒(v) are true. Suppose that (iii) holds and consider an arbitrary pair A, B of subsets of X such that $d(A,B) \neq 0$. Then $\mathrm{cl}_{u_d X} A \cap \mathrm{cl}_{u_d X} B = \varnothing$ by Theorem 2.5 of [14]. Since $u_d X \leq u_\rho X$, in the light of 4.2(h) of [10], we have $\mathrm{cl}_{u_\rho X} A \cap \mathrm{cl}_{u_\rho X} B = \varnothing$. This, together with Theorem 2.5 of [14], gives that $\rho(A,B) \neq 0$. Hence (iv) follows from (iii). Now, assume that (i) is not fulfilled. Then, with CC in hand, we deduce that, for some $\varepsilon \in (0,+\infty)$, there are sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ of X such that $\lim_{n\to+\infty} \rho(x_n, y_n) = 0$ and $d(x_n, y_n) \geq \varepsilon$ for all $n, m \in \omega$ (cf. hint to 8.5.19 of [3]). If $A = \{x_n : n \in \omega\}$ and $B = \{y_n : n \in \omega\}$, then $\rho(A,B) = 0$, while $d(A,B) \neq 0$. Therefore, (i) is a consequence of (iv). That (v) implies (i) follows from Proposition 2. □

Corollary 1. *Let d and* ρ *be compatible metrics on X. If* ρ *is totally bounded and* ρ*-convergence in* μ *implies d-convergence in* μ, *then d is totally bounded.*

Proof. It is clear that if d is uniformly smaller than ρ and the metric ρ is totally bounded, then d is also totally bounded. To complete the proof, it suffices to use the equivalence of (i) and (v) of Theorem 1. □

Theorem 2. *Assume that d is a totally bounded compatible metric on X. Then d-convergence in* μ *is equivalent with* $\mathcal{U}_d^*(X)$ *-convergence in* μ.

Proof. It is obvious that d-convergence in μ implies $\mathcal{U}_d^*(X)$ -convergence in μ. Since d is totally bounded, $u_d X$ is a metrizable compactification of X. By, for example, Propositions 3.4 and 3.5 of [11] or by Theorem 7 of [12], there is a countable collection $F \subseteq \mathcal{U}_d^*(X)$ such that $e_F X = u_d X$ and, moreover, $\phi(X) \subseteq [0;1]$ for each $\phi \in F$. Let $F = \{\phi_i : i \in \omega\}$ and define $\rho(x,y) = \sum_{i\in\omega} \frac{1}{2^{i+1}} |\phi_i(x) - \phi_i(y)|$ for all $x, y \in X$. Then ρ is a totally bounded metric on X such that $u_d X = u_\rho X$. Hence, in view of Theorem 1, d-convergence in μ is equivalent with ρ-convergence in μ. However, one can easily check that F-convergence in μ implies ρ-convergence in μ. In consequence, F-convergence in μ implies d-convergence in μ, which concludes the proof. □

Question 1. If d is a compatible but not totally bounded metric on X, must $\mathcal{U}_d^*(X)$ -convergence in μ imply d-convergence in μ?

A familiar theorem of ZFC states that a metrizable space X is compact if and only if every compatible metric on X is totally bounded. The standard proof to this theorem involves CC. However, one can easily prove in ZF that if X is a metrizable space such that every compatible metric on X is totally bounded, then X is closed in every metrizable space that contains X as a subspace. Indeed, let (Y, d) be a metric space and let $X \subseteq Y$ be not closed in (Y, d). Choose a point $x_0 \in (\mathrm{cl}_Y X) \setminus X$ and, for $x, y \in X$, define

$$\rho(x,y) = d(x,y) + \left| \frac{1}{d(x,x_0)} - \frac{1}{d(y,x_0)} \right|$$

to get a compatible but not totally bounded metric ρ on X in ZF (cf. 4.3.E.(c) of [3]). Unfortunately, this does not give a satisfactory answer to the following interesting question:

Question 2. Is it consistent with ZF+¬CC that there exists a non-compact metrizable space X such that every compatible metric on X is totally bounded?

3. Functional convergence in measure

It has not been explained so far why it is assumed here that, for each function $f \in \mathcal{M}(E,X)$, there exists $B_f \in \mathfrak{B}_s(X)$ such that $\mu[f^{-1}(X \setminus B_f)] = 0$. In fact, this assumption was needless in the previous section; however, it is helpful to get the following theorem:

Theorem 3. *Let us suppose that* $F \in \mathcal{E}(X)$, *while* $\langle f_n \rangle$ *is a sequence of functions from* $\mathcal{M}(E,X)$ *such that* $\langle f_n \rangle$ *is F-convergent in* μ *to functions* $f, g \in \mathcal{M}(E,X)$. *Then the following conditions hold:*

1. $\mu(\{t \in E : f(t) \neq g(t)\}) = 0$;
2. *each subsequence of* $\langle f_n \rangle$ *contains a subsequence that pointwise converges* μ*-almost everywhere on* E *to* f;
3. *if* $G \in \mathcal{E}(X)$ *is such that the sequence* $\langle f_n \rangle$ *is not* G*-convergent in* μ *to* f*, then there does not exist a function* $h \in \mathcal{M}(E,X)$ *such that* $\langle f_n \rangle$ *is* G*-convergent in* μ *to* h.

Proof. Using CC, we deduce that there is a sequence $\langle B_n \rangle$ of separable Borel subsets of X and there are sets $B_f, B_g \in \mathfrak{B}_s(X)$, such that the sets $X_0 = B_f \cup B_g \cup \bigcup_{n\in\omega} B_n$ and $E_0 = E \setminus [f^{-1}(X \setminus X_0) \cup g^{-1}(X \setminus X_0) \cup \bigcup_{n\in\omega} f_n^{-1}(X \setminus X_0)]$ have the properties that $\mu(E \setminus E_0) = 0$ and all functions f_n, f, g restricted to E_0 transform E_0 into the separable Borel in X set X_0. It follows from Theorem 6 of [12] that there exists a countable collection $H \subseteq F$ such that the restriction to X_0 of the evaluation map e_H is a homeomorphic embedding of X_0 into $\mathbb{R}^H$. Let $H = \{\phi_i : i \in \omega\}$. For each $i \in \omega$, choose a positive real number a_i such that $|\phi_i| \leq a_i$ and, for $x,y \in X$, define $\rho(x,y) = \sum_{i\in\omega} \frac{|\phi_i(x) - \phi_i(y)|}{a_i 2^{i+1}}$. Then ρ is a compatible metric on X_0. It is not difficult to check that the sequence $\left\langle f_n|_{E_0} \right\rangle$ of the restrictions $f_n|_{E_0}$ of functions f_n to E_0 is ρ-convergent in μ to $f_n|_{E_0}$ and $g|_{E_0}$. Hence, in view of Proposition 3, $\mu(\{t \in E : f(t) \neq g(t)\}) = 0$. Now, to conclude that (ii) holds, it suffices to use Proposition 5. Condition (iii) follows from (ii). □

Theorem 4. *Let* αX *be a compactification of* X *and let* $F \in \mathcal{E}(X)$ *generate* αX*, i.e.* $\alpha X = e_F X$*. Then* F*-convergence in* μ *and* $C_\alpha(X)$*-convergence in* μ *are equivalent.*

Proof. Since $F \subseteq C_\alpha(X)$, it is obvious that $C_\alpha(X)$-convergence in μ implies F-convergence in μ. Now, assume that a sequence $\langle f_n \rangle$ of functions from $\mathcal{M}(E,X)$ is F-convergent in μ to a function $f \in \mathcal{M}(E,X)$. Consider an arbitrary function $\phi \in C_\alpha(X)$ and a positive real number ε. By Theorem 4 of [13], there exist a non-void finite set $H \subseteq F$ and a positive real number δ, such that if

$$d_H(x,y) = \max\{|\psi(x) - \psi(y)| : \psi \in H\}$$

for $x,y \in X$, then $|\phi(x) - \phi(y)| < \varepsilon$ whenever $d_H(x,y) < \delta$. It follows from the F-convergence in μ of $\langle f_n \rangle$ to f that

$$\lim_{n\to+\infty} \mu(\{t \in E : d_H(f_n(t), f(t)) \geq \delta\}) = 0.$$

In addition,

$$\{t \in E : |\phi[f_n(t)] - \phi[f(t)]| \geq \varepsilon\} \subseteq \{t \in E : d_H(f_n(t), f(t)) \geq \delta\}$$

for all $n \in \omega$. In consequence,

$$\lim_{n\to+\infty} \mu(\{t \in E : |\phi[f_n(t)] - \phi[f(t)]| \geq \varepsilon\}) = 0.$$

This means that $\langle f_n \rangle$ is $C_\alpha(X)$-convergent in μ to f. □

We consider the set $C_\beta(X)$ as the metric space $(C_\beta(X), \sigma)$ where the metric σ on $C_\beta(X)$ is defined as follows: $\sigma(f, g) = \sup\{|f(x) - g(x)|; x \in X\}$ for $f, g \in C_\beta(X)$. In view of, for example, Theorem 7 of [12], when $F \in \mathcal{E}(X)$, then the compactification $e_F X$ of X is metrizable if and only if F is second-countable in $(C_\beta(X), \sigma)$. In what follows, every subset of $C_\beta(X)$ is equipped with the topology inherited from the topology on $C_\beta(X)$ induced by the metric σ.

Theorem 5. *Let αX and γX be compactifications of X such that αX is metrizable and $C_\alpha(X)$-convergence in μ implies $C_\gamma(X)$-convergence in μ. Then γX is also metrizable and $\gamma X \leq \alpha X$.*

Proof. Since αX is metrizable, there exists a totally bounded compatible metric ρ on X such that $u_\rho X = \alpha X$. Consider any function $\phi \in C_\gamma(X)$ and let $F = C_\alpha(X) \cup \{\phi\}$. Of course, $F \in \mathcal{E}(X)$. The compactification $e_F X$ is metrizable because F is second-countable. There is a totally bounded metric d on X such that $e_F X = u_d X$. It follows from Theorem 2 that ρ-convergence in μ implies d-convergence in μ. Therefore, $u_d X \leq u_\rho X$ by Theorem 1. This implies that $F \subseteq C_\alpha(X)$ and, in consequence, $C_\gamma(X) \subseteq C_\alpha(X)$. Then $\gamma X \leq \alpha X$ and $C_\gamma(X)$ is second-countable. Hence γX is metrizable. □

Corollary 2. *Let αX and γX be metrizable compactifications of X. Then $\alpha X \leq \gamma X$ if and only if $C_\gamma(X)$-convergence in μ implies $C_\alpha(X)$-convergence in μ.*

From Theorems 4 and 5 we immediately deduce the following:

Corollary 3. *Suppose that $F, G \in \mathcal{E}(X)$ are such that F-convergence in μ implies G-convergence in μ. If F is second-countable, then G is second-countable and $e_G X \leq e_F X$.*

Our final theorem is a nice conclusion from Theorem 5.

Theorem 6. *If there exists a metrizable compactification αX of X such that $C_\alpha(X)$-convergence in μ implies $C_\beta(X)$-convergence in μ, then X is compact.*

Proof. Let us assume that αX is a metrizable compactification of X such that $C_\alpha(X)$-convergence in μ implies $C_\beta(X)$-convergence in μ. Since $\alpha X \leq \beta X$, it follows from Theorem 5 that βX is metrizable and $\beta X = \alpha X$. If X were non-compact, βX would be non-metrizable (cf. 3.6. G of [3]). □

Corollary 4. *A metrizable space X is compact if and only if there exists a totally bounded metric d on X such that d-convergence in μ implies $C_\beta(X)$-convergence in μ.*

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