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THE CONCEPT OF A QUASI-MICROPOLAR FLUID MODEL

QUASI-MIKROPOLARNY MODEL CIECZY

A b s t r a c t

This paper presents a micropolar fluid model that direct applies Cosserat's continuum to hydrodynamics. The corresponding system of equations describing isotropic micropolar fluid is obtained by assuming lack of symmetry of the Cauchy stress tensor and taking into account the conservation of angular momentum. This turns out to be an extension of the Navier-Stokes fluid but containing turbulent effect built in.

Keywords: micropolar fluid, turbulent effect

S t r e s z c z e n i e

W artykule przedstawiono mikropolarny model cieczy stanowiący bezpośrednie zastosowanie kontinuum Cosseratów w hydromechanice. Zakładając brak symetrii tensora naprężenia Cauchy'ego oraz uwzględniając zasadę zachowania momentu pędu otrzymano układ równań opisujący izotropową ciecz mikropolarną. Układ równań jest uogólnieniem równań Naviera-Stokesa poprzez uwzględnienie efektu turbulentnego.

Słowa kluczowe: ciecz mikropolarna, efekt turbulentny

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1. Introduction

Over one hundred years ago, the Cosserat brothers published fundamental work containing a new version of continuum mechanics [8]. This was based on the idea of considering rotational degrees of freedom of material particles to be independent variables and corresponding couple stresses. This material model was later named the Cosserat or micropolar continuum. The basic ideas of this approach were first presented in [7].

The general nonlinear theory of the micropolar continuum was developed by Truesdell [44, 45]. The problem of finite deformation was considered by Grioli [19, 20], Toupin [43], Green & Rivlin [18], Eringen & Kafadar [14, 24], Stojanovic [39, 40, 41], Besdo [5] and Reissner [33, 34, 35]. The linear Cosserat theory is presented in the original papers, by inter alia Günther [21], Aero & Kuvshinskii [3, 4], Toupin [42], Mindlin & Tiersten [28], Koiter [25], Palmov [32], Eringen [15, 16], Schaefer [38], and Ieşan [23].

In the case of micropolar fluids, the review of achievements starts with pioneering papers by Aero et al. [2] and Eringen [17] as well as monographs by Migoun & Prokhorenko [27], Łukaszewicz [26], Eremeyev & Zubov [9], the micropolar continuum is applied to model magnetic liquids, polymer suspension, liquid crystals, and other types of fluids with a microstructure. In particular, Rosensweig uses magnetic fluids [36], named also ferrofluids, developing the micropolar hydromechanics where a magnetic field induces voluminous couples. Compared to micropolar elasticity, micropolar hydrodynamics is a more extensive part of mechanics with well-established experimentally constitutive equations. Some generalizations of the viscous micropolar constitutive model are presented by Eremeyev & Zubov [10, 47] and also Eringen [11, 12, 13].

The Cosserat brothers considered a simplified version of the micropolar continuum called quasi micropolar theory. This is based on the assumption that the rotation of local particles is equal to the average rotation of displacement field. The quasi micropolar continuum is well developed and comprises several general theorems, methods of integration and solutions of fundamental problems, see Hamel [22], Koiter [25], Mindlin and Tiersten [28], Muki & Sternberg [29], Bogy & Sternberg [6], Sawin [37].

2. Micropolar fluid model

Classical hydromechanics is based on an idealised model of a continuum in which the transmission of transitions between both sides of a surface element is only described by the Cauchy stress $t_i = \sigma_{ji} n_j$. This approach leads to symmetrical states of stress and strain which properly describe majority of solid and fluid materials. However, essential differences between the model and experimental evidence arise in the case of high stress gradients, vibrations excited by high frequencies, granular media and polymers. The above discrepancies between the theory of symmetric continuum and experimental data were the subject of investigations by Voigt [46] who first introduced additional transmission by a couple traction $m_i = \mu_{ji} n_j$. Such an assumption leads to the existence of the couple stress tensor μ_{ij} as well as a lack of symmetry of Cauchy's stress tensor σ_{ij}. The general theory of non-symmetrical continuum was developed by the Cosserat brothers [8]. According to their concept, the kinematics of the continuum point is described by the displacement vector u_i

and the micropolar rotation vector φ_i. Since the present section deals with the micropolar fluid model let us adapt Cosserat's formalism to our purpose as was done by Ostoja-Starzewski [31]. First of all, both vectors u_i and φ_i are replaced by their respective time rates $\mathrm{v}_i = \dot{u}_i$ and $\omega_i = \dot{\varphi}_i$. Additionally, the microinertia tensor of angular momentum per unit mass J_{ij} is introduced. The system of balance equations is as follows:
the conservation of mass

$$\frac{D\rho}{Dt} + \rho \mathrm{v}_{i,i} = 0 \tag{1}$$

the balance of linear momentum

$$\rho \frac{D\mathrm{v}_i}{Dt} = \sigma_{ji,j} + \rho X_i \tag{2}$$

the balance of angular momentum

$$\rho \frac{D(J_{ij}\omega_j)}{Dt} = \mu_{ji,j} + \rho Y_i + \epsilon_{ijk}\, \sigma_{jk} \tag{3}$$

In the case of an isotropic micropolar fluid, $J_{ij} = J\delta_{ij}$, where J is the microinertia of a continuum fluid particle. The above assumption comprises the isotropy of the geometric shape of fluid particles and has nothing to do with the isotropy of constitutive equations − these will be discussed separately.

Taking advantage of kinematic equations

$$\begin{gathered} \dot{\gamma}_{ji} = \mathrm{v}_{i,j} - \epsilon_{kji}\, \omega_k \\ \dot{\kappa}_{ji} = \omega_{i,j} \end{gathered} \tag{4}$$

one can perform constitutive equations of the linear micropolar, isotropic and centrosymmetric fluid

$$\begin{gathered} \sigma_{ji} = (\mu + \mu_r)\dot{\gamma}_{ji} + (\mu - \mu_r)\dot{\gamma}_{ij} + \left(-p + \lambda\dot{\gamma}_{k,k}\right)\delta_{ij} \\ \mu_{ji} = (c_d + c_a)\dot{\kappa}_{ji} + (c_d - c_a)\dot{\kappa}_{ij} + c_0\dot{\kappa}_{k,k}\delta_{ij} \end{gathered} \tag{5}$$

in the format proposed by Łukaszewicz [26]

$$\begin{gathered} \sigma_{ji} = \left(-p + \lambda \mathrm{v}_{k,k}\right)\delta_{ij} + \mu\left(\mathrm{v}_{j,i} + \mathrm{v}_{i,j}\right) + \mu_\mathrm{r}\left(\mathrm{v}_{i,j} - \mathrm{v}_{j,i}\right) - 2\mu_\mathrm{r}\, \epsilon_{mij}\, \omega_m \\ \mu_{ji} = c_0\omega_{k,k}\delta_{ij} + c_d\left(\omega_{j,i} + \omega_{i,j}\right) + c_a\left(\omega_{i,j} - \omega_{j,i}\right) \end{gathered} \tag{6}$$

It is worth noting here that the term centrosymmetry plays an analogous role to the term isotropy in case of classical continuum. Therefore, the governing equations (2-3) become

$$\begin{gathered} \rho \frac{D\mathrm{v}_i}{Dt} = \rho X_i - p_{,i} + (\lambda + \mu - \mu_\mathrm{r})\mathrm{v}_{j,ji} + (\mu + \mu_\mathrm{r})\mathrm{v}_{i,kk} + 2\mu_\mathrm{r}\, \epsilon_{ijk}\, \omega_{k,j} \\ \rho J \frac{D\omega_i}{Dt} = \rho Y_i + 2\mu_\mathrm{r}\left(\epsilon_{ijk}\, \mathrm{v}_{j,k} - 2\omega_i\right) + (c_0 + c_d - c_a)\omega_{j,ji} + (c_d + c_a)\omega_{i,kk} \end{gathered} \tag{7}$$

If there are no body forces and couples, the governing equations (1, 7) may be rewritten in absolute notation to the format

$$\begin{gathered}\frac{D\rho}{Dt} + \rho\,\mathrm{div}\mathbf{v} = 0\\ \rho\, {}^{D\mathbf{v}}\!/_{Dt} = -\mathrm{grad}p + (\mu + \mu_\mathrm{r})\nabla^2\mathbf{v} + (\lambda + \mu - \mu_\mathrm{r})\mathrm{grad}(\mathrm{div}\mathbf{v}) + 2\mu_\mathrm{r}\mathrm{rot}\boldsymbol{\omega}\\ \rho J\, {}^{D\boldsymbol{\omega}}\!/_{Dt} = (c_d + c_a)\nabla^2\boldsymbol{\omega} + (c_0 + c_d - c_a)\mathrm{grad}(\mathrm{div}\boldsymbol{\omega}) + 2\mu_\mathrm{r}\mathrm{rot}\mathbf{v} - 4\mu_r\boldsymbol{\omega}\end{gathered} \tag{8}$$

in which the viscosity coefficients assuring the positive definiteness of the entropy growth are

$$\begin{array}{cc} \mu \geqslant 0 & 3\lambda + 2\mu \geqslant 0\\ c_d + c_a \geqslant 0 & 3c_0 + 2c_d \geqslant 0\\ -(c_d + c_a) \leqslant c_d - c_a \leqslant (c_d + c_a) & \mu_r \geqslant 0\end{array} \tag{9}$$

According to Eq. (8), the motion of micropolar fluid can be treated as turbulent. However, when the micropolar effects tend to vanish, the fluid becomes classical Newtonian and in the special case of vanishing bulk viscosity $\lambda + 2/3\mu \longrightarrow 0$ it simplifies to a Navier-Stokes fluid.

3. Quasi-micropolar fluid model

Apart from the general micropolar theory, the Cosserat brothers also considered a simplified theory, according to which, couple inertia terms vanish $D\boldsymbol{\omega}/Dt = 0$ in Eq. (8_3) and the rotation of a local particle is equal to the average rotation of the displacement field, see Nowacki [30]. It is assumed that Eq. (4_1) reduces to

$$\dot{\boldsymbol{\gamma}}^A = \frac{1}{2}\mathrm{rot}\,\mathbf{v} - \boldsymbol{\omega} = 0 \tag{10}$$

nevertheless, the transmission of tractions and couple transitions through an arbitrary surface is done by stress tensor $\boldsymbol{\sigma}$ and couple stress tensor $\boldsymbol{\mu}$, and obviously, both tensors are still unsymmetrical.

Introducing $\boldsymbol{\omega}$ calculated from Eq. (10) into Eqs ($8_{2\text{-}3}$) we obtain

$$\begin{gathered}\rho\, {}^{D\mathbf{v}}\!/_{Dt} = -\mathrm{grad}p + \mathcal{L}(\mathbf{v}) - 2\mu_\mathrm{r}\mathrm{rot}\dot{\boldsymbol{\gamma}}^A\\ (c_d + c_a)\nabla^2\dot{\boldsymbol{\gamma}}^\mathrm{A} + (c_0 + c_d - c_a)\mathrm{grad}\,\mathrm{div}\dot{\boldsymbol{\gamma}}^\mathrm{A} + 4\mu_\mathrm{r}\dot{\boldsymbol{\gamma}}^A - \frac{1}{2}(c_d + c_a)\nabla^2\mathrm{rot}\mathbf{v} = 0\end{gathered} \tag{11}$$

in which the differential operator

$$\mathcal{L}(..) = (\mu + \mu_\mathrm{r})\nabla^2(..) + (\lambda + \mu - \mu_\mathrm{r})\mathrm{grad}\,\mathrm{div}(..) + \mu_\mathrm{r}\mathrm{rot}\,\mathrm{rot}(..) \tag{12}$$

may be simplified to the format

$$\mathcal{L}(..) = \lambda\nabla^2(..) + (\lambda + \mu)\mathrm{grad}\,\mathrm{div}(..) \tag{13}$$

The above reduction yields of known relation $\nabla^2(..) - \text{grad div}(..) + \text{rot rot}(..) = 0$.

Unsymmetrical stress tensor defined as

$$\boldsymbol{\sigma}^{\mathrm{A}} = 2\mu_{\mathrm{r}}\dot{\boldsymbol{\gamma}}^{\mathrm{A}} \tag{14}$$

serves for the reduction of $\dot{\boldsymbol{\gamma}}^{\mathrm{A}}$ in Eq. (11) yielding

$$\begin{gathered} \rho\, {}^{D\mathbf{v}}/_{Dt} = -\text{grad}p + \mathcal{L}(\mathbf{v}) - \text{rot}\boldsymbol{\sigma}^{\mathrm{A}} \\ \boldsymbol{\sigma}^{\mathrm{A}} = -\tfrac{1}{4}(c_d + c_a)\nabla^2\text{rot}\mathbf{v} \end{gathered} \tag{15}$$

Finally, applying differential operator (13) and introducing (15_2) into Eq. (15_1) we find

$$\begin{gathered} \frac{D\rho}{Dt} + \rho\text{div}\mathbf{v} = 0 \\ \rho\, {}^{D\mathbf{v}}/_{Dt} = -\text{grad}p + \mu\nabla^2\mathbf{v} + (\lambda + \mu)\text{grad div}\,\mathbf{v} - \tfrac{1}{4}(c_d + c_a)\nabla^2\text{rot rot}\,\mathbf{v} \end{gathered} \tag{16}$$

In the case of vanishing bulk viscosity $\lambda + 2/3\mu \longrightarrow 0$, we get the system of equations

$$\begin{gathered} \frac{D\rho}{Dt} + \rho\text{div}\mathbf{v} = 0 \\ \rho({}^{\partial\mathbf{v}}/_{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}) = -\text{grad}p + \mu\left(\nabla^2\mathbf{v} + \tfrac{1}{3}\text{grad div}\,\mathbf{v}\right) - \underline{(c_d + c_a)\nabla^2\text{rot rot}\,\mathbf{v}} \end{gathered} \tag{17}$$

which is the generalisation of the Navier-Stokes equations by the underlined term. Eq. (17_2) include not only the conventional coefficient of dynamic viscosity μ but also the sum of two micropolar viscosity coefficients $c_d + c_a$. All above coefficients of viscosity are constants according to the assumption of isotropy and linearity of constitutive equations (5) or (6). It is also worth to noting that the underlined term can be treated as a specific integral of the Navier-Stokes equations. Moreover, since according to Eq. (17_2) the turbulent effect is activated from the very beginning, we suggest that it be preceded by a specific continuous switch function dependent on Reynold's number

$$f(Re) = \frac{Re - Re_{min}}{Re_{max} - Re_{min}} \tag{18}$$

as is shown in Fig. 1. Reynold's number includes only the conventional coefficient of

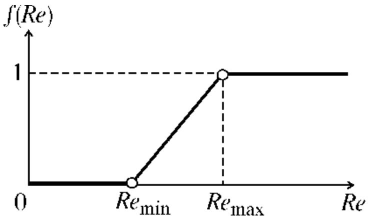

Fig. 1. Switch function *f*(*Re*) preceding turbulent effect term in Eqs (17)

dynamic viscosity μ, hence the sum of two micropolar viscosity coefficients $c_d + c_a$ serves as the length scale.

Example of application of the above quasi-micropolar fluid model will be a subject of separate paper.

Nomenclature

c_0, c_d, c_a	– micropolar viscosity coefficients
D/Dt	– absolute differential with respect to time
J_{ij}	– microinertia tensor of a fluid particle
n_i	– outer normal unit vector
m_i	– couple traction vector
p	– pressure
Re	– Reynold's number
t	– time
t_i	– traction vector
$u_i, \boldsymbol{u}$	– displacement vector
$\mathrm{v}_i, \mathbf{v}$	– time differential of displacement vector
ω_i	– time differential of microrotation vector
X_i, Y_i	– body force per unit mass and body torque per unit mass
$\partial/\partial t$ or $\cdot$ over a symbol	– partial differential with respect to time
δ_{ij}	– Kronecker's symbol
$\in_{ijk}$	– Levi-Civita's symbol
$\dot{\gamma}_{ij}, \dot{\kappa}_{ij}$	– strain rate tensor and couple strain rate tensor
$\dot{\boldsymbol{\gamma}}^{\mathrm{A}}$	– unsymmetrical part of strain rate tensor
λ, μ	– conventional viscosity coefficients
μ_r	– dynamic microrotation viscosity
ρ	– mass density
σ_{ij}, μ_{ij}	– Cauchy's stress tensor and couple stress tensor
$\boldsymbol{\sigma}^{\mathrm{A}}$	– unsymmetrical part of Cauchy's stress tensor
$\varphi_i, \boldsymbol{\varphi}$	– micropolar rotation vector
$\omega_i, \boldsymbol{\omega}$	– time differential of micropolar rotation vector

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