TECHNICAL TRANSACTIONS | **CZASOPISMO TECHNICZNE**
MECHANICS | MECHANIKA

4-M/2016

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THE FUZZY INTERPRETATION OF THE STATISTICAL TEST FOR IRREGULAR DATA

ROZMYTA INTERPRETACJA TESTU STATYSTYCZNEGO DLA NIEREGULARNYCH DANYCH

Abstract

The well-known statistical tests have been developed on the basis of many additional assumptions, among which the normality of a data source distribution is one of the most important. The outcome of a test is a *p*-value which may is interpreted as an estimation of a risk for a false negative decision i.e. it is an answer to the question "how much do I risk if I deny?". This risk estimation is a base for a decision (after comparing with a significance level α): reject or not. This sharp trigger – *p*-level greater than α or not – ignores the fact that a context is rather smooth and evolves from "may be" through "rather not" to "certainly not". An alternative option for such assessments is proposed by a fuzzy statistics, particularly by Buckley's approach. The fuzzy approach introduces a better scale for expressing decision uncertainty. This paper compares three approaches: a classic one based on a normality assumption, Buckley's theoretical one and a bootstrap-based one.

Keywords: statistical test, normality of distribution, fuzzy statistics, bootstrap

Streszczenie

Powszechnie znane testy statystyczne były opracowane przy wielu dodatkowych założeniach. Jednym z najważniejszych jest normalność rozkładu populacji źródłowej. Wynikiem testu jest wartość *p*, która jest interpretowana jako ocena ryzyka decyzji fałszywie negatywnej, tj. jest to odpowiedź na pytanie „ile ryzykuję jeżeli neguję?". Ta ocena ryzyka jest podstawą do podjęcia decyzji (po porównaniu z krytycznym poziomem istotności α): odrzucić czy nie. Takie ostre przełączenie – wartość *p* większa od α czy też nie – ignoruje fakt, że kontekst jest raczej gładki i ewoluuje od „może tak" przez „raczej nie" do „zdecydowanie nie". Alternatywą dla takich ocen jest statystyka rozmyta, a szczególnie podejście Buckleya. Podejście rozmyte wprowadza lepszą skalę do wyrażenia niepewności decyzji. Niniejszy artykuł porównuje trzy podejścia: klasyczne zakładające normalność, teoretyczne Buckleya i bootstrapowe.

Słowa kluczowe: test statystyczny, normalność rozkładu, statystyka rozmyta, bootstrap

DOI: 10.4467/2353737XCT.16.242.5991

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1. Introduction

The typical statistical approach leads from a certain knowledge about a sample to an uncertain knowledge (more precisely: hypothesis) about a general population from which the sample was taken. The non-parametric test is the first step, the parametric test for the selected and previously not rejected distribution is the second step. Computed estimators define the most probable distribution and its value is the limit to which a long series frequency goes. This definition involves simultaneously Kolmogorov axiom-based probability [1] and von Mises frequency-based probability [2].

As the beginning of a fuzzy approach to statistical problems, statistical hypothesis testing, in particular, the article of Casals and Giles [3] may be recognized. They generalized Neymann-Pearson lemma, where a concept of a fuzzy information system proposed by Tanaka et al. [4] was utilized. It allowed to construct the uniformly most powerful test for point-based hypotheses with the assumed precise significance level. It is characteristic of this approach that a fuzzy event taken from a limited set of fuzzy observations is considered and values of a membership for all considered events must sum to strict 1. In 2004, Buckley [5] proposed and then in 2006 [6] extended a different approach introducing fuzzy conditions, particularly fuzzy significance level. In 2006 Grzegorzewski [7] proposed a general classification of a possible fuzzy generalization for the classic theory of a statistical hypothesis testing, where three main elements constituting a decision system were distinguished. The theory of a statistical hypothesis testing is only the specific element of the system. The three main elements are: data, hypothesis and conditions. Each of these elements may be described precisely or imprecisely. The traditional theory requires the precise description for each of these elements, however all possible combinations are eight.

In the terms of Grzegorzewski's model, the Casals and Giles's proposal may be described as the triplet: fuzzy data, crisp hypothesis and crisp conditions. Buckley's proposal may be treated as: crisp data, crisp hypothesis and fuzzy conditions. To authors' knowledge, other combinations have not been proposed until now (Table 1).

In the further section, the authors evaluate the test for the equal of means and compare the results from classic, Buckley's and bootstrap approaches.

Table 1

Triplets of possible crisp/fuzzy combinations of Grzegorzewski's model

	HYPOTHESIS			
	crisp		fuzzy	
	DATA		**DATA**	
CONDITIONS	crisp	fuzzy	crisp	fuzzy
crisp	classic	Casals & Giles	*not known*	*not known*
fuzzy	Buckley	*not known*	*not known*	*not known*

2. Materials and Methods

2.1. Raw Data

The raw data was taken from the biochemistry investigation [8]. The dataset was divided into two subsets, each of 20 values (Table 2).

Table 2

The raw data

Data source	Values
Series H	4.50; 3.96; 4.11; 5.01; 4.87; 4.31; 3.20; 5.09; 4.61; 4.70 4.49; 3.51; 3.19; 3.29; 2.55; 3.10; 4.11; 3.10; 3.21; 4.67
Series M	4.00; 3.06; 5.90; 4.86; 5.70; 5.21; 3.51; 4.29; 4.61; 5.49 4.30; 4.90; 5.51; 4.39; 4.01; 3.90; 3.70; 2.81; 5.42; 3.50

The data are values of the callus tissue growth factor related to two variants H and M of the special medium modified with various additives. The test of means equality should detect possible differences between effects of these two variants.

2.2. Classic test

The means and the standard deviations for both variants were calculated using classic formulas:

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \tag{1}$$

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n} (x_i - \overline{x})^2. \tag{2}$$

The test for the equality of means with an unknown but mutually equal variances was performed with test statistic t calculated from the formula:

$$t = \frac{\overline{x}_M - \overline{x}_H}{\sqrt{\frac{(n_H - 1)s_H^2 + (n_M - 1)s_M^2}{n_H + n_M - 2}\left(\frac{1}{n_H} + \frac{1}{n_M}\right)}}, \tag{3}$$

where distribution of t goes asymptotically to t Student distribution with $(n_H + n_M - 2)$ degree of freedom.

2.3. Buckley's approach

The detailed calculations related to the above-mentioned data were described by Pietraszek and Skrzypczak-Pietraszek [9]. In short, the test statistic has been fuzzified according to formula:

$$\widehat{T}[\alpha] = \frac{\Delta \widehat{x}[\alpha]}{\sqrt{\frac{\widehat{\sigma}_1^2[\lambda]}{n_1} + \frac{\widehat{\sigma}_2^2[\lambda]}{n_2}}}, \tag{4}$$

where both variances and the means difference are of specific triangular form. It should be noted that Eq. 4 is specifically designed for unknown means and known variances but unfortunately Buckley did not a present formula for the variant with the unknown variances.

2.4. Bootstrap approach

The bootstrap procedure was conducted according to Shao and Tu [10] instructions. Both of the raw subsets were treated as sources for bootstrap draw, i.e. both subsets were randomly replicated and the test statistic t was evaluated from (Eq. 3) and then collected. After 40 000 draws, the collected values were transformed into descriptive statistics and a histogram.

3. Analysis

3.1. Classic approach

The descriptive statistics were evaluated for both subsets (Table 3). The test statistic t has the value of 1.789 at 38 degrees of freedom. It led to the p-value of 0.082.

Table 3

The descriptive statistics for raw data series

Data source	Mean	Variance	Standard deviation	Sample size
Series H	3.98	0.59	0.77	20
Series M	4.45	0.82	0.90	20

3.2. Buckley's approach

The fuzzified estimators of descriptive statistics were calculated for both the subsets. Next, the fuzzified estimator of the test statistic t was calculated according to Eq. 4. The result is presented in Fig. 1. as a triangular-like form of the fuzzy number.

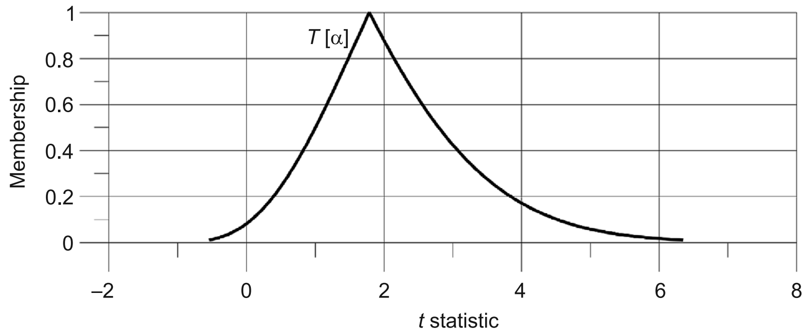

Fig. 1. Fuzzy estimator of test statistic *t* obtained for the fuzzified test of means equality

3.3. Bootstrap approach

The bootstrap approach was made in the following manner: the draws were made from both subsets creating new subsampled bootstrap subsets. Then, the test statistic *t* was evaluated based on descriptive statistics calculated for both subsets. The value of the test statistic was collected and next iteration of the bootstrap process was initiated. After 40 000 iteration, the test statistic *t* collection was processed and its descriptive statistics (Table 4) and the histogram (Fig. 2) were created. The *p*-value evaluated from the bootstrap was 0.067.

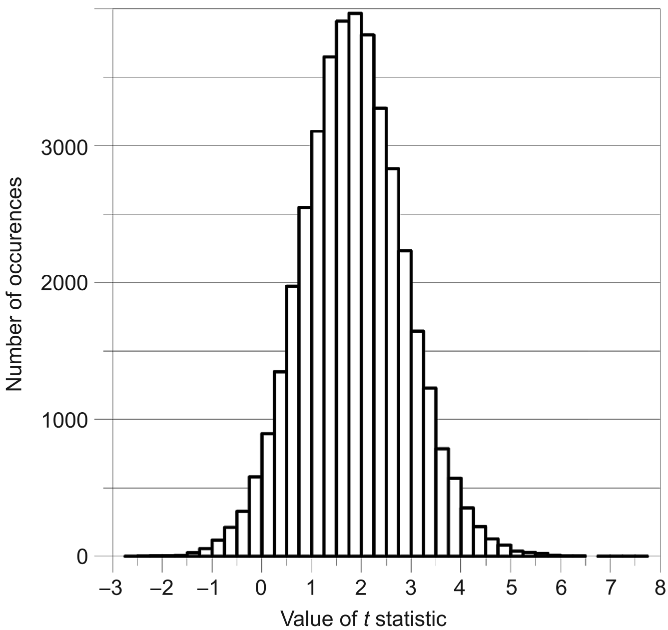

Fig. 2. Distribution of the test statistic *t* obtained from the bootstrap approach

Table 4

The descriptive statistics for the bootstrapped distribution of the test statistic *T*

Statistic	Value
Minimum	–2.72
Maximum	7.74
Mean	1.85
Median	1.83
Variance	1.06

4. Conclusion

The main characteristics of all approaches are consistent and the decisions which may be made on these results are the same: do not reject equality of the data subset means. On the contrary, the contexts of these approaches are different. The classic one assumes the normality of data and asymptotic equality of the test statistic t with Student distribution. The bootstrap approach is non-parametric and reveals the shape of distribution derived from raw data without additional assumptions. The Buckley's fuzzy approach expresses the subjective opinion about the possibility of such a value for the test statistic t. While the classic and the bootstrap approaches have the same deeper foundations, the Buckley's is far more different because of subjective foundations. The Buckley's approach appears to be useful if the description of data is vague.

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