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THE PATHS OF MATHEMATICS IN LITHUANIA: BISHOP BARANAUSKAS (1835–1902) AND HIS RESEARCH IN NUMBER THEORY

ŚCIEŻKI MATEMATYKI NA LITWIE: BISKUP BARANOWSKI (BARANAUSKAS) (1836–1902) I JEGO BADANIA Z TEORII LICZB

A b s t r a c t

Bishop Antanas Baranauskas is a prominent personality in the history of the Lithuanian culture. He is well known not only as a profound theologian, a talented musician creating hymns, a literary classicist and an initiator of Lithuanian dialectology, but also as a distinguished figure in the science of mathematics. The author of this article turns his attention to the mathematical legacy of this prominent Lithuanian character and aspires to reveal the circumstances that encouraged bishop Antanas Baranauskas to undertake research in mathematics, to describe the influence of his achievements in the science of mathematics, to show the incentives that encouraged him to pursue mathematical research in Lithuania as well as to emphasize his search for a connection between mathematics and theology.

Keywords: *number theory*, *geometry*, *infinity and theology*

S t r e s z c z e n i e

Biskup Antoni Baranowski (Antanas Baranauskas) należy do prominentnych osobistości w historii kultury litewskiej. Znany jest jako istotny teolog, utalentowany muzyk tworzący hymny, literacki klasyk i inicjator dialektologii Litwy; zajmował się również matematyką. W artykule przedstawiono okoliczności, w związku z którymi biskup A. Baranowski zajął się badaniami matematycznymi. Zarysowane zostało znaczenie jego wyników w rozwoju badań matematycznych na Litwie. Podkreślono również związki pomiędzy matematyką i teologią.

Słowa kluczowe: *teoria liczb*, *geometria*, *nieskończoność i teologia*

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1. Introduction

Bishop Antanas Baranauskas (Antoni Baranowski, 1835–1902) is a prominent personality in the history of the Lithuanian culture. He is well known not only as a profound theologian, a talented musician creating hymns, a literary classicist and an initiator of Lithuanian dialectology, but also as a distinguished personality in the science of mathematics.

Fig. 1. Bishop Antanas Baranauskas (Antoni Baranowski, 1835–1902), Kaunas, end of XIX. Source: Vilnius University Library

There were a number of authors who had written about the bishop's merits to mathematics in Lithuania. One of the most important of the authors was a writer and an encyclopedist prelate Aleksandras Jakštas-Dambrauskas (1860–1938). He knew the bishop personally and in 1906 wrote a journal article *Bishop Antanas Baranauskas as a Mathematician* [6, 7]. This article, which also appeared as a separate publication, was based exclusively on the correspondence between the author and the bishop. It presented the beginnings of Baranauskas' interests in mathematics through his articles that discuss various aspects of number theory and geometry as well as the concept of infinity in the context of the philosophy of mathematics.

Another author and a priest, Juozas Tumas-Vaižgantas (1869–1933), published a book about Baranauskas, as a writer when printing in the Lithuanian language was prohibited. The book was based on his Lithuanian literature lectures and included a chapter called *Sins of Mathematics* [16], where he described the bishop's biggest achievements in the field of mathematics.

Viktoras Biržiška (1886–1964), a professor of mathematics in the interwar Lithuania, acknowledged Baranauskas's achievements in mathematics by writing an article about him for the Lithuanian encyclopedia [5].

In 1970s Baranauskas' creative works, including those on mathematics, attracted interest of many researchers again. Regina Mikšytė (1923–2000), a researcher in literature, dedicated a big part of her life to studying the creative legacy of this prominent Lithuanian figure. She published a monograph where she also briefly discussed the bishop's activities in mathematics. In 1993, the year of revival, the book was improved, augmented and published again [12, 13].

The first people after the Second World War to return to the discussion of Baranauskas' works in mathematics for the Lithuanian reader were Petras Rumšas (1921–1987), a specialist in the didactics of mathematics, and Aleksandras Baltrūnas (1949–2005), an expert in the history of mathematics, who published a number of articles in the Lithuanian press [14, 4].

In 1985 the celebration of the 150th anniversary of Baranauskas' birthday raised a new wave of inquiries into the bishop's life and work. The writer Rapolas Šaltenis (1908–2007) published the book *Our Baranauskas* in which the chapter "The Inferno and the Swallow in Mathematics" recounted the reasons for the bishop's interest in mathematics [15]. The academician Jonas Kubilius (1921–2011), after having researched archival material and various texts on mathematics, prepared a scientific study which he first published in the book *Literature and Language*. The same study was later published as a collection of separate articles and finally, after some modificatios in 2001 it was issued as a special edition *Antanas Baranauskas and Mathematics* [8–10]. The distinction of this study is a professional evaluation of the bishop's mathematical legacy by a mathematician who worked on number theory. Eugenijus Manstavičius, a mathematician at Vilnius University, also wrote a number of articles about Antanas Baranauskas and his achievements in mathematics [11].

It is evident that in the 20th century Antanas Baranauskas and his works in mathematics were given quite a lot of attention. The articles published by various mathematicians evaluated his mathematical achievements and defined his contribution to the science of mathematics.

The author of this article turns his attention to the mathematical legacy of this prominent Lithuanian figure and aspires to reveal the circumstances that encouraged bishop Antanas Baranauskas to undertake research in mathematics, to describe the influence of his achievements in the science of mathematics, to show the incentives that encouraged him to pursue mathematical research in Lithuania as well as to emphasize his search for a connection between mathematics and theology.

2. Mathematics and Theology

The future bishop showed an inclination to mathematics already in his childhood. One of his first biographers J. Daubaras wrote, that "Antanukas (Little Antanas) was good at school, especially at *sums*" [7, p. 7-8] (i.e. mathematics). The basic arithmetic operations like counting to a big number, addition, subtraction and multiplication he learned at Anykščiai parochial school; other parts of mathematics he learned independently. His fascination with mathematics from early years can also be illustrated by his determination to solve a mathematical problem that he heard in the parochial school: "you have 100 rubles and you need to buy 100 animals. How many bulls, cows and calves can you buy if a bull costs 10 rubles , a cow 5 rubles and a calf half a ruble?" [16, p. 66]. It took him 2 weeks to solve the problem but he did it. Later "(…) he learned algebra at the Academy (Saint Petersburg Roman Catholic Theological Academy – J.B.) and from Thomas Aquinas"[1, p. 3]. He maintained his interest in mathematics through his studies of theology at the Catholic universities in Munich, Rome and Louvain. After forty years , in 1884, when Baranauskas became a suffragan bishop of Samogitia and settled in Kaunas, he turned to a deeper study of mathematics. He immersed himself in the world of mathematical calculations after having broadened his knowledge in algebra by reading textbooks by Russian authors such as Konstantin Burenin (К.П. Буренин, 1836–1882), Alexander Malinin (А.Ф. Малинин, 1835–1888) and August Davidov (А.Ю. Давидов, 1823–1885) and having got acquainted with the theory of geometry from an unnamed school-textbook. The secretary of the bishop, Rev. Juozas Laukaitis (1873–1952), remembered that once during the time of *recollectio* (i.e., spiritual retreat), when the bishop was meditating on the inferno, a question came into his mind: "how many people would the inferno accommodate?" Knowing that "the thickness of the Earth crust is 50 kilometers "he worked out the capacity of the inferno and after two years of calculation made a remark: "if from the beginning of the world all people had gone to the inferno, inside the Earth, they would have occupied only a small corner of it" [15, p. 146]. Such remarks in no way should be associated with the teachings of the church because, according to the writer Rapolas Šaltenis, he simply, "as if being under the spell, chased an uncatchable swallow" [15, p. 147]. His inclination to mathematics could also have been caused by its universality, "because the mathematical fields are free from institutional politics" [1]. On the other hand, he was attracted to mathematics because "indulging in mathematics feels like swimming into the middle of the sea and diving to the very depth; understanding that you are getting further and further from the shore as well as deeper and deeper towards the bottom. However, when you look back at the work you have done, you see that you are standing close to the shore in the water no more than up to your ankles" [15, p. 147]. This happened to Baranauskas when he got captivated by the operation of raising to a power, which, as he admitted himself, "absorbed all his efforts". Being highly inspired he made "many discoveries" in this field, as he remarked himself: "discoveries that were new for me, but known in mathematics for centuries" [2]. Getting deep into the mathematical intricacies he proved for himself the correctness of mathematical statements that were already known to mathematicians. For example, he re-created the proof of Newton's Binomial Formula.

3. Leisure Time Dedicated to the Number Theory

He continued to deepen his knowledge about the powers of numbers and started creating tables of numbers to the power of two, three and higher. He did not make merely mechanical calculations but used them to describe his mathematical insights. He was able to notice that the difference of the squares of two adjacent natural numbers equals the sum of these numbers [10, p. 14][1]. Antanas Baranauskas shared his mathematical insights with his pen-friend, the German linguist Hugo Weber (1832–1904). The linguist helped the bishop to establish

[1] A. Baranauskas rediscovered himselfa method already known in mathematics at the time.

a relationship with a teacher of mathematics at Eisenach gymnasium Carl Hossfeld, who provided Baranauskas with the new textbook in number theory written by Gustav Wertheim (*Elemente der Zahlentheorie*, issued in 1887). This encouraged Antanas Baranauskas to explore further the secrets of prime numbers as a branch of his favorite number theory. At that time, as he admitted in his letter to Weber "an obsessive question stuck in my mind" – does numeration have an end? And what end?" [13, p. 223-224]. So he got involved into calculating prime numbers, first in the hundreds of thousands and then in the first million. After having finished these tiresome direct calculations he was planning to calculate how many of them are there in ten million(10^7) but remarked that" the calculating process turned out to be so complicated that in half a year only one tenth of the plan was accomplished" [6, p. 5]. He understood that if he continued with his calculations he would need, for example, a few years to search in the range of ten to the power of eight (10^8), tens of years if to talk about ten to the power of nine (10^9) and several hundred years for ten to the power of ten (10^{10}). After discussing the improvement of calculations with Hossfeld, he discovered that higher mathematics does not have an easier way to do this. So he had to rely on his own resources and noticed some regularities, i.e. the symmetry that exists between the spaces of certain numbers. In the already mentioned textbook by Wertheim he found a formula determining the number of prime numbers that do not go beyond a given bound. After the discussions with Hossfeld he grasped the idea of the formula and presented his conclusion. He shared the results with Carl Hossfeld ,who made a mistake, while rewriting the text , and in addition he published the article *The Remark about a Formula of the Number Theory* (*Bemerkung über eine Zahlentheoretische Formel*) in the journal "Zeitschrift für Mathematik" (1890, No. 25, p. 382–384) [6, p. 6]. Afterwards Baranauskas continued his calculations and discovered that his formula is simpler than the one in Wertheim's textbook, which was created by the German mathematician Ernst D.F. Meissel (1826–1895). The bishop reviewed the process of his reasoning and prepared the article *About formulae used to calculate the number of prime numbers that do not go beyond the given bound* (*O wzorach służących do obliczenia liczby liczb pierwszych nie przekraczających danej granicy*). With the help of the Polish linguist Jan Baudouin de Courtenay (1845–1929), the article was published in "Rozprawy Wydziału Matematyczno-przyrodniczego" (1895, Vol. 28, p. 192-210) [6, p. 6] issued by the Academy of Sciences in Krakow. It was important that in the article Antanas Baranauskas included the table listing the values of the function that determines the number of prime numbers [4, p. 30-32]. Though this work by Baranauskas does not contain a rigorous proof, it got his name mentioned in the scientific work *History of the Theory of* Number (New York, 1952) by the American mathematician Leonard E. Dickson (1874–1954) [10, p. 34].

4. A Glance at Geometry

About 1891 the bishop turned from the theory of numbers to solving geometrical problems. He directed his attention to one of the three oldest mathematical problem – squaring the circle. In other words, the bishop was preoccupied with the challenge "of constructing a square with the same area as a given circle". Though back in 1882 the professor Carl L.F. von Lindemann (1852–1939) from Munich had proved the number π is transcendental, the bishop, possessing knowledge only in elementary geometry, got an interesting expression, presented as follows: $\pi = 3 + 0{,}1\sqrt{2}$. Shortly afterwards he shared his thougts about this new discovery with Rev. Dambrauskas. However, the priest explained to the bishop that the result is approximate and that an identical formula was already found in the 14th century by the great Italian Dante Alighieri. This news stopped the bishop‘s ambitions to write a thesis in Latin on the above-mentioned subject and dedicate it to the Pope Leo XIII [6, p. 7].

At the end of the 19th century, when the Lithuanian language was still banned, American Lithuanians did a great job by issuing Lithuanian textbooks that were in short supply; however, there was a problem with the Lithuanian terminology – some words did not exist in the language. With the mediation of the Rev. Dambrauskas, Most Rev. Antanas Baranauskas, who had already distinguished himself in Lithuanian language research, started to create terminology of geometry. These are a few examples of the terminology proposed by him after a discussion with the linguist Weber: the terms *smailus kampas* (acute angle), *status kampas* (right angle), *daugiakampis* (polygon), *trikampis* (triangle), *lankas* (arc), *erdvė* (space) were created on the basis of Lithuanian words and the terms such as *punktas* (point), *linija* (line), *kvadratas* (square), *kubas* (cube) were created on the basis of international words. Some of the words proposed by him did not take root in the Lithuanian language and were subsequently changed, e.g. *ratas* to *skritulys* (circle), *ratlankis* to *apskritimas* (circumference), *skersinis* to *skersmuo* (diameter), *stipinas* to *spindulys* (radius), *skerskampė* to *įstrižainė* (diagonal), *gija* to *styga* (chord), *trapezas* to *trapecija* (trapezoid), etc. [12, p. 258-259].

5. About the Limits of Mind and the Infinity

After geometry the bishop started to examine complicated sequences of numbers called transcendental progressions: a_1, a_2, a_3, ..., a_n, $(a_1 = a^a, a_2 = a_1^{a_1}, \ldots, a_n = a_{n-1}^{a_{n-1}})$. Analyzing the problem he noticed that, when a =2, the first three numbers of the progression can be easily calculated, when a = 3, two numbers can be easily calculated, when a = 5, only one number can be easily calculated, and when a equals more than 5, such a possibility ceases to exist altogether. Such observations lead the bishop to recognizing the limits of the human mind. He reflected all this in his study *About Transcendental Progression and about the Limits and Power of the Human Mind* (*O progresji transcendentalnej oraz o skali i siłach umysłu ludzkiego*), which was issued as a separate publication in 1897 in Warsaw [6, p. 9].

The end of 1897 brought big changes into Antanas Baranauskas' life – on the 23rd of October he was appointed to the position of the bishop of Seinai (Sejny). Therefore on Christmas, on the 25th of December, he moved to this peripheral Lithuanian-Polish town that had become the episcopal see. After having settled in a new place the bishop continued his predilection for mathematics and looked at the "Queen of Science" as a philosopher and theologian.

In the above-mentioned article about transcendental progression the bishop wrote: "There are truths that surpass the human mind; and therefore, there are minds that are more powerful than the mind of a human being. There is an infinite set of truths. And hence, there is the mind that has the infinite power of comprehension. The infinity of the object for comprehension indicates that there must be the infinite subject that comprehends” [6, p. 9] This confirmed the theological trend of the article; the return to the proof of the existence of God.

The considerations about the transcendental progression and the book on infinity by the French mathematician Renè de Clèrè encouraged the bishop to define the concept of infinity for himself. The bishop had an excellent education in philosophy and a naturally insightful mind that helped him to perceive the existence of two infinities – the actual one and potential one. The reasoning went as follows: “If we label actual infinity “*a*”, nothingness “0” and infinity “∞”, then these three characters will mean three different spheres, not only separate and having nothing in common, but also not coming in contact with each other at all. Between actual infinity and nothingness there lies the sphere of infinite potential $p = \infty$. Also, between actual infinity and infinity there is a sphere of infinite potential. It leads to the conclusion that actual infinity does not come into contact either with infinity “∞” or with zero, which means nothingness. Every existence (apart from God) is limited from all sides and all aspects by limitless potential” [3]. From this we can see that Antanas Baranauskas distinguished the actual infinity, in other words – the infinity in itself and infinite sequence of numbers, and the potential infinity, which we understand in terms of a finite process.

In addition, the bishop remarked that comprehending the contact of the mentioned spheres means that “the creation from nothingness is the creation from potential: the potential is born from nothingness and the existence is born from potential”. The Rev. Dambrauskas commented on this by saying “that Baranauskas had a deeper understanding of the problem of infinite creation than many other theologians” [6, p. 10].

6. Closing Remarks

The works in various fields of mathematics, such as number theory, geometry and linear order theory were based on thorough and sound calculations as well as on considerations of philosophical concepts. His accomplishments show that Antanas Baranauskas had a talent in mathematics. On the other hand, the bishop used calculation as a very important tool of mathematics, not only to develop his mind but also to solve the problems of morality and faith.

He was one of those Lithuanian amateur mathematicians who encouraged others to pursue the study of mathematics in the international environment. We can also say with full confidence that the bishop’s mathematical works proclaimed the unity of science and faith.

Translated by Jūratė Marchertaitė, Lithuanian University of Educational Sciences

Appendix. Math works of bishop A. Baranauskas

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KALINA BARTNICKA*

HOW TO STUDY MATHEMATICS – THE MANUAL FOR WARSAW UNIVERSITY 1ST YEAR STUDENTS IN THE INTERWAR PERIOD

JAK STUDIOWAĆ MATEMATYKĘ – PORADNIK DLA STUDENTÓW PIERWSZEGO ROKU Z OKRESU MIĘDZYWOJENNEGO

A b s t r a c t

In 1926 and in 1930, members of Mathematics and Physics Students' Club of the Warsaw University published the guidance for the first year students. These texts would help the freshers in constraction of the plans and course of theirs studies in the situation of so called "free study".

Keywords: *Warsaw University*, *Interwar period*, *"Free study"*, *Study of Mathematics*, *Freshers*, *Students' clubs*, *Guidance for students*

S t r e s z c z e n i e

W 1926 r. i w 1930 r. Koło Naukowe Matematyków i Fizyków Studentów Uniwersytetu Warszawskiego opublikowało poradnik dla studentów pierwszego roku matematyki. Są to teksty, które pomagały pierwszoroczniakom w racjonalnym skonstruowaniu planu i toku ich studiów w warunkach tzw. „wolnego stadium".

Słowa kluczowe: *Uniwersytet Warszawski*, *okres międzywojenny*, *„wolne stadium"*, *studiowanie matematyki*, *pierwszoroczniacy*, *studenckie koła naukowe*, *poradnik dla studentów*

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This paper is focused primarily on the departure from the “free study” in university learning in Poland after it regained its independence in 1918. The idea of the “free study” had been strongly cherished by professors and staff of the Philosophy Department of Warsaw University even though the majority of students (including the students of mathematics and physics) were not interested in pursuing an academic career. The concept of free study left to the students the decision about the choice of subjects they wished to study and about the plan of their work. The young people entering a university had difficulties deciding what subjects and in what order they should select, as they usually set their minds on preparing themselves to teach in secondary schools. To help them make the right choices, the elder students and young researchers active in students’ scientific clubs and circles at Warsaw University published guidances about what university study is, how to organize the work, and what subjects and in what order to select. The first of these guidances was published in 1926.

The concept of “free study” gradually collapsed, under the pressure of the urgent needs of that period in the history of Poland, which specifically required the training of professionals at schools of higher learning. These needs, along with the necessity to consider the cost of running the university, the economical planning of study, and devising of the ways of entrance into the university, required creation, sometimes through the back door, of more rigid curricula, and a more regulated course of study. The latter was reflected in the way the Ministry introduced its regulations on teacher examinations, the Master’s and Doctor’s examinations, and the academic staff’s acceptance of this process was reluctant and slow.

* * *

After the First World War, developing the Polish system of higher learning required swift measures and decisions from the newly emerging state authorities and academic clubs, the consequences of which could hardly be foreseen. Many learning and educational issues had been discussed during the War, but no unanimous position was established. Various concepts and visions clashed, some quite beautiful, but simply unreal. Few seemed to fully realise the actual economic, organisational and scientific capacities of the state and its society. The ambitious plans to create a homogenous law and a modern learning system were to be thwarted by the material hardships of both the state and the society. The newly emerging state administration had no clear vision of the system of higher learning, still less experience in running one.

As far as universities were concerned, in each of the former partitions the traditions, experiences and expectations were different. During the 19th century Polish people undertook multiple ineffective attempts at creating a Polish or Polish-German university in Poznań. The Polish youth went to study at German universities, of which the nearest ones were in Berlin and Wrocław (Breslau) [4, p. 247]. This was the decisive factor for why the Humboldt model of a university was the most familiar to the Polish people in the German partition: one which formally observed academic freedoms, such as the freedom to teach; professors were required to combine research and didactic work; and students were given the freedom to choose what subjects and in what order they were to study; whereas (in theory) the ultimate goal of study was to prepare students for independent academic research.

In the second half of the 19th century, during the autonomy of Galicia, the best conditions for the development of Polish schools of higher learning prevailed in the Austrian partition. At universities, Polish was made the official language. In Lwów (Lviv) and Kraków, buoyant learning and scientific institutions thrived. Both the Lwów and Jagiellonian universities were state institutions, under direct authority of the ministry in Vienna. First, the regulations of Leon Thun's state university reform of 1849 applied there, but then, along with other Austrian schools, they began to gradually emulate the Humboldt university model. The philosophy department was made equal in rank with other university departments and had lost its preparatory function. The universities were granted an extensive degree of autonomy. The principle of freedom to study and teach (the so-called "free study" [8, pp. 115-143]) was introduced. The students had the right to choose[1]. However, professorial teams of particular departments were gradually putting in order the sequence of studying subjects in specific years of study, and were recommending them to the students[2].

The principle of freedom to teach was adhered to, most fully, at philosophy departments, whose key goal was to train secondary school teachers. No specific learning curriculum was set, however. It was up to the student to choose freely the sequence of completing classes and taking exams. The course of studies of prospective teachers was "regulated" by the gradually expanding requirements of the state-level teacher examination involving specialised subjects (major and minor), the related exercise and discussion classes, and the obligatory subjects for all teachers: philosophy, Polish literature and the German language; as well as pedagogy (philosophy and pedagogy test or confirmed attendance at the Pedagogy or Philosophy seminar). The way to employment and scholarly career was made available when the student obtained the degree of doctor of philosophy, and then habilitation, also regulated by specified requirements[3].

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1 "Students may choose their teachers and lecturers irrespective of what will be required of them as they sign up for state examinations or specialised doctoral examinations", *O naukach. Przepisy ogólne o przyjęciu na uniwersytet ogłoszone w rozporządzeniu Ministra Wyznań i Oświaty z 1 październi-ka 1850…*, [in:] *Zbiór najważniejszych przepisów uniwersyteckich*, Ed. by K. Kumaniecki, Kraków 1913, p. 16, Article 44; [7, pp. 151-152].

2 An item of information states on p. 83 that "a group of professors" put together a study curriculum which is not binding, but a failure to comply with it may result in the student's inability to complete his or her studies within 10 semesters; See: *Wskazówki dla studentów Wydziału Lekarskiego przy zapisywaniu się na wykłady, wydane przez grono profesorów tego Wydziału na Uniwersytecie Lwowskim in: Zbiór ustaw uniwersyteckich, które uczniowie c.k. Uniwersytetów otrzymują przy immatrykulacji*, Lwów 1903, pp. 82-83.

3 "The instruction advised choosing those lectures in the first year of study which were fundamental for the given scientific disciplines. (…) in the first year (…) students were advised to attend the classes on the introduction to advanced mathematical analysis, analytical geometry, and experimental physics, and take part in seminars of mathematics, analysis and geometry; in the second year (…) calculus, the function theory, differential geometry; and in the third and fourth years differential equations and calculus of variations". The students were advised to start the study of physics with the course on theoretical mechanics and thermodynamics. Later on, the students might choose subjects in any order: begin the study of astronomy and geophysics with the course on differential and integral calculus; in the first year they might attend lectures on spherical astronomy, in the first or second years also lectures on chemistry. In natural sciences the first two

The professors of the Jagiellonian University's Philosophy Department observed that students of the mathematics and natural science departments, especially those in their 1st year of study, "were making gross mistakes in selecting their subjects". "The benefit a student gets from his studies depends to a large extent on whether he is properly prepared to understand the lectures he attends; therefore, it is of paramount importance that the student first of all should acquire the skills required to comprehend them". "Enrolling in the lectures in an improper order immensely decreases the benefits one acquires from studying". Therefore, in order to make it easier for students to select rationally the sequence of subjects they were to study, the philosophy department's professorial staff decided to "set out certain guidelines they could follow when enrolling in lectures and classes"[4].

In the early 20th century a set of guidelines was prepared for students: *Tips for Philosophy Department Students Who Enrol in Mathematical and Natural Science Lectures* [7, pp. 179--184][5]. The reason why this set of rules was drawn up was the necessity to make more efficient the process of learning of mathematical and natural science subjects as a result of their quick development and the progressing specialisation. In Austrian schools and those in Galicia, the academic freedoms introduced by law in 1850 had been gradually limited. This process varied and proceeded at different speeds in different departments. In principle, the reason for this was that students should use more efficiently the studying time they needed to embrace the material of their scholarly discipline. The development and diversification of science in the second half of the 19th century made it more difficult for the student to sensibly select and rationally set out the lectures and classes offered by the university departments over the semesters and years of his study.

The free study was intended to scientifically train prospective independent scholars, which was certified by their doctoral degree. However, young people were coming to study at universities in order to obtain the knowledge they needed to teach in secondary schools, fill positions in administration, obtain medical certificates etc., and their employment was dependent on whether they passed the state examination that certified they had obtained the knowledge necessary at the particular position. An inept selection of the subjects, especially in the first year of study, led to decreased efficiency and waste of time. There were certain differences between the universities in Lwów and Kraków in the numbers of students and terms of organisation, study courses, and studying objectives. Whereas Kraków was more oriented toward research and scholarly work, Lwów focused on training teachers.

During the First World War, universities in Galicia did not discontinue their work for long, all the while retaining their Polish character. Many future professors of schools of higher learning of post-war Poland were educated there. Therefore, their models of working and the opinions of the professorial staffs had a profound impact on the emerging Polish

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years were common for all specialisations, and the curricula would separate from the third year on; [7, pp. 176-177].

[4] *Wskazówki dla studentów Wydziału filozoficznego, zapisujących się na wykłady matematyczne i przyrodnicze*, [in:] *Zbiór najważniejszych przepisów uniwersyteckich*, edited by K.W. Kumaniecki, Kraków 1913, pp. 419-424.

[5] See also: *Wskazówki*, [in:] *Zbiór najważniejszych przepisów*, op. cit.

system of schools of higher learning and the *Act on Schools of Higher Learning* of 1920 (reprinted in [8, pp. 248-267, Annex 4])[6].

In the Russian partition, the issue of tradition, organisation and patterns of learning was quite complex. The Annexed Territories cherished the fine tradition of the Wilno University shut down in 1832. The Kingdom of Poland revered the traditions of Polish schools of higher learning: the Royal University of Warsaw closed down in 1831 as a result of repressions following the collapse of the November Uprising; the Warsaw Main School existed between 1862 and 1869. The Royal University of Warsaw was a state school. It comprised five departments: Theology, Philosophy, Law and Administration, Medical, and Fine Science and Arts. For that time, it was a modern school, with extensive freedoms in the area of learning, awarding of scientific degrees, research, uncensored work of its academic staff, and focus on practical learning and didactics. The Philosophy Department, of an equal status with other departments, comprised the Divisions of: Proper Philosophy, Mathematics, and Natural Sciences. The length of study and the outline of the curriculum were governed by official regulations, but the course of study was set out by the management of the particular department. After 1823 the supervision of and interference with the curricula by the educational authorities were becoming stronger and stronger.

Students were required to take exams in all subjects (major and minor) for the particular year of study. The final exam, at least two hours long, encompassed the whole material of the major subjects and was held in public in the summer session after the thesis had been handed in and reviewed. By passing this exam, the student obtained the Master's degree, and was "fit for national service having acquired skills and knowledge at University"[7]. There was no sequence of subjects arranged by the university authorities for the successive years of study because some of the subjects were taught by professors alternately every second year. The least detailed curriculum was set out at the Philosophy Department's Mathematics Division. The only recommendation was that the students ought to study geometry, algebra, and elementary physics [2, pp. 87-88, 109-110, 165].

The Warsaw Main School was a state school with four departments: Law and Administration, Medicine, Philology and History, and Mathematics and Physics. It was under the control and authority of the education ministry in Petersburg. The students were required after the first and second years of study to take the so-called professorial examinations. This was merely a formality, which really amounted to collecting professors' signatures in the student book every semester, which certified the attendance at the lectures. As a matter of fact, the students took only two oral exams: the middle exam after two years on the subjects attended, and the final exam. In 1866, a third exam in Russian was introduced. Next, the student submitted his thesis on the subject of his choice [2, pp. 356--357]. The Royal University and the Main School were oriented toward training teachers, administrative staff, and other specialists, according to specified curricula and study plans.

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[6] *Ustawa o szkołach akademickich*, 3 July 1920, "Official Journal of the Republic of Poland" 1920, vol. 72, item 494.

[7] *Tymczasowe urządzenie wewnętrzne Uniwersytetu Królewsko-Warszawskiego, 1818*, Tytuł VIII, Egzaminy; reprinted by R. Gerber, [in:] *Księga protokołów Rady Ogólnej Uniwersytetu Warszawskiego*, Warszawa 1958, pp. 190-191.

The Russian Imperial University, which in 1869 replaced the Polish Main School was organised according to patterns and regulations for Russian universities (mainly of 1882). The curriculum, plan and course of study were rigid. “Students were not required to complement their classes with additional reading: some professors tended to look at this with contempt, and at the exam they only required an ability to demonstrate the knowledge of the class programme, oftentimes with the actual phrases used there. The exams were highly formalised”. The studies were to be completed with a degree exam but after each year the students had to take promotion exams. The questions were drawn from a question box, and the lists of questions could be obtained by the students at the Dean’s office. The studies were usually completed with a certificate of a real (“rzeczywisty”) student, which certified the graduate’s ability to take up professional work. The best students (with examination average of 4.5 or higher) could write and defend a candidate thesis. By obtaining the status of candidate, the student was given the opportunity to obtain the Master’s degree and embark on a scientific career [3, pp. 484-490].

Other patterns of learning were introduced in Warsaw by the secret school operating in the last two decades of the 19th century, the so-called Flying University, and its offshoot from 1907, a private school run by the overtly operating Society of Science Courses (Towarzystwo Kursów Naukowych – TKN) [6]. This was a kind of free university, which exercised the freedom of learning and teaching. Lectures and other forms of classes depended to a large extent on the school’s ability to take on lecturers. The students were not subjected to state examination rigours. They obtained certificates confirming the subjects they passed, which were not officially recognised in the Kingdom of Poland and the Russian academic system.

TKN professors and students’ expertise must have had an impact on the didactic staff of the reinstated Warsaw University because a lot of TKN lecturers became later, in the restored Poland, professors of the Warsaw University and Warsaw University of Technology. Moreover, it was within the TKN circles that conceptual work on the reestablishment of Warsaw University (UW) was initiated. This work was underway already before the outbreak of and during WWI, and after the Russians withdrew from Warsaw in the summer 1915, the organisational work started in connection with the relaunching of both Warsaw University and Warsaw Technical University [3], pp. 14-15]. The operations of Warsaw University, which opened as early as 15 November 1915, and of its didactic staff, were to be of temporary nature all the way until 1920.

Although the *Act on Schools of Higher Learning* passed on 13 July 1920 laid down the conditions for putting the universities in order, it was rather general and required more detailed work on the schools’ statutes; it soon turned out that the visions and dreams of university professors did not go in line with the directions set out for the schools of higher learning by the Ministry for Religions and Public Education (MWRiOP).

Apart from drawing from the Polish university traditions, rejected under the Russian sway, it was probably through self-learning, by using the *Manual for Self-Learners*, that the views of the academic world in Warsaw and school managers in the restored Poland were being shaped on how the schools of higher learning and the organisation of studying should be arranged. It may be interesting to look at Zygmunt Janiszewski’s specific opinion expressed in the *Manual’s* Volume 1, which focused on mathematics: “We call a self-learner everyone who in his studies is dependent on his own initiative. The antonym of the self-learner is a 'pupil' of one school or a teacher, subject to foreign influence, curriculum and control. A student of mathematics need not be a pupil; at the best and most popular universities the students are to a large extent self-learners: they must set out their own plans of studies, select the lectures, coursebooks etc." [6, pp. 115-116]. In a footnote, the author explains that the evidence for such an understanding of studying is the "distribution by some German universities of printed guidelines, or '*ratschläge*' to all new students. Whereas the students of polytechnics and most universities in Italy, England, and particularly in Russia, are to a larger or smaller extent 'pupils', subjected to a strict curriculum, examinations etc." [6, p. 116].

However, it was also understood that self-learning required following a certain order which made the acquiring of new knowledge simpler. The sequence of contents in the subsequent volumes of the *Manual* can, it seems, be interpreted as an order suggested by the authors, or a course of study of the particular divisions of the given science[8]. It must be stressed that both studying at a TKN school and self-learning using the *Manual* required the students to possess substantial maturity, consistence, and independence. It seems that the fragment quoted above quite well represents the views of most professors of the re-emerging Warsaw University on what university studies should be like and what type of European university they sought to take as a model for Polish universities: the Humboldtian university and the "free study".

The work to restore or create Polish schools of higher learning on the territories of the former Russian (Warsaw, Wilno, and soon Lublin) and German partitions (Poznań) conveyed the features of a spontaneous uprising, particularly in Warsaw, where it started already during the war. Among the matters to be resolved quickly, the most urgent ones involved drawing up the curricula and courses of study, and recruiting the didactic staff. Both issues were closely connected with the concept of university, and were further linked to the principal goals set for the universities by the educational authorities, that is, the Ministry for Religious Denominations and Public Education.

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Only in Galicia did the universities – the Jan Kazimierz University in Lwów and the Jagiellonian University in Kraków – have a relatively stable (inasmuch as it was possible after the war ended) professorial staff and curriculum of study, and were therefore able to continue their pre-war activity which they had developed during the autonomy of Galicia (admittedly, inasmuch as the universities' financial capacity and the change of the seat of the educational authority from Vienna to Warsaw allowed it) [1, pp. 61, 69].

The professorial staff of Galicia universities and Warsaw's scholarly community heavily influenced the 1920 *Act on Schools of Higher Learning*. The Introduction to the *Act* stipulated

[8] The mathematical sciences unit in Volume 1 of the *Manual* was prepared by Z. Janiszewski, W. Sierpiński, S. Zaremba, and S. Mazurkiewicz. The chapters were as follows: General Introduction (description of mathematics and its breakdown), learning degrees (I – elementary, II – secondary school, methodology of mathematics teaching), III – higher mathematics and its sections.

that the goals of the schools of higher learning, particularly universities, would be to serve science and the homeland; cultivate and spread the knowledge; seek and find the truth "in all branches of human knowledge, as well as lead the way for the academic youth to explore the truth"; on the other hand their goal would be to "prepare the youth for practical professions, the execution of which requires a scientific capacity of various branches of knowledge and an independent judgement of its theoretical and practical aspects". Schools of higher learning obtained an extensive degree of autonomy in such areas as drawing up their own statutes (Art. 1) and granting academic, scientific and professional degrees (Art. 3), as well as "the right to freely learn and teach". Professors and *docents* had a guaranteed "right to present and discuss any issues, authoritatively and in line with their own conviction, and by scientific means" in the realm of science they represented, as well as a "complete freedom in whatever methods and exercise they chose to employ" (Art. 6). The ultimate self--governing power was held by the academic senate (Art. 18), which was responsible among other things for approving of the programme of lectures. The responsibility for "keeping watch of the development of learning" and the right organisation of teaching, lectures, speeches, and publishing were handed over to the department boards (Art. 32), but their statutes and programmes had to be submitted for approval to the Ministry.

Schools of higher learning were supposed to grant professional degrees (Art. 95). Two scientific degrees were introduced: a lower one, usually the Master's degree, and a higher one – the Doctoral degree. The mode of obtaining *veniam legendi* (habilitation) was laid down in clear terms. In the transitional period, until 1926, students who had begun their studies before 1920 could take exams and apply for doctoral degrees according to the existing regulations (Art. 112). The academic year was split into three trimesters. The *Act* did not include any guidelines regarding the course of study or examination requirements with regard to the students. These matters were to be settled by departmental statutes and regulations, and the continually modified ministerial decisions.

After 1920, subsequent regulations were put in place to regulate the curricula of study including those in medicine, pharmacy, and law, in the transitional period till 1926, upholding to some extent Galicia regulations. The studies at the Philosophy Department, which since 1916 included studies in mathematics and natural science, were also organised according to the regulations existing at pre-war universities of Galicia: the student selected the subjects of his own choosing (lectures and classes; preliminary seminars and regular seminars). "This selection was confined by the requirements of the final exams, but still it was quite extensive. The lectures informed the students of the up-to-date knowledge in a given discipline of science. Practical skills were practised at classes, and preliminary and regular seminars. The student's independent work was highly acknowledged, especially when conducted in some of the University's multiple students scientific clubs [3, pp. 76-77, 200-209].

The Warsaw University's Philosophy Department had no strict learning curricula. "It was recommended that the student choose subjects according to such groups as: philosophical, historical, mathematical, and natural science". The student was free to choose his or her major and minor subjects. The learning course was supposed to take four years, and the student was obliged to attend at least eight hours of class a week. Those students who had begun their studies before 1920, were bound to take only two examinations before they obtained a doctoral degree: one in the major and minor subjects, and the other in philosophy.

According to the new curricula, those who studied 11 trimesters had to take an obligatory Master's exam in one of a wide array of subjects including such as mathematics, physics, chemistry, and pedagogy. In order to make a Doctor's degree, the candidate had to submit his or her doctoral thesis and pass the exam. Those intending to become secondary school teachers had to undergo quite a complex procedure of the teacher examination, which they could take after three years of study. The teacher exam was on the one hand, "scientific", and on the other hand, "pedagogical". The first kind comprised two parts: a) written (two compositions: „domowe", prepared at home and „klauzulowe" – prepared under surveillance); b) oral, in the subject of one's teacher specialisation and an optional subject. The "pedagogical exam was oral and included Polish and optionally a foreign language if the subject was to be taught in this language; philosophy, pedagogy, and didactics of the major subject. It was also obligatory to conduct a trial lesson, the topic of which was set by the examination board. The pedagogical exam was meant to verify the candidate's qualifications to teach the subject of their choosing, educate secondary school students, and to prove that they knew the didactics of their subject and possessed the theoretical foundations of pedagogy" [3, pp. 204-205][9].

The scope of freedom in terms of subjects the students chose and workload varied across different departments. A student of the Philosophy Department (after 1927 divided into two departments: Mathematics and Natural Science, and Humanities) had in the years 1921–1933 an increase in the number of classes which they had to verify with signatures in their student books: at least 15 hours for the first three years and at least 8 hours in the fourth year, but the free selection of subjects was left to the student. As the post-war university operations were becoming more and more stabilised, the unfavourable consequences of free study for students were becoming even more evident.

Although the MWRiOP stressed the scientific goals of universities, it was primarily interested in training teachers and specialists. It was also evident that the ministry was seeking to extend its control over universities and, consequently, the professorial staff. University students, who were rather impoverished and in most cases poorly prepared for university studies by the post-war secondary schools, entered universities in order to obtain qualifications they needed for professional work, in a relatively short time. Whereas the country had a huge demand of well-educated secondary school teachers, lawyers, doctors, and other specialists, university professors sought to uphold the principle that universities train unselfish learned scholars. The curriculum and mode of learning did not take into account these needs, capacities, as well as ambitions of the studying youth. Scholarly training in the form of free study, which was so much in line with the ambitions of the academic community, was out of touch with the needs of the majority of students.

As the primary goal of universities, the 1920 *Act* mentioned research, rather than the necessity to train professionals, but failed to specify how these two goals ought to

[9] See: *Decree of the Minister of Religions and Public Education* [WRiOP], 1 September 1920, "The Official", Journal of the WRiOP Ministry, 1921, No. 2, [in:] *Studium Historyczne na Uniwersytecie Warszawskim. Poradnik dla studentów pierwszego roku*, Warszawa 1924, p. 63.

be reconciled, as an effective implementation of these two goals required a different organisation of the course of study. As the principle of free study was formally upheld, the university faced a grave problem. The latter was addressed by the Head of State Józef Piłsudski in May 1921 on the occasion of gifts donated by craft guilds to Warsaw University. He spoke openly of the goal to "provide the nation and the state with sufficient numbers of professional", and at the same time seek "to be anti-utilitarian (!), to be the shrine of pure science, strive for the absolute truth, whilst ignoring utilitarian aspects", which is fairly self-contradictory[10].

The Ministry attempted to bring some order into the learning system by means of regulations and examination requirements, but specifying this proceeded very slowly, and was determined on *ad hoc* basis by subsequent ministerial decrees. The government paid immense attention to the training of teachers of comprehensive secondary schools and teacher training colleges. It was the procedures of taking teacher examinations at universities in front of state examination boards that were developed in detail relatively quickly, although they were complemented several times. The UW statutes were passed as late as 1925, and brought into effect with the Minister's approval in the academic year 1925/26. The following year saw the split of the UW Philosophy Department into two separate departments: Mathematical and Natural Sciences; and Humanities. This led to the development by the new department boards of regulations setting out the learning organisation and course of study. In the following year, the Ministry issued new regulations on examinations.

Formally, students could freely plan their course of study: select the subjects, enter them into their student books, and do so for the following years of study. However, rash decisions, often due to the lack of information what to study, when and in what order, made the acquiring of knowledge harder, led to a waste of time, and, because the classes were paid, raised the cost of studying. This was a serious problem also because of the sharp rise in the students' population at UW. In the period between 1920/21 and 1923/24, the number of students rose from 6116 to 9419, and then remained steady at roughly nine thousand, which was 25% of all Polish students and about 21% of students at philosophical departments. The UW Philosophy Department, which until the academic year 1926/27 comprised mathematical and natural science subjects, as well as humanities, was the second most populous department after the Law Department, with some 1/3 of all UW students studying there, 2/3 of whom were in the humanities section (see [3, pp. 129-131]. This should give the picture of some 900 disoriented young people starting their studies every autumn and made their choices of subjects without realising the consequences of their decisions.

At the UW Philosophy Department it was the elder students working in students' scientific research clubs who offered the freshers help and advice. They drew up and published a guide for how to manage one's studies, giving such advice as: how to select the subjects (lectures, classes, seminars), in what order to study them and in which years, how to deepen one's knowledge with independent work and reading, without useless waste of one's energy or

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[10] See: T. Manteuffel, *Uniwersytet Warszawski w latach 1915/16–1934/35. Kronika*, Warszawa 1936, p. 40.

time[11]. In comparison to the "guidelines" at schools of Galicia or the rough information in the lists of lectures mentioned above, the publications of Warsaw University Students' Scientific Club (students of humanities, mathematical-physical or natural sciences departments alike) in the first decade of the interwar period were much more extensive and precise. These were little compositions clarifying what free study is, describing its pros and cons, what difficulties there may be, how to avoid them, and lastly, how to hold both: a high scientific level of study and the professional one. The authors understood the difficulties of the young people studying for material reasons, who made up a significant majority at the university, but they evidently cared about the high scientific level of the studies which were to open the way to research work.

In 1925 the Senate passed the *Warsaw University Statutes*[12]. Art. 38 of the *Statutes* stipulates that the Department Board "keeps watch of the development of science and promulgation of knowledge through proper organisation of the teaching process, lectures, public speeches, and publications"; Art. 97 discusses the organisation of the academic year, splitting it into 3 trimesters consisting of 10 weeks each (autumn trimester 1.X–15.XII; winter trimester 8.I–20.III; spring semester 20.IV–30.VI). Art. 99 § 2 included certain limitations on the registration of students and auditors in some classes. By registering in some department, the student could also select lectures and classes in another department. However, in case the student was suspected of attending a lecture that was not related to his or her course of study, the dean might reject the registration. The Statutes came into effect in the academic year 1925/26. The departments, in line with the 1920 *Act on Schools of Higher Learning*, were supposed to prepare their own internal regulations. The division of the Philosophy Department in the following year led to further changes.

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After the split of the UW Philosophy Department, Tadeusz Manteuffel explained in his article *How to Study History* that in most schools of higher learning, even at some university departments, the primary goal of studying was merely practical. They trained professionals who would be able to use their theoretical knowledge in their practical lives; teaching involved passing on to the student the scientific truths that had already been discovered on the one hand, and the encyclopaedic knowledge in a given discipline on the other. Whereas in the case of the humanities and Mathematics and Physics departments, "as a matter of fact,

[11] In 1924, the UW History Students Club, which was supervised by Marceli Handelsman, prepared the 80-page-long *UW History Study. The Manual for First Year Students*. After the Philosophy Department was divided into the Humanities Department and the Mathematics and Natural Science Department, this same club undertook to arrange for a series of lectures to be first delivered and next published, Part 1 in 1927 and Part 2 in 1928, under the common title *How to Study at the Humanities Department*. The authors of the lectures included some of the future luminaries of Polish science: Maria Ossowska, Tadeusz Manteuffel, Jerzy Manteuffel, Zofia Szmydtowa, Czesław Leśniewski, Zofia Podkowińska, and Michał Walicki.

[12] *Warsaw University Statutes*, [in:] *Ustawy i niektóre rozporządzenia dotyczące szkól akademickich*, Warszawa 1925, pp. 265-311.

they do not set themselves any practical goals and deal with unapplied sciences, i.e. with theoretical knowledge, science for science; they seek to train scientific researchers that are supposed to work on, deepen and enlarge a given branch of science". Students were to familiarise themselves with research methods and learn to think in "scientific categories". However, both at the humanities and the mathematical and physics departments the "staggering majority" of students were candidates for teachers with no scientific aspirations at all. With their needs in mind, the University was set to be reorganised: "the free study was bound to be discarded and a system of examinations was to be introduced. This system was eventually introduced last year" (i.e. 1926/27 – K.B.)[13].

In order to help the young people starting their mathematical studies understand the working and organisation of the Department, as well as select the classes and rationally plan their work, the students of the Warsaw University Club of Mathematics and Physics Students prepared in 1926 r. *Studium Matematyczno-Fizyczne na Uniwersytecie Warszawskim. Informator dla nowowstępujacych*[14], guide for 1st year students. They followed the example of the Warsaw University Club of History Students' publication of 1924.

Mathematical-Physical Study Guide of 1926 turned out to be extremely useful. 300 copies of the publication were distributed within two years. Meanwhile, further changes occurred in the structure of UW departments, in that a separate Mathematics and Philosophy department had been created, and in the Ministry's examination regulations. The Publishing Committee UW of the Club of Mathematics and Physics Students decided to publish a new, extended issue of the guide, which would consider not only the changes and new regulations, but embrace the perspective of the entire, 4-year course of mathematics and natural science studies. The 260-page-long publication *Mathematics, Physics and Astronomy Institute of Warsaw University. The Manual for Students* was published in 1930[15].

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[13] T. Manteuffel, *How to study history*, [in:] *Jak studiować na Wydziale Humanistycznym*, Warszawa 1927–1928, Part I, pp. 19-21.

[14] The facsimile of this publication comprises some 80 pages. Editorial Committee: Vice-President of the Warsaw University Club of Mathematics and Physics Students Hilary Kon, Chairman of the Publishing Committee Arkadiusz Piekara, and member of the Science Committee Jan Edward Szpilrajn; partners of the Editorial Committee: Physics Institute Assistant Witold Bernhardt, Vice-President of the Union of the Polish Academic Youth Science Clubs Eugeniusz Geblewicz, Vive-President of the Philosophy of Mathematics Section Adolf Lindenbaum, Chairman of the Warsaw University Club of Mathematics and Physics Students Jan Morgentaler, Astronomy Observatory Assistant Jan Rybka, Physics Institute Assistant Andrzej Sołtan, Head of the Marine Section of the Gdańsk Astronomy Observatory dr Edward Stenż, Physics Institute Assistant dr Szczepan Szczeniowski, Warsaw University of Technology Assistant dr Kazimierz Zarankiewicz, and Warsaw University Assistant Professor dr Antoni Zygmund. The Committee took advantage of the advice provided by Assistant Professor dr Kazimierz Kuratowski, Assistant Professor Bronisław Piekarski, Ksawery Świerkowski, Antoni Rajchman, Assistant Bolesław Iwaszkiewicz. They used the lists of lectures sent in by the Department Offices of other Polish universities.

[15] *Mathematics, Physics and Astronomy Institute of Warsaw University. The Manual for Students.* The 2nd run was altered completely by the Publishing Committee of the UW Club of Mathematics and Physics Students, Warszawa 1930, p. 263, in lithographic print. The names of the co-authors and the people who had contributed to the creation of the manual were thoroughly accompanied

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In the Preface in the *Mathematics and Physics Study* (1926), the author signed as EJS [most likely Edward Jan Szpilrajn] remarks that every year a crowd of 1st year students are looking for guidelines for "what they ought to do with themselves, which lectures to attend, from what and how to study", and advises that the candidates can obtain information if they turn for help to the Union of Clubs of Mathematics and Physics Students, whose Information Office, based in Kraków at 20 Gołębia str., responds to such enquiries by post. And in Warsaw, the information about the studies organisation and system can obtained from the Board of the Warsaw University Club of Mathematics and Physics Students, based at 72 Nowy Świat str. [the Staszic Palace – K.B]. Therefore, the members of Warsaw University Club of Mathematics and Physics Students decided to issue an extensive publication on the organisation of mathematics studies, especially that the *Warsaw University Statutes*, and examination regulations were at last passed.

These publications on mathematical studies (of 1926 and 1930) vary in terms of content and volume but the starting point in both is the distinction between the mathematics as a subject and the university mathematics. One should not take up university studies without a deeper interest in it or the necessary skills, as even As and Bs at the high school final exams were no sufficient proof of such skills. It is not a good reason to enrol in these studies because it is easy to get there or because one hopes to get a job easily after graduating. There is no use hoping, either, in the event of failing to get into a polytechnic, medicine or a natural science department, that after one year of studies at the Mathematics Department the candidate can better prepare him- or herself for the entrance exams there because studying at this department does not deal with high school subjects[16].

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by notices about their scientific degrees and functions. Two new people joined the co-authors of the 1926 publication.

The Editorial Board: Vice-Presidents and activists of the Mathematics and Physics Club: K. Billich and M. Jakimowiczówna, as well as J.A. Szpilrajn, at the time the Chairman of the Union of the Polish AcademicYouth's Mathematics, Physics and Astronomy Clubs. The publication used the texts of A. Zygmund and K. Zarankiewicz, the co-authors of the first publication of 1926, and the Collection of Regulations on University Studies (*Zbiór ustaw i rozporządzeń o studiach uniwersyteckich*), edited by T. Czeżowski (Wilno 1926). The other contributors to the Manual included S. Fogelson. MA, senior assistant of the Mathematics Seminar B. Iwaszkiewicz, dr M. Kerner, dr S. Lubelski, senior assistant of the Experimental Physics Institute dr S. Szczeniowski, Assistant of the Theoretical Mechanics Institute A. Wundheiler MA, Z. Marynowski, and J. Wasiutyński. The following professors and assistant professors endorsed the publication: S. Dickstein, J. Łukasiewicz, M. Kamieński, O. Nikodym, and A. Tarski; doctors: I. Bobrówna, W. Kapuściński, E. Rybka, A. Lindenbaum, S. Nikodymowa, S. Szczeniowski, M. Żebrowska; Masters of Science: B. Iwaszkiewicz, W. Ścisłowski, A. Wundheiler; "fellows": I. Kępińska, B. Piekarski, and A. Trojecka.

[16] S. Zaremba discussed this problem in response to a questionnaire sent by the Mianowski Fund in 1917 on the most urgent needs of the Polish learning system. "The significance of mathematics among other subjects of the Polish learning system (…) is all too often underestimated..." – he wrote. "It is very common to identify one's knowledge of mathematics with the proficiency in figures or graphic structures". However, "a deeper understanding is out of the question without the proper study of mathematics (…) The ability to properly understand mathematics (…) belongs to the basic features

Between studying at a secondary school and university there is also a huge leap in terms of exploring and understanding of such concepts as "section ('sekcja'), limit, convergence, continuity, and uniform convergence" at the lecture on the "Introduction to analysis" – this must not be ignored but one cannot be put off by first obstacles. Both publications recommend that beginners study *The Manual for Self-Learners* as an indispensible complementation of the tips and guidelines they provide, as it will help them learn the skill to study things. Both dedicated a lot of space to explaining to the first year students entering the university what university is and what is the subject they chose to study, mathematics.

University was not an institution where they taught you something, but an establishment where you could learn many things. Aside from a certain set facts, the student is provided with something far more precious, "ways of understanding and exploring, and later methods of scientific research". By learning them, the student will comprehend the goal of research, "which does not so much lie with seeking the truth, as with seeking the beauty", and the latter may induce one to work creatively. The student will face both research obstacles and ways leading to discovering new truths. Therefore, it is not the lectures, but classes and seminars, as well as work in research clubs, that are the most crucial during university studies. A seminar is a forge of knowledge, a place where the student makes personal contact with the professor and discusses scientific problems. Working in research clubs is slightly less formal; it is easier to make contacts and acquaintances with older colleagues and feel the research atmosphere[17].

* * *

The *Mathematics and Physics Study* of 1926 contains tips regarding the intellectual work. It advises taking notes during lectures but says students should use them and learn from them carefully as one can make mistakes while note-taking. It is important to read coursebooks, but not too many at the same time. It is sensible to start with one book, which is neither too extensive nor too difficult, study it carefully, even twice, and only skim other books. It is imperative that students take on an active attitude: work through sets of exercises and attempt to prove the theorems. In case of problems with understanding something, it is useful to ask someone more competent. It is wrong, however, to read more than one can take in, and in case of fatigue, it is good to take a rest. "It is wrong to force oneself to read or research as it is ineffective". It is imperative that students study foreign languages. They should start with German, for the time being read using a dictionary, and after 2–3 months they will need no dictionary at all[18].

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of a mind that is truly educated (…) It is one of those features which are in one's living practice the most useful (…) in any profession". Therefore, the teaching of mathematics must not be conducted in a merely utilitarian way, but rather in the specifically scientific way". He added, "universities (…) can only then properly educate professors of mathematics of secondary schools and seminaries when they are equipped with the right number of professional teachers themselves" (S. Zaremba, *O najpilniejszych potrzebach nauki w Polsce ze szczególnym uwzględnieniem matematyki*, „Nauka polska. Jej potrzeby, organizacja i rozwój", Vol. I, Warszawa 1918, pp. 4-7).

[17] *Studium Matematyczno-Fizyczne*, 1926, pp. 8-9.

[18] *Studium*…, op. cit., pp. 10-11.

The publication contains a four-year program of mathematical study, outlined by, as it may be guessed by looking at the initials K.Z., Kazimierz Zarankiewicz. The program divides the disciplines studied in the 1st and 2nd years into two groups: group A contains the more important subjects; group B contains the subjects that may be moved on to other years. It was also recommended that students should compare the program they were setting out on their own with the suggestions of the *Manual for Self-Learners*[19]. K.Z. recommended choosing:

In year 1: A) Analysis I (Infinitesimal calculus); Analytic geometry; Set theory. B) Abstract algebra with group theory; Number theory.

In year 2: A) Theory of functions of a real variable; Theory of analytic functions; Ordinary differential equations. B) Synthetic geometry.

In year 3: A) Partial differential equations; Differential geometry; Infinite sequences and series; Calculus of variations; Groups. B) Not specified.

In year 4: the studies were supposed to involve only lectures and work in the selected specialisation[20].

What is characteristic are the following comments made by the program's author: The proposed plan and sequence of studies may be set out differently "as far as studying the specific science does not require any knowledge of another [science]".

1) The Year 1 subjects should be studied together, by reading one coursebook from each section;
2) The program mentions sections that contain knowledge every mathematician must possess;
3) Every student with scientific ambitions should explore in detail and in depth the section he or she selected to study, and which will later be the field of his or her own creative work;
4) The choice of specialisation is made according to one's liking;
5) The topics of lectures and seminars at every university are dependent on the individuality of the professors working creatively there.

* * *

A separate part of the manual of 1926 deals with the sequence in which students should study the particular sections. The author signed as Dr A. Z., i.e. Antoni Zygmund, answers three questions: What to study? How to study? Which coursebooks to use? The formal assessment of this text including the coursebooks it recommends and their assessment can be judged by someone specialising in the history of mathematics. But what is important for the historian of education and higher learning are the more general questions, such as students' workload.

Dr A. Zygmund argues that studying mathematics is hard and involves hard work. Analysis I and Analytic geometry, lectures and classes take a Year 1 student 11 hours a week.

[19] See: Z. Janiszewski, *Wstęp do Stopnia III, Poradnik*, op. cit., Vol. I, pp. 122-123.

[20] The Editorial Committee added a disclosure that the proposed study curriculum is theoretically justified but not very practical under the current regulations (probably due to the order of taking teacher examinations); See *Studium*…, op. cit., p. 12.

They cannot be delayed because without embracing these sections it is impossible to study the Year 2 subjects. One should add to this 5 hours of the lecture in practical physics and, optionally, the completion of Practical Lab I in the spring semester. In total, it is 16 hours a week. It is the maximum of what can be done effectively but even this leads to overwork, especially when students also need to earn their living. Aside from this, there is individual work: doing the exercises assigned in classes and others. This is necessary for embracing and memorising the material. Attending lectures alone is not enough; their contents need to be worked on at home, too. Students must therefore also read. The author then reviews and evaluates the coursebooks for each subject[21].

* * *

Szczepan Szczeniowski presented problems connected with studying physics along with recommendations for an arrangement of subjects and reviews of coursebooks. In year 1, students need to complete the mathematical subjects they will need in studying physics: Abstract algebra and the Determinant theory; Analytic geometry; Analysis. At the same time, as their major subjects they must study Experimental physics, complete classes in Lab I, and in the chemical lab along with lectures. In year 2, their major subjects should be Theoretical physics and Lab II exercises (attending Lab II is conditional on attending Lab I for one year and completing these classes in year 1). This arrangement of subjects cannot be altered or moved to a later period.

The author argues that as few as 10% of students take notes at lectures, whereas physics is pure science and employs the mathematical apparatus. It involves precise note taking, particularly because there are no instructors' notes published in Polish or good coursebooks. The best way is to analyse the notes using a coursebook immediately after the lecture; any unclear problems need to be consulted with coursebooks. It is, however, arduous; it is necessary to attend seminars, and, from Year 2 analyse and present the assigned topics, not using coursebooks but by working on one's own. It is possible because the topics of the subjects are specified earlier on. The topics of classes in Lab II are also specified well in advance; they take three hours, and preparing them involves the assistants' help. In the subsequent years of study, the students know enough about arranging their workload to do this by themselves. Bearing in mind that the sciences are mutually connected and dependent, no delays are allowed in studying any of the subjects[22].

Edward Stenż wrote two short texts on studying and coursebooks used in meteorology and geophysics; Jan Morgentaler wrote on astronomy. They were quite superficial, probably because they required some foundations of mathematics and physics studied with all the Mathematics students during their first year in the university. However, a much more extensive part of the manual discussed issues of more general nature regarding scientific degrees (Master and Doctor) and the conditions of receiving qualifications to teach at secondary schools and teacher training colleges. It provided information on the ongoing

[21] *Ibidem*, pp. 15-21.
[22] *Ibidem*, pp. 22-32.

work on Master studies in Astronomy and on Pedagogy studies. As a matter of fact, regulations were being continually amended[23].

* * *

The Decree by the Minister for Religious Danominations and Public Education of 26 November 1925 on Master's degree examinations at the universities' philosophical departments, that is humanities and mathematical and physical, were analysed very thoroughly. Obtaining the lower scientific degree, that is Master's degree, was defined as a proof of completing higher studies and became the indispensable condition of obtaining a doctoral degree (Act 1 of the Decree). No doubt, this raised the value of the Master's degree and opened up to its holders two ways of using it: either seeking to obtain the desired teacher's certificate, or treating is as a threshold to the potential scientific career. Irrespective of how one looks at it, it was clearly an attempt to reconcile university's scientific goals with those involving training professionals. It also enabled universities to include modules in the curricula which aimed to train researchers, in a way pushing them into the system of teacher training. The candidate could obtain a Master's degree in a specific field within a philosophy department's learning framework. This field also designated a prospective teacher's major subject. However, by the time a candidate received his or her teacher certificate, they had to prove their methodological qualifications in theory and practice by taking the teacher examination before a separate Examination Board. The examination procedure was complex and comprised many stages.

In order to be eligible for the final phase of the Master's qualification process, the candidate had to prove passing all the subjects that the Ministry deemed relevant for Master's competency within the specific specialisation (e.g. in physics, mathematics, history, etc.) There were 8 or 9 examinations, and the regulations specified the subjects and sequence of completing them quite strictly. There was little choice left to the student. The examination--bound subjects were divided into three groups of subjects and sciences that complemented one another. The sequence of taking the exams was to a large extent dictated by the sequence of studying them.

Group A comprised six basic subjects (differential and integral calculus, and the introduction to analysis; analytical geometry; higher algebra with elements of number theory; theoretical mechanics; experimental physics; main principles of philosophy with special focus on logic. Group B comprised two exams on "separate" (oderwanej) or "applied" mathematics, and several sets of subjects recommended by the Departmental Council. Group C comprised complimentary subjects: the student had a choice of some sections of theoretical physics, astronomy, crystallography, logic or graphic methods and numerical calculation.

Formally, the candidate had some freedom in terms of the dates and sequence of taking the exams, but in fact the regulations aimed to force the student to work systematically and to gain control over the stages of his or her learning process. This was a serious, even decisive,

[23] *Ibidem*, pp. 32-38.

step towards eradicating the free study[24]. If a candidate's Master's thesis was considered sufficiently good, the examination board designated the date to discuss this thesis. It could be conducted along with the last exam for the Master's degree. During the discussion the student was supposed "to demonstrate his or her general knowledge in a given field of his or her specialisation".

The regulations regarding re-taking the exam or re-writing the thesis in case of failing to pass the exam are quite puzzling. The candidate could re-take the exam no earlier than in the following examination session. One was allowed to re-take the exam three times, and the fourth attempt had to be approved by a special decision of the department's council. These regulations were to be binding from the academic year 1926/27. However, earlier, in March 1926, the Ministry's decree came into effect regulating the curriculum and examinations in the field of mathematics for the degree of Master of Philosophy.

The total number of lectures and classes of Group A subjects, in which the student declared his or her attendance, could not be lower than 10 hours a week so that the trimester could be approved as completed for the student to be eligible for the Master's exam. The theme of the Master's thesis could be obtained after the student passed all Group A exams. The discussion was also supposed to involve testing the level of the candidate's mathematical knowledge and could be combined with the exam that finalised the Master completion course, i.e. Exam 8 from Group B.

* * *

The *Mathematics and Physics Study* publication of 1926 contained information about lectures on mathematics, physics, and astronomy at the Polish universities for the academic year 1926/27, with most detailed information on Warsaw University. It contained even a tip where to look for the bulletin board informing Warsaw University students of the time of a lecture, its place, and the number of hours it took every week. It clarified who and when (that is, in what order), could/should listen to specific lectures. There was no information about the seminars because these were intended in principle for 3rd or 4th year students. The proposed plan and distribution of hours was agreed as a result of the negotiations between Jerzy Kazimierz Bilich, representing the Maths-Physics Club, and the rector and professors. Although it "did not contain major collisions", some inconveniences could not be avoided due to the shortage of classrooms and lack of space in them faced by the growing numbers of lecturers and students.

The *Study* presented in quite great detail the institutes of mathematics, physics, and astronomy at Warsaw University: it gave their addresses, informed about how long they existed, whether they possessed libraries or other collections, who was the head of the institute, the names of professors on duty hours, what classes were held there. It provided

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[24] According to *Act 5* of the Minister for Religious Denominations and Public Education's *decree of 26 November 1925*, the student was obliged to complete each subject that was part of the given Master's program, in the appropriate sequence. The subject of the Master's thesis was to be explored in the course of two trimesters.

the addresses of the major libraries and listed the key scientific associations in Warsaw and all of Poland.

Lastly, the *Study* presented the information on the students scientific Club working at the Philosophy Department, the Warsaw University Mathematics and Physics Club, who initiated the *Mathematics and Physics Study* publication. The Club was housed in the Staszic Palace in Warsaw. In 1926, the Club celebrated its 10th anniversary, which was a good occasion to recapitulate its activities and demonstrate its successes. The University Rector, the Philosophy Department's Dean, as well as professors attended the celebrations. It was usually the senior years students, graduates, and candidates seeking to obtain scientific degrees, and even junior scholars that were active participants in the Club. Prof. S. Dickstein exercised the academic care over the club, and its first president was Kazimierz Kuratowski. The Club' patrons and associates oftentimes included distinguished professors. It was a member of the Union of Mathematical, Physical and Astronomers' Clubs of the Polish Academic Youth. The Club operated in many fields, such as research, publishing, and students' mutual help; it cooperated with other scientific associations and students' research caucuses all over Poland; its library contained more than 2000 books including expensive and rare coursebooks; it published its own coursebooks, and it sold coursebooks. The club undertook to awaken, support, and develop students' scientific interests; held regular scientific meetings, where papers were presented and debated.

Any student could join the Mathematics and Physics Club by paying 1 złoty of entrance fee and a monthly fee of 30 groszys. In 1926, the Club had 325 members and 10 senior members. The *Study's* authors recommended joining and working for the Club stressing that it was an important element in a students scientific development; it allowed making new contacts, dealing with older and more experienced mathematicians, participating in less formal discussions than those in class or at seminars. This helped create a truly scientific atmosphere, which is extremely important for the development of both students and researchers, for the former it was even more important "than lectures, a book, or seminars".

The formal problems related to the discipline, curricula, presentation of the subjects, coursebooks and other publications recommended for further reading, the reputation of the professors and other academic staff, as well as other people that had contributed to the publication of the guidances of 1926 and 1930, can all be analysed and evaluated by experts, historians of mathematics. However, it is worth noting that the free study as a principle in university study organisation was clearly decreasing in importance, and this process was reflected in the content of guidances for Philosophy Department students, both in mathematics and natural science, and in humanities. The publishers and authors of both the *Mathematics and Physics Study* of 1926 and the *Mathematics, Physics and Astronomy Institute* of 1930 decided to deal with the tips and guidelines regarding mathematical studies in a way more broad than just include lists of lectures. Both these publications contained a lot of information about university studies in general, about mistakes, problems and threats caused by wrong planning and selection of subjects, about Master's, Doctor's and teacher examinations. These explanations are interesting for a researcher of today as they cast light on the changes that were underway.

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LWÓW PERIOD OF S. ULAM'S MATHEMATICAL CREATIVITY

LWOWSKI OKRES TWÓRCZOŚCI MATEMATYCZNEJ S. ULAMA

A b s t r a c t

We provide an outline of Stanisław Ulam's results obtained in the framework of the widely understood Lvov school of mathematics.

Keywords: *Stanisław Ulam*, *Lvov School of mathematics*, *Borsuk-Ulam theorem*, *Kuratowski--Ulam theorem*, *topological groups and semigroups*

S t r e s z c z e n i e

W artykule przedstawiono zarys wyników Stanisława Ulama, które uzyskał w ramach szeroko rozumianej działalności Lwowskiej Szkoły Matematycznej.

Słowa kluczowe: *Stanisław Ulam*, *Lwowska Szkoła Matematyczna*, *twierdzenie Boruska-Ulama*, *twierdzenie Kuratowskiego-Ulama*, *grupy i półgrupy topologiczne*

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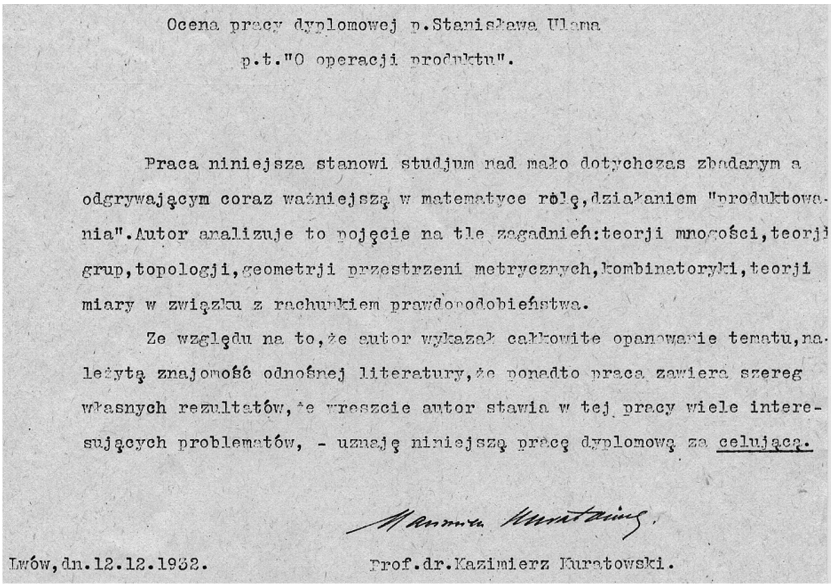
Lvov period of the life of outstanding Polish mathematician Stanislaw Ulam lasted from his birth (1909) until 1939, when he moved to the United States. The aim of this note is to provide a review of Ulam's mathematical results obtained in Lvov.

Fig. 1. Stanisław Ulam (1909–1984)

We will start with a short biographical information. Born in a Jewish family in Lvov, Ulam entered the Polytechnical school in 1927. As a student, he was influenced by K. Kuratowski and started to work in set theory.



Ocena pracy dyplomowej p.Stanisława Ulama
p.t."O operacji produktu".

Praca niniejsza stanowi studjum nad mało dotychczas zbadanym a odgrywającym coraz ważniejszą w matematyce rolę,działaniem "produktowania".Autor analizuje to pojęcie na tle zagadnień:teorji mnogości,teorji grup,topologji,geometrji przestrzeni metrycznych,kombinatoryki,teorji miary w związku z rachunkiem prawdopodobieństwa.

Ze względu na to,że autor wykazał całkowite opanowanie tematu,należytą znajomość odnośnej literatury,że ponadto praca zawiera szereg własnych rezultatów,że wreszcie autor stawia w tej pracy wiele interesujących problematów, - uznaję niniejszą pracę dyplomową za celującą.

Lwów,dn.12.12.1932. Prof.dr.Kazimierz Kuratowski.

Fig. 2. Kuratowski's report on Ulam's diploma thesis

His first mathematical paper was published in Fundamenta Mathematicae in 1929. It contains one of Ulam's results obtained in students years. In his report about Ulam's diploma thesis, Kuratowski remarked that the author provides an analysis of the notion of product on the background of set theory, group theory, topology, geometry of metric spaces, combinatorics, measure theory in connection to probability theory.

In 1933 Ulam earned his PhD at the Lvov Polytechnical School (now Lviv Polytechnical University).

He was one of the most active members of the Lvov School of mathematics. Ulam participated in sessions of Lvov mathematicians in the Scottish café (Kawiarnia Szkocka, in Polish) and contributed to the Scottish book, a list of open problems formulated by these mathematicians.

Fig. 3. Modern picture of the Scottish café; it is again named "Szkocka"

In 1935, Ulam accepted J. von Neumann's invitation and moved to Princeton. According to his published memoirs, Ulam spent in Lvov all the summers in the years 1936–39. When in Lvov, he contacted local mathematicians and continued collaborating with them. However, his joint paper with Oxtoby [19] gives Cambridge, USA, as his affiliation.

In 1944, Ulam joined the Manhattan project. Before that he worked for the University of Wisconsin at Madison. After the war, Ulam became a professor at the University of Southern California in Los Angeles. In 1946 he returned to Los Alamos, where he kept his position until 1965. During these years Ulam was a visiting professor at various universities: Harvard, MIT, the University of California, San Diego, the University of Colorado at Boulder. Since 1967.

Ulam was a Graduate Research Professor at the University of Florida in the period 1974–1984. In 1998/99, the Department of Mathematics at the University of Florida initiated the Ulam colloquium in memory of Stanisław Ulam.

Ulam died in 1984 in Santa Fe.

There is vast literature devoted to Stanisław Ulam and his mathematical achievements (see, e.g. [7, 8, 18]). We believe, however, that the material of the present note, which concentrates on the Lvov period of activity, is not completely covered by other publications.

Borsuk-Ulam theorem. This well-known result was conjectured by Ulam and proved by Karol Borsuk [4]. It states that for any continuous map of an n-dimensional sphere into an n-dimensional Euclidean space there are two antipodal points with coinciding images.

Satz II [7]). *Ist* $f \in R^{n^{S_n}}$ *(d. h. bildet* f *die Sphäre* S_n *auf einen Teil von* R^n *ab), so gibt es einen derartigen Punkt* $p \in S_n$, *dass* $f(p) = f(p^*)$ *ist.*

[7]) Dieser Satz wurde als Vermutung von St. Ulam aufgestellt.

Fig. 4. Fragments of Borsuk's paper

The Borsuk-Ulam theorem is known to be equivalent to the Brouwer fixed point theorem as well as other statements. It has numerous variations as well as applications in topology and other parts of mathematics (see the book [16]). In particular, N. Alon [1] and D.B. West [2] applied the Borsuk-Ulam theorem and its generalization to solving the so called Necklace splitting problem. In [25] it is shown how this theorem can be used in constructing consensus halving, i.e. division of an object into two parts so that n people believe that these parts are equal.

T. Banakh and I. Protasov [3] used this theorem for finding the number of decomposition of Abelian groups into non-symmetric parts.

Kuratowski–Ulam theorem. In [13] Kuratowski and Ulam proved a theorem which is a counterpart of the Fubini theorem for categories: if C is a set of the first category in the product of Baire spaces X and Y (the space Y is supposed to be separable), then the set $C \cap (x \times Y)$ is of the first category in $x \times Y$ for all x except the set of the first category.

These results of [13] were generalized in subsequent papers. In particular, in [10] the notion of Kuratowski-Ulam pair is introduced. Namely, such is a pair of spaces (X, Y) for which the conclusion of the Kuratowski-Ulam theorem holds. A space Y is a universally Kuratowski-Ulam space if (X, Y) is a Kuratowski-Ulam pair for every space X. The article [10] is devoted to properties of universal Kuratowski-Ulam spaces.

Measurable cardinals. In 1930, in his paper [26] published in Fundamenta Mathematicae, Ulam introduced the notion of a measurable cardinal. An uncountable cardinal number κ is measurable if there exists a κ-additive, non-trivial, 0-1-valued measure on the power set of κ. Here, κ-additivity means that the measure if the union of any disjoint family of cardinality $< \kappa$ is the sum of measures of its members. It is known that any measurable cardinal is inaccessible. The theory of large cardinals is now an important part of the set theory.

In [26] Ulam also introduced an object which is now called the Ulam matrix.

Remark that some consequences of one result of [26] are derived in [27]. In particular, it is proved in the latter paper that in every set of positive outer (Lebesgue) measure there exists an uncountable disjoint family of subsets of positive outer measure.

Topological groups. The theory of topological groups and semigroups was also developed in the Lvov school of mathematics. Stanisław Ulam published few papers in this direction; most of them were coauthored by Józef Schreier. As Ulam remarked, the published joint paper were a result of collaboration taking place almost every day.

A basis of a topological (semi)group X is a subset S of X such that the sub(semi)group generated by S is dense in X. It is proved in [22] that the semigroup $C([0, 1]^m)$ of the continuous selfmaps of the m-dimensional cube $[0, 1]^m$, $m \geq 1$, has a basis consisting of five elements.

In [23] the following statement is proved: almost every (up to a set of the first category) pair of elements of a compact metrizable connected group is a basis of this group.

In [20] Ulam and Schreier consider the topological group S_∞ of bijections of the set N of natural numbers. The topology on S_∞ is generated by a complete metric. It is proved that a proper normal subgroup of S_∞ consists of finite permutations (i.e. those moving only finite number of elements of N). In other words, the group S_∞ is topologically simple.

One of the main results of the paper is finding a basis in S_∞ that consists of three elements. This paper is supplemented by a short note [24] in which it is shown that there is no outer automorhism of S_∞.

The paper [21] deals with auto-homeomorphisms of the circle. One of the result of this note is that the group of order-preserving autohomeomorphisms is simple.

Some additional information concerning the activity of Józef Schreier in the theory of topological (semi)groups can be found in [11].

Borsuk-Ulam functor and geometric topology. In the paper [5] K. Borsuk and S. Ulam considered the subset $X(n)$ of nonempty subsets of cardinality $\leq n$ in a metric space X. The set $X(n)$ is endowed with the Hausdorff metric. Although Borsuk and Ulam did not notice this explicitly, the construction X(n) determines a functor in the category of metric spaces and continuous maps.

Let $I = [0, 1]$. It is proved in the mentioned paper that $I(n)$ is an n-dimensional, locally connected Cantor manifold. Also, $I(n)$ is homeomorphic to the space I^n, for $n = 1, 2, 3$, while for $n > 3$ the space $I(n)$ cannot be embedded into I^n. Some formulated problems concerning spaces $X(n)$ were solved by another authors. In particular, Jaworowski [12] considered the problem of preserving ANRs by the functor of symmetric product. His proof, however, contained a gap. This was first noticed by V. Fedorchuk [9] who found a correct proof that uses the theory of Q-manifolds.

In the joint paper [14] with K. Kuratowski the authors study the number $\tau(A, B)$, where A, B are compact metric spaces, defined as follows:

$$\tau(A,B) = \min_{f(A)=B} \max_{f(x)=f(y)} d(x,y),$$

where d denotes the metric on A. If **B** is a class of metric spaces, then $\tau(A, B)$ denotes the infimum of $\tau(A, B)$, for all spaces B in **B.** For an n-dimensional space A and the class **B** of smaller dimension the number $\tau(A, B)$ is proved to coincide with the Urysohn constant. Also if A is a non-unicoherent continuum and **B** consists of unicoherent continua, then $\tau(A, B)$ is proved to be positive.

In [6], some invariants of maps with small preimages are found. In particular, it is shown that the non-contractibility of a map into a given ANR-space is such an invariant of a compact metric space.

Remarks. We do not pretend to be complete. In particular, we did not mention above the joint paper with Łomnicki [15] containing important results (although with an erroneous proof) concerning products of probability measures (e.g. [8]).

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MATHEMATISCHE KRÄNZCHEN IN PRAG – A FORGOTTEN GERMAN MATHEMATICAL SOCIETY

MATHEMATISCHE KRÄNZCHEN IN PRAG – ZAPOMNIANE NIEMIECKIE TOWARZYSTWO MATEMATYCZNE

Abstract

The most important and interesting phenomena from the history of the association Mathematische Kränzchen in Prag (the Prague German Mathematics Community), which operated in Prague between spring 1913 and spring 1934, will be introduced, on the basis of the study of surviving archive sources available in Czech country and abroad, original professional journals mathematical works, and diverse secondary literature. We will try to clarify the position of the German mathematical community in the Czech lands, respectively in Central Europe in the 1920s and 1930s. We will try to capture its specifics resulting from the Prague *genius loci*, to describe its contributions to the development of science, to indicate its links to the surrounding German scientific world and to show its relations with Czech and foreign professional associations and societies.

Keywords: Prague German mathematics community (1913–1934)

Streszczenie

W artykule przedstawione zostały najważniejsze i najbardziej interesujące zjawiska z historii towarzystwa Mathematische Kränzchen w Pradze, które funkcjonowało w latach 1913–1934, w oparciu zarówno o badania zachowanych źródeł archiwalnych dostępnych w Czechach i za granicą, jak i o analizę czasopism i innych materiałów z okresu działalności Towarzystwa. Zaprezentowany został wpływ niemieckiej społeczności matematycznej z Pragi na ośrodki w Czechach i w Europie Środkowo-Wschodniej w latach 20. i 30. XX w. Poczyniono starania uchwycenia specyfiki działalności Towarzystwa w oparciu o swoisty *genius loci* Pragi po to, aby ukazać wkład Towarzystwa do rozwoju matematyki w ogóle, wskazać jego powiązania z niemieckim światem naukowym oraz uwypuklić relacje Towarzystwa z czeskimi i zagranicznymi stowarzyszeniami naukowymi.

Słowa kluczowe: Niemieckie Towarzystwo matematyczne w Pradze (1913–1934)

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1. Short historical background

In order that the readers could better understand the development of German mathematical community in the Czech lands, its scientific and association activities which are partly presented in other parts of this article, we give a short historical background.

In the second half of the 19th century, due to the rise of nationalistic movements, the Czech and German communities which naturally existed in Bohemia for some centuries separated. This separation was also reflected in culture, science and education. An important feature of that period was the process in which Czech science was "becoming independent". First professional societies were bilingual and Czechs and Germans collaborated with each other. In the 1870s, the Czech community was getting stronger and more active and finally prevailed. This process was accompanied, on the one hand, by protracted national conflicts and, on the other hand, by expensive constructions of new schools and the establishment of new associations.

In the 1860s, the efforts of Czech political representatives and intellectuals as well as the movement of university students to have their courses of studies in Czech language required an establishment of Czech mathematical lectures at the Prague Technical University (1864). At first they existed in parallel with German ones, which had better teachers and more funding. The arrival of better qualified Czech teachers and students who have been educated at Czech secondary schools led to the strengthening of the positions of Czech mathematical lectures at the Prague Technical University[1] and the establishment of similar lectures at the Prague University (1871).

The German University in Prague was founded in 1882 as an academic institution favored by Vienna government. It was given better buildings, larger part of libraries, all seminars and better organized places than the Czech University in Prague, which was established in the same year. Therefore, it did not need to compete with the Czech University and to demonstrate the significance of its professional activities. But the Czech mathematical lectures had comparable professional standards to the German ones and even began to exceed them in student enrolment. At the end of the 19th century the importance of the Czech mathematical community was increasing, because of the growth in the number of the Czech teachers and students. On the other hand, the number of German students was decreasing, because most of the German professors considered Prague to be merely a temporary place on the way to Vienna or Germany. Gradually, the German University in Prague got into a position of isolation and it acquired a character of a provincial school[2].

After creation of Czechoslovakia (1918) the German University in Prague was not abolished; on the contrary, it became an equal, recognized and respected state university not subdued, oppressed or financially restrained by the new republic. Actually, it was the only official state minority university in the inter-war Europe divided into states based on supranational principle. It maintained its position until the beginning of World War II.

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[1] It should be noted that the independent Czech Technical University in Prague was established in 1869. For more information on the Czech Technical University see [16].

[2] For more information on the development of the mathematical community in the Czech lands in the second half of the 19th century and at the beginning of the 20th century see [1].

The Faculty of Science of the German University in Prague (founded in 1920) was relatively small but significant scientific and pedagogical workplace in Europe. It attracted German-speaking students from the Czech and Slovak countries who, just at the beginning of the 20th century, were headed to Berlin, Dresden, Vienna and Budapest because diplomas from German, Austrian or Hungarian universities had to be validated or completed by other Czechoslovak state exams. It attracted foreign Jewish students and democratically thinking students from Lithuania, Latvia, Ukraine, Hungary, and Poland, and – since the mid 1930s – also from Germany. Rather low school fees and costs of living in Prague and its good accessibility played a positive role in its development. The renown of certain professors (e.g. L. Berwald, R. Carnap, C.I. Cori, Ph. Frank, A. Kirpal, A. Lampa, K. Löwner, A. Naegle, G.A. Pick, E.G. Pringsheim, and F. Spina) also played an appreciable role.

The period from 1920 until 1939 saw some national, religious, economic and social problems resulting not only from economic crisis but in particular from increasing strength of fascism, increase in domestic conflicts between liberal and social democratic groups (supported also by German speaking multicultural and Jewish circles) on one hand and national and anti-Semitic groups on the other hand, escalation of conflict between certain German professors and Czechoslovak government at the end of 1930s, expulsion of Jewish professors from the University in 1938 and 1939, inclusion of the school into Reich German universities and war transformation (1939) and end of the German University in Prague (1945). In spite of these problems, this was the time of the largest bloom and boom of the German mathematical community in Prague – the time when the Prague German mathematics was at its peak and when its results achieved worldwide fame and recognition. However, it is necessary to emphasize that the German mathematical community was not directly affected by the abovementioned negative phenomena until 1939, because most pedagogues in the high schools and other members were Jewish or had democratic mindsets. People of various nationalities (German-speaking citizens of Czechoslovak Republic, Austrians and Germans, other European and American citizens), people of different religion (Catholics, Protestants, Jews and people without religion), people of various political affiliation (democrats, communists, Sudeten German Party members, Zionists and people with no interest in politics), people of varied social background and with different relation to Czech countries or Czechoslovakia actively and effectively collaborated with one another and for them, the most important thing was their love for mathematics, mathematical studies, results and achievements, which fascinated, filled and associated them much more than other matters could divide them[3].

2. The way to the birth of the *Mathematische Kränzchen in Prag*

German mathematicians and physicists, mostly from Prague, more or less regularly met from 1913 until 1934 to give lectures, round tables and discussions within the Mathematical

[3] For more information on the development of the German University in Prague from 1920 until 1939 see [7]. For more information on the German mathematics and German mathematical community in Prague see [2, 4].

Circle [Mathematische Kränzchen in Prag][4]. Their meetings took place on the premises of the Mathematics Seminar of the German University (Viničná street No. 3, Prague 2) or in the study rooms of the German Technical University (Dominikánská street No. 5 (nowadays Husova), Prague 1). It is not clear who established the "circle"and when and how.

Fig. 1. Faculty of Science of the German University in Prague[5]

In the first decade of the 20th century only a handful of university professors of mathematics worked in Prague. Josef Anton Gmeiner (1862–1927) taught at the German University from 1901 until 1906 and Josef Grünwald (1876–1911) taught there from 1906 until 1911, however, they were not creative mathematicians and their education and approach to mathematics was rather representative of the 19th century. From 1880 until 1929 Georg Alexander Pick (1859–1942) taught mathematics at the German University. Pick was a significant persona of Prague-German mathematics. Anton Karl Grünwald (1838–1920), who connected his life with the Prague-German mathematical community, gave mathematics lectures at the German Technical University from 1869 until 1909. From 1896 until 1904 Wilhelm Weiß (1859–1904) taught at the German Technical University and Karl Ernst Zsigmondy (1867–1925) taught there between 1905 and 1906. Both Weiß and Zsigmondy approached Prague only as a temporary transfer station within the journey to better positions in the monarchy. Karl Carda (1870–1943) taught at the German Technical University from 1906 until 1939 and Anton Grünwald (1873–1932) from 1909 until 1932. These two educated a number of technicians but they were not top-tier productive mathematicians. In the years 1901–1915 descriptive geometry was taught by Eduard Janisch (1868–1915).

For the sake of completeness, we should add that the German University had two positions of a full professor of mathematics; the German Technical University had two

[4] Mathematische Kränzchen could also be translated as Mathematical Round Table or Mathematical Ring.

[5] Today the building of the Faculty of Science, Charles University in Prague,Viničná street 3, Prague 2.

positions of a full professor of mathematics and one position of a full professor of descriptive geometry. A few private docents lectured at these schools as well and an assistant position was assigned to each full professor position. The German University also had one position of a professor of general (theoretical) physics and one position of a professor of experimental physics. The German Technical University also had two positions of a full professor of physics and one position of a professor of engineering mechanics (theoretical mechanics and applied mathematics). Both schools offered lectures by private docents and assistants, respectively technical experts and laboratory workers. Prague also had a number of high schools with approximately twenty teachers of mathematics, physics and descriptive geometry[6].

Shortly before the First World War the situation changed significantly because in a short period a group of young, energetic, talented professors, docents, assistants and students educated at foreign universities who belonged to the world's best arrived in Prague at their young age. In 1909 Gerhard Hermann Waldemar Kowalewski (1876–1950) obtained a professorship at the German Technical University and in 1912 Theodor Michael Friedrich Pöschl (1882–1955) became the professor of mechanics and applied mathematics at the same school. Philipp Frank was appointed a professor of physics at the German University in 1912 upon recommendation from Albert Einstein. In 1913 Kowalewski's position was taken by Wilhelm Johann Eugen Blaschke (1885–1962), who worked in Prague only for a short time – until 1915. In 1913 Paul Georg Funk (1886–1969) became a regular assistant of mathematics at the German Technical University. He gave lectures as a professor there until 1939. From 1915 until 1939 P. G. Funk gave special optional lectures also at the German University.

In 1916 Karl Mack (1882–1943) was appointed a full professor of descriptive geometry at the German Technical University. He connected his entire life with Prague. He officially lectured at the German Technical University until 1943; apart from that he provided regular lectures and seminars in descriptive geometry at the German University. In 1918 Roland Weitzenböck (1885–1955) obtained the position of a full professor of mathematics at the German Technical University but already in 1920 he left to work at the Technical University in Graz. In 1919 Ludwig Berwald (1883–1942), a young assistant, arrived at the German University and he immediately underwent Habilitation and became a respectable

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[6] We should remember the most important representatives of Prague-German physics, some of whom were engaged in the Mathematical Circle's activities: Carl Ferdinand Lippich (1838–1913), Ernst Lecher (1856–1926), Anton Lampa (1868–1938), Albert Einstein (1879–1955), Philipp Frank (1884–1966), Heinrich Rausch-Traubenberg (1880–1944), and Reinhold Heinrich Fürth (1893–1979).

Mathematics and physics was also intensively developed by professors of astronomy, geodesy and meteorology who, nevertheless, did not participate much in the activity of the Mathematical Circle. Prague astronomical, geophysical and meteorological researches were involved by Ladislaus Weinek (1848–1913), Rudolf Ferdinand Spitaler (1859–1946), Adalbert Prey (1873–1949), Leo Wenzel Pollak (1888–1964), and Erwin Finlay-Freundlich (1885–1964).

Basic information on the aforementioned persons of the German University in Prague can be found for example in [7, 11, 13, 17] and on http://www-history.mcs.st-andrews.ac.uk and http://de.wikipedia.org.

successor of G.H.W. Kowalewski. In 1922 L. Berwald was appointed an adjunct and in 1927 a full professor of mathematics and eventually, he connected his entire life with Prague and Prague-German mathematical community.

3. *Mathematische Kränzchen in Prag* 1913–1934

Young professors with passion for mathematics and private docents spiced up the life in the Prague-German mathematical community in 1913. They understood that regular professional meetings, which they were used to from their experiences abroad, would be very useful and important for their future work[7]. They were the people who influenced the very first activity of the Mathematical Circle. L. Berwald, Ph. Frank, P.G. Funk, and G.H.W. Kowalewski were among the most active participants.

G.A. Pick was a representative of the older generation of Prague mathematicians who joined the group[8]. From 1915 young assistants and doctoral candidates (R.H. Fürth, H. Hahn, K. Löwner, E. Nohel, A. Winternitz, and J. Wanka) become involved in lectures. The number of young lecturers once again grew at the end of 1920s and at the beginning of 1930s, when students, doctoral candidates and young assistants (W. Fröhlich, W. Glaser, B. Goldschmied, P. Kuhn, E. Lammel, H. Löwig, A.E. Rössler, K. Sitte, W.E. Stein, and E. Winter) started to report their first results.

Professors L. Berwald, R. Carnap Ph. Frank, P.G. Funk, R.H. Fürth, K. Löwner (1893–1968), W.C. Gottlieb Müller (1880–1968) and A. Winternitz (1893–1961) were also among the active participants of the “circles”.

The activity of the Mathematical Circle was not limited to Prague-German mathematicians but it also involved German mathematicians in Brno (lectures were given by Heinrich Tietze (1880–1964), Emil Waelsch (1863–1927) and Friedrich Schoblik (1901–1944)), Czech mathematician Vojtěch Jarník (1897–1970) and a Russian mathematician Eugen Bunickij (1874–1952) also had their own lectures. The precise share

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7 See reports on activity of similar clubs and seminars in Berlin, Göttingen, Hamburg, Leipzig Munich, Vienna, etc. published in the magazine *Jahresbericht der Deutschen Mathematiker-Vereinigung*. Note that those who came to Prague had experience from their studies or employment in schools in Berlin, Bonn, Erlangen, Innsbruck, Greifswald, Göttingen, Heidelberg, Königsberg, Leipzig, Munich, Oslo, Paris, Pisa, Graz, Strasbourg, Vienna etc.

8 Let us quote interesting words on the creation of the Prague mathematical circle: “*In Prague, new influences, connected at first with names such as G. Pick and P. Funk, had large significance for Berwald’s creative activity. P. Funk who studied at Hilbert, brought the newest variation theory from Göttingen and G. Pick developed continuation of the “Erlangen Program” by development of differential geometry with groups of transformations without motion invariants, especially by the idea of affine differential geometry as the closest case. It is known how this program was very successfully executed under scientific and organizational leadership of W. Blaschke. L. Berwald contributed significantly to this activity. It was a time when the “Prague circle” was established at Vltava river, a circle of Prague mathematicians and physicists of extraordinary scientific and social quality*” (*the designation circle comes from W. Blaschke, who reduced the status and rules of this club to empty set*) [15, p. 234].

of participation of German mathematicians from Brno and Czech mathematicians in the lectures of the Mathematical Circle could not be satisfactorily reconstructed[9].

Active members of the Mathematical Circle realized from the beginning that German mathematical community in the Czech countries, and subsequently in Czechoslovakia, was not large and they had to try to engage in activities in other German-speaking countries. Therefore from 1913 until 1937 they regularly informed the “world” of their activities through brief and well-arranged reports published in the magazine *Jahresbericht der Deutschen Mathematiker-Vereinigung*[10]. From 1914, foreign mathematicians (Germany, the Netherlands, Poland, Austria, Ukraine and USA) started to report at the meetings[11].

In spring 1934 Mathematische Kränzchen in Prag finished its activity and was transformed into Deutsche physikalisch-mathematische Gesellschaft in Prag, which initiated regular lectures in two sections – mathematical and physical section[12].

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[9] No information about the existence and activities of the Mathematische Kränzchen in Prag were published, either in the journals *Časopis pro pěstování mathematiky a fysiky* [Journal for Cultivation of Mathematics and Physics] and *Rozhledy matematicko-přírodovědecké* [Scopes of Mathematics and Nature Sciences], in the bulletin *Zprávy ze zasedání Královské české společnosti nauk* [Reports of the Royal Czech Scientific Society], in the annual reports *Výroční zprávy Jednoty českých mathematiků a fysiků* [Annual Report on the Union of Czech Mathematicians in Prague] and *Almanach České akademie věd* [Annual Report on the Czech Academy of Sciences] or in other Czech journals and newspapers.

[10] Active members of “*circles*” were also involved in the reviewing activity for the international report journal *Jahrbuch über die Fortschritte der Mathematik*. They reported on articles published in German, French and Italian journals, and on monographs and textbooks issued in the aforementioned languages. Thus they obtained certain prestige in the European mathematical community, free-of-charge access to the newest literature (to a certain degree) and overview of current results in their fields. For example, Ph. Frank in the years 1916–1923 reported on 284 works, P.G. Funk in the years 1916–1918 on 13 works, A. Winternitz in the years 1917–1924 on 31 works, K. Löwner in the years 1918–1922 on 25 works, L. Berwald in the years 1921–1937 on 150 works, F. Schoblik in the years 1930–1941 on 132 works. W.J.E. Blaschke, L.G.E.M. Bieberbach, M. Pinl, H. Tietze, R. Weitzenböck and E. Waelsch, who spent a rather short time of their professional careers in Prague, respectively Brno, were also regular reviewers for a long period. For example, E. Waelsch in the years 1893–1896 reported on 49 works, W.J.E. Blaschke in the years 1910–1941 on 223 works, L.G.E.M. Bieberbach in the years 1912–1941 on 692 works, H. Tietze in the years 1916–1936 on 96 works, R. Weitzenböck in the years 1926–1941 on 379 works, M. Pinl in the years 1924–1941 on 1190 works.

[11] Interesting memories describing the atmosphere in the Prague German mathematical community are written in [5, 9].

[12] *Das mathematische Kränzchen in Prag ist in der neu gegründeten Deutschen physikalisch--mathematischen Gesellschaft in Prag aufgegangen. Es setzt als mathematische Abteilung dieser Gesellschaft unter neuem Namen seine bisherige Tätigkeit fort* (Jahresbericht der Deutschen Mathematiker-Vereinigung 45, 1913, p. 49).

4. Brief description of professional activities

From the spring 1913 until the spring 1934, 365 lectures, discussion contributions and reports on the newest discoveries, results and works published in German, French and Italian journals were given in the "mathematical circles".

For the sake of clarity we hereby provide the number of lectures in individual years: 1913 – 19, 1914 – 22, 1915 – 13, 1916 – 22, 1917 – 22, 1918 (spring) – 11, school year 1918/1919 – 13, 1919 (autumn) – 7, 1920 – 17, 1921 – 29, 1922 – 12, 1923 – 20, 1924 – 20, 1925 – 16, 1926 – 13, 1927 – 8, 1928 – 11, 1929 – 8, 1930 – 18, 1931 – 17, 1932 – 22, 1933 – 20, 1934 (winter) – 5[13]. They were delivered by 49 different lecturers, out of whom 13 were foreigners[14].

Lectures were provided from October until mid-December and from mid-January until mid-June, sometimes until the end of June (in exceptional cases until first half of July). There usually were 15–20 lectures per year. Their number decreased for a short period in 1915 and 1922, which could have been related to war events and departures of W.J.E. Blaschke, G.H.W. Kowalewski, R. Weitzenböck and K. Löwner to Germany and also to an increase in pedagogical and administrative work related to the creation of the Faculty of Science of the German University in Prague. A more significant decline at the end of 1920s (1926, 1927, 1928 and 1929) could have been caused by the fact that the full professors G.A. Pick and L. Berwald limited their lecture activities in the Mathematical Circle due to the increase in university students and doctoral candidates and they freed space for younger colleagues (H. Löwig, W. Glaser, P. Kühn), who gradually became more and more engaged participants. It is remarkable that not even the economic crisis led to major decrease in the number of lectures and to termination of the "circles": this was because they were organized on a voluntary basis, the lecturers did not receive any payment and rooms for meetings were provided free of charge by the German University in Prague, i.e. the activities were not financially dependent on support from state, school or other institution. On the contrary, in the first half of 1930s the most active foreign participants arrived at Prague to be engaged in the Mathematical Circle.

Participants of the Prague "mathematical circles" knew the development of their fields, they were in contact with foreign colleagues; they knew the newest results, journal publications and books and they were able to obtain, study, understand them and report on them. They closely monitored works of their German, French, Italian, Dutch, Romanian and Hungarian colleagues which were published in German, French or Italian language. They usually reported on the issues they had been actively focusing on. In view of the fact that the Prague-German mathematical community was not large, it could not include the entire spectrum of the mathematics of that time. However, it is interesting that its members did

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[13] Several interesting experiments also took place, for example K. Mack in 1918 explained functioning of the so-called perspektograf, R.H. Fürth in 1921 presented the working of diffuse and condensation pump, resp. electronic piano.

[14] In the case of foreign or generally non-Prague lecturers, their workplace is given in brackets following their names. We should note that R. Carnap and M. Pinl reported as foreign participants at first; later they became Prague university pedagogues.

not pay almost any attention to the results of theory of sets, probability theory and statistics, topology, logic etc., achieved by Russian and Polish mathematicians (for example P.S. Aleksandrov, D.F. Egorov, A.N. Kolmogorov, N.N. Luzin, A.A. Markov, L.S. Pontryagin, M.Ya. Suslin, P.S. Urysohn, S. Banach, S. Mazurkiewicz, Z. Janiszewski, S.M. Mazur, W. Sierpiński, H. Steinhaus, A. Tarski), even though a number of their works had been published in German or French language. Almost unnoticed was the development of modern algebra and theory of numbers, i.e. the results of works of B. L. van der Waerden or E. Noether. Most likely insufficient contacts with English and American mathematical community, lower availability of English literature in Prague and imperfect knowledge of the English language caused that Prague-German mathematicians did not report on English articles and monographs. It is not easy to comment in detail on cycles and individual lectures because no preparatory materials of lecturers, lecture texts, their written or audio records, notes and recollections of students have been preserved; we also do not have any abstracts, syllabuses or brief summaries.

In lectures and reports mathematicians paid attention to the following then current mathematical topics: differential, difference and integral equations, function theory and potential theory, functional analysis, geometric function theory, special functions, axiomatic theory of probability, classification of substitutions, fundamentals of statistics, fundamentals of mathematical logic, calculus of variations and optimalization, mathematical theory of games, elementary, differential, affine and integral geometry, Minkowski geometry, Hilbert spaces, special geometric transformations, geometric extremal problems and inequalities, history of mathematics. Physicists focused on the following modern issues: theory of relativity, "ether" theory and Michelson experiment, continuum theory, gravitational field theory, Brownian motion, quantum mechanics and its mathematical basis, theory of radiation and atom model, nuclear physics, microscopy, statistical physics, application of calculus of variations in physics, hydrodynamics and its application in engineering practice and optics (beam propagation in anisotropic media).

5. Some typical and interesting examples[15]

G.A. Pick, who gave 50 lectures and reports in the years 1913–1927, was among the most active mathematical participants of the "mathematical circles". Function theory and potential theory were represented most often in his lectures (24), followed by differential geometry (7), matrix theory (5), real analysis (5) and also 9 reports on various topics. In the short, most active period from 1913 until 1917, he gave 26 lectures, in 1920 until 1923 he gave 15 lectures. G.A. Pick presented his most important works in geometric function theory from 1915 until 1918. In January 1914 he presented the report *Distanzschätzungen*

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[15] The pictures of L. Berwald, R. Carnap, P.G. Funk, G.H.W. Kowalewski and G.A. Pick are taken from the their passports which are deposited in their personal files (Archive of the Czech Republic, Prague). The picture of K. Löwner is taken from the private Loewner family archive (USA) and is reprinted with the permission of Löwner's daughter Marian Tracy.

im Funktionenraum[16], which provoked a reaction from W.J.E. Blaschke in February 1914 in the form of the report *Neue Distanzschätzungen im Funktionenraum*. Joint work by W.J.E. Blaschke and G.A. Pick called *Distanzschätzungen im Funktionenraum II*[17] is remarkable. Through the methods of classical analysis (the extremal convex function theorem) an integral representation of convex functions is proved. It is a result which "was ahead of its time". In its nature the work falls within the context of functional analysis: standard proofs of analogical representations are usually based on Krein-Milman theorem (1940) on representation of convex subsets through extremal points[18].

G.A. Pick (in 1930s) G.H.W. Kowalewski (1917) L. Berwald (1938)

P.G. Funk (1921) K. Löwner (in 1930s) R. Carnap (1932)

Fig. 2. Photos of Prague German leading mathematicians

The overview of lectures delivered within the "mathematical circles" shows that modern mathematical analysis was not among the preferred topics. Lecture *Der neue Beweis von F. Rieß für seinen Satz über die Erzeugung der linearen Funktionaloperationen durch*

[16] Ph. Frank, G. Pick, *Distanzschätzungen im Funktionenraum I*, Mathematische Annalen 76, 1915, 354-375.

[17] Mathematische Annalen 77, 1916, 277-300.

[18] For example [10, 564 and 605].

die Integrale von Stieltjes (W.J.E. Blaschke, 1914) reacting to Riesz theorem (1909) on representation of bounded linear functionals on the space of continuous functions was an exception. Lebesgue integral occurred in two lectures of A. Winternitz (1920) and H. Löwig informed of metric linear spaces.

K. Löwner was also an active participant of the "mathematical circles". From 1916 until 1922 he delivered 17 lectures; after returning from Germany during 1930 and 1933 he delivered 7 lectures. Two lectures focused on physical topics, two were dedicated to Chebyschev polynomials, other lectures focused on two topics close to Löwner's interests – functions of complex variables and matrix functions. K. Löwner reported on complex analysis in seventeen presentations. He dedicated ten lectures to results of other mathematicians and he based seven reports on his own scientific results. They were related to conformal mapping and he delivered them from 1916 until 1918 and in 1921. We emphasize these are results of permanent value, results that significantly influenced the development of geometric function theory. They still meet with response nowadays, because they are frequently quoted. Special attention should be paid to the three lectures on the topic *Erzeugung von schlicht abbildenden beschränkten Funktionen durch infinitesimale Transformationen* (1921). K. Löwner published the reported results in 1923 under the name *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises. I*[19]. This major work presents the first nontrivial results on the Bieberbach conjecture ($|a_3| < 3$), but it also brought new approach into geometric theory of functions based on so-called *Löwner's differential equation.* This method was surprisingly used not only by L. de Branges within complete solution of the Bieberbach conjecture but it also unexpectedly entered modern mathematics after 2000 through so-called SLE (stochastic Loewner evolution)[20].

Löwner's lectures from 1932 and 1933 were dedicated to monotone matrix functions and their relations with Pick's functions. Löwner's work called *Über monotone Matrixfunktionen*[21] introduced new mathematical issues which are still alive nowadays[22].

W.J.E. Blaschke, who delivered 13 lectures, was an important persona of the Prague "mathematical circles" from 1913 until 1915. It was W.J.E. Blaschke who brought to Prague new, productive and modern geometric topics – conformal mappings, their properties and applications, theory of convex bodies, isometric problems and affine differential geometry. These topics later dominated in his famous Hamburg seminar[23]. The lecture *Konforme*

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[19] Mathematische Annalen 89, 1923, 103-121.

[20] SLE is the fundament of the excellent results achieved by W. Werner and S. Smirnov, who were awarded the Fields medal in 2006, respectively 2010. See for example *2006 Fields Medals awarded*, Notices Amer. Math. Soc. 53, 2006, 1037-1044, A.M. Vershik, J. Bourgain, H. Kesten, N.Yu. Reshetikhin, *The mathematical work of the 2006 Fields medalists*, Notices Amer. Math. Soc. 54, 2007, 388-404, R. Malhotra, *Fields medalists 2010*, Current Sci. 99, 2010, 1647-1653, *Fields Medals awarded*, Notices Amer. Math. Soc. 57, 2010, 1459-1465, T. C. Hales, B. Weiss, W. Werner, L. Ambrosio, *The mathematical work of the 2010 Fields medalists*, Notices Amer. Math. Soc. 58, 2011, 453-468.

[21] Mathematische Zeitschrift 38, 1934, 177-216.

[22] For more information see [3].

[23] W. Blaschke, *Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie, II. Affine Differentialgeometrie*, Springer, Berlin 1923, VI + 259 pages.

Abbildung einfach zusammenhängender schlichter Gebiete[24] given in autumn 1913 was interesting and inspiring for Prague-German mathematical environment. Its topic was inspired by Lewent's monograph *Konforme Abbildung*[25]. At the end of 1913 and in 1914 and 1915 W.J.E. Blaschke delivered six lectures discussing modern issues of the convex bodies theory, isometry and properties of curves and surfaces related to his numerous articles[26]. It is probable that already from 1914 he reported on the results which he later summarized in the book *Kreis und Kugel*[27]. This book had enormous influence on the study of properties of convex bodies and so called "in Größen" (i.e. in "large-scale geometry")[28].

In the preface (V–VI), W.J.E. Blaschke wrote on his Prague inspiration and collaboration with his colleagues: *Die erste, ehrfurchtvollste Verbeugung Herrn F. Klein! Von ihm stammt die auf dem Begriff der stetigen Transfer-mationsgruppen beruhende geometrische Denkart, die allem Folgenden zugrunde liegt.*

Der nächste, freundschaftlichste Gruß dem mathematischen Kränzchen in Prag! 1916 hat Herr G. Pick gemeinsam mit einem von uns die ersten Untersuchungen zur affinen Flächentheorie veröffentlicht, später haben sich A. Winternitz und L. Berwald dem affinen Verein beigesellt, und insbesondere Herrn Berwald haben wir beim Zustandekommen dieses Buches viel zu danken…

...Bei der Korrektur haben uns insbesondere die Herren E. Artin, L. Berwald, A. Duschek, G. Thomsen unterstützt.

In his famous monograph, he quoted many times excellent results of his Prague colleagues (L. Berwald, G.A. Pick, A. Winternitz).

24 These popular problems became the main topic of some lectures delivered by G. A. Pick (from 1913 until 1921) and K. Löwner (from 1917 until 1920).

25 L. Lewent, *Konforme Abbildung. Herausgegeben von E. Janke. Mit einem Beitrag von W. Blaschke*, Teubner, Leipzig und Berlin 1912, VI + 118 pages. W.J.E. Blaschke wrote for this monograph Chapter 5. See also E. Study, W. Blaschke: *Vorlesungen über ausgewählte Gegenstände der Geometrie. Zweites Heft. Herausgegeben unter Mitwirkung von W. Blaschke: Konforme Abbildung einfach zusammenhängender Bereiche*, Teubner, Leipzig und Berlin 1913, IV + 142 pages.

26 W. Blaschke, *Eine isoperimetrische Eigenschaft des Kreises*, Mathematische Zeitschrift 1, 1918, 52-57, *Beweise zu Sätzen von Brunn und Minkowski über die Minimaleigenschaft des Kreises*, Jahresbericht der Deutschen Mathematiker-Vereinigung 23, 1914, 210-234, *Kreis und Kugel*, Jahresbericht der Deutschen Mathematiker-Vereinigung 24, 1915, 195-207, *Ein Beweis für die Unverbiegbarkeit geschlossener konvexer Flächen*, Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen, Math.-phys. Klasse, 1912, 607-610, *Über den grössten Kreis in einer konvexen Punktmenge*, Jahresbericht der Deutschen Mathematiker-Vereinigung 23, 1914, 369-374, *Einige Bemerkungen über Kurven und Flächen von konstanter Breite*, Ber. Verh. königl. Sächs. Ges. Wiss. Leipzig, Math.-phys. Kl. 67, 1915, 290-297, *Konvexe Bereiche gegebener konstanter Breite und kleinsten Inhalts*, Mathematische Annalen 76, 1915, 504-513, *Über Raumkurven von konstanter Breite*, Ber. Verh. königl. Sächs. Ges. Wiss. Leipzig, Math.-phys. Kl. 66, 1914, 171-177.

27 W. Blaschke, *Kreis und Kugel*, Veit und Co., Leipzig 1916, X + 169 pages.

28 On the influence of Blaschke's book see for example [8].

W.J.E. Blaschke was succeeded in the "mathematical circles" in Prague by **G.A. Pick,** who was attracted by the topic of conformal mapping and its properties[29], convex bodies[30], affine differential geometry and infinitesimal geometry[31].

Geometric topics were a long-term interest of **L. Berwald,** who in 1916 until 1934 gave 34 lectures. The first lecture named *Geschlossene algebraisch rektifizierbare Kurven* was delivered probably during his short visit to Prague; the lecture focused on rectification of curves in non-Euclidean spaces[32]. He then paid attention to affine geometry, which was the subject of several lectures from the turn of the first and second decade of the 20th century[33]; he also focused on theory of convex bodies and projective differential geometry[34]. In the mid-twenties he focused on differential geometry of curves, which was discussed through six lectures[35]. Since 1930s he dealt with the issues of Finsler spaces[36].

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[29] From 1913 until 1921, G.A. Pick spoke on these topics in more than 10 lectures. He was inspired by the results of W.J.E. Blaschke, L.G.E.M. Bieberbach, G. Faber, P. Koebe and E.L. Lindelöf. He published his achievements in the articles named *Zur Theorie der konformen Abbildung kresförmiger Bereiche*, Rendiconti Circolo Matematico di Palermo 37, 1914, 341-344, *Über eine Eigenschaft der konformen Abbildung kreisförmiger Bereiche*, Mathematische Annalen 77, 1915, 1-6, *Über den Koebeschen Verzerrungssatz*, Ber. Verh. königl. Sächs. Ges. Wiss. Leipzig, Math.--phys. Kl. 68, 1916, 58–64, *Über die konforme Abbildung eines Kreises auf ein schlichtes und zugleich beschränktes Gebiete*, Sitzungsber. Akad. Wiss. Wien, Math.-naturw. Kl. 126, 1917, 247-263, *Zur schlichten konformen Abbildung*, Ber. Verh. königl. Sächs. Ges. Wiss. Leipzig, Math.--phys. Kl. 81, 1929, 3-8.

[30] In the autumn 1914, G.A. Pick had two lectures named *Gebietsbestimmungen für konvexe Kurven*, See G. Pick, *Über das Gebiet, welches von konvexen Kurven in der Ebene bedecktwird*, Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen, Math.-phys. Klasse, 1915, 113-118.

[31] From 1913 until 1923 G.A. Pick talked on some questions of differential geometry four times; in 1919 he gave one lecture on projective and infinitesimal geometry. His fundamental contributions to differential geometry are summarized in two articles *Über affine Geometrie IV: Differentialinvarianten der Flächen gegenüber affinen Transformationen*, Ber. Verh. königl. Sächs. Ges. Wiss. Leipzig, Math.-phys. Kl. 69, 1917, 107-136 and *Über affine Geometrie XV: Affingeometrie der Kurven höherer Räume*, Ber. Verh. königl. Sächs. Ges. Wiss. Leipzig, Math.-phys. Kl. 70, 1918, 76-90.

[32] L. Berwald, *Über die algebraisch rektifizierbaren Kurven im nichteuklidischen Raum*, Sitzungsber. Math.-phys. Kl. Bayer. Akad. Wiss. München, 1916, 1-18.

[33] See for example Berwald's articles *Über affine Geometrie. XXVII: Liesche F_2. Affinnormale und mittlere Affinkrümmung*, Mathematische Zeitschrift 8, 1920, 63-78 and *Über affine Geometrie. XXX. Die oskulierende Flächen zweiter Ordnung in der affinen Flächentheorie*, Mathematische Zeitschrift 10, 1921, 60-172.

[34] See L. Berwald, *Zur projektiven Differentialgeometrie der Ebene*, Jahresbericht der Deutschen Mathematiker-Vereinigung 30, 1921, 110-121.

[35] L. Berwald, *Über Parallelübertragung in Räumen mit allgemeiner Massbestimmung*, Jahresbericht der Deutschen Mathematiker-Vereinigung 34, 1926, 213-220 and *Parallelübertragung in allgemeinen Räumen*, Atti del Congresso internazionale dei Matematici, Bologna volume 4, 1931, 263-270.

[36] For more information see [6, 14].

G.H.W. Kowalewski delivered 11 lectures from 1913 until 1919. They spanned a wide range, from analysis (e.g. Du Bois Reymond criterion on the convergence of Fourier series (1913), properties of special functions (1918/1919)), to algebra (e.g. Cartan's theory of finite groups (1914), symmetric kernels (1914), classification of linear substitutions (1919)), fundamentals of "natural geometry" (1914), geometric extremal problems (1918/1919), inequalities (1918/1919), to elementary geometry (e.g. Euler's formula (1918/1919)). He paid attention to equally interesting problems in the "mathematical circles". *Das Boss Puzzle* (1918) lecture diverged from the mentioned issues, it was inspired by mathematical rebuses, puns, games and their applications[37].

Eduard Winter (1896–1982) introduced the Prague-German mathematical community to the newest results of Bolzano researches in 1932. Winter intensively focused on these issues already since mid-twenties[38]. In the same year **R. Carnap** lectured on Hilbert's foundations of mathematics and Gödel's major results in logic and foundations of mathematics. In 1934 he spoke about the problems of axiomatization of mathematical theories[39].

Thanks to Ph. Frank and R.H. Fürth a number of lectures dedicated to applied mathematics and mathematical physics were available to the "mathematical circles". For example, **Ph. Frank** delivered lectures on the basics of statistical physics (1915), hydrodynamic lift and its applications in engineering (1916, 1918; see for example Frank's paper *Eine Anwendung des Koebeschen Verzerrungssatzes auf ein Problem der Hydrodynamik*, Mathematische Zeitschrift 3, 1919, 78-86), determination of size of ultramicroscopic particles (1917), calculus of variations and its application in physics (1917, 1922, 1927), optimalization and its application in physics (1918), scattering of

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[37] See G. Kowalewski, *Mathematica delectans. Ausgewälte Kapitel aus der Mathematik der Spiele in gemein-verständlicher Darstellung. Heft 1, Boss-Puzzle und verwandte Spiele*, W. Engelmann, Leipzig 1921, 72 pages. It can be noted that G.H.W. Kowalewski gave at the German University in Prague in the summer semester 1916/1917 a special lecture titled *Über einige Spiele und ihre mathematische Theorie* (one hour weekly) and in the winter semester 1919/1920 a lecture titled *Mathematik der Spiele* (one hour weekly). For more information see *Ordnung der Vorlesungen an der k. k. Deutschen Karl-Ferdinand Universität zu Prag in Sommersemester 1917*, Prag 1917 and *Ordnung der Vorlesungen an der Deutschen-Karl-Ferdinand Universität zu Prag in Wintersemester 1919–20*, Prag 1919.

[38] E. Winter, *Bernard Bolzano und sein Kreis*, Verlag von Jakob Hegner, Leipzig 1933, *Der böhmische Vormärz in Briefen B. Bolzanos an F. Příhonský (1824–1848)*, Deutsche Akademie der Wissenschaften, Berlin 1956, *Wissenschaft und Religion im Vormärz. Der Briefwechsel Bernard Bolzanos mit Michael Josef Fesl 1822–1848*, Berlin 1965 (editors: E. Winter and W. Zeil, preface: E. Winter), E. Winter, P. Funk, J. Berg, *Bolzano. Ein Denker und Erzieher im österreichischen Vormärz*, Wien 1968; See also *Bernard Bolzano – Gesamtausgabe*, Friedrich Frommann Verlag ve Stuttgartu – Bad Connstatt (Jan Berg, Friedrich Kambartel, Jaromír Loužil, Bob van Rootselaar and Eduard Winter), for more information see *The Bernard Bolzano Pages at the FAE*: http://www.sbg.ac.at/fph/bolzano.

[39] For more information see D. Hilbert, W.G. Ackermann, *Grundzüge der theoretischen Logik*, Springer, Berlin 1928, K. Gödel, *Über formal unentscheidbare Sätze der Principia mathematica und verwandter Systeme I*, Monatshefte für Mathematik und Physik 38, 1931, 173-198, P. Bernays, D. Hilbert, *Grundlagen der Mathematik I, II*, Springer, Berlin 1934, 1939.

particles on a lattice (1918), adiabatic invariant perturbation theory (1927), beam path in anisotropic media (1932; see for example Frank's paper *Lichtstrahlen und Wellenflächen in allgemein anistoropen Körpern*, Zeitschrift für Physik 80, 1933, 4-18, supercritical speeds (1933). **R.H. Fürth** spoke about evaluation of systematic errors in physics (1921), heat conduction, fluid flow and cooling (1923), diffusion in a gravitational field (1926), analyzing errors and asymmetrical distributions (1926), the foundations of statistical mechanics (1928), wave mechanics (1928) and fluctuation phenomena in degenerate gases (1928)[40]. **P.G. Funk** presented the basic methods of calculus of variations used in physics (1917) and physical continuum theory (1917). They were among the few Prague-German mathematicians who noticed the significance of von Mises's works in probability theory and understood his efforts to develop the theory of probability on precise mathematical basis. We should note that Ph. Frank gave 66 lectures within "mathematical circles" in the years 1913–1932, R.H. Fürth gave 21 lectures in the years 1917–1931 and P.G. Funk gave 10 lectures in the years 1914–1933.

The preserved archive materials deposited in the Archive of Academy of Sciences of the Czech Republic and the Archive of Charles University in Prague clearly show that German and Czech mathematical communities were not hostile and mutually isolated but they respected each other and cooperated. The Czech-German activities, which developed promisingly not only in mathematics, were completely destroyed by Nazi persecution, occupation and war.

Appendix

The chronological list of 365 lectures which were delivered between 1913 and 1934 at the regular meeting of Mathematische Kränzchen in Prag given below is based on the reports published in the German journal *Jahresbericht der Deutschen Mathematiker--Vereinigung* (JDMV). The quotations were verified and corrected.

Year 1913[41]

Ph. Frank: *Die neueren Theorien der spezifischen Wärme* (30. 4., 8. 5.)
W. Blaschke: *Die Minimalzahl der Scheitel einer konvexen Kurve und verwandte Probleme* (22. 5.)
W. Blaschke: *Bieberbachs Beweis für den Jordanschen Satz* (30. 5.)
G. Kowalewski: *Du Bois Reymonds Kriterium für die Konvergenz der Fourierreihen* (6. 6.)
Th. Pöschl: *Über das Prinzip der kleinsten Formänderungsarbeit und über die Anwendung des Hamiltonschen Prinzips auf nicht holonome Systeme* (13. 6.)
G. Kowalewski: *Das Beispiel Hölders zum Satz von Carathéodory* (20. 6.)
G. Pick: *Einfache Auswertung der Hölderschen Determinante* (27. 6.)
W. Blaschke: *Hölders Behandlung des Lagrangeschen Problems* (27. 6.)
G. Pick: *Carathéodorys Satz über Fourierreihen* (27. 6.)
G. Pick: *Über einen Matrizensatz* (24. 10.)

[40] For more information see [12].
[41] JDMV 22, 1913, 124-125, 207.

W. Blaschke: *Konforme Abbildung einfach zusammenhängender schlichter Gebiete* (31. 10.)
G. Pick: *Differentialinvarianten gegenüber konformen Abbildungen* (31. 10.)
Ph. Frank: *Differentialgeometrische Anwendungen der Vektoranalysis* (7. 11.)
P. Funk: *Flächen mit lauter geschlossenen geodätischen Linien* (14. 11.)
Th. Poeschl: *Über dynamische Äquivalenzprobleme* (19. 11.)
Ph. Frank: *Verlauf der Bahnkurven der Mechanik* (28. 11.)
W. Blaschke: *Beweise für die isoperimetrische Eigenschaft des Kreises* (3. 12.)
G. Pick: *Affine Differentialgeometrie* (10. 12.)

Year 1914[42]

G. Pick: *Distanzschätzungen im Funktionenraum* (16. 1.)
W. Blaschke: *Die Minimaleigenschaft der Kugel* (23. 1.)
Ph. Frank: *Statisch unbestimmte Systeme* (30. 1.)
G. Kowalewski: *Cartans Theorie der Zusammensetzung kontinuierlicher Gruppen* (6. 2.)
G. Kowalewski: *Der erste Fundamentalsatz von S. Lie und die natürliche Geometrie* (13. 2.)
W. Blaschke: *Beweis für die Unmöglichkeit von Montierungsspannungen in einer geschlossenen konvexen Fläche* (19. 2.)
Ph. Frank: *Abschätzung der Eigenwerte, die zu einer konvexen Eigenfunktion gehören* (26. 2.)
W. Blaschke: *Neue Distanzschätzungen im Funktionenraum* (26. 2.)
G. Kowalewski: *Über schiefsymmetrische Kerne* (5. 3.)
Th. Poeschl: *Die Methode von Ritz* (14. 3.)
W. Blaschke: *Ein Beweis für den Satz von Brunn über die Flächeninhalte paralleler Querschnitte eines konvexen Körpers* (1. 5.)
G. Pick: *Beweis eines neuen Satzes über konvexe Funktionen* (9. 5.)
W. Blaschke: *Der neue Beweis von F. Rieß für seinen Satz über die Erzeugung der linearen Funktionaloperationen durch die Integrale von Stieltjes* (23. 5.)
E. Nohel: *Birkhoffs Beweis für den Satz von Poincaré über die Fixpunkte bei flächentreuer Abbildung eines Kreisrings* (6. 6.)
G. Pick: *Konkave Funktionen* (13. 6.)
H. Hahn: *Die charakteristischen Eigenschaften des eindeutigen und stetigen Abbildes einer Strecke* (22. 6.)
G. Pick: *Gebietsbestimmungen für konvexe Kurven* (7. 11.)
J. von Geitler (Czernowitz): *Zur Theorie der Resonanzstrahlung* (13. 11.)
W. Blaschke: *Über den größten Kreis in einer konvexen Punktmenge* (19. 11.)
G. Pick: *Gebietsbeschränkungen für konvexe Kurven (Fortsetzung)* (26. 11.)
P. Funk: *Kugelfunktionen und Integralgleichungen* (5. 12.)
P. Funk: *Eine geometrische Anwendung der Abelschen Integralgleichung* (11. 12.)

Year 1915[43]

Ph. Frank: *Über die Kontroverse zwischen Hilbert und E. Pringsheim* (16. 1.)
G. Pick: *Verkürzung der nichteuklidischen Längen bei konformer Abbildung* (23. 1.)

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[42] JDMV 23, 1914, 29, 99, 127.
[43] JDMV 24, 1915, 28, 46-47, JDMV 25, 1917, p. 32.

M.P. Rudzki (Krakau): *Erdbebenwellen* (30. 1.)
W. Blaschke: *Eine Minimumaufgabe über Kurven konstanter Breite* (6. 2.)
G. Pick: *Über die Bestimmung einer analytischen Funktion durch vorgegebene Wertepaare* (13. 2.)
G. Pick: *Über die Bestimmung einer analytischen Funktion durch vorgegebene Wertepaare (Fortsetzung)* (20. 2.)
W. Blaschke: *Einige Folgerungen aus den Sätzen von Pick* (20. 2.)
Ph. Frank: *Über die Einsteinsche Gravitationstheorie* (27. 2., 6. 3.)
G. Pick: *Über Integralabschätzungen* (8. 5.)
Ph. Frank: *Die Brownsche Bewegung* (15. 5.)
Ph. Frank: *Grundlagen der statistischen Mechanik* (12. 6.)
G. Pick: *Hurwitz' Beweis eines Fatouschen Satzes* (20. 11.)

Year 1916[44]

Ph. Frank: *Einsteins Gravitationstheorie* (22. 1., 29. 1.)
L. Berwald: *Geschlossene algebraisch rektifizierbare Kurven* (5. 2.)
G. Pick: *Determinantensätze von Bendixson und Hirsch* (12. 2.)
Ph. Frank: *Einsteins Gravitationstheorie (Forts)* (19. 2.)
J. Wanka: *Elementare Berechnung des Krümmungshalbmessers ebener algebraischer Kurven* (26. 2.)
Ph. Frank: *Einsteins Theorie des Merkurperihels. Schwarzschilds Herleitung der Einsteinschen Differential-gleichungen der Planetenbewegung* (11. 3.)
G. Pick: *Eine neue Herleitung und Verschärfung des Koebeschne Verzerrungssatzes* (18. 3.)
Ph. Frank: *Das Wasserstoffatommodell von Debye* (12. 5.)
G. Pick: *Extremaleigenschaften bei konformer Abbildung* (19. 5.)
G. Pick: *Über Radons konvexe Funktionen p-ter Stufe* (26. 5.)
G. Pick: *Neue Beweise und Sätze zur konformen Abbildung (Faber, Bieberbach)* (2. 6.)
K. Löwner: *Verzerrungssatz für die Abbildung eines Kreises auf einen konvexen Bereich* (9. 6.)
A. Winternitz: *Neue Abschätzungen bei konvexen Funktione*n (16. 6.)
P. Funk: *Einfacher Beweis, daß für jedes Legendresche Polynom* $P(x)$, $|P(x)/P(1)|<1$ *für* $|x|<1$ (24. 6.)
L. Berwald: *Integralfreie Lösung der Gleichung* $dx_1^2+dx_2^2+\ldots+dx_n^2=0$ *durch Schluß von n auf n* + 1 (30. 6.)
Ph. Frank: *Hydrodynamischer Auftrieb und konforme Abbildung* (28. 10.)
Ph. Frank: *Hydrodynamischer Auftrieb (Schluß)* (4. 11.)
G. Pick: *Konforme Maßbestimmung und Eigenschaften der Greenschen Funktion* (11. 11.)
G. Pick: *Beweis von Lindelöf für die Eindeutigkeit der Randzuordnung bei conformer Abbildung* (18. 11.)

[44] JDMV 25, 1917, 32, 82, 113.

Ph. Frank: *Das Bohrsche Atommodell und die Quantentheorie* (25. 11.)
G. Pick: *Differentialgeometrie der Flächen gegenüber der Gruppe der inhaltstreuen Affinitäten* (2. 12.)

Year 1917[45]

A. Winternitz: *Verallgemeinerte konvexe Funktionen und konvexe Funktionale* (13. 1.)
Ph. Frank: *Virialsatz und Brownsche Bewegung* (20. 1.)
L. Berwald: *Der konforme Raum als abgeschlossenes Kontinuum* (27. 1.)
G. Pick: *Affine Flächengeometrie* (3. 3.)
G. Pick: *Konforme Abbildung schlichter und zugleich beschränkter Bereiche* (10. 3.)
Ph. Frank: *Variationsprobleme der Fluglehre* (28. 4.)
P. Funk: *Variationsproblem beim Segeln. Beweis des Du Bois-Reymondschen Lemmas der Variationsrechnung* (5. 5.)
A. Winternitz: *Erhaltung der Dimensionszahl* (12. 5.)
K. Löwner: *Das allgemeine Problem der konformen Abbildung nach Koebe (Crelles J. Bd. 147)* (19. 5.)
A. Winternitz: *Neuer Beweis des Jordanschen Kurvensatzes* (9. 6.)
R. Fürth: *Über Brownsche Bewegung* (16. 6.)
K. Löwner: *Abbildung einer körperlichen Ecke auf einen ebenen Bereich nach Koebe* (23. 6.)
G. Pick: *Vollständige partielle Differentialsysteme mit homogener Lösung* (23. 6.)
K. Löwner: *Konforme Abbildung einer körperlichen Ecke (Schluß*) (30. 6.)
P. Funk: *Statik der Kontinua als Grenzfall der Fachwerkstatik* (7. 7.)
G. Pick: *Abschätzung positiver harmonischer Funktionen* (20. 10.)
R. Fürth: *Physik der kleinen Teile* (27. 10.)
K. Mack: *Vertauschung der Risse* (3. 11.)
Ph. Frank: *Größenbestimmung ultramikroskopischer Teilchen* (10. 11.)
L. Berwald: *Geometrie der zentrisch symmetrischen Punktepaare* (17. 11.)
K. Löwner: *Beziehungen zwischen beschränkter und schlichter Abbildung* (24. 11.)
Ph. Frank: *Fouriersche Integrale und Reihen bei Fourier* (1. 12.)

Year 1918[46]

A. Winternitz: *Sätze über den Schwerpunkt konvexer Bereiche* (25. 1.)
K. Mack: *Ein Perspektograph* (2. 2.)
Ph. Frank: *Fortpflanzungsgeschwindigkeit der Diffusion* (16. 2.)
G. Kowalewski: *Klassifikation der linearen Substitutionen* (2. 3.)
A. Winternitz: *Unverbiegbarkeit konvexer Polyeder nach Cauchy* (4. 5.)
A. Winternitz: *Unverbiegbarkeit geschlossener konvexer Flächen (Weyl)* (1. 6.)
L. Berwald: *Affingeometrische Sätze* (1. 6.)
L. Berwald: *Größtdreiecke eines konvexen Bereichs* (8. 6.)
K. Löwner: *Schwerpunkt und "konformer Schwerpunkt" konvexer Kurven* (15. 6.)
Ph. Frank: *Zyklische Fehlerrechnung (v. Mises)* (22. 6.)
G. Kowalewski: *Das Boss Puzzle* (6. 7.)

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[45] JDMV 25,1917, 113, JDMV 26, 1918, 71, JDMV 27, 1918, 47-48.
[46] JDMV 27, 1918, 47-48.

Winter semester 1918/1919[47]

K. Löwner: *Neue Abschätzungssätze zur konformen Abbildung*
G. Kowalewski: *Ableitung der Eigenschaften der Exponentialfunktion aus ihrer Konvexität*
A. Winternitz: *Über zwei Abschätzungssätze von Hamel*
P. Funk: *Auflösung eines Systems dreigliedriger Differenzengleichungen*
G. Kowalewski: *Extremumaufgaben bei ebenflächigen Körpern*
G. Pick: *Konforme übertragung der kreisgeometrischen Maßbestimmung auf ein einfach zusammenhängendes Gebiet*
G. Kowalewski: *Über die Eulersche Summenformel*
R. Weitzenböck: *Über den Winkel zweier Ebenen im* R_4
Ph. Frank: *Hydrodynamische Anwendungen der Verzerrungssätze*
G. Kowalewski: *Über die Ausgleichsgerade von n Punkten der Ebene*

Summer semester 1918/1919[48]

H. Tietze (Brünn): *Über die Analysis situs*
Ph. Frank: *H. Weyls Reine Infinitesimalgeometrie*
R. Weitzenböck: *Zur Invariantentheorie der Galilei-Newton-Gruppe*

Year 1919[49]

L. Berwald: *Zur affinen Flächentheorie* (7. 11.)
A. Haas (Leipzig): *Der Nernstsche Wärmesatz und die Quantentheorie* (14. 11.)
K. Löwner: *Der Bieberbachsche Drehungssatz* (21. 11.)
Ph. Frank: *Über das Gesetz der rationalen Indizes* (28. 11.)
G. Pick: *Über das Gesetzt der rationalen Indizes* (5. 12.)
G. Pick: *Zur projektiven Infinitesimalgeometrie* (5. 12.)
R. Fürth: *Über Wahrscheinlichkeitsnachwirkung* (13. 12.)

Year 1920[50]

Ph. Frank: *Über den Bau der Atome* (16. 1.)
G. Pick: *Abschätzungen bei orthogonalen Matrizen* (23. 1.)
L. Berwald: *Zur projektiven Differentialgeometrie der Ebene I.* (30. 1.)
Ph. Frank: *Über die Entstehung von Strahlungsfrequenzen durch Zufall* (20. 2.)
L. Berwald: *Zur projektiven Differentialgeometrie der Ebene II.* (27. 2.)
G. Pick: *Bericht über die Abhandlung: "Über Potentialtheorie und konforme Abbildung" von G. Faber* (21.5.)
K. Löwner: *Über Tschebytscheffsche Polynome* (4. 6., 11. 6.)
Th. Pöschl: *Spannungsverteilung längs des Umfanges eines elliptischen Loches (Grenzfälle: Kreis und Schlitz) in einer unendlich ausgedehnten Platte* (18. 6.)
L. Berwald: *Die oskulierenden Flächen zweiter Ordnung in der affinen Flächentheorie* (25. 6.)

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47 JDMV 28, 1919, 56. The dates of the lectures are not known.
48 JDMV 28, 1919, 56. The dates of the lectures are not known.
49 JDMV 28, 1919, 56-57.
50 JDMV 29, 1920, 32-33, JDMV 30, 1921, 32.

Ph. Frank: *Über Plemeljs Theorie der Leiterbelegung* (2. 7.)
G. Pick: *Abschätzungssätze von G. H. Hardy und M. Riesz* (9. 7.)
L. Berwald: *Bericht über die geometrischen Vorträge in Bad Nauheim* (19. 11.)
A. Winternitz: *Affinlänge und Lebesguesches Integral* (19. 11.)
A. Winternitz: *Affinlänge und Lebesguesches Integral (Fortsetzung und Schluß)* (26. 11., 3. 12.)
L. Berwald: *Zur Affingeometrie der Kurven auf Flächen* (10. 12.)

Year 1921[51]

Th. Pöschl: *Bestimmung aller Bipotentiale, die nur von einer Veränderlichen abhängen* (21. 1.)
A. Winternitz: *Die Knoppsche Erzeugungsweise der Kurven von Peano, Osgood und v. Koch* (4. 2.)
K. Mack: *Über reelle Bilder von Dyaden und nullteiligen Kreisen durch Umkehrung der stereographischen Projektion* (11. 2.)
Ph. Frank: *Referat über die Abhandlung: "Über die Eigenwerte bei den differential-gleichungender mathematischen Physik" von R. Courant (Math. Zeitsch. 7(1920))* (18. 2., 25. 2.)
A. Winternitz: *Eine neue Definition des Krümmungstensors affin zusammenhängender Mannigfaltigkeiten* (4. 3.)
G. Pick: *Abschätzungssätze bei konformer Abbildung* (29. 4.)
?. Görig[52]: *Eine neue Theorie der Invalidenversicherung* (6. 5., 20. 5.)
K. Löwner: *Erzeugung von schlicht abbildenden beschränkten Funktionen durch infinitesimale Transformationen* (27. 5., 3. 6.)
K. Löwner: *Erzeugung von schlicht abbildenden beschränkten Funktionen durch infinitesimale Transformationen (Schluß)* (10. 6.)
K. Mack: *Vorführung des Perspektographen* (10. 6.)
Th. Pöschl: *Über das Torsionsproblem* (17. 6.)
K. Löwner: *Bericht über die Arbeit von E. Trefftz: über die Torsion prismatischer Stäbe von polygonalem Querschnitt (Math. Ann. 82(1920))* (24. 6.)
R. Fürth: *Demonstrationen mit der Diffusionspumpe* (24. 6.)
R. Fürth: *Über eine Quelle systematischer Fehler bei physikalischer Statistik* (1. 7.)
R. Fürth: *Vorführung von Versuchen über Erzeugung von kurzen Wellen mit Glühkathodenröhren* (1. 7.)
G. Pick: *Extremumfragen bei Funktionen komplexer Variabeln* (8. 7.)
R. Fürth: *Vorführung des Simonschen Lichtbogens in Verbindung mit der Glühkathodenröhre* (8. 7.)
G. Pick: *Bemerkung über Extreme positiver quadratischer Formen bei linearen Nebenbedingungen* (27. 10.)
K. Löwner: *Besprechung der neueren Arbeiten über Potenzreihen mit ganzzahligen Koeffizienten* (4. 11.)

[51] JDMV 30, 1921, 32, 51, JDMV 31, 1922, 53.
[52] His first name is not known.

K. Löwner: *Besprechung der neueren Arbeiten über Potenzreihen mit ganzzahligen Koeffizienten (Schluß)* (11. 11.)
R. Fürth: *Demonstration der Vollmerschen Kondensationspumpe* (11. 11.)
G. Pick: *Bericht über die Abhandlung "Über Potenzreihen mit vorgeschriebenen Anfangsgliedern" von F. Riesz (Acta Math. 42(1919))* (18. 11.)
Ph. Frank: *Über das Schursche Kriterium für die Hurwitzschen Gleichungen* (25. 11.)
R. Fürth: *Vorführung eines elektrischen Klaviers* (25. 11.)
Chr. von Ehrenfels: *Das Primzahlgesetz* (2. 12.)
Ph. Frank: *Über das Schursche Kriterium für die Hurwitzschen Gleichungen (Schluß)* (9. 12.)

Year 1922[53]

G. Pick: *Abschätzung der Wurzeln von Determinantengleichungen* (19. 1.)
K. Mack: *Eine Fadenkonstruktion perspektiver Bilder* (27. 1.)
–: *Diskussion über "Das Primzahlgesetz" von Ehrenfels* (17. 2.)
L. Berwald: *Über orthogonales Rechnen im Raum von Riemann* (24. 2.)
G. Pick: *Funktionen mit kleinstem Mittel des absoluten Betrages auf dem Einheitskreis* (3. 3.)
K. Löwner: *Existenzbeweis dazu* (3. 3.)
A. Winternitz: *Über die Formel von Lagrange* (12. 5.)
Th. Pöschl: *Über die Stabilität rotierender Wellen* (26. 5.)
L. Berwald: *Über die Starrheit der Eiflächen* (23. 6.)
Ph. Frank: *Neuere Arbeiten von Carathéodory über Variationsrechnung* (20. 10.)
G. Pick: *Zur Eulerschen Gleichung bei Variationsaufgaben n-ter Ordnung in der Ebene* (1. 12.)
Th. Pöschl: *Gasströmungen in Düsen mit Berücksichtigung der inneren Reibung* (15. 12.)

Year 1923[54]

Ph. Frank: *Zur Differentialgeometrie der reellen Bahnkurven* (19. 1.)
L. Berwald: *Bemerkungen zur Differentialgeometrie der reellen Bahnkurven* (26. 1.)
L. Berwald: *Über gewisse Identitäten, die bei Variationsproblemen höherer als erster Ordnung auftreten* (26. 1.)
G. Pick: *Affingeometrie der ebenen rationalen Kurven dritter Ordnung* (2. 2.)
G. Pick: *Über einige neuere Arbeiten* zur Theorie der Transformationsgruppen (9. 2.)
Th. Pöschl: *Über achsensymmetrische Stromfunktionen* (27. 4.)
L. Berwald: *Sätze über Eilinien* (4. 5.)
G. Pick: *Über einige neuere Arbeiten zur Theorie der Transformationsgruppen (Schluß)* (11. 5.)
E. Waelsch (Brünn): *Vektoranalysen* (1. 6., 7. 6.)
Chr. von Ehrenfels: *Über das Primzahlgesetz* (15. 6.)
Ch. von Ehrenfels, A. Winternitz: *Zum Primzahllehre* (26. 10.)
R. Fürth: *Über Wärmeleitung durch Diffusion* (2. 11.)

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[53] JDMV 31, 1922, 53, 70, JDMV 32, 1923, 50.
[54] JDMV 32, 1923, 50, JDMV 33, 1925, 35.

R. Fürth: *Über Abkühlung fester Körper durch strömende Flüssigkeiten nach Boussinesq* (9. 11.)
Ph. Frank: *Über adiabatische Invarianten* (16. 11.)
A. Winternitz: *Affingeometrie der Kurven des* R_n (23. 11.)
G. Pick: *Abschätzungen symmetrischer Funktionen* (30. 11.)
G. Pick, A. Winternitz: *Über die Abhandlung von W. Sternberg: Einige Sätze über Mittelwerte (Leipz. Ber. 71(1919))* (7. 12.)
L. Berwald: *Über einen Satz von O. Fort* (7. 12.)
A. Winternitz: *Bemerkungen zur Grundlegung der Analysis* (7. 12.)

Year 1924[55]

G. Pick: *Einige Bemerkungen über rationale Funktionen* (18. 1.)
L. Berwald: *Über einige Ungleichungen für bestimmte Integrale* (25. 1.)
A. Winternitz: *Über die Abhandlung von Stieltjes: Sur un algorithme de la moyenne géométrique* (Werke I, Abh. VIII) (25. 1.)
Th. Pöschl: *Zur graphischen Statik zusammengesetzter Fachwerke* (1. 2.)
Th. Pöschl: *Elastische Linie eines auf nachgiebiger Unterlage gelagerten Balkens* (8. 2.)
R. Fürth: *Das n-fach iterierte Fehlerintegral* (8. 2.)
Ph. Frank: *Bericht über Tomascheks Michelsonversuch mit Fixsternlicht* (8. 2.)
Ph. Frank: *Über die Arbeit von O. Onicescu: Campo newtoniano viciniore ad un campo vettoriale assegnato (Lincei Rend. (5) 29. I. (1920))* (7. 3.)
A. Winternitz: *Über einen Zusammenhang zwischen der Theorie der Transformationsgruppen und der Theorie der Parallelübertragung I.* (14. 3.)
G. Pick: *Bericht über die Abhandlung von I. Schur: über eine Klasse von Mittelbildungen ... (Sitzungsber. Berl. Math. Ges. 22(1923))* (21. 3.)
A. Winternitz: *Über einen Zusammenhang zwischen der Theorie der Transformationsgruppen und der Theorie der Parallelübertragung II.* (28. 3.)
Ph. Frank: *Über Levi-Civitas Theorie der stationären Bewegung* (16. 5., 23. 5.)
Th. Pöschl: *Über die Hencky-Prandtlschen Kurven* (13. 6.)
A. Winternitz: *Bericht über die Arbeit von H. Bohr: Zur Theorie der fastperiodischen Funktionen I. (Acta Math. 45(1924)* (24. 10., 31. 10.)
L. Berwald: *Über Parallelübertragung in Räumen mit allgemeiner Maßbestimmung* (14. 11.)
R. Fürth: *Über Glühkathodenröhren* (21. 11., 28. 11.)
Ph. Frank: *Über mehrfach periodische Systeme* (5. 12.)

Year 1925[56]

Ph. Frank: *Periodizität und Quasiperiodizität von Bahnkurven* (16. 1.)
A. Winternitz: *Zahlentheoretisches zur Theorie der mehrfach periodischen Systeme* (23. 1., 30. 1.)
Ph. Frank: *Die adiabatischen Invarianten vom Standpunkte der Störungstheo*rie (20. 3.)

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[55] JDMV 33, 1925, 35, JDMV 34, 1926, 6-7.
[56] JDMV 34, 1926, 6-7, 105-106, JDMV 35, 1927, 98-99.

Ph. Frank: *Die adiabatischen Invarianten vom Standpunkte der Störungs theorie (Fortsetzung)* (27. 3.)
G. Pick: *Integration elliptischer Differentiale durch Logarithmen* (24. 4.)
A. Winternitz: *Über den Satz von der Gebietsinvarianz* (8. 5.)
A. Winternitz: *Ein neuer Beweis des Jordanschen Kurvensatzes* (15. 5.)
A. Winternitz: *Ein neuer Beweis des Jordanschen Kurvensatzes (Fortsetzung)* (22. 5.)
Ph. Frank: *Über die charakteristischen Eigenschaften von n in Involution befindlichen Integralen in der Theorie der mehrfach periodischen mechanischen Systeme* (20. 6.)
L. Berwald: *Parallelübertragung und Maßbestimmung in allgemeinen Räumen* (23. 10., 30. 10.)
L. Berwald: *Über eine invariante Einteilung der zweidimensionalen allgemeinen Räume und die Untersuchungen von Landsberg (Math. Ann. 65(1908))* (6. 11., 7. 11.)
R. Fürth: *Über die Verwendung der Differentialgleichung* $\Delta u = e^u$ *in der Theorie der Glühkathodenröhren* (4. 12.)
R. Fürth: *Über die Anwendung der Potentialtheorie bei der Glühkathodenröhre* (11. 12.)

Year 1926[57]

Th. Pöschl: *Über Spannungen und Deformationen von elastischen Flächen* (15. 1., 22. 1.)
Ph. Frank: *Über die Heisenberg-Bornsche Quantenmechanik* (4. 1., 4. 2., 26. 2.)
G. Pick: *Konvergenz von Reihen, die nach Matrizen fortschreiten* (5. 3)
Ph. Frank: *Über die Schrödingersche Wellenmechanik* (22. 10., 12. 11., 19. 11., 26. 11.)
R. Fürth: *Über ein Problem der Diffusion im Schwerefelde* (3. 12)
R. Fürth: *Anwendung der Fehlerrechnung auf ein Problem unsymmetrischer Verteilung* (3. 12.)
R. Carnap (Wien): *Über die topologische Struktur des Raum-Zeit-Kontinuums* (10. 12.)

Year 1927[58]

Ph. Frank: *Optische Deutung der Carathéodoryschen Methode der Variationsrechnung* (28. 1.)
G. Pick: *Konforme äquivalenz von Funktionen zweier Veränderlichen* (4. 2.)
L. Berwald: *Über die Arbeiten von G. Kowalewski zur natürlichen Geometrie ebener Transformationsgruppen* (4. 3., 18. 3.)
A. Winternitz: *Ziele der mathematischen Grundlagenforschung* (6. 5.)
A. Winternitz: *Bemerkungen zu Brouwers intuitionistischer Mathematik* (20. 5.)
G. Pick: *Über die Absolutbeträge der Wurzeln algebraischer Gleichungen* (27. 5.)
Th. Pöschl: *Über achsensymmetrische elastische Probleme* (2. 12.)

Year 1928[59]

R. Fürth: *Über neuere Probleme der statistischen Mechanik* (3. 2.)
R. Fürth: *Über Wellenmechanik in Systemen, die aus sehr vielen gleichen Teilsystemen bestehen* (10. 2.)

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[57] JDMV 35, 1926, 98-99, JDMV 37, 1928, 42-43.
[58] JDMV 37, 1928, 42-43, JDMV 38, 1929, 83-84.
[59] JDMV 38, 1929, 83-84.

R. Fürth: *Schwankungserscheinungen bei entarteten Gasen* (14. 2.)
L. Berwald: *Über die Abhandlung von Levi-Civita "Sur l'écart géodésique" (Math. Ann. 97(1927)) und einige daran anschließende Arbeiten* (2. 3.)
Ph. Frank: *Quantenmechanik und Hermitesche Formen* (9. 3., 16. 3.)
H. Löwig: *Die Differentialgleichungen der Extremalen eines Mayerschen Problemes in der Variationsrechnung als Gleichungen einer infinitesimalen Berührungstransformation* (18. 5., 8. 6.)
P. Kuhn: *Zur Viggo Brunschen Methode in der Zahlentheorie* (15. 6.)
Ph. Frank: *Mathematische Grundlagen der Quantenmechanik* (23. 11., 30. 11.)

Year 1929[60]

Ph. Frank: *Mathematische Grundlagen der Quantenmechanik* (19. 4.)
K. Menger (Wien): *Der Euklidische Raum* (10. 5.)
W. Glaser: *Über die Arbeiten von H. Geppert: "Sugli invarianti adiabatici di un generico sistema differenziale" (Lincei Rendiconti 6. 8. 1928)* (31. 5., 7. 6., 14. 6.)
A. Winternitz: *Zur geometrischen Auffassung der transitiven Transformationsgruppen* (11. 11., 25. 11.)
P. Funk: *Über Geometrien, in denen die Geraden die Kürzesten sind* (9. 12.)

Year 1930[61]

W. Glaser: *Bericht über die Abhandlung von H. Geppert: Theorie der adiabatischen Invarianten allgemeiner Differentialsysteme (Math. Ann. 102(1929))* (28. 1., 3. 2.)
Ph. Frank: *Über J. v. Neumanns Arbeit zur wahrscheinlichkeitstheoretischen Grundlage der Quantenmechanik* (3. 3.)
Ph. Frank: *Bericht über die Abhandlung von J. v. Neumann: Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren (Math. Ann. 102 (1929))* (10. 3., 17. 3.)
L. Berwald: *Die neue einheitliche Feldtheorie Einsteins* (24. 3., 31. 3.)
A. Brauer (Berlin): *Die Verteilung der quadratischen Reste* (29. 3.)
W. Glaser: *Bericht über die Abhandlung von W. Mayer: Beitrag zur geometrischen Variationsrechnung (Jahresbericht der Deutschen Mathematiker-Vereinigung 38(1929))* (12. 5., 19. 5.)
A. Winternitz: *Grundlegung einer Infinitesimalgeometrie und Herstellung ihrer Beziehung zur Variationsrechnung* (26. 5.)
P. Kuhn: *Mittelwertbildungen der zahlentheoretischen Funktionen* (2. 6.)
H. Löwig: *Lineare Differenzengleichungen mit Koeffizienten von gemeinsamer Periode* (7. 11.)
H. Löwig: *Zur Theorie der nichtlinearen Differenzengleichungen* (21. 11.)
P. Funk: *Numerische Methoden zur Berechnung von Eigenwerten* (28. 11.)
L. S. Ornstein (Utrecht): *Neue Untersuchungen über die Brownsche Bewegung* (2. 12.)
K. Löwner: *Über Maßbestimmungen, die mit analytischen Abbildungen invariant verknüpft sind* (5. 12., 12. 12.)

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[60] JDMV 38, 1929, 83-84, JDMV 40, 1931, 107-108.
[61] JDMV 40, 1931, 107-108, JDMV 42, 1933, 133-134.

Year 1931[62]

E. Bunickij: *Kontinuierliche Spektra der gewöhnlichen Differentialgleichungen und verwandte Probleme* (16. 1., 6. 2.)
E. Bunickij: *Über eine Klasse von rationalen Zahlen mit ähnlichen Teilbarkeitseigenschaften wie die ganzen Zahlen* (23. 1.)
W. Fröhlich: *Beiträge zur Kinematik einer speziellen C-Geometrie der Ebene* (27. 2.)
L. Berwald: *Bericht über die Arbeit von A. Haar: "Über einige Eigenschaften der orthogonalen Funktionensysteme" (Math. Zeitschr. 31(1929))* (13. 3.)
R. Fürth: *Über die Unschärferelationen der Quantenmechanik* (20. 3.)
W. Fenchel (Göttingen): *Curvatura integra von gewissen Riemannschen Räumen* (8. 4.)
L. Berwald: *Über die Analoga von Egerváry und Lipka zum Satze von Kakeya (St. Lipka, Acta der Univ. Szeged 5(1931); E. Egerváry, ebenda)* (17. 4.)
K. Löwner: *Über das Zentrumproblem* (23. 4.)
Ph. Frank: *Über die Arbeiten von Herzberger zur geometrischen Optik* (8. 5.)
R. Fürth: *Über die Noten von E. Fermi zur Quantenelektrodynamik (Lincei Rendiconti (6) 9(1929), 12(1930))* (15. 5.)
L. Berwald: *Über die Arbeit von L. Fejér: "Ein trigonometrisches Analogon eines Kakeyaschen Satzes" (Jahresbericht der Deutschen Mathematiker-Vereinigung 38(1929))* (19. 6.)
W. Fröhlich: *Bericht über die Abhandlung von G. Thomsen "Un teorema topologico sulle schiere di curve e una caratterizzazione geometrica delle superficie isotermoasintotitche" (Boll. Unione Mat. Ital. 6(1927), 80-85)* (30. 10.)
W. Fröhlich: *Bericht über die Arbeiten von W. Blaschke: "Topologische Fragen der Differentialgeometrie I. und II." (Math. Zeitschr. 28(1928), 150-160)* (6. 11.)
H. Löwig: *Über das Summationsproblem* (13. 11.)
A. G. Silverman (Hannover, USA): *Über gewisse Definition der Summabilität* (20. 11.)
A. Winternitz: *Über einen Landauschen Beweis des Picardschen Sätze* (4. 2.)

Year 1932[63]

N. Wiener (Cambridge, USA): *Taubersche Sätze* (4. 1.)
E. Winter: *Bericht über den wissenschaftlichen Nachlaß Bolzanos, hauptsächlich die Mathematik betreffend* (15. 1.)
R. Carnap: *Hilberts Grundlegung der Mathematik* (22. 1.)
R. Carnap: *Bericht über Gödels Arbeit: "Über unentscheidbare Sätze ..." (Monatsh. f. Math. u. Phys. 38(1931))* (5. 2.)
L. Berwald: *Die Sätze von A. Cohn (Math. Zeitschr. 14(1922)) und St. Lipka (Szeged Acta 3(1927)) über die Abgrenzung der Wurzeln einer algebraischen Gleichung* (26. 2.)
Ph. Frank: *Über den Strahlengang in anisotropen Medien* (4. 3.)
F. Schoblik (Brünn): *Über belastete Integralgleichungen* (11. 3.)
W. Müller: *Ausbreitungsvorgänge in der zähen Flüssigkeit (Eine Anwendung der Theorie der Besselschen Funktionen)* (15. 3.)

[62] JDMV 42, 1933, 133-134.
[63] JDMV 42, 1933, 133-134, JDMV 45, 1935, 48–49.

H. Löwig: *Über die Exponentialfunktion und den natürlichen Logarithmus von Matrizen* (8. 4.)
B. Goldschmied: *Über Wechselstromschaltungen* (15. 4.)
K. Löwner: *Über mathematische Fragen bei Wechselstromschaltungen* (22. 4.)
W. Glaser: *Über adiabatische Invarianten* (29. 4.)
—: *Gemeinsam mit der Gesellschaft für angewandte Mathematik und Mechanik, Ortsgruppe Prag* (6. 5.)
L. Prandtl (Göttingen): *Neuere Ergebnisse der Turbulenzforschung* (6. 5.)
K. Löwner: *Über monotone Matrixfunktionen* (27. 5.)
Ph. Frank: *Dörge und die Misessche Behandlung der Wahrscheinlichkeitsrechnung* (3. 7., 10. 7.)
A. Winternitz: *Bestimmung aller wesentlich verschiedenen Erklärungen einer "Spiegelung" von Punkten* $z_i = F_i(x_1, \ldots, x_n | y_1, \ldots, y_n)$, $i = (1, 2, \ldots, n)$, *welche gegenüber einer "Spiegelung" an einem beliebigen Punkte* u_i *invariant ist. (Durchgeführt für n* < 3 *unter Differenzierbarkeitsannahmen über* F_i) (4. 11., 11. 11.)
W. Mayer: *Eine neue Axiomatik der ebenen Affingeometrie* (18. 11.)
E. Foradori (Innsbruck): *Axiomatik des Teilbegriffes* (25. 11.)
H. Löwig: *Über die Anzahl der Leitgleichungen einer eingliedrigen Gruppe von Berührungstransformationen* (9. 12.)

Year 1933[64]

K. Löwner: *Monotone Matrixfunktionen und Cauchysches Interpolationsproblem* (13. 1., 20. 1.)
E. Lammel: *Über Werteverteilung bei regulären analytischen Funktionen* (4. 2.)
A. Rössler: *Über Affinminimalflächen und Affinsphären* (3. 3.)
H. Löwig: *Zur Theorie der metrischen Vektorräume* (10. 3., 17. 3.)
W. Glaser: *Über optische Abbildung durch Elektronenstrahlen* (24. 3., 31. 3.)
F. Pollaczek (Berlin): *Geometrische Wahrscheinlichkeiten in der Fernsprechtechnik* (5. 5.)
Q. Beck: *Die mathematische Methode der heutigen Atommechanik* (12. 5.)
V. Jarník: *Simultane diophantische Approximationen* (19. 5.)
L. Berwald: *Über die Anzahl der Wurzeln einer algebraischen Gleichung in der oberen Halbebene und auf der reellen Achse (Bericht über die Dissertation des Herrn S. Benjaminowitsch)* (26. 5.)
W. E. Stein (Brüx)[65]: *Lineare projektive Geometrie in mehrdimensionalen Räumen* (2. 6.)
L. Berwald: *Über die Lage der Nullstellen gewisser Linearkombinationen von rationalen Funktionen* (9. 6.)
W. Fröhlich: *Theorie der Zöpfe (Bericht über die gleichnamige Abhandlung von E. Artin (Hamb. Abh. 4(1926))* (16. 6.)
M. Pinl (Berlin): *Krümmungseigenschaften totalisotroper Flächen* (20. 10.)
P. Funk: *Über die Heavisidesche Operatorenrechnung* (27. 10.)

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[64] JDMV 45, 1935, 48-49.
[65] Brüx is German name for Czech town called Most.

P. Funk: *Über kritische Drehzahlen* (27. 10.)
W. Fröhlich: *Beweis für die Lösung des Wortproblems bei den Zöpfen n-ter Ordnung nach Artin* (17. 11.)
K. Sitte: *Theorie der gegenseitigen Diffusion von Elektrolyten* (1. 12.)

Year 1934[66]

L. Berwald: *Über die Auffassung der Differentialgeometrie nichtholonomer Räume bei Cartan* (12. 1.)
R. Carnap: *Über allgemeine Fragen der Axiomatik* (19. 1.)
L. Berwald: *Cartans neue Theorie der Finslerschen Räume* (26. 2., 2. 3.)
A. Winternitz: *Zurückführung der affinen Grundlage eines regulären Variationsproblems auf den Mittelpunktsbegriff* (2. 3.)

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2-NP/2015

MARTINA BEČVÁŘOVÁ*

THE STUDY OF HISTORY OF MATHEMATICS IN THE CZECH REPUBLIC

STUDIA NAD HISTORIĄ MATEMATYKI W CZECHACH

A b s t r a c t

In the Czech lands, there is a long and fruitful tradition of research and study of the history of mathematics which began in the second half of the 19th century. The most important papers and books were written by J. Smolík, F.J. Studnička, J. Úlehla, K. Rychlík and Q. Vetter. But from the 1950s to the 1980s only a few professionals from the Institute of History of the Academy of Sciences of the Czech Republic, along with a few university professors, devoted their attention to the history of mathematics. For example, we can mention two historians of mathematics, Jaroslav Folta and Luboš Nový, whose papers and activities became well-known in Europe. However, on account of various professional and political circumstances, no new generation of historians of mathematics was raised. The first step to the new development of research in the history of mathematics was made in the 1980s, when the special commission on history of mathematics was created at the Faculty of Mathematics and Physics of Charles University in Prague, thanks to activities of Jindřich Bečvář, Ivan Netuka and Jiří Veselý.

Keywords: *history of Mathematics in the Czech Republic*

S t r e s z c z e n i e

Już od II połowy XIX w. na ziemiach czeskich obserwujemy długą tradycję badań i studiów nad historią matematyki. Ważne znaczenie w tej dziedzinie mają m.in. prace przygotowane przez następujących autorów: J. Smolík, F.J. Studnička, J. Úlehla, K. Rychlík i Q. Vetter. Po II wojnie światowej, w latach 1950–1980, kilku specjalistów z Instytutu Historii Akademii Nauk Republiki Czeskiej wraz z profesorami uniwersyteckimi zwróciło swoją uwagę na rozwój historii matematyki. Wśród nich byli: Jaroslav Folta i Luboš Nový, których dokonania znane są Europie. We wspomnianym okresie z różnych powodów nie powstała nowa generacja historyków matematyki. W kolejnym okresie krystalizowania się dyscypliny naukowej – historii matematyki – w Czechach utworzono w 1980 r. na Wydziale Matematyki i Fizyki Uniwersytetu Karola w Pradze komisję ds. badań z historii matematyki. Jindřich Bečvář, Ivan Netuka i Jiří Veselý należeli do osób wydatnie i owocnie wspierajacych działność wspomnianej komisji

Słowa kluczowe: historia matematyki w Republice Czeskiej

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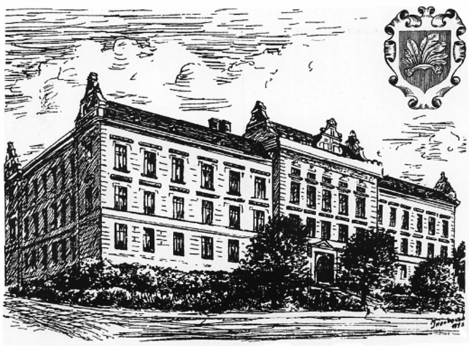
The situation changed only after 1990, when the new discipline of postgraduate studies, **History and Didactics of Mathematics and Computer Science**, was accredited at the Faculty of Mathematics and Physics of Charles University in Prague and at the Faculty of Science of Masaryk University in Brno. In Prague, the PhD studies were opened in the academic year 1992/1993. An individual study plan is prepared for each student. It contains a large background and deeper parts in the chosen area as well as a section directly connected with the proposed thesis topic. At present, there are 19 PhD students and 43 already defended dissertations, most of them specialised in the history of mathematics[1].

35. MEZINÁRODNÍ KONFERENCE

HISTORIE MATEMATIKY

Velké Meziříčí, 22. až 26. 8. 2014



Praha **2014**

Fig. 1. The title page of the proceedings from the conference in 2014

A further important stimulation is the possibility of reporting on one's work and presenting one's results at special events, such as the regular **International Conference on the History of Mathematics** (this year it was held for the 35th consecutive time), and at the **Seminar in the History of Mathematics**, which takes place in Prague or at the special methodological seminars focused on the specific problems which appear in the course

[1] For more information see: http://www.karlin.mff.cuni.cz/~becvar/pgs/pgs.htm.

of work on the history of mathematics[2]. The conference was established in 1980 by the School on the History of Mathematics and it was predo-minantly meant for the professors at universities preparing future teachers of mathematics for secondary schools and high schools. Its main aim was to supplement and increase their knowledge and to give them enough material so that they would be able to teach the newly created course *Philosophical Problems of Mathematics* (later on titled History of Mathematics), which became a compulsory part of the curricula for the innovative education of future teachers.

In the 1990s the conference noticeably changed its character because of completely new participants (undergraduate students, PhD students, teachers at universities as well as professional historians of mathematics) who participated at the conference to present their works and results. Since 2003, the conference is titled *International Conference on the History of Mathematics* and every year more than 60 participants from the Czech Republic, Slovakia and Poland, sometimes also from Germany, Russia, Ukraine and Italy attend the conference and present their research or give invited plenary lectures on the development of some disciplines of mathematics or some mathematical problems from the historical perspective. The conferences take places in Velké Meziříčí (a nice historical town in Moravia) or Poděbrady (a famous Czech spa not far from Prague) and they are organized by Jindřich Bečvář, Martina Bečvářová, Magdalena Hykšová, Martin Melcer and Irena Sýkorová. In order to document the contents of the conference, the proceedings containing extended versions of the individual contributions are published every year (since 2006, the contributions are in Czech, English, Slovak or Polish languages) thanks to the financial support of the Faculty of Mathematics and Physics of Charles University in Prague[3].

Teachers of mathematics in Czech secondary schools, student-teachers or PhD students in history of mathematics can take part in the bi-annual seminar on the history of mathematics titled *History of mathematics for teachers in secondary schools*. Its main aim is to give them new information, sources, materials and examples to increase their knowledge so that they could improve and enhance their lessons and lectures with the historical aspects, be able

Fig. 2. Velké Meziříčí

Fig. 3. Poděbrady

[2] For more information see: http://www.karlin.mff.cuni.cz/~becvar

[3] For more information see: http://www.fd.cvut.cz/personal/becvamar/konference/hlavnindex.html; Also [1, 2, 5, 6, 8].

to show the role of mathematics in the development of modern science and technology and be able to make mathematics nice and attractive for students, among other things. The 12th seminar was held in August 2015 in Poděbrady. Its program is prepared by Jindřich Bečvář, Martina Bečvářová, Zdeněk Halas and Martin Melcer. Euclid of Alexandria and his Elements are planned as its main topic[4].

A great encouragement for a young incipient researcher is the possibility of having their results published. In the case of more extensive work in the history of mathematics, this is often a problem. Thanks to Jindřich Bečvář from the Faculty of Mathematics and Physics of Charles University in Prague and to Eduard Fuchs from the Faculty of Science of Masaryk University in Brno, a publication series entitled *History of Mathematics* was established in 1994; this series makes it possible to publish both shorter and longer works, as well as entire monographs and textbooks on the history of mathematics in the Czech, English and Slovak languages.

The editorial board is composed of Czech mathematicians and historians of mathematics: Jindřich Bečvář, Antonín Slavík and Ivan Netuka (all from the Faculty of Mathematics and Physics of Charles University in Prague), Martina Bečvářová, Magdalena Hykšová and Miroslav Vlček (all from the Faculty of Transportation Sciences of the Czech Technical University in Prague), Vlastimil Dlab (the School of Mathematics and Statistics, Carleton University, Ottawa, Canada), Eduard Fuchs (the Faculty of Science of the Masaryk University in Brno), Jiří Hudeček (the Faculty of Arts of Charles University in Prague).

At present, the series has 58 volumes (50 in Czech or Slovak and 8 in English), which can be loosely divided into seven groups. The first one consists of 14 monographs devoted to the evaluation of scientific and pedagogical work of leading Czech mathematicians of the second half of the 19th century and the first half of the 20th century (František Josef Studnička, Emil Weyr, Eduard Weyr, Jan Vilém Pexider, Karel Rychlík, Vladimír Kořínek, Ladislav Svante Rieger, Jan Sobotka, Karel Zahradník, Wilhelm Matzka and Heinrich Löwig, the monographs on Jan Vilém Pexider and Heinrich Löwig are also available in the English version).

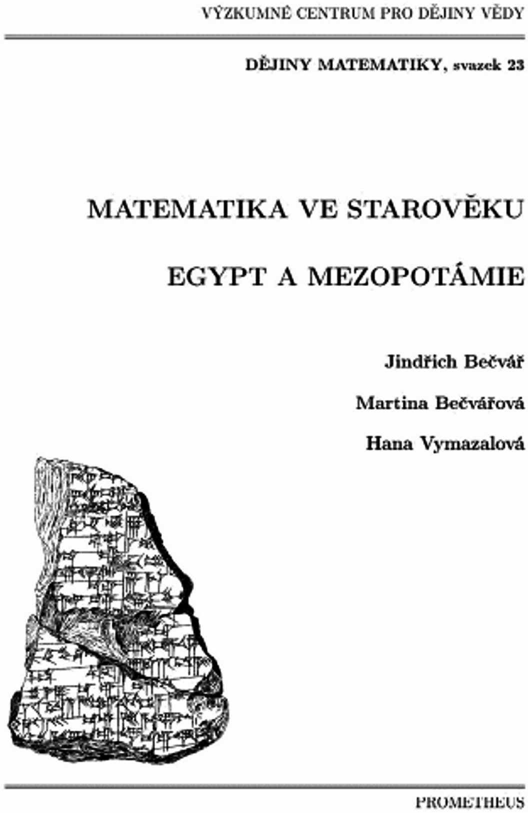
The second group presents several interesting topics from the history of mathematics (mathematics in Egypt, Mesopotamia and old China, Greek mathematics, mathematics in Middle Ages and Renaissance in Europe, European mathematics in the 16th and 17th century).

The volumes in the third group analyze the development of certain mathematical disciplines and issues (integral calculus, graph theory, probability theory, number theory, product integration, geometry of curves, discrete optimization, lattice theory, linear algebra, linguistics, geometric transformations, etc.).

The fourth group describes the development of mathematical research, schools and mathematical education, teaching methods and textbooks, the establishment and evolution of some communities and associations in the past in the Czech, Austrian and Polish lands (mathematics at the Jesuit Clementinum in the years 1600–1740, the birth and the first decade of the Union of Czech Mathematicians, the development of the German Technical University in Brno, the Czech mathematical community from 1848 to 1918, the role of some Czech mathematicians in the development of mathematics in the Balkans, the history of

[4] For more information see: http://www.fd.cvut.cz/personal/ becvamar/seminar_ss.

financial mathematics in Czech textbooks, some Czech teachers of geometry, philosophical conception of probability in the works of Czech thinkers, the growth of mathematical culture in the Lvov area).


VÝZKUMNÉ CENTRUM PRO DĚJINY VĚDY

DĚJINY MATEMATIKY, svazek 23

MATEMATIKA VE STAROVĚKU

EGYPT A MEZOPOTÁMIE

Jindřich Bečvář

Martina Bečvářová

Hana Vymazalová

PROMETHEUS

Fig. 4. The title page of the volume 23

The fifth group contains the commented transcriptions of unknown or forgotten mathematical manuscripts (Czech versions of Euclid's Elements from the 1880s and the first decade of the 20th century, Jarník's notebook with Göttingen mathematical lecture course given by P.S. Aleksandrov in the academic year 1927/1928).

The sixth group presents the Czech translations of classic mathematical works (the old Egyptian hieratic mathematical texts, the Mathematics in Nine Chapters, the letters of Gerbert of Reims) or some unique personal memories (Em. Weyr's diary describing his study visit in Italy, his mathematical work done there and his contacts with Italian mathematicians around 1870).

The seventh group offers proceedings of some national conferences on the history of mathematics showing various relations and connections between mathematics and arts, architecture, geography, techniques etc.

CHARLES UNIVERSITY IN PRAGUE
UNIVERSITY OF TASMANIA IN HOBART
UNIVERSITY OF ALBERTA IN EDMONTON

HISTORY OF MATHEMATICS, volume 52

THE FORGOTTEN MATHEMATICIAN

HENRY LOWIG

(1904 – 1995)

Martina Bečvářová et al.

PRAGUE 2012 MATFYZPRESS

Fig. 5. The title page of the volume 52

It should be mentioned that some volumes are extended versions of PhD or habilitation theses, others contain results of many research projects accomplished in the last twenty years. Most are unique contributions to our understanding of the development of mathematics and would be of interest not only to mathematicians, but also to historians, linguists and anyone who wants to learn about mathematics and mathematical thinking in the past[5].

The series has a non-commercial character. It was and it is supported in part by projects financed by the Czech Ministry of Education, the Czech Science Foundation, the Grant Agency of the Czech Academy of Science, the Grant Agency for the Development of Czech Universities, the Research Center for the History of Sciences and Humanities, the Faculty of Mathematics and Physics of Charles University, the Faculty of Transportation Sciences of the Czech Technical University in Prague and the Czech Mathematical Society[6].

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[5] All volumes are available on: http://www.dml.cz

[6] For more information on the series *History of Mathematics* (for example the contents and the front pages of volumes); See: http://www.fd.cvut.cz/personal/becvamar/Edice/Edice.htm; Also [3, 4, 7].

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SERGEY S. DEMIDOV*

WORLD WAR I AND MATHEMATICS IN "THE RUSSIAN WORLD"

I WOJNA ŚWIATOWA I MATEMATYKA W IMPERIUM ROSYJSKIM

A b s t r a c t

The First World War marked a turning point in the Russian history. The country entered the war in August 1914 as an empire, and in 1918, when the war ended, its name was: the Russian Soviet Federative Socialist Republic. In 1917 it confronted two revolutions – the February and the October Revolutions. As a result of the October Revolution, the Bolsheviks ruled the country and began the construction of a new type of state. In 1918 a civil war broke out, which was largely over in 1920, but in some areas continued until 1922. In the end of 1922 the USSR was formed – the Union of the Soviet Socialist Republics. In this article we analyze the impact which these events had on academic and mathematical life. We discuss the mathematical schools of St. Petersburg and Moscow, mathematical centers in Kazan Kharkov, Kiev and Odessa, academic institutions relocated inland (University of Warsaw ,Riga Polytechnics) and others. We also mention mathematicians immigrants from Russia, who became a common phenomenon in mathematical communities of other countries.

Keywords: *history of mathematics in the Russian empire at the turn of 19th and 20th centuries, changes in mathematical centers in the Soviet Russia*

S t r e s z c z e n i e

Pierwsza wojna światowa wyznaczyła punkt zwrotny w historii Rosji. Kraj do początku wojny w sierpniu 1914 roku funkcjonował jako Imperium, a w czasie jej zakończenia w 1918 roku jego nazwa brzmiała: Rosyjska Federacyjna Socjalistyczna Republika Radziecka. W wyniku rewolucji październikowej bolszewicy rządzili krajem i rozpoczęli budowę nowego typu państwa. W 1918 roku wybuchła wojna domowa, która w wielu miejscach trwała do roku 1920, zaś w niektórych do roku 1922, z którego końcem powstał ZSRR – Związek Socjalistycznych Republik Radzieckich. W artykule analizujemy wpływ wspomnianych wydarzeń na matematyczne życie naukowe. Omówione zostaną szkoły matematyczne w Petersburgu i Moskwie, ośrodki matematyczne w Kazaniu Charkowie, Kijowie i Odessie, a także instytucje akademickie ewakuowane w głąb Rosji (np. Carski Uniwersytet Warszawski, Politechnika Ryska) i inne. Wspomnimy również matematyków imigrantów z Rosji, którzy zasłużyli się w ośrodkach matematycznych innych krajów.

Słowa kluczowe: *historia matematyki w imperium rosyjskiego na przełomie XIX i XX wieku, zmiany w ośrodkach matematycznych w Rosji Sowieckiej*

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1. Introduction

On July 19 (August 1) 1914 Germany declared war against Russia, and the conflict between Austria-Hungary and Serbia, of a seemingly local significance, turned into a major war, later called First World War . The official completion of that war was on June 28, 1919 – the date of the signing of the Treaty of Versailles, which summed up the war's preliminary results. One of the results was a collapse of four empires –the Russian, German, Ottoman and Austro-Hungarian empires. This collapse changed radically the political map of the world. The balance of forces in the world changed completely and, last but not least, the spirit of international relations changed. The methods of warfare marked the unprecedented importance of a technological factor, the equipment of troops based on advanced technologies, which was determined by scientific and technological level of adversaries. Science and technology, which at the turn of the century seemed the basis for the future progress of humanity, became a source of military power aimed at the destruction of the enemy and the victory that promised domination. Russia entered the First World War having the rank of an empire, and by the date of its formal completion was already a republic – the Russian Socialist Federative Soviet Republic. During the war, she experienced two revolutions – the February 1917 one and the October 1917 one, and the fratricidal civil war, which lasted until 1923. It was only the end of this war that became the actual completion of the First World War for Russia. The country has entered its next break between the First and the Second World Wars as the Union of Soviet Socialist Republics. This Union, founded December 29, 1922, was proclaimed as the state of a new type – a republic of workers and peasants, built on the ideological foundations of Marxism, in which the power was actually carried out by the Bolsheviks (VKP (b) – All-Union Communist Party of Bolsheviks) headed by V.I. Lenin.

The aim of our report is to try to trace the evolution of mathematics (i.e. of the mathematical community, of its institutions, of mathematical research and of mathematical education) during the war in the "Russian world", that is, in the Russian-speaking world of people who met the beginning of the First World War as citizens of the Russian Empire, but many of whom later found themselves far beyond its borders as a result of the dramatic historical events of those years. Let us try to answer the question of what role the war played in this development.

To understand and appreciate the scale of changes which occurred during the war, try to start to look at the Russian mathematics in early summer 1914, as it met the beginning of war. Let us go back to May – beginning of June of that year – to the end of the last semester of peace in the history of the higher education of the Russian Empire.

2. Mathematics of the Russian Empire by the summer of 1914

The main centers of mathematical life of the Empire were its capital – St. Petersburg and Moscow. In St. Petersburg the centers were the Imperial Academy of Sciences and the University. These institutions were connected with the activity of the school known in Russia and in Europe as the Petersburg mathematical school or school of P.L Chebyshev which grew in the last third of the XIX century into one of the most important in Europe. Its main areas of studies became probability theory, constructive theory of functions, analytical mechanics, mathematical physics and number theory, which at the beginning of the twentieth century were represented by such names as A.A. Markov, A.M. Lyapunov and V.A. Steklov. The orientation of studies towards applications dominated in St. Petersburg circles (the exception was the theory of numbers – a traditional area for the northern capital since L. Euler), along with the desire for a rigorous and at the same time effective solution of problems, for the construction of algorithms which permitted one to complete the solution either with a numerical answer or with a suitable approximation, as well as the desire for simplicity and using elementary means. In St. Petersburg the general understanding of mathematics and of its place in the world was positivist. Characteristic for it was a negative attitude toward idealistic philosophy, toward religion, and of course toward the monarchy. The policy of the school was defined by the leader of the school – the academician A.A. Markov – an outstanding mathematician, famous for his results in the theory of probability, approximation theory and number theory. His convictions were shared by the academician A.M. Lyapunov, the author of classical results in the qualitative theory of differential equations, mathematical physics and probability theory. Among their supporters was a student of Lyapunov, the academician V.A. Steklov, who began to gain popularity in the world for its results in the field of mathematical physics. All three were wonderful teachers, who lectured at university and other educational institutions of St. Petersburg. If one counts in addition such known scientists and teachers as Yu.V. Sokhotskii, K.A. Posse, D.F. Selivanov, I.I. Ivanov, N.M. Gyunter and Ya.V. Uspenskii, it is not surprising that St. Petersburg of that time was positioned in the mathematical world as one of the most important scientific and educational centers. The leading role in the upbringing of the new generation of mathematicians belonged to Steklov, who created one of the most prominent schools in the first third of the twentieth century. The most known of its representatives are: V.I. Smirnov, Ya.D. Tamarkin, A.A. Fridman. All of them stayed at the university in the years 1910 – 12 “to prepare for a professorship” (which was equivalent to the position of the modern post-graduate). In the same years, a pupil of Markov, A.S. Bezikovich, also stayed at the university. The explosion of scientific activity of young people in St. Petersburg in those years was largely prepared by the work that St. Petersburg mathematicians (Markov et al.) carried out among the student youth. A mathematical seminar for high school students was working with great success.

In Moscow, the focus of the mathematical life were the university and the Moscow Mathematical Society, functioning by the university since 1864, whose activities in the last third of the XIX century took the nationwide (all-Russian) character. In 1866 the Society started to publish the journal “Matematicheskii Sbornik”(Mathematical Collection), which became a platform for the developing Russian mathematical community. In Moscow, mathematics developed in a paradigm different from that of St. Petersburg. And although the Muscovites, like St Petersburg mathematicians, had inherent interest in the applied subjects (this is how the drive for the development of intellectual and industrial forces of the Empire, common to all the Russians of that time, manifested itself), all their other mathematical tastes radically diverged: Muscovites had special interest in the geometric studies and the aptitude for the idealist and even religious philosophy. This tendency became a reason why the school, which developed in the last third of XIX – the beginning of XX century, was named a philosophical and mathematical one. The Moscow mathematical community welcomed the Orthodox Christianity and even the monarchism.

By the beginning of the century studies on mechanics (especially on aerodynamics and fluid mechanics) and differential geometry (in the direction laid by the works of K.M. Peterson) flourished in Moscow. The most important achievements of Muscovites of that time are associated with the names of N.E. Zhukovskii and D.F. Egorov. The difference in worldviews which characterized the mathematical communities of two capitals created a confrontational relationship between these communities. Petersburgers looked down at their Moscow colleagues and did not miss the opportunity to put "illiterate Muscovites" in place. Muscovites who were offended by this attitude of the academic Petersburg were looking for topics (if possible, far away from the interests of St. Petersburg) which would allow them to take a position defining the face of the modern mathematics (differential geometry and the mechanics of continuum did not count as fashionable at that time). At the beginning of the century they found such a theme – it was the theory of functions of a real variable, a new section of mathematics initiated by the French – E. Borel , H. Lebesgue and R. Baire. In 1911 in *Comptes Rendus* of the French Academy of Sciences, the article of Egorov "On a sequence of measurable functions" was published, which contained the well-known theorem that bears his name, and in 1912 in the same journal the article of his pupil N.N. Luzin on C-property of measurable functions appeared. These works began the history of one of the most influential mathematical schools of the twentieth century – of the Moscow school of function theory. So in the spring of 1914 the face of mathematics in Moscow was determined as follows: in applied mathematics, by Zhukovskii and his pupils, in differential geometry, by Egorov and B.K. Mlodzeevskii. S.S. Byushgens and S.P. Finikov started to work on their master's theses (magister thesis) on the theory of surfaces, while V.V. Golubev, V.V. Stepanov and I.I. Privalov worked on various questions of the theory of functions. But the main event of the summer of 1914 was the return from a long trip to Göttingen and Paris of the rising star of the Moscow school, Luzin. In the fall semester he announced at the University a course on the theory of functions of a real variable and a seminar on the same topic. From this seminar, in the first years of its existence, grew the first generation of the famous Luzitania – D.E. Men'shov, M.Ya. Suslin, A.Ya. Khinchin, P.S. Aleksandrov.

These successes did not change the overall negative attitude of St. Petersburg to the Muscovites, especially since the set-theoretic works of Cantor and research on the theory of functions of a real variable based on these works (in the words of Uspenskii "Cantor's and Lebesgue's trash") were met by them with a point-blank hostility. The confrontation between Muscovites and Petersburgers created the tension throughout the Russian mathematics community: we must not forget that the majority of teaching staff in Russian universities were graduates of the metropolitan universities.

And in the Russian province in the late XIX – the early XX century there was a sharp rise in mathematical activity. Although traditionally the whole social and cultural life of the Russian Empire was rigidly centralized, and the top-down power pervaded all fields of activity, at the distance from the capitals the impact of the managing hand became nevertheless noticeably weaker.

In particular, in the province it was possible to develop freely the ideas coming from the West which did not find support among the capital's mathematicians. So, in Kazan

University, besides the traditional (since N.I. Lobachevsky) geometrical topics studied by A.V. Vasil'ev, A.P. Kotel'nikov et al., there was the avant-garde research in mathematical logic of P.S. Poretskii and N.A. Vasil'ev.

The mathematical logic became a topic of research in a young Novorossiisk University in Odessa, with I.V. Sleshinskii and S.I. Shatunovskii. There V.F. Kagan began his geometrical research with the questions in non-Euclidean geometry and the foundations of geometry.

In general, at the turn of the centuries the mathematical life in the south of the Russian Empire considerably quickened. At Kharkov University, one of the oldest in the country, the high level of teaching was established by the efforts of Lyapunov and Steklov. In the prewar years such mathematicians as D.M. Sintsov, N.N. Saltykov, A.B. Psheborskii, and finally one of the greatest mathematicians of the twentieth century S.N. Bernshtein, worked there. Kiev University had very moderate mathematical achievements in the XIX century, but thanks to the endeavors of a remarkable representative of the St. Petersburg school D.A. Grave, who moved there in 1901, it sharply raised its mathematical level. By Grave's efforts a school was created in 1908–1914, which had mainly algebraic character. Such famous mathematicians as B.N. Delone, O.Yu. Shmidt, N.G. Chebotarev, who laid the foundation of the Soviet school of algebra, and also M.F. Kravchuk and A.M. Ostrovskii.came out from this school in those years. Among the mathematicians of the Warsaw University (where such well-known scientists as N.Ya. Sonin, V.A. Anisimov and G.F. Voronoi worked before) we can name two pupils of the St. Petersburg school – D.D. Mordukhai-Boltovskoi and V.I. Romanovskii. Although Yuryev (formerly Dorpat) University was at that period going through not the best of times, however, among its professors in 1914 we can see such known scientists as the alumni of Moscow University V.G. Alekseev and L.S. Leibenzon. Of the outstanding mathematicians working in those years in other educational institutions of the Russian Empire, we can name the outstanding algebraist professor of the Tomsk Institute of Technology F.E. Molin and one of the pioneers in the development of qualitative methods of the theory of differential equations, a professor of the Riga Polytechnic Institute P.G. Bohl. The listed names of the first-class mathematicians, the wide range of their research, the importance of the mentioned schools in the science of the twentieth century show that mathematics on the eve of the events of the World War I in the Russian Empire experienced a period of rapid growth. The Russian mathematical community also developed at extreme speed. Mathematical societies worked actively: besides the oldest Moscow mathematical society the Mathematical branch of Novorossiysk society of scientists in Odessa (founded in 1876), the Kharkov mathematical society (founded in 1879), the Kazan physical and mathematical society (from 1880 it existed as physical and mathematical section of the Kazan society of scientists, and from 1890 as independent society), the Kiev physical and mathematical society (founded in 1889).

A large number of participants gathered at the mathematical section of the All-Russian congresses of scientists and physicians, the first of which took place in January 1868 in St. Petersburg, and the last one on the 13th of June, 1913, in Tiflis. While at the First congress there were only 6 mathematical reports, at the 13th their number increased to 31. While at the first congresses the number of participants of mathematical sections was somewhere about 50, it rose by the last congresses to 500. At these congresses problems of school mathematical education were put forward and actively discussed. The leading Russian scientists took part in these discussions together with teachers of high schools, who made the majority at the congresses. The importance of these problems became the reason for the organization of special All-Russian congresses of teachers of mathematics. The first such congress was carried out in St. Petersburg in January of 1912, the second in Moscow in January of 1915. The central theme of these congresses was the teaching reform; the movement towards reform, headed by F. Klein, was supported passionately by the Russian mathematical community. The training of school students in the functional thinking, and also the introduction to the school program of elements of "higher mathematics" became a goal of this reform. Russia took active part in the work of the International commission on teaching mathematics created in 1908. N.Ya. Sonin became the chairman of its Russian section (subcommittee).

In general the Russian mathematicians appeared as active participants in the all large international undertakings of the end of XIX century and the beginnings of the XX century. They were active participants of the international congresses of mathematicians, starting from the first in 1897 in Zurich, then in Paris (1900), Heidelberg (1904), Rome (1908), finally, in the last pre-war congress of 1912 in Cambridge. They apprehended with enthusiasm the beginning of the large project carried out by the Berlin mathematician C. Ohrtmann – "Books of achievements of mathematics in a year" (Jahrbuch über die Fortschritte der Mathematik), whose first volume was issued in 1871 – and took an active part in it. A.N. Korkin, E.I. Zolotarev, K.A. Posse, D. M. Sintsov cooperated with this year--book. As A.V. Vasil'ev wrote: "It is very difficult, I think, to estimate that enormous benefit which it brought; in particular, of course, the Russian science is especially obliged to it. With amazing ignorance of our language by foreigners … only thanks to this Jahrbuch the Russian mathematical literature could become known to the mathematicians of other countries" [1, p. 323]. The Russian mathematicians took part in implementation of the international project "Enzyklopädie der mathematischen Wissenschaften" organized in Germany – .D.F. Selivanov wrote the section on the calculus of finite differences (1901), A.N. Krylov in cooperation with C. Müller wrote the section on the theory of the ship (1906–1907), T.A. Afanas'eva-Ehrenfest in a co-authorship with her husband P. Ehrenfest wrote on statistical mechanics (1909–1911).

The Russian mathematicians became frequent visitors in Paris and Göttingen and actively published in the French, German and Italian mathematical journals. Long scientific trips of the persons "prepared for a professorial rank" in the leading European mathematical centers became a usual practice of the Russian system of mathematical education. Already mentioned Luzin, still being a student, at the end of 1905 was sent to Germany and France, from where he returned only in the summer of the next year. And while working on his master thesis he stayed in these countries since the end 1910 until the beginning of summer of 1914.

Many leading Russian professors spent a considerable time in the West. Some of them met the beginning of the World War I in such study tours.

One can say that the Russian mathematics met the World War I as an integral part of the European mathematical world. And though the Russians were not among its leaders – those were still the French and Germans, with Italians snapping at their heels – nevertheless the Russian mathematical community was one of its most successful and dynamically developing groups.

3. Mathematics and mathematicians in the first years of the war (before the February revolution of 1917)

When during the military operations there was a real danger of the entry of the German troops on the territory of the Russian Empire, the Russian government made the decision on the evacuation of higher educational institutions from the western territories far inland. So in 1915 the Kiev university was evacuated to Saratov, the Warsaw university to Rostov--on-Don.

Only in 1918 did the Warsaw polytechnical institute find a haven in Nizhny Novgorod, the Yuryev university in Voronezh, the Riga polytechnical institute in Ivanovo-Voznesensk.

The Kiev university returned home in the fall of 1916.Later the Warsaw and Riga polytechnical institutes, and also the Yuryev university (which became Tartu university) returned to the already independent states. The last three institutions returned respectively to Warsaw, Riga and Tartu already not in full strength: the Russian professors, with few exceptions, preferred to remain in Russia. So professors of the Warsaw polytechnical institute joined the structure of the Nizhny Novgorod university, and professors from Riga and Yuryev laid the foundation of the Ivanovo Voznesensk polytechnic institute and the Voronezh university, respectively. The former Warsaw university remained in Rostov-on-Don ever since.

Perhaps only at these schools, which were settling down at the western boundaries of the Empire, the war seriously broke the normal course of pedagogical process and scientific researches.

At all other universities – in Petrograd, Moscow, Kazan and even in Kharkov – the tide of life proceeded in a habitual rhythm. Lectures were given, scientific seminars were conducted (research seminars already were coming into fashion: for example in Moscow such seminars were conducted by Egorov and Luzin), the theses were prepared and defended.

It was very important that, by then-effective legislation of the Russian Empire, the youth studying at the higher schools, the persons staying at universities "for preparation for a professorial rank", the privat-dozents and the professors were not subject to a draft for an active military duty. This created necessary conditions for preservation of the scientific capacity of the country.

In Moscow Zhukovskii worked successfully with his pupils (S.A. Chaplygin, etc.). Their works on aerodynamics acquired special value because of the prospects of use of aeronautic equipment which were opened by the war. Mlodzeevskii and especially Egorov with his pupils continued to develop successfully the traditional (for Moscow) directions in differential geometry and in the geometric theory of partial differential equations. In the spring semester of 1914 Egorov announced a seminar on the theory of functions. Then, as Men'shov remembered later [2, p. 188], "just appeared … Lebesgue's integral and the so--called metric theory of functions". In the summer of that year Luzin returned to Moscow from Paris and announced his course on the theory of functions of a real variable. "Precisely this special course given from year to year and the seminar accompanying it … were the center from which the Moscow school of the theory of functions – a remarkable monument to scientific activity of N.N. Luzin – grew", – Men'shov said [3, p. 475]. Luzin then was preparing for the defense of his thesis, which was titled "Integral and trigonometric series";

it was published in 1915 and defended in May 1916. The work was so successful that the Academic board decided, as an exception, to grant immediately to the author of the thesis the degree of the doctor of pure mathematics, bypassing the master's degree. The thesis included also a result of his pupil A.Ya. Khinchin, who introduced a notion of asymptotic derivative. In the same year remarkable results of another of his pupils, Men'shov, appeared, and in 1915 yet another of his pupils, P.S. Alexandrov, proved a continuum hypothesis for the Borel sets (B-sets) – the sets which, according to a belief of that time, exhausted all stock of sets really used in mathematics. The belief, as it soon became clear, was incorrect: in 1916 the student of Luzin M.Ya. Suslin introduced a new type of the sets which were not B-sets: A-sets.

This class of sets, also called Suslin sets or analytical sets, turned into the main object of research in descriptive theory of sets for many years, becoming the trademark of Luzin school. Results of Luzin's pupils were immediately printed in the Parisian Comptes Rendus (we should not forget that Russia and France were allies in that war) and became known in Europe. Moscow was becoming one of the leading centers of mathematical research in Europe.

Petrograd (the name that St. Petersburg received after the beginning of the war with Germany) kept its position as a recognized European mathematical center. The academicians Markov, Lyapunov, Steklov kept working, N.M. Gyunter's and Ya.V. Uspenskii's research talents ripened, Yu.V. Sokhotskii, K.A. Posse, I.L. Ptashitskii, Selivanov, I.I. Ivanov continued their scientific and pedagogical activity, finally, the whole cohort of remarkable pupils grew – Smirnov, Tamarkin, Fridman prepared their master theses. A.S. Bezikovich and I.M. Vinogradov obtained their first-class results. The war affected the activity of the Petersburg school almost only in some delays in publishing and in postponing the dates of defending of theses. Only Friedman, who left for the front as a volunteer and in 1914–1916 served in air units, was involved directly in military operations.

Normal activity proceeded in other Russian educational institutions located in the territories which were not affected by military operations.

4. Mathematics and mathematicians in the era of the collapse of the Russian Empire

The situation started to change sharply in the era of the revolutionary events of 1917, especially after the October revolution, which led to the cardinal change of the system of the civic life, to the demolition of the old state machinery, to the establishment of the new orders unknown hitherto and to a fratricidal civil war. All territory of the former Empire flared. Of course, such events extremely negatively affected the life of the scientific and the educational institutions.

The termination of the normal functioning of the institutions of power and the disastrous situation with food and fuel put the university professoriate on the edge of survival. Old and sick persons quickly descended to a grave. In 1918 Lyapunov committed suicide, in 1921 Zhukovskii, died and in 1922 did Markov. For the younger and vigorous there came time of search for daily bread. Especially grave situation developed in both capitals. Luzin with his pupils (Men'shov, Suslin, Khinchin) went over to Ivanovo-Voznesensk where in 1918, as we already mentioned, the Polytechnical institute was organized. The mathematicians of Petrograd (Tamarkin, Fridman, Bezikovich, Vinogradov) safeguarded themselves in Perm, where in 1916 a branch of Petrograd university was opened (in 1917 this branch became an independent university).

The events in the south of the European part of the Empire were developing in an unpredictable way and with an extraordinary speed – the detachments of white and red armies, parts of the regular German army, the soldiers of unexpectedly appearing and also quickly disappearing "states", finally, numerous gangs (the most known of which was headed by the legendary Father Makhno) operated there. Despite all lawlessness created there in those years D.A. Grave, who settled in Kiev already in 1899, could create the well-known school (O.Yu. Schmidt, N.G. Chebotarev, B.N. Delone), connected primarily with the beginning of national research on modern algebra.

Such famous mathematicians as A.M. Ostrovskii, later M.F. Kravchuk, N.I. Akhiezer and M.G. Krein also were his pupils. In Kharkov the activity of the higher school and the mathematical society also continued. Its high scientific level was kept and supported by such mathematicians as D.M. Sintsov, N.N. Saltykov, S.N. Bernstein. Despite grave situation in Odessa mathematicians tried to arrange the educational and the scientific life there as much as it was possible.

The severe conditions of life in the country caused by the events of revolution and civil war were aggravated by the uncertainty of the relation of the new authorities to establishments of science and education. The unwillingness to reconcile with such situation and to live in the world operated by the new Soviet power pushed many persons, including mathematicians, to the emigration. So from Petrograd in 1922 Ya.A. Shokhat[1] went to Poland, and then in a year to the USA, in the same year Selivanov[2] was deported on well--known "philosophical steamship", in 1924 Bezikovich[3] and Tamarkin[4] escaped (having crossed the border with one of the Baltic countries), and in 1929 the academician Uspenskii[5], who was on a study tour, decided not to return home. A number of mathematicians from the universities of the South of the Empire emigrated.

So in 1919, from Kharkov, N. N. Saltykov left at first to Tiflis and later, when in 1921 Bolsheviks came to the power there, to Belgrade[6]; in the same year the young statistician Yu.Ch. Neyman left Kharkov and moved to Poland[7]. In 1922 A.P. Przeborski left

[1] J.A. Shohat (1866–1944). For many years he was a professor of the University of Pennsylvania.

[2] D.F. Selivanoff (1855–1932). He lived and worked in Prague.

[3] A.S. Besicovitch (1891–1970 Initially he worked with H. Bohr in Copenhagen, later he was a professor in Cambridge. Member of the London Royal Society.

[4] J.D. Tamarkin (1888–1945). He settled in the USA. Since 1929 professor of Brown University. In 1942–1943 the vice-president of the American mathematical society.

[5] J.V. Uspensky (1883–1947). Since 1929 until his death he worked at Stanford University.

[6] N.N. Saltykov (1872–1961). He at the Belgrade university. The full member of the Serbian Academy of Sciences.

[7] Jerzy Neyman (1894–1981). In 1938 he was invited in the Californian university in Berkeley with which all his further activity was connected. Member of the National Academy of Sciences of USA.

Kharkov and moved to Poland[8]. In 1920 from Odessa A.D. Bilimovich[9] moved to Belgrade, and in 1922, also from Odessa, E.L. Bunitskii moved at first to Belgrade and then to Prague[10].

The events of the revolution and of the civil war which followed became the time of a radical break-up of the old institutions and of the old mentality. The public life changed rapidly. What took years in the usual course of life, could be accomplished almost instantly in such a period. The Tomsk university, founded in 1878, for a long time comprised only one faculty – the medical faculty. The long-term persistent efforts towards organization of other faculties did not yield any results and only in 1917 the physical and mathematical faculty opened. The Saratov university, founded in 1909, acquired, at last, physical and mathematical faculty only in 1917. In 1918 universities opened in Simferopol (Tavrian university), Tiflis and Tashkent (Turkestan university), in 1919 in Baku and Erevan, in 1920 in Yekaterinburg and in Vladivostok (Dalnevostochnyi university), in 1921 in Minsk. In those years the history of many higher educational institutions, including teacher training colleges and institutes of technical profile, began. New forms of scientific activity started to appear. So in 1918 in Kiev the All-Ukrainian Academy of Sciences was created. V.I. Vernadskii was elected its first president. In February 1922 at the initiative of Steklov the Physical and mathematical institute of the Russian Academy of Sciences was organized, which received his name in 1926 (from this institute in 1934 the V.A. Steklov Mathematical institute emerged). In the same 1922 a number of research institutes started their activities at Moscow university. Among them there was the Research institute of mathematics and mechanics, whose first director was Mlodzeevskii, replaced by Egorov the next year.

The first post-revolutionary decade became the time of migration of the pedagogical and scientific personnel on an unprecedented scale. Moscow, which in 1918 got the status of the capital of the state, became the main point of attraction. There V.F. Kagan moved from Odessa in 1923, Schmidt did from Kiev in 1920, A.P. Kotelnikov in 1924, and E.E. Slutskii in 1926. Some mathematicians moved to Petrograd: at the very beginning of the war G.M. Fikhtengolts, from Odessa and in 1922 Delone from Kiev moved there. The university centers (first of all Moscow and Petrograd) attracted the studying youth, whose social and ethnic composition radically changed: people of worker and peasant origin, and also numerous Jewish youth, for whom receiving the higher education in imperial Russia was extremely complicated, came to the higher schools.

In the first post-war years the Russian Academy of Sciences endured the most dramatic period in its almost bicentennial history. Its mathematical class in 1923 consisted of three full members: Steklov, Krylov and Uspenskii. In 1926 Steklov died, and soon Uspenskii left the country forever and the mathematical class shriveled to one full member – Krylov.

The national commissariat of education, under whose authority the Academy fell, at first did not count it among the priorities at all. Moreover, many of the Bolshevik leaders

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8 A.B. Przeborski (1871–1941). He worked a few months at the Vilno university, afterwards he moved to Warsaw.

9 A.D. Bilimovich (1879–1970). He worked in the Belgrade university. A full member of the Serbian Academy of Sciences and in 1936–1940 the secretary of the Department of natural and mathematical sciences of this Academy.

10 E.L. Bunitzky (1874–1952). He worked in the Charles University.

considered Academy the obsolete heritage of an old regime. In their opinion the new Socialist Academy created in 1918 in Moscow (later renamed the Communist Academy) should take the place of the old Academy "forgotten" in the old capital. The return of Academy of Sciences to the number of the state-forming institutions of the country was to a considerable extent a merit of Steklov, who was elected its vice-president in 1919. A person of the leftist view who accepted Bolshevist revolution and established good relations with the people's commissar of education A.V. Lunacharskii as well as with V.I. Lenin himself, Steklov acccomplished that the Academy of Sciences gained the status of the head scientific institution of the USSR. But this status went into effect only in the later 1920s.. In 1923 the country only started recovering after the end of a long civil strife. The era of the Soviet state construction (including construction of the system of national education, at the school and higher level) began. The dreams about the world revolution of the Bolshevist ideologists who brought about the revolution were consigned to the past[11]; it became obvious that the country was destined to build a new society while living in a hostile environment. The collectivization and the industrialization of the country was coming. It was necessary to have educated personnel and consequently, efficient schools – the primary, the secondary and the highest. The educational system and, first of all, the school system existing in the Empire, collapsed in the first years of the Soviet rule. And for the accomplishment of the tasks facing the USSR it was necessary to solve an ambitious problem – to build the mass schooling, which did not exist in imperial Russia, to create a branched system of polytechnic education, the construction of which merely began before the revolution. It was necessary to carry out all this in a mobilization order: as we know, there were less than two decades remaining before a new war and the most shrewd politicians already felt its breath. The construction of a scientific and technical and educational complex which was necessary for the country required, in particular, great mathematics: a highly professional mathematical community and a well-built system of mathematical education. Was it possible to solve such problems in a foreseeable future? To answer this question we will consider the situation which developed in the Soviet mathematical community in 1923–1927.

5. Mathematics in the USSR during 10 years

As we already said, by the time of the revolutionary events of 1917 mathematical research in the country was on the rise. The level of this research was such, that even the difficulties of the five-year period which saw the trials of the devastating wars (one of which was the civil war) and the bloody revolution did not stop its progress. As Khinchin wrote in an article devoted to development of mathematics in the country during ten years of the Soviet

[11] In 1918 V.I. Lenin wrote: "…The international revolution was coming nearer … at such a distance that it should be reckoned with as an event of the next few days" [4, p. 185]. In the Constitution of the USSR of 1924 we still read: "the new union state … will serve as a right stronghold against the world of capitalism and a new decisive step on the way of association of workers towards the World Socialist Soviet republic". But already the Stalin Constitution of 1936 does not even mention this republic.

power, [5, p. 41]: «Perhaps in those first difficult years of the revolution, mathematics, for purely external reasons, was put under rather special conditions, which allowed it to develop more intensively than other exact sciences did: a mathematician does not need laboratories or reactants; paper, a pencil and creative power are the prerequisites for his scientific work; and if to this one adds an opportunity to use a more or less solid library and a certain share of scientific enthusiasm (which almost every mathematician has), no devastation can stop his creative work. The lack of the modern literature was to a certain extent compensated by continued scientific communication which was organized and supported during these years». Mathematical Moscow, which loudly publicized itself in 1911–1916, generally managed to cross safely the rough waters of history: the death in 1919 of typhus of ingenious Suslin became the sole terrible loss. In 1922 Luzin returned to Moscow from his trip to Western Europe and the regular meetings of his seminar resumed. In this seminar students – N.K. Bari, V.I. Glivenko, L.G. Shnirelman, later A.N. Kolmogorov, and still later M.A. Lavrent'ev, L.V. Keldysh, E.A. Leontovich, P.S. Novikov and G.A. Seliverstov participated with their teachers (Stepanov, Alexandrov and Urysohn. The "old men" – Privalov, Men'shov and Hinchin – returned to Moscow and got into gear [6]. The studies on the set theory and the theory of functions proceeded successfully, wherein attention was concentrated on the problems of the theory of analytical sets. However, already at that time in Egorov--Luzin school the tendency to expanding the scope of research distinctly showed itself.

As Stepanov wrote later: "Every scientific school with the specialized subject is, in the course of its development, in danger of epigonism … when the main problems are resolved and settled by the works of a number of talented scientists, the same scientists and their pupils gather remaining bits. The Moscow school in general overcame this danger by expansion of its area of research and by application of methods of the theory of functions and theories of sets to other branches of mathematics" [7, p. 51]. The school's own achievements in the metric theory of functions became a starting point for work in new directions. The metric theory of function in many respects also defined the methods used in new areas.

Even in the years of revolution Luzin and his pupils (Privalov, Golubev, Men'shov, Khinchin) started to investigate problems of the theory of functions of a complex variable.

In 1925 M. A. Lavrent'ev joined them. Later he brought up a remarkable pupil – M.V. Keldysh.

Alexandrov's and Urysohn's achievements of 1921–1924 marked the first steps of the Soviet topological school. In 1925 under the leadership of Alexandrov the seminar on topology started to work. In this seminar such outstanding mathematicians as A.N. Tikhonov and L.S. Pontryagin grew up.

In 1923 the first important results of Khinchin on probability theory appeared, and at the end of the years 20th one of the greatest mathematicians of the XX century – Kolmogorov – started his studies on this discipline.

In 1922–23 Khinchin also started the studies on number theory. In 1925/26 academic year he organized a special seminar on this subject, in which A.O. Gelfond and Shnirelman participated.

Research in the directions traditional for Moscow proceeded: in differential geometry (Egorov, S.P. Finikov), in the theory of partial differential equations (Egorov), in applied mathematics (S.A. Chaplygin). If one adds the works on tensor differential geometry by

Kagan and his pupils, on the theory of integral equations by Egorov and V.A. Kostitsyn, on the theory of almost periodic functions by Stepanov, on probability theory and statistics by E.E. Slutskii, and, finally, on the theory of groups by O.Yu. Shmidt and his pupils, one can say that by the mid-1920s Moscow became an important and quickly developing center of mathematical research. This center was formed around the Research institute for mathematics and mechanics of Moscow university and the Moscow mathematical society. Egorov, who tried to do everything for revival of the normal life of the Russian mathematical community, was at the head of both institutes. The publication of "Matematicheskii Sbornik" (Mathematical collection) resumed, now as an all-union and even an international mathematical journal – the journal started to publish articles not only in Russian, but also in the German, French and Italian languages[12].

Muscovites got to work on the edition of Complete works of N.I. Lobachevskii. At last, they prepared the All-Russian mathematical congress and conducted it from April 27 to May 4, 1927. This meeting revived regular activity of mathematical community in the country, now the USSR. At this congress the decision was made to organize the First All-Union congress of mathematicians in 1930 at Kharkov.

Thus Moscow became the center of the life of the national mathematical community de facto and de jure. But in Petrograd, renamed Leningrad in 1924, there remained the Russian Academy of Sciences, which since 1925 was called the Academy of Sciences of the USSR. As we said before, the Academy needed considerable efforts to keep its place among the national state institutions. Steklov played a major role in this. It was under his leadership that the text of the new statute of the Academy, adopted in 1927, was drafted, according to which the Academy assumed the place of the main scientific institution of the Soviet Union. But this happened already after his death, which took place in 1926.

The mathematical community of the old capital took the period from 1917 to mid--1920s very hard. We already said that the academicians Lyapunov and Markov died, and a number of young talented mathematicians – Uspenskii, Tamarkin, Shokhat, Bezikovich – emigrated to the west. Tragicaly, the brilliant Fridman died (from a typhus). A situation began to improve only in mid-1920s. The mentioned academic Physical and mathematical institute started to play an essential role. Mathematical physics (Steklov, Gyunter, Smirnov), the theory of differential equations – ordinary (Krylov, Smirnov, I.A. Lappo-Danilevskii) and partial (Steklov, Gyunter), number theory (I.I. Ivanov, B.N. Delone, I.M. Vinogradov, R.O. Kuzmin, B.A. Venkov) became the most important directions of research of Leningrad mathematicians. We would like to remark that the young Petrograd mathematicians also dealt with the questions of the theory of a real variable (Bezikovich, G.M. Fikhtengolts) – a subject absolutely forbidden by leaders of he old Petersburg school. At the end of the 1920s also S.L. Sobolev's and L.V. Kantorovich's first studies appeared. So by the mid-1920s mathematical Leningrad possessed a serious creative potential.

Among other points of creative growth it is necessary to mention, first of all, the cities of Ukraine: Kiev, Kharkov and Odessa. Bernshtein continued to publish excellent results

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[12] As a result, foreign authors, including E. Cartan, M. Frechet, B. Gambier, J. Hadamard, H. Hopf, S. Lefschetz, R. Mises, E. Noether, W. Sierpinski, L. Tonelli, started actively to publish their articles in the journal [8].

on the theory of differential equations, constructive theory of functions and probability theory.

Grave's pupils (Kravchuk, N.I. Akhiyezer, M.G. Krein) worked successfully. The activity of a school for nonlinear oscillation of N.M. Krylov – N.N. Bogolyubov developed. D.M. Sintsov continued his geometrical research.

An important center of mathematical researches was, as before, Kazan, where since the times of Lobachevskii work in the field of geometry was successfully conducted and where in 1928 the famous algebraist Chebotarev moved from Odessa. Tiflis became a new point on the mathematical map of the country (G.N. Nikoladze, A.M. Razmadze, N.I. Muskhelishvili), where in January of 1918 the university was opened. Important results in the fields of probability theory and mathematical statistics were obtained by a professor of the Warsaw – Rostov-on-Don university who got stuck in Tashkent and became one of founders of Turkestani university, V.I. Romanovskii.

Summing up the achievements of mathematics in the USSR by the end of the first decade of the Soviet power, Egorov wrote [9, p. 231-232]: "the works of mathematicians of the USSR take a worthy place among works of the European scientists and contribute their share to development and improvement of various mathematical disciplines".

As a result, the mathematics of our country left the war decade on the rise. Having suffered some losses – some mathematicians who could not stand the hardship of the wartime died early, a number of scientists emigrated from the country, many projects were frozen for a long time or even completely stopped, some institutes and universities found themselves outside of the country – in general the Soviet mathematics by the mid-1920s became a phenomenon extremely noticeable in the world and, as the subsequent events testify, was preparing for a powerful leap forward. Its implementation was initiated by "the travel from Leningrad to Moscow" of the presidium of Academy of Sciences and of the V.A. Steklov mathematical institute, which took place in 1934. This move put an end to the conflict of mathematicians of two capitals, which kept the national mathematical community in tension, and marked the start of the process of formation of the Soviet mathematical school. But this is already a subject for another report.

6. Conclusions

Certainly, any war causes a significant damage to the society, especially, if it is a world war – millions of the dead and crippled, sufferings of civilians, etc. It also brings great losses to the development of science, to the scientific community and to the system of national education. Not being able to stand the severe conditions of life developing in the conditions of the war, the old and the sick die prematurely, and some gifted young people who could grow into serious researchers perish on fronts (Fortunately, by the Russian legislation with which the country entered the war, the pupils of higher educational institutions, persons staying "for preparation for a professorial rank", and also privat-dozents and professors of the higher schools were not subject to conscription). The higher educational institutions located in the west of the Empire were evacuated to the east and their normal activity was broken. As a result of reduction of financing many projects were slowed down or even stopped completely. But the mankind did not yet learn to solve many problems facing the society without resorting to such a surgical tool as a war. As hard as it is, a war helps with the solution of some public problems[13]. So World War I helped with the solution of some problems facing the Russian scientific community. So to say, those problems could be solved also under peace and certainly they would have been solved without any war, but their solution would require a considerably longer time under peace. The World War I, more distinctly than ever, uncovered opportunities which the war industry relying on advanced technologies (which, in turn, were based on the last scientific achievements) could provide for its success.

So the utilization of the aeronautic equipment in military operations showed importance of the aerodynamic research which was successfully conducted in Moscow by the group of Zhukovskii. In 1901 in his manor Kuchino near Moscow the millionaire D.P. Ryabushinskii created an Aerodynamic laboratory, which was directed by Zhukovskii. However only the war prompted the state (already the Soviet state!) to organize a big state institute: On December 1, 1918 in Moscowthe Central aero hydrodynamic institute (TsAGI) , under the leadership of Zhukovskii, was opened. The importance of problems of mechanics became undisputable. Hence the appearance at the Moscow university in 1922 of the Research institute of mathematics and mechanics[14]. The creation of such establishments of absolutely new type in the usual routine of public affairs turns into an extremely slow process of any kind of coordination of numerous departments, the search for free resources, etc. The wartime situation changes everything. What in a time of peace can take many years becomes sometimes a matter of days in war conditions. So, the organization of new universities and of new faculties at already existing universities which went on for years and years, was carried out extremely quickly during the war. Even the compelled evacuation of higher education institutions helped with this process (This evacuation also promoted geographical expansion of the scientific and educational institutions of the country).

For the solution of problems of the rising industrial society professionals were necessary. It was necessary to change the whole education system, which had to become a mass one. It was necessary to build a new mass (not elite, as earlier) high school, it was necessary to create and expand the system of educational institutions training specialists for the developing industry. It was necessary to open the doors of the educational institutions to the broad masses, and not just to representatives of the highest strata of society. All this happened in the USSR in the late 1920s–1930s. As a result the youth of working and peasant background and the numerous youth from small Jewish towns came to universities[15].

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[13] The regaining of independence by Poland as a result of the war led to the birth of one of the most brilliant mathematical schools of the XX-th century. Of course, at any development of events Poland would gain independence sooner or later. However, the World War I accelerated this event and so helped in the successful development of the Polish mathematical school [10, 11].

[14] Along the same line also lies the creation in 1933 of the Faculty of Mathematics and Mechanics at the Moscow university, which in the post-war Soviet Union became one of the central institutes of the country and of the world in the development of mechanics of flight and of theoretical astronautics.

[15] One must not forget the restrictions of the rights to the higher education introduced by the Soviet power for “descendants of exploiting classes” of the Russian Empire. However, in real life there were ways allowing one to bypass these restrictions. So among the graduates of Moscow university there were a descendant of a noble family A.N. Kolmogorov or a son of a merchant I.G. Petrovskii.

Considering the high starting level at which mathematics in the Empire was at the beginning of World War I„all this created prerequisites for a real explosion of scientific activity in the field of mathematics which happened in the 30th–40th years in the USSR.

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TECHNICAL TRANSACTIONS | **CZASOPISMO TECHNICZNE**
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STANISŁAW DOMORADZKI*

RIEMANN SURFACES IN PUZYNA'S MONOGRAPH: *TEORYA FUNKCYJ ANALITYCZNYCH*

POWIERZCHNIE RIEMANNA W MONOGRAFII PUZYNY: *TEORYA FUNKCYJ ANALITYCZNYCH*

A b s t r a c t

In the paper, we discuss the exposition of material on the theory of surfaces in J. Puzyna's monograph "Teorya funkcyj analitycznych" [*Theory of Analytic functions*] (published at the turn of XIX and XX centuries) which is necessary for consideration of the Riemann surfaces of analytic functions. Though the monograph contains elements of the set theory, the author preferred a descriptive exposition.

Keywords: *history of mathematics in Poland*, *Lvov university*, *Riemann surfaces*

S t r e s z c z e n i e

W artykule przedstawiono treści dotyczące powierzchni Riemanna w dziele Józefa Puzyny *Teorya fynkcyj analitycznych* z przełomu XIX i XX w. Warto zauważyć, że chociaż monografia zawiera elementy teorii mnogości, to jednak autor preferuje opisowy sposób prezentacji materiału.

Słowa kluczowe: *historia matematyki w Polsce*, *Uniwersytet Lwowski*, *powierzchnie Riemanna*

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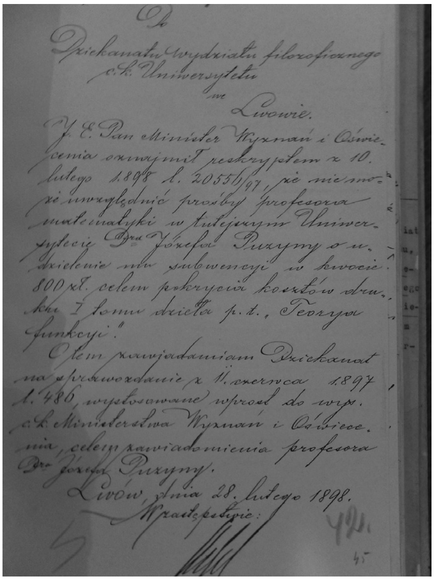
1. Introduction

In his paper *On teaching mathematics* (see [1]), the famous mathematician V. Arnold wrote that the classification theorem for surfaces is one of most important statements that the students of mathematics should necessarily know: “it is this remarkable theorem (which asserts, for example, that any compact connected oriented surface is a sphere with a number of handles) that gives the correct idea of what modern mathematics is …”. Therefore, he asserts, this theorem should be an inevitable part of any university course of mathematics. Also, Arnold recalls that classical course of (complex) analysis by Hermite begins with the notion of a Riemannian surface and this is an important feature which makes it more *modern* that many other contemporary textbooks.

In connection with this it is interesting to look at the role and place of the Riemann surfaces, and surfaces in general, in Józef Puzyna’s monograph *Teorya funkcyj analitycznych* [*Theory of Analytic functions*] published in 1898–1900.

2. Information about J. Puzyna

Józef Puzyna (1856–1919), a longtime professor of mathematics at the University of Lvov, was born on 19 March 1856 in Nowy Martynów. In 1875 he graduated from the famous Franz Joseph Gymnasium in Lvov, then he enrolled at the Faculty of Philosophy of the University of Lvov. After staying at the University of Berlin he received his doctorate in 1883 at the University of Lvov. After his habilitation in 1885 Puzyna delivered lectures in mathematics as a docent.



Do
Dziekanatu wydziału filozoficznego
c.k. Uniwersytetu
we
Lwowie.

J. E. Pan Minister Wyznań i Oświecenia oznajmił reskryptem z 10. lutego 1898 l. 20556/97, że nie może uwzględnić prośby profesora matematyki w tutejszym Uniwersytecie Dra Józefa Puzyny o udzielenie mu subwencyi w kwocie 800 zł. celem pokrycia kosztów druku I tomu dzieła p.t. „Teorya funkcyj".

O tem zawiadamiam Dziekanat na sprawozdanie z 11. czerwca 1897 l. 486, wystosowane wprost do wys. c.k. Ministerstwa Wyznań i Oświecenia, celem zawiadomienia profesora Dra Józefa Puzyny.

Lwów, dnia 28. lutego 1898.
W zastępstwie:

Fig. 1. Letter from the Ministry in Vienna informing on funding of the 1st volume of Puzyna’s monograph (Lvov District Archive)

He directed the Department of Mathematics at the University of Lvov as an associate professor in the years 1889-1892 and as a professor in 1892 until the end of his life, 1919. He was a very good lecturer and taught many different branches of mathematics.

In 1898, Józef Puzyna published the first volume of his famous monograph *Teorya funkcyj analitycznych*. The book was highly valued by mathematicians from Poland and by foreign mathematicians. It was a pioneering book that completely covered the modern theory of analytic functions (see: [2–5, 7, 11]).

In connection with this it is interesting to observe that in his monograph Puzyna pays a lot of attention to the notion of surface and results related to classification of surfaces.

Strictly speaking, only branched covers of complex plane are considered in the book. In his exposition of the material Puzyna follows Riemann's original approach.

Some other information concerning Puzyna's monograph can be found, e.g., in [3, 4, 7, 11]. The present note extends the material from the author's monograph [2].

3. Riemann surfaces in Puzyna's monograph

Part V of Volume II of Puzyna's monograph is entitled *Connectivity of surface (Analysis Situs). Riemann surface*. Note that Puzyna subsequently uses the term "compactness" instead of "connectedness" in his monograph. In modern terminology, the Riemann surfaces are one-dimensional complex manifolds. The notion of a Riemann surface is used in complex analysis in order to make the multi-valued analytic functions single-valued. The topological properties of the Riemann surfaces play therefore an important role in the theory of analytic functions of the one variable.

The exposition starts with the definition of closed surface and studying topological properties of surfaces by means of their sections by connected simple curves. However, the definition of surface is necessarily not rigorous as the author avoids using charts, i.e. homeomorphisms onto domains of euclidean spaces. According to Puzyna a surface is a section of a plane or an arbitrary surface (sic!) bounded by one or few closed curves.

The simple connected surfaces are introduced by means of an intuitive definition. These are the surfaces that satisfy the following properties:

1) Every curve connecting two points of the surface can be transformed into another one so that it does not leave the surface in the process of transformation. The endpoints of the curve either stay the same or change;
2) Every connected curve contained in the surface can be shrunk to an arbitrary point, while remaining on the surface in the process of shrinking;
3) If the surface possesses the boundary, then every simple (non-self-intersecting) curve that connects two distinct points of the boundary divides the surface into two separate parts.
4) Actually, if a closed surface is considered, it is supposed that its boundary consists of a chosen point in this surface.

In modern terms, the author implicitly uses the notion of homotopy (isotopy) of continuous maps in this definition.

Then n-connected surfaces are introduced. These are the surfaces in which one can make $n - 1$ cuts such that the result of cutting is a simply connected surface.

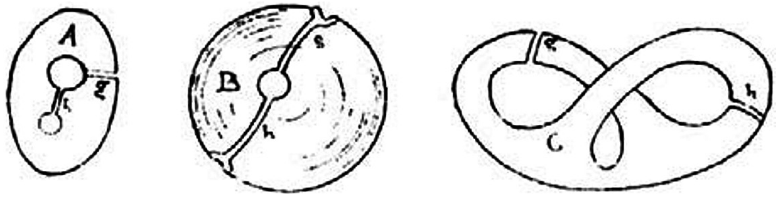


Fig. 2. Examples of 2-connected surfaces are: a planar annulus, sphere with two holes etc. The third figure is an example of a 3-connected surface

The proofs of statements on the surfaces are based on intuitive approach as well as descriptive arguments. Therefore, the level of rigor here is necessarily strictly lower that in exposition of set-theoretic or algebraic material. Note that even simply formulated and intuitively evident statements of the planar topology can have complicated proofs, and the famous Jordan theorem is a good example supporting this statement.

Next, the notion of the genus of a surface is defined.

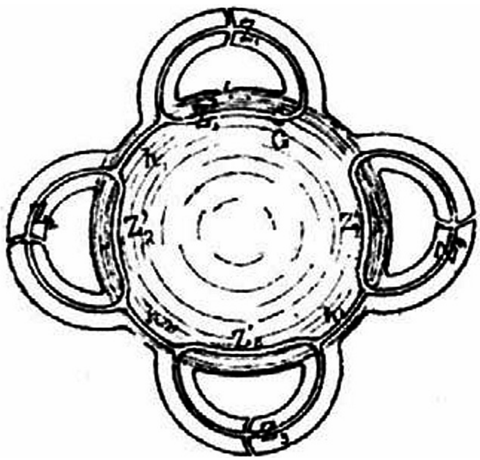


Fig. 3. A figure from Puzyna's monograph: Sphere with handles

Also, maps (transformations) of surfaces are described in the following, rather intuitive, way:

1) any infinitely close points remain so in the mapped surface;
2) any points that are finitely distant remain so in the mapped surface. Note that Puzyna does not use here language of set-theoretic topology.

The one-sided surfaces are introduced and a classification theorem (i.e. that every (oriented) surface is homeomorphic to a sphere with handles) for them is presented. The exposition of this proof is again based on an intuitive approach.

A generalization of Euler's theorem onto (triangulable) surfaces is also given (The author calls the result l'Huilier's theorem). This allows the author to consider the Euler characteristic of a surface.

The following section XIV contains a description of the construction of Riemann surfaces, first, at a neighborhood of a branching point. This construction is illustrated by the following pictures.

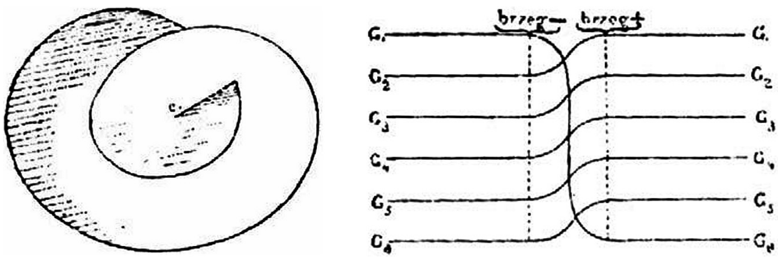


Fig. 4. Neighborhood of a branching point

It is proved that the algebraic notion of the genus of a Riemann surface can also be described in topological terms. Actually, the genus is a topological invariant of a surface.

The harmonic functions on Riemann surfaces (both open and closed) are considered in Section XX. The following Section XXI contains, in particular, a construction of a rational function on a given Riemannian surface by means of the 2nd order Abel integral.

Also, Riemannian surfaces appear in Part VIII, Section XXII devoted to maps of circular polygons in a half-plane (see the picture below).

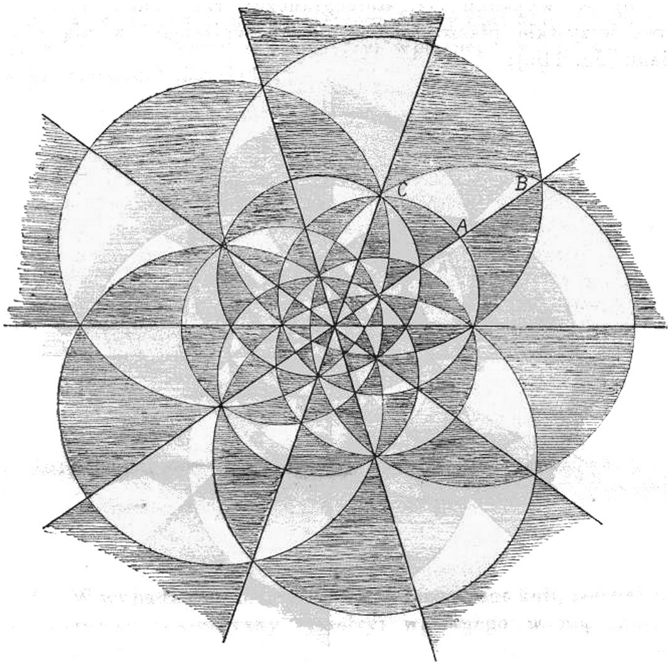


Fig. 5. Exemplary Riemannian surface

As a conclusion we emphasize the prevalence of descriptive exposition of the material concerning Riemannian surfaces despite of set-theoretic language elaborated in the initial chapters of the monograph.

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STANISŁAW DOMORADZKI*, MAŁGORZATA STAWISKA**

DISTINGUISHED GRADUATES IN MATHEMATICS OF JAGIELLONIAN UNIVERSITY IN THE INTERWAR PERIOD. PART I: 1918–1925

WYBITNI ABSOLWENCI MATEMATYKI NA UNIWERSYTECIE JAGIELLOŃSKIM W OKRESIE MIĘDZYWOJENNYM. CZĘŚĆ I: 1918–1925

A b s t r a c t

In this study, we present profiles of some distinguished graduates in mathematics of the Jagiellonian University from the years 1918–1925. We discuss their professional paths and scholarly achievements, instances of scientific collaboration, connections with other academic centers in Poland and worldwide, involvement in mathematical education and teacher training, as well as their later roles in Polish scientific and academic life. We also try to understand in what way they were shaped by their studies and how much of Kraków scientific traditions they continued. We find strong support for the claim that there was a distinct, diverse and deep mathematical stream in Kraków between the wars, rooted in classical disciplines such as differential equations and geometry, but also open to new trends in mathematics.

Keywords: *history of mathematics in Poland*, *Jagiellonian University*, *Cracow*

S t r e s z c z e n i e

W niniejszym artykule przedstawiamy sylwetki niektórych wybitnych absolwentów Uniwersytetu Jagiellońskiego w zakresie matematyki z lat 1918–1925. Omawiamy ich drogi zawodowe i osiągnięcia naukowe, przykłady współpracy naukowej, związki z innymi ośrodkami akademickimi w Polsce i na świecie, zaangażowanie w nauczanie matematyki i kształcenie nauczycieli oraz ich późniejsze role w polskim życiu akademickim. Próbujemy także zrozumieć, w jaki sposób zostali oni ukształtowani przez swoje studia i na ile kontynuowali krakowskie tradycje naukowe. Znajdujemy mocne dowody na poparcie tezy, że w Krakowie międzywojennym istniał wyraźny, zróżnicowany i głęboki nurt matematyczny, zakorzeniony w dyscyplinach klasycznych, takich jak równania różniczkowe i geometria, ale również otwarty na nowe trendy w matematyce.

Słowa kluczowe: *historia matematyki w Polsce*, *Uniwersytet Jagielloński*, *Kraków*

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1. Introduction

Mathematical traditions at Jagiellonian University in Kraków (Cracow) are several centuries old. Founded in 1364, the university got its first chair for mathematics and astronomy in 1405. It functioned during the rule of the Jagiellonian dynasty and elected kings, as well as during the 123-year period of occupation by the Austro-Hungarian empire [9, 17, 29]. World War I brought some interruptions to its activity. Kraków was a fortress, important for the war operations. In anticipation of a Russian attack, emergency evacuation was ordered and one-third of the population left the city. A few university buildings, including its clinics, were requisitioned for the military needs. The university was closed for the winter semester of the academic year 1914/15. However, the offensive was thwarted before the Russians approached Kraków and soon the classes resumed. They continued until the bloodless liberation of Kraków on October 31, 1918, when Polish soldiers disarmed an Austrian garrison stationed in the City Hall Tower and the occupying army capitulated. To commemorate the event, the words "Finis Austriae" were entered in the book of doctoral promotions at the Jagiellonian University [13].

On November 11, 1918 (the same day the armistice was signed in Compiègne), the Regency Council handed in the military command to Józef Piłsudski. On November 14, it passed all remaining authority to him and dissolved itself. On November 16, Piłsudski notified the Entente Powers about the emergence of the independent Polish state.

The beginnings of the newly proclaimed Republic of Poland were turbulent. Even though the Great War was over, the state had to fight for its borders against the competing interests of Germans as well as Czechs, Slovaks, Lithuanians and Ukrainians, who also aspired to independence, and finally, in 1919–1921, to defend itself against the Bolshevik Russia. Many students interrupted their courses to volunteer for the military service. Internally, the emerging republic also faced many difficulties. It had to forge, among other things, a common legal, administrative and educational system, in place of no longer relevant disparate systems of the three occupying powers. The existing Polish-language academic institutions – that is, the Jagiellonian University, the Lwów (Lvov; Lviv) University and the Lwów Polytechnics – played a major role in these endeavors, educating the future forces to staff Polish administration, courts and schools at all levels, including the new or revived academic schools in Polish territories formerly under Russian or German occupation[1]. Of course hardships of everyday life affected the educational process and the scientific research. At the opening of the academic year 1922/23, the vice-rector of the Jagiellonian University, Professor Stanisław Estreicher, a historian of law and a bibliographer, characterized the current conditions as follows:

> "It cannot be denied (...) that the current relations, resulting from the tremendous shock of civilization which was the Great War, do not create by any means conditions that are favorable for the development of scientific institutions. The funding of our institutes and laboratories, which was not great

[1] Since 1871, Polish was the language of instruction in the institutions of higher education in the Polish province of Galicia of the Austro-Hungarian empire.

> even before the war, in current conditions does not bear any proportion even to the most modest needs. Every year we lose more and more scientific contact with abroad, we are more and more isolated from intellectual trends, foreign books and journals reach our hands less and less frequently, with more and more difficulty. Even the most important scientific institutions lack the space or are unable to expand, and it is almost impossible to bring scientific staff from outside because of the difficulties in finding accommodations for them. The same difficulty affects the young people: our youth struggle not only with provisional difficulties and with dearness of scientific aids, but primarily with the difficulty of finding accommodations in Kraków. The number of youth does not increase at the rate at which it should, if one takes into account how big is the need of the society for intellectual forces, lacking in Poland in every way" [37] (translated by the second author).

A committee was formed to help the students. Institutions and more affluent private citizens answered the appeal and contributed cash and commodities. Despite all difficulties, young people were trying to pursue academic studies to satisfy their intellectual needs and prepare themselves for their roles in the public life of the new state. In the academic year 1921/22, the total number of students at the Jagiellonian University was 4580. Most of them, 2002, were enrolled in the faculty of philosophy (where mathematics also belonged). There were 976 women students, 763 of them in the faculty of philosophy [37].

In 1918 there were three professors of mathematics at the Jagiellonian University: Stanisław Zaremba, Kazimierz Żorawski and Jan Sleszyński. Lectures were also given by Antoni Hoborski, Alfred Rosenblatt and Włodzimierz Stożek, soon joined by Leon Chwistek, Witold Wilkosz and Franciszek Leja. The standard curriculum, established before 1918, comprised mathematical analysis (differential and integral calculus), analytic geometry, introduction to higher mathematics, differential equations, differential geometry, theory of analytic functions. Less frequently, lectures were given in number theory, power series, synthetic geometry, algebraic equations, higher algebra and projective geometry. Occasionally, there were also lectures in theory of elliptic functions, theory of transformations, theory of conformal mappings, spherical trigonometry, calculus of variations, analytic mechanics, algebraic geometry, elementary geometry, kinematics of continuous media, equations of physics, probability, theory of determinants, introduction to the methodology of mathematics, mathematical logic, integral equations. (see [29]) New subjects were added by Witold Wilkosz, who became an extraordinary professor in 1922: foundations of mathematics, set theory, theory of quadratic forms, theory of functions of a real variable, geometric topology, group theory. The course of studies could be concluded with either the teacher's examination or PhD examination. An additional subject was required at the examination (most commonly, physics was chosen).

A common view is that Kraków was lagging behind the mathematical centers of Lwów and Warsaw, neglecting developments in topology, set theory and functional analysis, in which the Polish school of mathematics achieved its most remarkable and lasting results. It has been claimed that the emphasis on classical mathematics, mostly on differential equations, caused the exodus from Kraków of younger mathematicians more interested in pursuing modern fields and eventually led to the dismantling of Kraków research group. Here are the impressions of the outstanding Russian mathematician Nicolai Lusin after his visit to Warsaw, expressed in his letter to Arnaud Denjoy, dated September 30, 1926:

> "It seems to me that the mathematical life in Poland follows two completely different ways: one of them is inclined to the classical parts of mathematics, and the other to the theory of sets (functions). These ways exclude one another in Poland, being the irreconcilable enemies, and now a fierce struggle is going on between them" (cited after [11]; translated by S. S. Demidov).
>
> "The classical side is currently represented only by the ancient (over 500-year old) Kraków University and Kraków Academy. Among Polish mathematicians the most stalwart supporter of this way is Professor Zaremba. Other supporters of this tendency stick close to Mr. Zaremba. (...) Thus currently only Kraków is a stronghold of classical mathematics. However, the Polish mathematicians whom I saw in Warsaw claim univocally that Mr. Zaremba's cause is doomed to failure, and this is why numerous colleagues and students are leaving him. Thus Mr. Zaremba's students, Dr. Kaczmarz and Dr. Nikliborc, are leaving Kraków for Lwów; Mr. Banach and Mr. Stożek already did so. A student of Mr. Zaremba, Mr. Leja, left for Warsaw. Mr. Nikodym also intends to leave Kraków and only material difficulties do not allow him to relocate to a different city. One can therefore speak about dissolution of the group of Kraków mathematicians" (cited after [17]; translated by the second author).

The truth is more complex than this. We already mentioned new subjects introduced (mainly by Wilkosz) to the students' curriculum which reflected the latest mathematical developments. These subjects also made their way into research of Kraków mathematicians, e.g. Ważewski did some significant work in topology and Gołąb in geometry of metric spaces (see the subsequent sections devoted to individual mathematicians for more detail). Let us now address the relocations of Kraków mathematicians. In the first years of the Republic of Poland, it was necessary to provide Polish-speaking staff to the revived or newly created institutions of higher education. Lwów and Kraków mathematicians took the opportunity to engage in building academic and scientific life where there was none. Kazimierz Żorawski, who worked mainly in differential geometry and in the academic year 1917/18 served as the rector of the Jagiellonian University, moved to Warsaw in 1919, to become a professor first at the Warsaw Polytechnic, then at the University of Warsaw (both reopened in 1915) and to serve as an official in the Ministry for Religious Denominations and Public Enlightenment. He remained in Warsaw until his death in 1953. Franciszek Leja, an assistant in Kraków, whose works concerned differential equations, functions of a complex variable, potential theory and approximation theory[2], obtained a chair of mathematics at the Warsaw Polytechnics in 1923. However, he returned to Kraków to become an ordinary professor

[2] Leja was also first to define explicitly a topological group. For more information on him, see e.g. the article by M. Kosek in this volume.

of mathematics in 1936. A few mathematicians who were based in Lwów before World War I – Wacław Sierpiński, Zygmunt Janiszewski and Stefan Mazurkiewicz – decided to take positions at the University of Warsaw. Their departures (as well as the death of Józef Puzyna in 1919) left the Lwów mathematics understaffed. Some of the gaps were filled by Hugo Steinhaus, Włodzimierz Stożek, Stefan Kaczmarz, Władysław Nikliborc and Stefan Banach. Without formal training in mathematics and university diploma, Banach had no chance for a standard academic employment. The invitation from Lwów to work as Antoni Łomnicki's assistant (issued at the initiative of Steinhaus) came as a unique opportunity. Steinhaus himself did not have an academic appointment for a few years following his PhD, even though he got his habilitation in Lwów in 1917. After World War I he worked as a mathematical expert for a gas company and considered himself a "private scholar" before joining the Lwów University. Stożek became a professor of mathematics at the Lwów Polytechnics, but soon ceased to do research and devoted himself to writing textbooks. Kaczmarz and Nikliborc went on to get their PhD degrees from Lwów and to build their mathematical careers, ended by their premature deaths (Kaczmarz fell in the September campaign of 1939 and Nikliborc was driven to suicide by Communist security forces in 1948). While Kaczmarz's research concerned real analysis, orthogonal series and numerical methods, Nikliborc continued to work on potential theory, differential equations and mechanics. Otto Nikodym held on to his (secure and relatively well-paid) job as a high-school teacher, but ultimately left Kraków to pursue an academic career, getting his PhD (in 1924) and habilitation (in 1927) at the University of Warsaw. One can thus say that the relocations were motivated not so much by individuals' wishes to pursue particular mathematical interests, but rather by available career opportunities[3]. While Zaremba remained faithful to classical mathematical disciplines, he did not prevent his colleagues from pursuing topics in real analysis, set theory, topology and other modern developments, or from teaching them to students. His main neglect was in not taking care to create positions for junior mathematicians. Here is an account by Andrzej Turowicz, a student in the years 1922–26:

> "When I came for my first year of studies at the university, there were no assistants. The so-called "proseminarium" [a pre-seminar], which preceded the recitations, was led by docent Leja. I was in this pre-seminar of his. He was a high school professor and had contract classes [at the university]. Only when I entered the second year, Wilkosz took care to have two deputy assistants nominated, that is Leśniak and Mrs. Wilkosz ([who was] a mathematician). Then also Turski made it. Even later, there was Krystyn Zaremba".

> "I taught high school for 10 years. At that time, there was one adiunkt and one deputy assistant at the university. That was it. There was no chance of getting

[3] There were also mathematicians who got their PhD in Kraków and were employed by other centers, e.g., Stefan Kempisty (PhD 1919), later a professor at the Stefan Batory University in Wilno, working mainly in real analysis, and Witold Pogorzelski (PhD 1919), later a professor of Warsaw Polytechnics, working mainly in differential and integral equations.

> into university [as junior faculty]. It was Zaremba's fault. He did not take care to acquire an assistant" [61].

In the initial years, however, the Jagiellonian University and the Academy of Mining (created in 1919) were able to accommodate the mathematical talents of Tadeusz Ważewski and Stanisław Gołąb. They became world-class researchers and later created their own scientific schools. There were a few other distinguished graduates of mathematics in the period 1918–1926 (before the unification of university system and introduction of the master's degree for all disciplines) who later made an impact on Polish intellectual and academic life. Below we present their profiles.

2. Profiles

2.1. Tadeusz Ważewski (1896–1972)

Born in the village of Wygnanka, in the Austro- Hungarian province of Galicia (later in Tarnopol Voivodeship in the II Republic of Poland, then in USSR, now Ukraine), he passed his *matura* in the 1st Gymnasium in Tarnów. In the years 1914–1920 he studied in the philosophical faculty of the Jagiellonian University He started studying physics, then, influenced by S. Zaremba, switched to mathematics. Military service interrupted his studies. When Poland regained independence in November 1918, he served on a citizen patrol in Kraków. In the years 1920–1921 he taught in St. Anne State Gymnasium in Kraków as a substitute teacher. In the years 1921–1923 he studied at the University of Paris, where in 1924 he obtained PhD degree on the basis of the thesis "Sur les courbes de Jordan ne renfermant aucune courbe simple fermeé de Jordan" concerning dendrites, i.e., locally closed continua which do not contain simple closed curves (later published as [65]). On the doctoral committee there were Émile Borel, Arnaud Denjoy and Paul Montel [12]. Paul Émile Appell is also signed on Ważewski's diploma, as the rector of the university of Paris at that time. Ważewski returned to Kraków, where he got his habilitation in 1927 for a thesis concerning rectifiable continua [66]. He became a deputy professor in 1926 and an extraordinary professor of the Jagiellonian University in 1933 [1, 2][4].

Ważewski was arrested along with other professors in the Sonderaktion Krakau on November 6, 1939, imprisoned in the Sachsenhausen concentration camp and released

[4] The Council of the Philosophical Faculty of Jagiellonian University, of which mathematics was part, proposed Ważewski as a candidate for extraordinary professorship already in 1929. However, in a poll among mathematicians from other centers, Ważewski's name was not mentioned. Instead, other names appeared: of Władysław Nikliborc, Otto Nikodym, Antoni Zygmund, Bronisław Knaster, Alfred Rosenblatt, Alfred Tarski, Juliusz Schauder, Witold Hurewicz and Aleksander Rajchman. The ministry concluded that Ważewski's candidacy was inferior and rejected the proposal [17].

in February 1940. He stayed in Kraków until the end of the Nazi occupation, officially lecturing at the Men School of Commerce while also engaging in clandestine teaching and conducting an illegal mathematical seminar with many participants who presented their results. In 1945 he became an ordinary professor and took active efforts to rebuild academic life in Kraków, not only at the Jagiellonian University, but also at the State Pedagogical College and Academy of Commerce as well as at the State Mathematical Institute (later Mathematical Institute of the Polish Academy of Sciences). He lectured, published papers, edited journals (e.g. *Annales Polonici Mathematici*), headed academic units (a university chair and a division of the Mathematical Institute), presided the Polish Mathematical Society (1959-61) and, perhaps most importantly, supervised PhD students, many of whom later became distinguished mathematicians. He gave short communications at the International Congresses of Mathematicians, in Oslo (1936) and Stockholm (1962), and a plenary address in Amsterdam (1954) [35].

Initially Ważewski was interested in point-set topology and obtained important results in this area. In his PhD thesis he constructed a universal dendrite, i.e., a dendrite containing a homeomorphic image of any other dendrite. This set, known as Ważewski's dendrite, still attracts interest of mathematicians, either in its own right [5] or as a tool in the study of Berkovich analytic spaces, lying at the intersection of algebraic geometry, number theory and topology [34]. In 1923 Ważewski proved a theorem on properties of the hyperspace of a locally connected metric continuum [64]. This result was obtained independently and published amost simultaneously by Leopold Vietoris [63] and later generalized by Menachem Wojdysławski [77] and other mathematicians [8]. In later years he occasionally returned to purely topological topics, e.g. in [73]. However, differential equations became his main interest, the area of his highest achievements and a springboard for new directions of research. He started working on them in 1930s and published over 25 papers concerning them before World War II. Two of these works he wrote jointly with Stanisław Krystyn Zaremba [74, 75], and in at least one he addressed questions inspired by the work of Adam Bielecki [67]. He also worked on differential inequalities. In 1947 Ważewski published first version of his fundamental result, later called Ważewski's retract principle [69] – a creative application of topology to differential equations. The result has many variants now [58]. Roughly speaking, it says that certain properties of the solutions of a differential equation (or a system of equations) on the boundary of a given domain imply that some solutions to the equation must stay in the domain. The case of systems of equations requires significant use of the topological notion of retract. Versions for dynamical systems were also developed and led to the theory of so-called Conley index [54], which is a tool for analyzing topological structure of invariant sets of smooth maps or smooth flows. In 1948 [70], investigating the domain of existence of an implicit function, Ważewski introduced and studied a linear differential equation, later called Ważewski equation. The existence of polynomial solutions to this equation was later shown [44] to be equivalent to the famous Keller Jacobian Conjecture (which is still open in the complex case). In 1960s Ważewski wrote a series of papers in the optimal control theory, in which he created a new direction in the area by using the notion of a differential inclusion, introduced by S. K. Zaremba and A. Marchaud [31, 48].

Ważewski had a great pedagogical talent. He was ranked very highly by students as an instructor. He published lecture notes [71] and in his research he sometimes took up topics of educational value. For example, he gave a unified proof of all cases of the de l'Hôspital rule in calculus ([72]; later he also published a version for Banach spaces). He also had a strong sense of mission. During the occupation he entered twice the Kraków ghetto to talk to an amateur mathematician named Rappaport (first name unknown). The meetings were arranged by Tadeusz Pankiewicz [49, 51], a Pole who was allowed by the Nazis to run a pharmacy inside the ghetto until its end. Ważewski risked his own life, as he was a former concentration camp prisoner and the entrance was illegal, but he had discussions with Rappaport, who gave an approximate solution to the angle trisection problem. Rappaport did not survive the war; his result was published by Ważewski in 1945 [68]. Another memorable instance of Ważewski's doing what he considered the right thing was the following: his former student Andrzej Turowicz became a Benedictine monk and a priest (Fr. Bernard) after the war. It became hard for Turowicz to participate actively in the scientific life, as the communist authorities kept limiting the influence of the Catholic church in the public life. But thanks to Ważewski's firm stand, he was employed in the Mathematical Institute of the Polish Academy of Sciences since 1961, had his habilitation approved in 1963 and became an extraordinary professor in 1969. (For more information about Turowicz, see [14] and the references therein).

2.2. Władysław Nikliborc (1899–1948)

Born in Wadowice, he finished high school there and passed his *matura* examination in 1916. His university studies were interrupted by military service. Released from the military in December 1920, he studied mathematics at the Jagiellonian University, graduating in 1922. He attended lectures in mathematics (by Stanisław Zaremba, Kazimierz Żorawski, Jan Sleszyński, Antoni Hoborski, Alfred Rosenblatt, Włodzimierz Stożek, Witold Wilkosz), physics (by Władysław Natanson, Czesław Białobrzeski, Konstanty Zakrzewski, Stanisław Loria) and astronomy (by Tadeusz Banachiewicz). In October 1922 he became an assistant in the Chair of Mathematics headed by Antoni Łomnicki at the Faculty of Mechanics of the Lwów Polytechnics. He also taught mathematics at a private Ursuline gymnasium for women.

In 1924 Nikliborc obtained his PhD degree at the Faculty of Philosophy of the Lwów University, passing exams in mathematics, astronomy and philosophy and presenting a thesis "On applications of the fundamental theorem of Cauchy on the existence of solutions of ordinary differential equations to boundary value problems for the equation $y'' = f(x, y, y')$". His habilitation at Lwów University took place in 1927 and was based on a two-part paper "Sur les fonctions hyperharmoniques", published in Comptes Rendus of the Paris Academy of Sciences in 1925 and 1926 ([45, 46]). In this paper Nikliborc considered the Dirichlet problem in a polydisk for a function which is the real part of a holomorphic function of two complex variables. Nowadays such functions are called pluriharmonic. The study of these functions was initiated in 1899 by Henri Poincaré and continued by Luigi Amoroso ([3]), who in 1912 also considered a Dirichlet problem (in a somewhat more general domain), but treated it as a problem in four real variables. It was Nikliborc who made subsequent advances in the theory of pluriharmonic functions. The referee of his habilitation, Hugo Steinhaus, praised not only his ingenuous methods but also his thorough knowledge of differential equations. In 1931 Nikliborc got habilitation in theoretical mechanics at the Lwów Polytechnics, on the basis of a paper about rotating fluid. He became interested in fluid mechanics and celestial mechanics and worked on these subjects. From 1937 to 1939 he was an extraordinary professor at the Warsaw Polytechnics. He wrote academic and high school textbooks (some of them jointly with Steinhaus or Włodzimierz Stożek). He spent the years of World War II in Lwów. During the Soviet occupation (1939–1941) he was a professor of mathematics in Lwów, and during the German occupation (1942–1944) he lectured at the Staatliche Technische Fachkurse, a school for vocational training created as partial replacement of the Polytechnics (which, like all Polish institutions of higher education, was closed by the Nazis). He took care of ailing Stefan Banach in his last days. In 1945 he went to Kraków, to take the Chair of Mathematics in the Faculty of Engineering at the Academy of Mining and Metalurgy and to give lectures at the Jagiellonian University, but soon he moved to Warsaw (Andrzej Turowicz was appointed to lecture in his stead, [61]). He became an ordinary professor there: first at the Polytechnics, then at the University. He committed a suicide in 1948 in Warsaw after being arrested and interrogated by the communist security forces [15]. His advanced textbook in differential equations was published posthumously in 1951 (see also [59]).

Nikliborc mastered differential equations as a student in Kraków. Rumor had it (as related later by Fr. Turowicz, [61]) that he and Stefan Kaczmarz, an earlier Jagiellonian student, moved to Lwów for their PhD in order to avoid taking further examinations with Zaremba. Because of lack of materials, we cannot confirm or deny personal factors playing a role, but both graduates were certainly well prepared for scholarly work in mathematics. While Kaczmarz's interests were closer to those of the Lwów mathematical school, Nikliborc continued working on classical analysis, differential equations and generalization of harmonic functions. However, he wrote a joint paper with Kaczmarz [36], and entered several problems in the Scottish Book, one as a "Theorem" (# 128, 129, 149 and 150 [39]). His later interests in mechanics were influenced mainly by Leon Lichtenstein, whom he visited in Leipzig in 1930–1931. Solid background in physics and astronomy acquired in Kraków helped Nikliborc in his work on equilibrium figures in hydrodynamics and on the three-body problem.

2.3. Stanisław Bilski (1893–1934(?))

Stanisław Bilski was born in Zgierz. He passed his *matura* examination in 1911 in Łódź Merchants' School of Commerce. In the years 1914–1918 he worked as a teacher in Zgierz. He started his studies in 1918 in Warsaw, then continued studying mathematics and philosophy in Kraków. In the school year 1925/26 he taught mathematics in the state gymnasium in Rybnik. In 1926 he obtained PhD in philosophy at the Jagiellonian University on the basis of required exams and the thesis “A priori knowledge in Bertrand Russell’s epistemology”, supervised by Władysław Heinrich. He translated into Polish some writings of the dialectical materialistic philosopher Joseph Dietzgen and poetic works by Schiller, Goethe and Heine.

Since 1916, Bilski was active in the Polish Socialist Party-Left. He established first communist structures in Zgierz and organized worker demonstrations. In 1929, to avoid arrest by Polish authorities, he escaped to the USSR under the name of Stefan Biernacki. He supervised the Polish section of the Central Publishing House of the Nations of USSR in Moscow, then he lectured on the history of Polish labor movement in Kiev. He was arrested at the beginning of the Great Purge in 1934, sentenced as an alleged Polish spy and executed [76].

2.4. Jan Józef Leśniak (1901–1980)

Born in Ropczyce, he attended a gymnasium in Jasło. He passed his *matura* with distinction in 1919, then he studied mathematics at the Jagiellonian University. In 1922 he became a scientific aide of the Mathematical Seminar, then he was made an assistant. In 1928 he passed the exam for high school teaching licence and took a position in H. Sienkiewicz gymnasium in Kraków. At the same time he gave contract lectures on issues of elementary mathematics at the Jagiellonian University and – since 1930 – lectures on didactics of mathematics at the Pedagogical Institute of Studies. In 1939 he was arrested by Gestapo and taken to the concentration camps in Wiśnicz and Auschwitz. Released in 1940, he returned to Kraków, where he taught mathematics in the School for Commerce and Industry. After the war he reassumed his position in a gymnasium and continued his lectures at the Jagiellonian University. In 1947 he obtained there his PhD degree (under the supervision of Franciszek Leja) on the basis of the thesis “Methods of solving equations”. In the same year he started working at the Pedagogical College (WSP) in Kraków. In 1951 he submitted a habilitation thesis “Educational values of instruction in mathematics and their fulfilment in high school” and became an extraodinary professor. In 1963 he was promoted to the level of a professor of mathematics at the Faculty of Mathematics, Physics and Chemistry at the Pedagogical College. In 1961, in the paper [43], he proposed a new formal approach to indefinite integrals. His output counts over 20 items, including several books on elementary mathematics, theoretical arithmetics and functions of one variable.

2.5. Stanisław Gołąb (1902–1980)

Stanisław Gołąb was born on July 26, 1902, in Travnik (Bosnia), where his father, Walenty, was a judge; his mother, Jadwiga neé Skibińska, was a teacher. In 1910 the family moved to Kraków. Gołąb attended the V Gymnasium, where in 1920 he passed his maturity examination. One of his gymnasium teachers was Antoni Hoborski, later a professor of mathematics in the Academy of Mining, who also conducted lectures at the Jagiellonian University. In the years

1920–1924 Gołąb studied mathematics at the Jagiellonian University. In 1923 he completed the Pedagogical Study at the Jagiellonian University and in 1926 he passed examinations in mathematics and physics for candidates for high school teachers.

In 1923 he started working as a deputy assistant to Hoborski, and after his graduation in 1924 was promoted to an assistant in the Chair of Mathematics at the Academy of Mining. Influenced by Hoborski (who wished to establish a scientific school in geometry in Kraków), he chose geometry as his area of interest and started research in it. In the years 1925–1932 he published 14 papers.

In 1928, Gołąb, with Hoborski's support, obtained a scholarship from the Division of Science of the Ministry of Religious Denominations and Public Education for studying abroad. He went to Delft (the Netherlands) to work with J. A. Schouten. They wrote a paper together [55]; Gołąb also started working on his PhD thesis there. In 1929–1930 he got another scholarship, from the Fund for National Culture. He spent 3 months again in Delft, finishing his PhD thesis and writing the second part of his paper with Schouten [56]. Then he went to Rome, where he learned absolute calculus from E. Bompiani and relativistic mechanics from T. Levi-Civita. The last three weeks he spent in Prague, in private discussions with L. Berwald and V. Hlavatý. His trip resulted in new publications [32]. In 1932, Gołąb took part in the International Congress of Mathematicians in Zürich, giving two short talks there [22, 23, 35].

Since 1930, Gołąb conducted contract lectures at the Jagiellonian University. In 1931 he obtained the PhD degree at the Jagiellonian University on the basis of the thesis *O uogólnionej geometrii rzutowej*, which was presented in Polish (and published as [19]) and was related to his joint work with Schouten. His supervisor was Stanisław Zaremba; the referee was Witold Wilkosz. In 1932 he obtained habilitation at the Jagiellonian University on the basis of the paper *Zagadnienia metryczne geometrii Minkowskiego*, published in 1932 in the Proceedings of the Academy of Mining [21]. The topic of his habilitation lecture was *Metryka kątowa w ogólnych przestrzeniach*, chosen by the committee out of three topics proposed by the candidate (which he developed in his congress talks and several other publications, including a joint paper with Adam Bielecki, [6]). The examination questions concerned, among other things, Minkowski geometry, convex functions (Wilkosz), differential equations (Zaremba) and general metric spaces (Hoborski), and Gołąb's answers were evaluated as very good. The ministry extended Gołąb's licence to lecture to the Faculty of Mining at the Academy of Mining, and he received the title of professor in 1939.

On November 6, 1939, Gołąb was imprisoned along with other Kraków professors and taken to the concentration camps, first in Sachsenhausen, then in Dachau. He tried to take care of his teacher Hoborski, but ultimately witnessed his death. Released[5] in December 1940

[5] According to [57], Wilhelm Blaschke claimed to intervene personally on Gołąb's behalf. In private communication with the second author, Zofia Gołąb -Meyer, Staniaław Gołąb's daughter, could not confirm whether this was the case. On the basis of documents in her possession, she brought to our

together with a group of junior scholars, he worked in forestry administration, taking part in clandestine teaching since 1943. After the liberation he continued working at the Academy of Mining, becoming an ordinary professor there in 1948. When the State Mathematical Institute (later the Mathematical Institute of the Polish Academy of Sciences) was created, he became the head of the Division of Geometry there (he headed the division until his retirement). In the years 1950–55 he was a contract professor at the Pedagogical College in Kraków, and in 1954 he transferred from the Academy of Mining and Metalurgy to the Jagiellonian University (however, until 1962 he remained as a half-time employee at the Academy). He was the head of the Chair of Geometry and the dean of the Faculty of Mathematics, Physics and Chemistry of the Jagiellonian University. Because of his worthy behavior as the dean (supporting the rights of protesting students and faculty) during the March Events of 1968 he was demoted from the position of the head of the Laboratory of Geometry by the communist authorities. He retired in 1972 and died in Kraków on April 30, 1980 (see [16, 18, 41, 50]).

Gołąb's scientific output comprises over 250 publications, including two monographs: *Tensor Calculus* ([28]) and *Functional Equations in the Theory of Geometric Objects* (joint with J. Aczél; [30]). His main research interest was geometry, in particular clasical differential geometry, tensor calculus (under the influence of Schouten), spaces with linear or projective connection, Riemann, Minkowski and Finsler spaces, general metric spaces and other areas. He supervised 28 PhD degrees during his career. While working on the theory of geometric objects, in 1939 Gołąb [27] gave the first exact definition of a pseudogroup of transformations [33, 41]. He also promoted the methods of functional equations in geometry, starting with the so-called translation equation, for which he found a solution of class C^1 [25] But his interest in functional equations dates earlier. In 1937 [24] he published a note in which he gave conditions characterizing the transformation $z \to \overline{z}$ of complex numbers as the only solution to the multiplicative equation $f(uv) = f(u)f(v)$ (in the same journal issue, different conditions were given by A. Turowicz, [62]). He went on to obtain other results in functional equations and supervise PhD candidates in this field, and is considered as the founding father of the Polish school of functional equations ([42)]. In metric geometry, he proved e.g. that, under certain mild assumptions, approximation of a point x of a Minkowski space by points in a given closed subset C is unique if the distance function to C is differentiable at x ([26]; later on, many variants and refinements of this result were obtained by various authors, cf. [78]) Another memorable result by Gołąb in this field concerns the unit disk in the plane with a norm. It says that the perimeter of the unit disk can take any value between 6 and 8, and the extremal values are taken if and only if the disk is a regular hexagon or a parallelogram [4, 21]. Gołąb also worked on problems in applied mathematics, mainly in mining and geodesy [7], in close collaboration with engineers. He was also interested in mathematics education and history of mathematics. He edited the monograph "Studies in the history of the chairs in mathematics, physics and chemistry of the Jagiellonian University", published in celebration of the University's 600th anniversary [29]. He lectured

attention the intervention of Hasso Härlen, one of Gołąb's coauthors. This intervention is described in detail in [52].

with lucidity and liked working with students. In turn, they were fond of him as an educator and treated him as an authority.

2.6. Zofia Krygowska (1904-1988)

Anna Zofia Krygowska (neé Czarkowska) was born in Lwów. She grew up in Zakopane, where she finished gymnasium. In the years 1923–1927 she studied mathematics at the Jagiellonian University. Then she passed the teacher's licence examination and taught in elementary and high schools. In 1931 she obtained the master's degree from the Jagiellonian University. She became interested in issues concerning the process of teaching mathematics, in particular curriculum development and bringing new trends in mathematics into school-level education. She engaged in the activities of the Methodological Center for Mathematics, participated in many conferences and discussions and published articles on the subject of teaching mathematics [10]. In 1937 she published a textbook *Mathematics for the 1st grade of high school*.

During the German occupation Krygowska was involved in clandestine teaching. Officially she worked as an accountant in a lumber company, while also travelling on behalf of the underground educational authorities to organize, teach and coordinate illegal classes (especially in the Podhale region), risking her life. In 1945 she resumed her work as a high school teacher. In the years 1948–1951 she headed the Methodological Center. In 1950 she obtained the PhD degree at the Jagiellonian University on the basis of the thesis "On the limits of rigor in the teaching of elementary geometry" prepared under the direction of Tadeusz Ważewski. In this thesis she developed and studied an original system of axioms of geometry and proved its equivalence to Hilbert's system [60]. She also took a full-time position at the Pedagogical College in Kraków. She began active efforts to modernize the education of teachers and to make didactics of mathematics a research discipline in its own right, while continuing her involvement in improving teaching of mathematics – especially geometry – in schools. In the years 67–71 she wrote or co-wrote five textbooks in geometry, which were subsequently adopted for high school use in Poland. She also published many articles and books on the issues of teaching of mathematics and its methodology.

In 1958 Krygowska became a professor in the newly created Unit of Methods of Teaching of Mathematics in the Pedagogical College in Kraków. She initiated post-graduate doctoral programs in didactics of mathematics. She supervised 22 PhD degrees and 4 habilitations in this field. She was a member of many national and international committees related to teaching mathematics, most importantly of the Commission for the Study and Improvement of Mathematics Teaching (CIEAEM), founded in 1950. She served as a president, then as a honorary president, of this organization. She gave an address at the International Congress of Mathematicians in Nice in 1970. She was also instrumental in creating a series of lectures for teachers which were broadcast by Polish state television in the years 1968–1977. She supported launching a journal devoted to didactics of mathematics by the Polish

Mathematical Society and in 1982 the publication of “Dydaktyka Matematyki” started. She was an editor of this journal until her death [47].

Krygowska became acquainted with modern developments in set theory, topology, mathematical logic and other subjects as a student in Kraków. She gave credit to Witold Wilkosz, who taught these subjects, for initiating her interest in them. The new mathematical notions as well as emphasis on rigor made their way to her school textbooks and were consistent with the trends (started in 1930s) of reformulating mathematics in a modern way, advocated by the Bourbaki group. Krygowska also acknowledged the role of psychology in teaching. She was familiar with the work of Jean Piaget (whom she met at an international conference in public education in Geneva in 1956) and followed some of his ideas in her paper “Methodological and psychological basis for the activity-based method of teaching mathematics” [38].

Since her youth, Krygowska had a passion for mountaineering. She made a few first routes in the Tatra mountains, either in all-women teams or with her husband Władysław Krygowski, a lawyer [53].

We thank Dr. Zofia Pawlikowska-Brożek, Dr. Zofia Gołąb-Meyer, Professor Roman Duda and Dr. Emelie Kenney for discussions on the topics related to Kraków mathematics between the wars, as well as for their support and encouragement. The photos come from the archives of Z. Pawlikowska--Brożek and S. Domoradzki. They were obtained from private individuals or UJ Archives. This work was partially supported by the Centre for Innovation and Transfer of Natural Sciences and Engineering Knowledge, University of Rzeszów.

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STANISŁAW DOMORADZKI*, MAŁGORZATA STAWISKA**

DISTINGUISHED GRADUATES IN MATHEMATICS OF JAGIELLONIAN UNIVERSITY IN THE INTERWAR PERIOD. PART II: 1926–1939

WYBITNI ABSOLWENCI MATEMATYKI NA UNIWERSYTECIE JAGIELLOŃSKIM W OKRESIE MIĘDZYWOJENNYM. CZĘŚĆ II: 1926–1939

A b s t r a c t

In this study, we continue presenting profiles of some distinguished graduates in mathematics of the Jagiellonian University. We consider the years 1926–1939, after the ministerial reform which allowed the students to graduate with a master's degree. We also give a list of master's theses in mathematics.

Keywords: *history of mathematics in Poland*, *Jagiellonian University, Cracow*

S t r e s z c z e n i e

W niniejszym artykule przedstawiamy sylwetki niektórych wybitnych absolwentów Uniwersytetu Jagiellońskiego w zakresie matematyki. Rozważamy lata 1926–1939, po reformie ministerialnej umożliwiającej studentom ukończenie studiów w stopniu magistra. Podajemy także listę prac magisterskich z matematyki.

Słowa kluczowe: *historia matematyki w Polsce*, *Uniwersytet Jagielloński, Kraków*

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1. Introduction

After regaining independence in 1918, the Second Republic of Poland had to build a unified modern state out of territories formerly under occupation of the three superpowers of Russia, Germany and Austro-Hungary. This required, among other things, creating a common educational system. Few academic schools existed continuously on partitoned territories; Jagiellonian University was one of them. Some schools established earlier were closed and then reviwed during World War I or after its end (University of Warsaw; Stefan Batory University in Vilnius). New institutions of higher educations were established, e.g. the Academy of Mining in Kraków (1919). The first legal bill concerning the higher education was issued in 1920. More detailed regulations followed. On March 12, 1926, a decree of the Minister of Religious Denominations and Public Education was issued, which concerned the curriculum of studies and examinations in the field of mathematics for the master's degree. The introduction of this degree was an innovation, at first optional for the students, but soon it became an educational standard. The ministerial decree stipulated that, during their course of studies, the students had to pass several exams (differential and integral calculus with introduction to analysis; analytic geometry; principles of higher algebra with elements of number theory; theoretical mechanics; experimental physics; main principles of philosophical sciences; a block of two exams in pure or applied mathematics to be determined by the Faculty Council; and additionally one of a few subjects designated as "auxiliary"). The final exam concerned general mathematical knowledge and was accompanied by the discussion of the master's thesis [26]; for an in-depth discussion of the formation of the higher education system in Poland between the wars see [1].

Andrzej Turowicz was the first at the Jagiellonian University to get the master's degree in mathematics:

> "I was the first in Kraków to get the master's diploma. When I enrolled [at the university], one could pursue the old course, pre-master. I decided to do the master's degree. The second master [in mathematics] in Kraków was [Stanisław] Turski. He was two years younger than I. [Zofia] Czarkowska (currently Mrs. Krygowska) was at the university along with me. I had a gifted classmate, Stefan Rosental, who however later became a physicist and finished his career as a vice-director of the Bohr Institute in Copenhagen" [68]

In the years 1926–1939 lectures were given by S. Zaremba, A. Hoborski, A. Rosenblatt, W. Wilkosz, T. Ważewski, S. Gołąb, O. Nikodym, L. Chwistek, J. Spława-Neyman, A. Rożański, F. Leja, J. Leśniak and S. K. Zaremba [30]. At the initiative of Wilkosz, two assistants (Jan Leśniak and Irena Wilkoszowa, cf. [23], were employed and more lectures were enhanced with the recitation classes. An important part of the course of studies was teachers' training. Many students chose the teaching career. Because of the shortage of academic jobs in Poland, a country with nearly 35 million of population and about

40 fully accredited academic schools in 1938[1], even those who had talent and inclination for research started out as high school teachers, sometimes continuing for many years (e.g. A. Turowicz). A course in elementary mathematics from the higher standpoint was offered to address the needs of future teachers. It was taught by Jan Leśniak.

As recalled by Kazimierz Kuratowski [48], at the First Congress of Polish Science in 1951 an assessment was issued of the achievements of Polish mathematics in the inter-war period. It stated that the greatest achievements were in functional analysis and topology; important contributions were made in real analysis, set theory and mathematical logic. Among other branches cited at the Congress were:

> "Differential equations (in particular the results concerned with harmonic functions, the existence of integrals of partial differential equations of the second order, the qualitative theory of ordinary differential equations and the properties of integrals of partial differential equations of the first order).
>
> Geometry, together with transformation theory (in particular, the results concerning the in- variants of surface bending, algebraic geometry, Finsler and Riemann spaces and the topology of geometric objects).
>
> The theory of analytic functions (in particular the results concerning the approximation of functions by polynomials, the convergence of series of polynomials in many variables, univalent and multivalent functions)".

These disciplines were precisely the strong points of mathematics at the Jagiellonian University, and moreover they were hardly represented anywhere else in Poland (although the report does not name any particular mathematical center in this context). Obviously these topics dominated not only the faculty's research, but also the students' master theses, although some topics in e.g. topology, measure theory or even functional analysis were represented, too (see the Appendix). In the period 1928–1939, over 135 people graduated from the Jagiellonian University with the master's degree in mathematics (see the Appendix for partial information; at the time of writing this article, we were not able to verify the data in full). Below, we present the profiles of those who made their mark on Polish scientific and academic life.

2. Profiles

2.1. Stanisław Krystyn Zaremba (1903–1990).

Born in Kraków, a son of mathematician Stanisław Zaremba (a professor of Jagiellonian University) and a Provençal woman Henrietta Leontyna neé Cauvin. After finishing high school with science-oriented curriculum in 1921, he started studying mathematics at Jagiellonian University. Following in his father's footsteps, he continued his studies at

[1] According to [75], there were in total about 800 professorial positions and 2700 junior faculty positions in 1939; in mathematics, according to [48], there were respectively 23 professorial chairs and 27 junior positions.

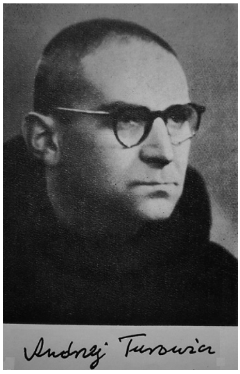
the Sorbonne in Paris in the years 1924–1927. Because of health problems he returned to Kraków, where he got master's degree in mathematics from Jagiellonian University in 1929. He edited lectures of Professor Jan Sleszyński, which were later published as a two-volume *Proof Theory* (in 1923 and 1929). Since 1929 he was an assistant at the Stefan Batory University in Wilno (Vilnius). There he got PhD degree on the basis of the thesis [81] "Sur l'allure des intégrales d'une équation différentielle ordinaire du premier ordre dans le voisinage de l'intégrale singulière" (supervised by Juliusz Rudnicki). He also mentored a distinguished student Duwid Wajnsztejn, who went on to obtain PhD in Kraków. In 1936 Zaremba got his habilitation at Jagiellonian University on the basis of the thesis "On paratingent equations" [79, 80], in which he introduced paratingent equations, a generalization of differential equations nowadays known as differential inclusions. About the same time similar relations were independently studied by André Marchaud [32]. This generalization later allowed Tadeusz Ważewski and others to build natural foundations of optimal control theory. Since 1937 he was back in Kraków, first as an *adiunkt*, later as a *docent* at Jagiellonian University.

When World War II broke out, Zaremba returned to Vilnius, then under Lithuanian control. After Lithuania was annexed by the Soviet Union in 1940, he went to Stalinabad (now Dushanbe, in Tajikistan), where he worked as a professor of mathematics in the Pedagogical Institute. Along with the Polish Army (formed from Polish nationals in the USSR under the command of General Władysław Anders) he went first to Persia, then to Palestine, where he taught in high schools for the army. In 1946 he worked at the University of Beirut. Fearing persecution or even death from the new communist Polish authorities, he decided to stay in the West. Since 1946 he lived in Great Britain. Until 1952 he was a professor at the Polish University College in London. Then he became a mathematical consultant for Boulton Paul Aircraft Ltd. in Wolverhampton. This position – as he acknowledged himself – gave him an opportunity to familiarize himself with the theory of stochastic processes and start research on this subject. In 1950s he collaborated with Zbigniew Łomnicki, a graduate in mathematics and physics of the Lwów University and a fellow emigré, on the theory of time series. They published 8 joint papers [22]. In 1954 he took part in the International Congress of Mathematicians in Amsterdam, where he gave a 15-minute talk "Spacing problems in Abelian groups" on pioneering application of group theory to communication theory [35]. In the years 1958–1969 (with a yearlong break in 1966/67) he lectured at the University of Wales. He spent the years 1969–1976 in North America (Madison, WI, and Montréal, Québec). In 1976 he returned to Wales. Before the martial law in 1981 he frequently visited Poland, in particular Warsaw and Kraków. He died in Aberystwyth (see also [25, 56, 83]).

In the years 1925–1937 Zaremba was an active mountaineer. He made many first routes and winter ascents as a pioneer of snow mountaineering in Tatra mountains. He was on boards of mountaineering societies. He published literary accounts of his expeditions in the journals "Wierchy", "Krzesanica" and "Taternik" (of which he was an editor in 1929–1930). While in emigration, he climbed e.g. in the Hindu Kush range. At his wish, his ashes were scattered over Tatra and the mountains of Wales [61].

In recent years there has been substantial interest in so-called Zaremba's conjecture [45]. Stated in [82], it was motivated by his search for lattice points suitable for quasi-Monte Carlo methods in numerical integration and postulates the following: there is an universal integer constant $K > 0$ such that for every integer $d > 0$ there exists an integer b co- prime with d, $1 \leq b < d$, such that all partial quotients of the continued fraction expansion $b/d = [0; a_1, a_2, \ldots, a_k]$ satisfy $a_i \leq K$. Zaremba also conjectured that $K = 5$. In the paper [14] the problem was reinterpreted in terms of properties of the orbit of the vector (0, 1) under some semigroup of matrices and the conjecture was proved for almost all d (in the sense of density) with $K = 50$.

2.2. Andrzej Turowicz (Fr. Bernard OSB; 1904–1989)



Born in Przeworsk, in the family of August, a judge, and Klotylda neé Turnau. Initially homeschooled, he finished King Jan Sobieski Gymnasium in Kraków. After his 'matura' exam he studied mathematics at Jagiellonian University, in the years 1922–1928. He was the first graduate to obtain the degree of master of philosophy in the area of mathematics. In 1931 he took a high school teacher qualifying exam and started teaching in schools in Kraków and Mielec. While in Kraków, he combined teaching school with academic activities. In the years 1929–1930 he was an assistant in the Chair of Mathematics of the Academy of Mining in Kraków, substituting for Stanisław Gołąb, who went to Netherlands on a scholarship. In 1937 he got a position of the senior assistant in the Chair of mathematics of Lwów Polytechnics, where he worked until 1939. Here is how he recalled the circumstances [68], cassette 1b) of his appointment:

> "After the Jędrzejewicz Brothers reform, [Antoni] Łomnicki submitted a geometry textbook for high schools. The Ministry gave me this text for refereeing, [as] I taught high school in Kraków. I did a very detailed report; I went over all problems, I wrote which ones were too difficult. The authors were anonymous, [but] I guessed it was Łomnicki. I remembered his geometry and trigonometry [textbooks] from Austrian times. However, the referees' names were made known to authors. When he read my report, [Łomnicki] sent me an offer to become an adiunkt. He wanted to bring me where he was for this report".

After the Soviet-style reorganization of the Polytechnics, with Ukrainian as the language of instruction, Turowicz taught mathematics at the Faculty of Architecture, then at the Faculty of Mechanics. In 1941 he returned to Kraków, where he worked as a clerk in the Chamber of Industry and Commerce until 1945. At the same time, he taught analytic geometry at the underground university and took part in clandestine sessions of the Polish Mathematical Society.

The day after the entrance of the Red Army to Kraków, January 18, 1945, Turowicz crossed the frozen Vistula river to the Tyniec Abbey. Ten days later he entered the Benedictine order, taking the name of Bernard. In the years 1946–1950 he studied theology. He was ordained a priest in 1949. In 1946 Turowicz obtained at the Jagiellonian University the PhD degree on the basis of the thesis "On continuous and multiplicative functionals" (published as [69]. Tadeusz Ważewski, the supervisor, noted in his report that Turowicz's thesis was an evidence "of deep mathematical culture and revealed rare philosophical sense of its author in treating a problem". The thesis answered a question posed by Stefan Banach and Meier Eidelheit. Here is how Turowicz related the story about his work on the problem:

> "I had an incident with Banach like this: for a meeting of the mathematical society, I proposed a talk on multiplicative and continuous functionals (my proposal was in spring 1939; I finished the work after the war). I am delivering the talk and Banach enters the room, slightly late, with an incredibly sullen face. I noticed that Banach was angry. He listened with extreme attention and his face changed. When I finished, Banach took the floor and said: "I also dealt with this problem; you did it in a totally different way, and you did it well." I received his opinion with gladness. The next day after this meeting, Stożek (who was not at the meeting) asked: "Was Banach there?" [I said] "Yes, he was." [He said] "I did not want to scare you in advance; he was very angry when he found out what you were to talk about. He said: 'I am dealing with this; I must have told someone, and [now] Turowicz is presenting it as his own.'" Banach came with the intention of giving me a hard time. Luckily the idea of the proof was completely different [from his], therefore [he] praised me [and] did not make a scene. (...) Since then, Banach was very friendly towards me" ([68], cassette 2b).

In the years 1946–1952 and 1956–1961 Turowicz gave lectures in mathematics at the Jagiellonian University, initially as the replacement for Władysław Nikliborc, who moved to Warsaw before the end of the term. In 1949 he lectured on algebra to the first-year students. Czesław Olech, who was taking the class, had the following memory of Turowicz [54]: "He commuted for our lectures from the monastery in Tyniec near Kraków, where he resided. He almost dashed into the lecture room, made the sign of the cross and filled the blackboard with legible text. If someone could take exact notes of it – and there were women in the class who could – then we had a 'textbook' for the exam". The lecture notes from the years 1946–1948 were indeed published (internally) as "The theory of determinants and matrices with applications to the theory of linear equations and the forms of 1st and 2nd degree" (1949), presenting some contents for the first time in the Polish mathematical literature.

At the beginning the new communist authorities did not object to Turowicz's appointment, as they took efforts to rebuild academic life in Poland facing the shortage of qualified scholars. (Later, Turowicz gave credit to his classmate Stanisław Turski, who worked for the Ministry of Education, for signing an appropriate permission). However, in the years

1952–1956 – the Stalinist period in Poland – there was no place for a priest at a state institution, so he taught mathematics for philosophers at the Catholic University of Lublin. In 1954 the institution applied for granting him the title of *docent* (i.e. an independent scientific worker) on the basis of his scholarly output. It was a legitimate procedure at that time. At the request of the dean of the Faculty of Philosophy evaluations were written by Hugo Steinhaus and Tadeusz Ważewski. Steinhaus wrote, among other things, that, when Turowicz was in Lwów, "(...) I, along with other Lwów mathematicians, had an impression that we dealt with a young mathematician who would develop his creative abilities in the right conditions." Mentioning Turowicz's publications, he wrote that "(...) all these works are an evidence of mathematical cultivation and creative capabilities of the author. (...) I add that all colleagues whose opinion I asked said without reservation that Dr. Andrzej Turowicz fully deserves the title of docent. I must also repeat here a general opinion about great personal qualities of Dr. Turowicz, who enjoys universal respect in the circles of his acquaintances and colleagues". Despite very good evaluations, the application took a long time and was ultimately denied in 1957. At Ważewski's insistence, in 1961 Turowicz obtained a position in the Mathematical Institute of the Polish Academy of Sciences. It was a research position; the employees of the Institute did not have direct contact with students. In 1963 he got his habilitation on the basis of a series of papers about "orientor fields" (i.e., differential inclusions) and their applications to control systems. He became an extraordinary professor in 1969. In the years 1970–1973 he taught in the doctoral study program organized by the Faculty of Electrotechnology, Automated Control and Electronics of the Academy of Mining and Metalurgy. The lecture notes for some classes he gave there were published in a book form under the title "Matrix Theory" [71]. At the Tyniec Abbey, for a few years he taught history of monasticism to candidates for the holy orders. He retired from academic positions in 1974.

Turowicz's scholarly output is very diverse and spans functional analysis, differential equations, control theory, probability, linear algebra, logic, game theory, convex geometry, algebra, functions of one complex variable and numerical analysis. In Lwów he wrote one joint paper with Stefan Kaczmarz [38]. He also collaborated with Stanisław Mazur, but their joint results were never published, even though Mazur found the manuscript after World War II. The work concerned a generalization of Weierstrass' theorem on approximation of continuous functions by polynomials, akin to what is now known as the Stone-Weierstrass theorem (Stone proved his versions in 1937 and 1948). The reasons for not submitting the paper for publication were twofold, and clear to those who knew Mazur, including Turowicz himself [19, 72]. First, Mazur was always striving for the best possible version of his results, and delayed submissions in hope of improving them. Second, he was a committed communist, so he distanced himself from a former colleague who became a priest. In the written evaluation of Turowicz's output in 1950s Ważewski mentioned his joint results with Mazur and expressed regret that they were unpublished. Turowicz successfully collaborated with other scholars, e.g. with H. Górecki on applications of mathematics to automated control theory. They published several joint papers and a monograph "Optimal Control" [31]. Another monograph by Turowicz, "Geometry of Zeros of Polynomials" [70] published in 1967, concerns polynomials in one complex variable and discusses the number of real zeros, localization of zeros of polynomials and their derivatives and other related topics. It was written with engineering applications in mind, but it contains many classic mathematical results.

Turowicz was active in the Polish Mathematical Society, starting from 1927. In the years 1973–1975 he was the president of the Kraków branch of the Society. Since 1978 he collaborated with the Committee for History of Mathematics by the General Management of the Polish Mathematical Society. He had an incredible memory and a gift of storytelling, without taking himself or the surrounding world too seriously. He was willing to meet with students and give interviews. He was also regarded as a moral authority (see also [25, 55]).

2.3. Stanisław Turski (1906–1986)

Born in Sosnowiec, he finished high school there. In the years 1924–28 he studied physics and astronomy at the Jagiellonian University. During his studies he worked as a schoolteacher. His diploma thesis "A new method of determining precession coefficients" was awarded a prize by the minister of education. Since 1927 he worked as a mathematician, starting at the level of an assistant, at the Jagiellonian University and at the Academy of Mining and Metalurgy. He also gave lectures in mathematics in Kraków Pedagogium. He obtained his PhD under the supervision of Witold Wilkosz, presenting a thesis "On a generalization of theorems on uniformity of integrals of a hyperbolic equation". Arrested in 1939 in Sonderaktion Krakau, he was imprisoned in the concentration camps of Sachsenhausen and Dachau. After his release in 1941 he took part in clandestine teaching at an academic level.

After the World War II, Turski engaged in rebuilding academic life in Poland, as a supporter of the communist party and its program. Nominated by Minister Stanisław Skrzeszewski (a recipient of PhD in logic from Jagiellonian University), he led a group dispatched by the Ministry of Education to reactivate the Gdańsk University of Technology in 1945 as a Polish-staffed institution replacing a German academic-level polytechnic school. He was an extraordinary professor of mathematics and a rector (president) in the years 1946–1949 [76]. He also became a parliament member in 1947. In 1949 he was called to work in Warsaw, at the University of Warsaw and in the Ministry of Education. He became an ordinary professor of mathematics in 1951 and got habilitation in 1953. In 1954 he took part in the International Congress of Mathematicians in Amsterdam as a delegate of the Polish Academy of Sciences. In the years 1952–1969 he served as the rector of the University of Warsaw. During his term, in March 1968, student protests against communist authorities erupted. As a result, 34 students were expelled and 11 were suspended from the university, and the professors were officially prohibited from participating in students' rallies. Other repressions followed.

Turski's work before World War II concerned partial differential equations, number theory and functions of a complex variable. Out of his publications of that period, three were joint with Alfred Rosenblatt, a 'docent' in Kraków [64–66]. After the war he published a few papers applying mathematical methods in mechanics of solids. His paper with Jerzy Nowiński [53] contained a numerical solution to a system of differential equations occurring in elasticity theory obtained with the use of ARR (Differential Equations Analyzer) – the first analog computer constructed in Poland in 1953. In 1963 Turski arranged for an exhibition, followed by purchase in 1964, of an ALGOL-running computer from Denmark and for staff-training courses, which lead to creation of the Unit of Numerical Computations at the University of Warsaw. To reflect the emergence of a new direction in research and education, his Chair of General Mathematics was renamed the Chair of Numerical Methods. These two institutions were later combined to give rise to the Institute of Computer Science [51]. Turski retired in 1976.

2.4. Antoni Nykliński (1906–1964)

Born in Kraków, he studied mathematics at the Jagiellonian University in the years 1925–1930, obtaining the master's degree in 1932. Afterwards he worked in high schools and collaborated with the Chair of Mathematics of the Academy of Mining on research in differential geometry. He translated into Polish a popular book on mathematics by Egmont Colerus [20]; the preface to the Polish edition was written by Stefan Banach. In 1939 Nykliński was arrested, imprisoned at the Montelupi Street in Kraków, then in Wiśnicz Nowy, and finally taken to the concentration camp in Auschwitz. After the World War II he resumed his work in high and academic schools. He conducted lectures in mathematics at the Preparatory Study of the Jagiellonian University since 1951. In 1956 he was nominated to the post of adiunkt in the Chair of Mathematics at the Faculty of Electrification of Mining and Metallurgy of the Academy of Mining and Metallurgy. In 1962 he got his PhD degree at the Faculty of Finance and Statistics of the Main School of Planning and Statistics in Warsaw. His research interests and educational activities focused on linear programming and probability. He died in Kraków [84].

2.5. Czesław Kluczny (1908–1979)

Born in Strzemieszyce Wielkie, he finished a gymnasium in Olkusz in 1927. In the years 1927–1932 he studied mathematics at the Jagiellonian University, obtaining a master's degree. In 1932 he started working as a high school teacher in Radom. In 1942 he was arrested by Gestapo and sent to concentration camps – first Auschwitz-Birkenau, then Mauthausen. He returned to Poland in 1945 in very poor health, recovering for a year. In 1946–1950 he was employed by the Silesian Technical Scientific Institutions in Katowice. In 1950 he started working for the Gliwice Polytechnic, where he remained until his retirement in 1976. He also lectured at Silesian University in Katowice and at the Pedagogical College in Częstochowa. In 1959 he obtained PhD degree in mathematics at the Jagiellonian University under the supervision of T. Ważewski. In 1961 he got habilitation at the Maria Curie-Skłodowska University in Lublin. He became an extraordinary professor in 1971.

Kluczny worked in qualitative theory of ordinary differential equations [40–42]. He supervised 6 PhD degrees (3 of which were interrupted because of his death) and

1 habilitation. He was a co-founder of the Gliwice (later Upper Silesian) branch of the Polish Mathematical Society.

2.6. Władysław Benedykt Hetper (1909–1941?)

Born in Kraków, he finished the King Jan Sobieski 3rd Gymnasium there. Then he studied mathematics at Jagiellonian University in 1927–1932, obtaining master's degree. His master thesis concerned integral equations, but soon he became interested in logic under the influence of Leon Chwistek. Along with Jan Herzberg and Jan Skarżeński, he collaborated with Chwistek on his program of establishing consistent foundations of mathematics [16–18][2]. When Chwistek took the Chair of Logic of the Jan Kazimierz University in 1933, his students followed him there. Hetper was the most active of them, publishing several papers, in which he paid attention to the latest developments in logic.

E.g. in [33], he proposed structural rules (in the style of sequent calculus of Gerhard Gentzen, introduced in 1934) for a propositional calculus written in so-called Polish notation, introduced by Jan Łukasiewicz in 1924. He proved consistency and completeness of his system, comparing his methods to those of Hilbert. In the introduction he credited Witold Wilkosz, who apparently worked on similar problems[3]. While working on his PhD (supported by a government scholarship), Hetper shared a room with Mark Kac, with whom he became friends. They had intellectual discussions, played chess and went cross-country skiing together. A devout Catholic, Hetper represented a positive example of Christianity to his secular Jewish friend. Hetper and Kac got their PhD degrees and had them conferred in a double ceremony on June 5, 1937 [37]. Soon Kac left Poland and Hetper went on to get his habilitation in 1939 (the thesis [34] was printed in 1938). When the World War II broke out, he fought in the September campaign in 1939 as an ensign of infantry reserves. He escaped from German captivity, but was arrested by Soviets at an attempt of illegal border crossing. Because he carried with him a mathematical manuscript, he was accused of espionage

[2] The following description of Chwistek's program can be found in [36] (see also [77]): "The first step was the creation of what Chwistek called elementary semantics, which, besides its name, has nothing in common with semantics in Tarski's sense. Chwistek claimed that in Hilbert's metamatematics there is contained intuitive semantics, i.e. the rules for the construction of the simplest possible expressions from given elements (letters or signs). This intuitive semantics is formalised and expanded into a system of syntax in terms of which the propositional calculus and the theory of classes are constructed. On this basis the axiomatisation of classical mathematics which assumes no non-constructive objects is finally undertaken. If successful, and this matter must be left to the mathematician to judge, it would provide a proof of the consistency of mathematics. In this, more than in anything else, lies the importance of Chwistek's system".

[3] "It has been known to me through private communication that similar problems were studied by Dr. W. Wilkosz, professor of Jagiellonian University; however, I do not know of any of his publications or results in this direction" [33].

and imprisoned [60]. His last known address was the Starobielsk camp and his last letter to his family was dated 1941 (We thank Professor Roman Duda for this information). Hetper died probably in 1941.

2.7. Danuta Gierulanka (1909–1995)

Born on June 30, 1909, in Kraków, in the family of Zofia neé Romanowska and Kazimierz Gierula, a civil servant in the Ministry of Communication until 1939. In the years 1927–1932 she studied mathematics in the Faculty of Philosophy at the Jagiellonian University. She obtained a master's diploma in philosophy in the field of mathematics for the thesis "Periodic solutions of differential equations". After graduation she enrolled again in the University, in its Pedagogical Study, in order to prepare herself properly for a teaching licence examination, which she passed in 1933. She taught mathematics, physics, chemistry and propaedeutics of philosophy in gymnasia: Landowners' Gymnasium of Benedictine Nuns in Staniątki near Kraków, Humanistic Gymnasium of Mary's Institute in Kraków. In 1938 she started doing research in psychology, under the supervision of Władysław Heinrich, on the problems of psychology of thinking, and more precisely, on forming geometrical notions. During World War II she supported herself by giving private lessons and working in offices of commerce. She took part in clandestine teaching along with her younger brother Jerzy, later a renowned physicist.

In 1945 Gierulanka was nominated to the post of the senior assistant in the Laboratory of Experimental Psychology of the Jagiellonian University. She combined these duties with teaching mathematics in one class in H. Kołłątaj Lyceum in Kraków. She initiated research on thought processes in mathematics. In 1947 she obtained a doctorate at the Humanistic Faculty of Jagiellonian University for the thesis "On acquiring geometrical notions" [28] (She published a book based on her thesis in 1958). The primary subject of the examination was psychology, the secondary one was mathematics. The examiners were professors W. Heinrich, Stefan Szuman and Tadeusz Ważewski (two psychologists and a mathematician). The mathematician's questions were interesting and showed his extremely favorable attitude to the problems of instruction: "Non-euclidean geometry", "On criteria of good elaboration of mathematical theorems", "Characteristics of mathematical talent". Stefan Szuman wrote in his report: "She investigated the psychological process of forming clear and rigorous notions in geometry by the pupils". Gierulanka passed the doctoral examination with distinction. In 1953 she was transferred to the post of adiunkt in the Chair of Mathematical Analysis in the Faculty of Mathematics, Physics and Chemistry at the Jagiellonian University. It was suggested that she would prepare there a candidate's thesis (then an equivalent of a PhD thesis) in mathematical analysis incorporating psychological research, which would be a counterpart of her PhD thesis regarding university teaching. The plan failed. In 1957 Gierulanka returned to the Laboratory of Experimental Psychology for a year. In 1958 she became an adiunkt in the Chair of Philosophy. She obtained habilitation in 1962 on the basis of the thesis "The problem of specificity of mathematical cognition" [29], but she did not get the position of a docent in the Chair of Philosophy. She was transferred to the Chair of Psychology, where she worked until her retirement in 1971. She retired with the sense of injustice: the program of doctoral studies in university teaching for assistants in various academic disciplines was not launched and the university authorities did not show recognition of her work and achievements. She died in Kraków on April 29, 1995.

Gierulanka's scientific path can be best described in her own words. In the "Information on my previous scientific work" attached to the application for habilitation she wrote:

> "Influenced by lectures and seminars conducted by Prof. R. Ingarden, in which I participated regularly since 1946, I was getting an even broader view of the philosophical problematics related to my psychological work. A problem in which I have taken stronger and stronger interest since 1948 is the problem of specificity of mathematics; in its solution I would see a natural complement of the work on acquiring geometrical notions. [This work] gave an idea– thanks to investigation of the course of suitable psychical thought processes– only about psychological sources of the paradoxical opposition between the fundamental clarity and comprehensibility of mathematics and the actual state of its being comprehended. To explain it fully it is necessary to realize what the specific character of mathematics and of the cognitive means it employs consists in. Seeing Cartesian mental intuition and what is called clara et distincta perceptio as a cognitive activity typical for mathematics, I analyzed this notion. (...) Another kind of cognitive processes very relevant for mathematical cognition are processes of understanding. I have been concerned with the problems of understanding since 1951, conducting for 2 years research in Laboratory of Experimental Psychology concerning primarily understanding of texts. With the subsidy from Scientific Pedagogical Society I conducted research parallel to this, concerning learning mathematics from textbooks, including analysis and criticism of school textbooks in geometry in use at that time. (...) Because of the financial difficulties of the Scientific Pedagogical Society the research ceased, and the partial results obtained were not published; those concerning textbooks became obsolete when change occurred".

The problematics undertaken by Gierulanka was very comprehensive. Later on, these issues were dominated by cognitivist psychology. The book [78] is an attempt at rereading Gierulanka's work. In her habilitation, Gierulanka addressed the problem of mathematical cognition as a philosopher, although she also used some previously collected psychological materials. The referees of her scientific output were Professors Zofia Krygowska, Izydora Dąmbska, Tadeusz Czeżowski and Roman Ingarden. Gierulanka analysed mathematical perception and deduction from the phenomenological standpoint and described attempts at systematization and unification of mathematics. She did not consider reduction to set theory as true solution of the problem of unification of mathematics. She criticized Bourbakist mathematics, blaming it for, among other things, being arbitrary in constructing systems of axioms and making unnatural generalizations. However, she saw some possibilities for applying the notion of mathematical structure [2]. Krygowska considered her presentation of the state of contemporary mathematics exaggerated and tried to defend the Bourbakist approach by pointing out that it can reflect an actual course of mathematical creative processes as reported by some distinguished mathematicians. Ingarden agreed with Gierulanka's reference to Descartes, considering his epistemology still relevant for contemporary mathematics. He wrote:

> "One needs to realize that mathematics is not only a certain set of theorems and methods used over the last decades, but that it is a certain historical creation, evolving over at least last three centuries, precisely since the Cartesian reform. During that time not only did mathematics encompass even more new domains of study, but it also significantly kept changing its methods, the understanding of its role and of its ultimate sense, undergoing a series of internal crises (e.g., emergence of non-euclidean geometries, antinomies at the end of 19th century, and finally Gödel's theorems in 1930s) as well as a series of external crises through attacks of various forms of modern European scepticism, e.g. Hume's attack, various forms of positivism and empiricism of 19th century, up to Vienna neo-positivists in 20th century, who made mathematics a system of tautologies. Because of this the problem of specificity of mathematical cognition is extremely complicated and does not allow one to restrict considerations of mathematics only to the form it has had in the latest decades. It cannot be excluded that the cognitive tendencies represented by Descartes – despite differing current views – do not lose their relevance".

Gierulanka was also an active editor and translator. She took part in translating some of Ingarden's works from German to Polish and in editing his collected works. She translated the first and second volume of Husserl's "Ideas of pure phenomenology and phenomenological philosophy" and (together with Jerzy Gierula) "On the problem of empathy" by Edith Stein.

2.8. Adam Bielecki (1910-2003)

Born on February 13, 1910, in Borysław (Drohobycz county, Lvov voivodship). In 1928 he finished Hoene--Wroński Gymnasium in Kraków. Simul- taneously with the high school course he studied piano and theory of music at the Kraków Conservatory. He enrolled in the Jagiellonian University to study mathematics. As a student, he taught acoustic in a private Żeleński School of Music in Kraków. He finished his studies in 1931, obtaining the master's degree in philosophy in the field of mathematics. In 1935 he obtained the doctorate on the basis of the thesis "On integral representation of *m*-dimensional surfaces contained in the *n*-dimensional euclidean space by implicit functions" [3]. The problem– of equivalence of the implicit and parametric representations – was posed by Witold Wilkosz, who supervised the thesis. Bielecki learned about it from Tadeusz Ważewski and solved it using the technique of C^∞ – partition of unity, later reintroduced and refined by Laurent Schwartz [39]. In the years 1935–1936 he worked in the Theoretical Physics Seminar at the Jagiellonian University, becoming a senior assistant in the Chair of Theoretical Physics in 1936. He collaborated with his colleagues, publishing 1 joint paper with Stanisław Krystyn Zaremba [14], 1 with Jan Weyssenhoff (a theoretical physicist; [13] and 1 with Weyssenhoff and Myron Mathisson [12]. He also published two volumes of poetry. He was arrested in the Sonderaktion Krakau and taken first to the Sachsenhausen-Oranienburg concentration camp, then to Dachau. Released in April 1940, he returned to Kraków. He supported himself first by giving private lessons, then– from September 1942 to January 1945 – by part-time teaching at the Vocational School of Construction in Kraków. At the same time, starting in 1942, he became active in organizing clandestine teaching in the underground Jagiellonian University, holding some classes in his private apartment, preparing lecture notes for students and lending them books from what remained of the Library of the Laboratory of Theoretical Physics. He was also a member of the underground research group in theoretical physics led by J. Weyssenhoff.

After the war, Bielecki decided to "give his strengths to the Mathematical Institute" [30]. In 1945, he worked first as a senior assistant, then as an adiunkt of the I Mathematical Laboratory at the Jagiellonian University. From 1945 to 1947 he was a deputy professor and a head of the Chair of Mathematics at the Faculty of Engineering of the Academy of Mining in Kraków. He was also strongly involved in the activities of the Polish Mathematical Society, to which he belonged since 1931 (taking part in clandestine scientific meetings during the occupation). In 1947 he was called to Lublin and assumed (as a deputy professor) the Chair of Mathematical Logic and Foundations of Mathematics at the Faculty of Mathematics and Sciences of the Maria Curie-Skłodowska University. In 1949 he got habilitation on the basis of the thesis concerning differential equations and differential inclusions [4]. In 1959, after the death of Mieczysław Biernacki, Bielecki took over as the head of the Collective Chair of Mathematics. In order to save the mathematics program from liquidation, he supervised a few PhD theses and supported 3 habilitations in a 3-year period of time. Some of the candidates started their research under the direction of Biernacki and worked in the theory of univalent functions in one complex variable. Not only did they finish their theses, but Bielecki was able to adapt his interests in a way that allowed him to write joint papers with them, concerning mainly subordination theory (e.g. [11]). This is still an active research topic, especially in relation with the Loewner differential equation (which was used in 1985 in Louis de Branges's proof of the Bieberbach conjecture and whose stochastic version, introduced by Oded Schramm in 2000 and later studied by him together with Gregory Lawler and the Fields medalist Wendelin Werner, found applications in statistical mechanics and conformal field theory). Besides university teaching and supervising PhD candidates (11 over the whole career), Bielecki was active in curriculum development and teachers' education. He organized post-graduate courses for mathematics teachers and qualifying exams for those teachers who did not have a master's degree in mathematics. In 1970s he delivered lectures and created lesson plans for teachers as a part of Radio and Television Teachers University educational program. His presentations were later followed by articles containing in-depth discussion of mathematical and educational issues, published in the biweekly "Oświata i Wychowanie". He retired in 1980, but continued part-time teaching until 1991. He died in Lublin on June 10, 2003.

Bielecki's main mathematical interest were differential equations and differential inclusions. His best-known and most-cited result [21] is a method of proving theorems on existence of solutions of differential and integral equations. The method consists in a suitable change of a norm in the relevant space of functions so that a certain operator becomes a contraction and Banach Fixed Point Theorem can be applied [5, 6]. In another paper [8] Bielecki extended Ważewski Retract Theorem to differential inclusions. He also worked on various aspects of geometry, publishing, among others, the paper [12] motivated by aplications to the general theory of relativity and a joint paper with S. Gołąb [10] concerning characterization of Riemannian space among Finsler spaces by the properties of the angular metric. His notable result in foundations of geometry [7, 9] concerns reducing the number of axioms given by Hilbert for Euclidean geometry (while weakening some of them) and the proof of independence of the system obtained.

2.9. Antoni Bielak (1910–1991)

Born on June 10, 1910, in Kraków, in the family of Antoni, a mathematician, and Jarosława neé Kowalow. After finishing The St. Hyacinthus II Gymnasium and Lyceum in Kraków he studied mathematics at the Jagiellonian University. He finished his studies obtaining the master's degree in 1939. During the occupation he took part in clandestine teaching as a member of the Underground State Examination Committee for the maturity exams. After the WWII he taught in gymnasia and lycea in Kraków. In the years 1951–1954 he taught mathematics at the Preparatory Study of the Jagiellonian University. From 1952 to his retirement in 1972 he taught in the A. Witkowski V Lyceum, where he also held classes for gifted youth.

His pupils remembered him fondly [52]. "Not too young, stooping, of short stature, he had all characteristics of a mathematics teacher, and was one indeed. (...) Mathematics, the nightmare and curse of generations of pupils, was perceived by us differently. To be sure, the majority in the class was far from admiration for "the queen of sciences" and from [the possession of] deeper mathematical knowledge, but at least it was not afraid of the lessons. (...) Those who liked mathematics and had no problems with it, learned a lot at school". Among Bielak's pupils were Zdzisław Opial, Andrzej Lasota, Włodzimierz Mlak, and Antoni Leon Dawidowicz, later distinguished mathematicians and university professors. Bielak was also active in the Polish Mathematical Society. He died on February 3, 1991, in Kraków.

2.10. Franciszek Bierski (1912–2002)

Born in Warszowice Śląskie, he graduated from the Jagiellonian University in 1936. He taught mathematics and physics in the gymnasium and lyceum in Piekary Śląskie.

After World War II he was employed by the Academy of Mining and Metalurgy in Kraków. In 1959 he got PhD in mathematics from the Jagiellonian University under the direction of Franciszek Leja. He chaired the Laboratory of Mathematical Analysis at the Academy of Mining and Metalurgy in the years 1970–1983, and in 1974–1983 he was the deputy director, then the director, of the Institute of Applied Mathematics. Zentralblatt für Mathematik lists 9 research publications authored or coauthored by Bierski, mostly in the field of analytic functions. He also wrote and published several academic textbooks. He died in Kraków.

2.11. Roman Leitner (1914–2008)

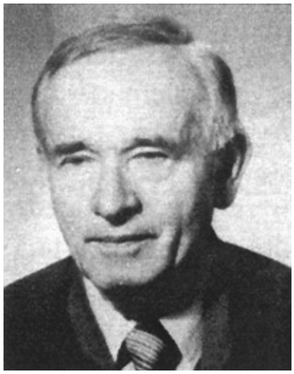
Born in 1914 in Radziechów near Lwów, he passed his maturity examination in Jasło in 1932. Then he studied mathematics at the Jagiellonian University, obtaining the master's degree and the high school teacher's diploma in 1937. He also studied physics [67]. He was employed by the III State Gymnasium in Kraków and worked voluntarily as an unpaid assistant at the Jagiellonian University, in Stanisław Zaremba's chair. In September 1939 he was in Lwów. During the occupation he was involved in the clandestine teaching and in the years 1943–1944 had to go into hiding. After the liberation of Lublin he joined the Polish Army. He became an officer of field artillery and a lecturer in the Officers' School of Shellproof Weapons (first in Chełm, then in Modlin). As a teacher, he was released from the army in 1946 and returned to the Jagiellonian University, where he became a senior assistant. He also gave lectures in mathematics to teachers studying for professional development at the Higher Pedagogical Study in Katowice and conducted summer professional development courses for high school teachers in Szklarska Poręba (1949) and Kołobrzeg (1950).

In 1949 Leitner got his PhD under the direction of Tadeusz Ważewski. In 1951 he was called to be a deputy head of the Chair of Mathematics of newly founded Military Academy of Technology in Warsaw. The head was Witold Pogorzelski, earlier a professor of mathematics at Warsaw University of Technology, and a holder of PhD from Jagiellonian University (received in 1919). Along with other employees, they conducted reseach in differential and integral equations and taught mathematics for applications in modern military technology. In 1954 Leitner became a 'docent'. In 1957 Pogorzelski returned to Warsaw University of Technology and Leitner took over the Chair. During his term (until his retirement in 1984) the Chair introduced new courses of studies, e.g., extramural and supplementary courses as well as programs for foreign students. Preparatory courses for applicants to technical universities were very popular.

Leitner coorganized the Television Technical University and televised preparatory courses for applicants. Together with Wojciech Żakowski (later a professor in Warsaw

University of Technology) he wrote a study guide in mathematics for applicants, reprinted many times. Part of this guide became a geometry textbook for lyceum. He also wrote other lecture notes and textbooks for students, as well as educational computer programs. In 1970 the Military Academy undertook supervision of the XXIV C.K. Norwid State Lyceum (a high school) and its personnel, among them Leitner, taught advanced classes for the students. Leitner took care to organize regular instructor training for the employees of his Chair, by visiting classes and discussing performance, organizing model lectures and recitations as well as courses and seminars in methodology of teaching. He was always thoroughly prepared for his classes and his lectures were considered beautiful. Leitner died in 2008 [43, 44].

2.12. Tadeusz Rachwał (1914–1992)



Born on May 29, 2014 in Kraków, in the family of Władysław, a craftsman, and Maria neé Kulpa. He finished A. Witkowski gymnasium in Kraków in 1932 and studied at the Jagiellonian University. He credited his high school in instilling in him good studying habits in mathematics. In 1936 he obtained the master of philoshophy degree in the field of mathematics. In October 1936 he started working in the Chair of Mathematics of the Academy of Mining in Kraków, also taking up the course of studies at the Faculty of Mining. Here is how he remembered his choice of a career [63]:

> "My dreams as a graduate were partially fulfilled. I set my sights on a different course of technical studies. I knew that I would need mathematics, so I applied myself to it with special care. But first of all I related it with my hobbies, which were painting and drawing. I wanted to take up architecture. But my fate directed me in such a way that I stayed in Kraków and took up mathematics. Because later my classmates Litwiniszyn and Wojtanowicz encouraged me to study at the Academy of Mining, I decided on those studies".

During the occupation Rachwał worked as a measuring technician at the Hydrological Subdivision in the region of Jasło, and then as an accountant in the "Dezet" company in Kraków. He also fought in the Home Army. After the war he resumed his work at the Academy of Mining, as a senior assistant. In 1950 he finished his studies at the Faculty of Mining of the Academy of Mining and Metalurgy (as the school became known since 1949) and a year later he was nominated for the post of a deputy professor and the head of the Chair of Descriptive Geometry. In 1955, on the basis of the thesis "A study of the order of tangency of a regular curve with a strictly tangent ball" he obtained the degree of candidate of sciences (which was at that time conferred instead of PhD). The thesis was supervised by Stanisław Gołąb. Rachwał got his habilitation in 1962 (at the Technical University of Kraków) on the basis of the thesis "On a certain mapping of one-sheeted hyperboloid onto a plane". In 1971 he received the title of an extraordinary professor.

He organized the Institute of Mathematics of the Academy of Mining and Metalurgy and was its first head. Until his retirement in 1984, he headed the Laboratory of Descriptive Geometry in the Institute. He initiated long-term collaboration between the Institute and the lignite mines in Turoszów and Konin and the sulphur mine in Grzybów. He published about 30 research works in differential and descriptive geometry and in applications of mathematics to mining, as well as 8 textbooks and sets of lecture notes in descriptive geometry (the 2-volume text [62] had many editions). He supervised 7 doctorates and 4 habilitations. He died in Kraków on April 18, 1992.

The drawing of Antoni Bielak made by K. Małachowski was reproduced from [52]. The recordings of Fr. Bernard Turowicz were made available by Dr Zofia Pawlikowska-Brożek.
The photos come from the archives of Z. Pawlikowska-Brożek and S. Domoradzki. They were obtained from private individuals or UJ Archives.
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3. APPENDIX: Master's theses in mathematics prepared in Kraków between 1928 and 1936

Translated by the second author from [46] and [47]. Some misprints of the Polish original titles were corrected. The spelling of proper names was preserved.

Up to 1930/31:

(1) Stanisław Turski: *Determining the magnitudes of quaternionic precessions by the Newcomb constants*
(2) Andrzej Turowicz: *On an application of iteration to solving differential and integral equations*
(3) Gerson Gottesfeld: *Fundamental properties of linear congruences*
(4) Michał Seidmann: *Definitions of lines of curvature*
(5) Etla Hornówna: *A short outline of tangential transformations in the plane from the viewpoint of Lie*
(6) Stanisław Krystyn Zaremba: *Differential equations and tangential transformations in the projective plane*
(7) Rev. Józef Stępień: *Kinematic method in the theory of surfaces*
(8) Jan Skarżeński: *On Prof. Łukasiewicz's theory of deduction*
(9) Regina Hausmannówna: *Principles of algebra and analysis of tensors*
(10) Lija Jankielowska: *On inflexibility of elliptic surfaces*
(11) Izrael Brumberg: *A few fundamental theorems in the theory of minimal surfaces*
(12) Aleksander Orłowski: *Curves of constant width*
(13) Roman Dniestrzański: *Vectorial method in the theory of surfaces*
(14) Józef Steczko: *The notion and properties of parallelism in a Riemann space*

1930/31

(1) Chaim Wasserfall: *Surfaces of constant curvature*
(2) Wolf Kestenblatt: *From the theory of analytic continuation (Mittag-Leffler star)*

(3) Anna Zofja Czarkowska: *Fundamental theorems in the theory of conformal transformations of planar domains*
(4) Hersz Händel: *The fundamental theorem on geodesic curves on a surface*
(5) Stanisław Malecki: *On birational transformations*

1931/32

(1) Adam Bielecki: *On integral representation of surfaces and curves by implicit functions*
(2) Władysław Hetper: *Abel-Laplace integral equations*
(3) Szymon Berg: *Fundamental theorems by Brill-Noether*
(4) Karol Kozieł: *A mathematical formulation of a problem in the theory of refraction*
(5) Sr. Prezepja Wilczewska: *The theory of general complex numbers as an application of the theory of Lie groups*
(6) Emma Epsteinówna: *Singular points of an analytic function given by a Taylor series on the circle of convergence of this series*
(7) Antoni Nykliński: *Fundamental properties of Weingarten surfaces*
(8) Klara Goldstoffówna: *The form of a homogeneous differential equation of Fuchs type*
(9) Janina Martini: *Asymptotic solutions of systems of differential equations*
(10) Stefan Piotrowski: *Parametric from of differential equations in partial derivatives of order one*
(11) Danuta Gierulanka: *On periodic ordinary integral in a real variable*
(12) Florjan Szozda: *Fréchet's natural parameters*
(13) Rozalja Nordówna: *Riemann's method in linear partial differential equations of order two, hyperbolic*
(14) Emil Reznik: *Set theoretic foundations of expandability of functions in the series of Bessel functions*

1932/33

(1) Bronisław Czerwiński: *Upper and lower Perron integrals and the question of uniqueness of solutions of a system of differential equations*
(2) Juljusz Keh: *Bessel differential equation and main properties of Bessel and Hankel functions*
(3) Zygmunt Sejud: *Frenet's formulas for an n-dimensional Riemann space*
(4) Janina Perausówna: *An estimate of the domain of existence of an integral of a linear nonhomogeneous partial equation of order one*
(5) Rozalja Gansówna: *Foundations of the theory of equivalence of planar figures*
(6) Józef Hetper: *Stokes' theorem from the topological viewpoint*
(7) Rudolf Wolf: *Brouwer's theorem on invariance of the number of dimensions*
(8) Leopold Haller: *The length of a set lying on a rectifiable continuum in relation with the counting function*
(9) Stefan Sedlak: *On the notion of invariant*
(10) Irena Wilkoszowa: *Convex functions and the functional equation* $f(x+1) = xf(x)$

(11) Czesław Kluczny: *(Essentially) two-parameter family of solutions of a differential equation* $F\left(x, y, z, \frac{\partial z}{\partial x} \frac{\partial z}{\partial y}\right)$
(12) Władysław Misiaszek: *The role of equations attached in the reduction of linear and homogeneous differential equations*
(13) Aron Teitelbaum: *Tangential transformations in relation with differential equations in partial derivatives of order one*
(14) Eugenjusz Ziemba: *Finite tangential transformations*
(15) Wincenty Łabuz: *Boundary problems of an ordinary differential equation of order two*
(16) Paweł Szabatowski: *Contingent and paratingent*
(17) Helena Mandelbaumówna: *On the behavior of an analytic function on the boundary of the disk of convergence*

1934/35

(1) Jadwiga Rättig: *Theory of Dirichlet series*
(2) Maria Kostka: *Fundamental properties of regular closed spatial curves*
(3) Danuta Stachórska: *Some sufficient conditions for integral existence of an inverse transformation to a transformation of class C*
(4) Mieczysław Warchoł: *A catalog of principles of geometry of Riemannian spaces*
(5) Sydonia Kleinerówna: *Principles of the theory of analytic sets*
(6) Helena Gelberówna: *Transformations of so-called euclidean motions and symmetries in the plane and their properties*
(7) Władysław Skrzypek: *On systems of completely integrable differential equations*
(8) Maria Holcherg: *A boundary problem for differential equations dependent on a parameter*
(9) Stanisław Kądzielawa: *Malmsten's method of seeking the integrating factor for differential equations*
(10) Jan Angress: *Solution of a completely integrable system of differential equations by Mayer's method*
(11) Antoni Bulanda: *On complete extension of of functional operators*
(12) Franciszek Ryszka: *Main principles of the theory of automorphic functions*

1935/36

(1) Franciszek Bierski: *Projective geometry in two-dimensional complex space*
(2) Antoni Bulanda: *Maximal extensions of Hermite operators in a Hilbert space*
(3) Jadwiga Dymnicka: *Surfaces with constant Gaussian curvature*
(4) Kazimierz Gurgul: *Whether a function of a complex variable corresponding to minimal surfaces according to Weierstrass' formula must be analytic*
(5) Wacław Juszczyk: *Transformations of euclidean motions and symmetries in the plane and their properties*
(6) Józef Janikowski: *On transformations of a differential equations in a neighborhood of a singular point*
(7) Józefa Konarska: *On fields of rays and differential equations of order I*
(8) Karol Kałuża: *On approximation of functions of one real variable*

(9) Jerzy Klimonda: *On a special class of natural equations for a surface in* R_3
(10) Maria Kostka: *Fundamental properties of regular closed spatial curves*
(11) Józef Kwieciński: *Development of the theory of curves in four-dimensional euclidean space* R_4
(12) Tadeusz Kamiński: *Fundamental theorems in the theory of tangentail transformations*
(13) Jerzy Łomnicki: *Implicit functions in the domain of complex variables*
(14) Franciszek Ryszka: *Main outlines of the theory of automorphic functions in one variable*
(15) Jadwiga Rättig: *Theory of Dirichlet series*
(16) Zofia Stockówna: *An outline of the theory of additive set functions*
(17) Herbert Welke: *Extremal properties of the circle and the ball*
(18) Jakub Zaręba: *Properties of the group of projective transformations of the plane*
(19) Jerzy Kuczyński: *Tentative proofs in Prof. Wilkosz's tribe logic taking into account cardinal set theory*
(20) Jan Karafiał: *On mechanical integration of certain differential equations*
(21) Józef Lesikiewicz: *On change of variables in certain differential equations*

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STANISLAW DOMORADZKI*, MYKHAILO ZARICHNYI**

ON THE BEGINNING OF TOPOLOGY IN LWÓW

POCZĄTKI TOPOLOGII WE LWOWIE

A b s t r a c t

We provide one of the first surveys of results in the area of topology by representatives of the Lvov School of mathematics and mathematicians related to the University of Lvov. Viewed together, these results show the importance of this school in the creation of topology.

Keywords: *topology*, *history of mathematics in Poland*, *Lvov School of Mathematics*

S t r e s z c z e n i e

W artykule dokonamy jednego z pierwszych przeglądów wyników Lwowskiej Szkoły Matematycznej z zakresu topologii w celu ukazania znaczenia tej szkoły w tworzeniu topologii.

Słowa kluczowe: *topologia*, *historia matematyki w Polsce*, *Lwowska Szkoła Matematyczna*

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By now, the history of the Lvov School of Mathematics has already been well studied by historians of mathematics. The monograph by R. Duda [18] recently translated into English is an exploit in this direction. On the other hand, there are many publications on the history of topology in Poland. This raises a natural question: can one find a description of the history of Lvov topology at the intersection of the above groups of publications? Is it enough, say, to provide topological study as a part of the history of Lvov mathematical school? We think that the answer to this question is negative. Therefore we focus on topological grounds of the Lvov mathematics (considered in the period from the late 19th century to the Second World War). We will draw attention to some points not previously observed and also add a certain consistency to the material. Clearly, mathematics, as do other sciences, has not only its history but also its geography. In the period of activity of the Lvov School of Mathematics the role of the latter was much higher than it is currently, when electronic means of communication greatly facilitate the cooperation of scientists at the distance and blur the links of scientific results to a specific territory.

Our presentation of the material is inevitably centered around personalities. Given the large amount of biographical literature, we focus mainly at mathematical results. Moreover, we are interested in the most important scientific contribution made in topology during the Lvov period of activity of the individuals discussed.

Józef Puzyna

Born in 1856 in Nowy Martynów, near Rohatyń. For over 30 years he taught mathematics at the University of Lwów. He served in various university administrative functions, was a rector of the university. He was the first who mentioned topology as a mathematical subject in Polish in his “Studya topologiczne” (“Topological studies”). The basic scientific direction for Puzyna was comprehensive analysis. The main achievement of Puzyna, namely, his monograph *Teorya funkcyi analitycznych* (Theory of analytic functions) contains the very first exposition in Polish of the foundations of set theory and set-theoretic topology. The contents of this book are described in more detail in [11–13].

Much of Puzyna’s book is concerned with topology of surfaces. Despite the fact that in the introductory chapters of the monograph he actively uses the language of set theory, the author prefers the traditional descriptive approach of topology of surfaces while presenting the theory of surfaces.

For the theory of surfaces in Puzyna’s monograph reader can refer to the article [12] in this volume.

Zygmunt Janiszewski

Born in Warsaw in 1888. He studied mathematics at the University of Zurich, Göttingen, Paris, Munich, Marburg, Graz. He attended lectures of famous mathematicians including J. Hadamard, E. Borel, H. Minkowski, Landau, C. Runge, D. Hilbert, Picard, E. Zermelo, A. Toeplitz, F.Bernstein, E. Goursat. His doctoral thesis *Sur les continues irreductibles entre deux points* was defended at the Sorbonne in 1911. Among the committee members were H. Lebesgue, H. Poincaré and Emile Borel. His fundamental work *On cutting the plane by continua* (O rozcinaniu płaszczyzny przez kontinua: [21]) is devoted to the topology of the plane. In particular, it contains the following statements on cutting of the plane:

1) The sum of two continua does not cut the plane provided that none of them cuts the plane and their intersection is either connected or empty.
2) The sum of two continua such that their intersection is disconnected cuts the plane.

The publication [21] in Polish is supplemented by an extended French abstract, which made the results more accessible to the Western readers. An interesting discussion on the influence of Janiszewski's results as well as results of other Polish topologists on the mathematical activity of some American mathematicians working approximately in the same period can be found in [2].

It was Janiszewski who suggested that only one direction of mathematical research should be chosen in Poland, namely, the set theory and (set-theoretic) topology.

Janiszewski died in 1920.

Wacław Sierpiński

Born in 1882 in Warsaw. Georgy Voronoy was one of his doctoral advisors. In 1908, W. Sierpiński was approved as a Privatdozent in mathematics at the University of Lvov.

One of his books written during the Lvov period, *Outline of the set theory* (in Polish), is closely connected to set-theoretic topology. His Lvov period is taken to last until the end of March, 1919; although Sierpiński was interned in the years 1914–1917 (in the list of lectures delivered at the University of Lvov in 1916/17 one can see the remark that "Prof. Sierpiński does not have lectures in this semester") and took the leave on demand in the Fall semester, 1918/19.

Therefore, the known important objects of fractal geometry such as Sierpiński gasket and Sierpiński carpet were introduced by a mathematician affiliated to the University of Lvov.

Among Sierpiński's topological results of this period one should mention also the universality of Sierpiński's carpet in the sense that it contains a topological image of any planar curve. A three-dimensional generalization of Sierpiński's carpet, namely, Menger curve, is a universal object for all three-dimensional curves. Further generalizations of Sierpiński's carpet are the m-dimensional sets μ_m^n in the n-dimensional Euclidean space. The sets μ_m^{2n+1} are universal spaces for n-dimensional sets; they are also model spaces for the so-called Menger manifolds [3].

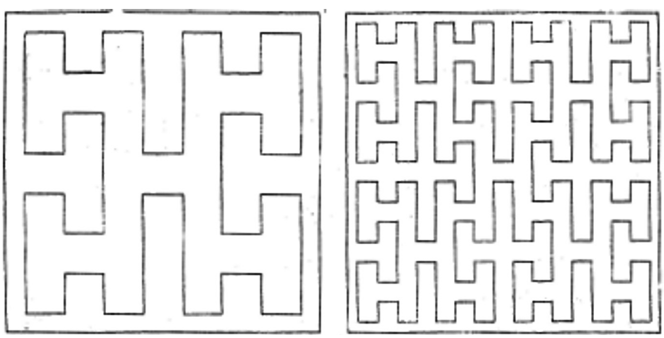

Fig. 1. Approximations of the universal curve. A picture from Sierpiński's paper

The modern theory of Menger manifolds was developed by many authors (see, e.g. [39, 40]). It is an *n*-dimensional counterpart of the theory of Hilbert cube manifolds, which itself provides an infinite-dimensional generalization of finite-dimensional manifolds.

One more result from this period concerns the space-filling curves, i.e. curves whose images contain squares. The object constructed in [48] is now called the Sierpiński curve. Note that Sierpiński continued investigations of Peano and Hilbert. The Sierpiński curve is an example of fractal curve; its approximations provide solutions of the Travelling Salesman Problem. Interesting information concerning space-filling curves can be found in [7].

W. Sierpiński died in 1969.

Józef Schreier

Born in Drohobycz in 1909. Schreier prepared his PhD thesis *On finite basis in topological groups* in 1932–1934. He was awarded his PhD in 1934, and Stefan Banach participated in the ceremony of award of degree as the advisor.

S. Ulam was the coauthor of eight publications with Schreier. They worked both in the topological group and topological semigroup theory. Stanisław Ulam recognized the importance of investigation of topological-algebraic objects.

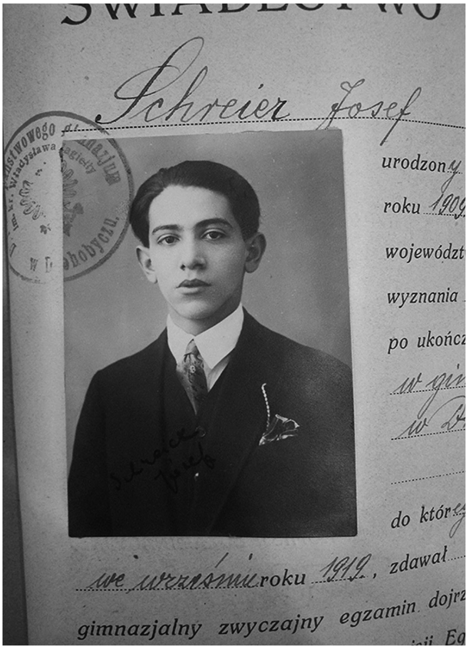

Fig. 2. Józef Schreier (Lviv Discrict Archiv)

In [46] Schreier and Ulam proved that every automorphism of the group S_ω of permutations with finite supports of a countable set is an inner automorphism, i.e. there exists an element *s* in *S* such that $x = sxs^{-1}$, for any *x* in *S*. Here, S_ω is regarded as a topological group with respect to the topology of pointwise convergence.

Some of publications by Schreier and Ulam concern the notion of base of a topological (semi)group. A detailed exposition of this direction of their investigation can be found in [19].

J. Schreier died in 1943.

Stefan Mazurkiewicz

Born in Warsaw, in 1888. Mazurkiewicz's doctorate (University of Lvov, 1913) was supervised by Wacław Sierpiński. The results concerned the square-filling curves. One of his results consist in a complete proof (for $n = 2$) of statements announced by H. Lebesgue [35] (the proof given therein turned out to be incomplete):

1) every planar curve which fills a 2-dimensional domain necessarily contains points of multiplicity 3;
2) there is a curve that fills a 2-dimensional domain and such that the multiplicity of its points is at most 3.

Also, it was proved in this paper that every planar nowhere dense continuum can be represented as the image of Cantor set such that the preimage of every point consists of at most 2 points.

Mazurkiewicz left Lvov for Warsaw in 1919 but still kept in contact with mathematicians from Lvov afterwards.

He died in Grodzisk Mazowiecki, in 1945.

Stefan Banach

Born in Kraków, in 1892. His Ph.D thesis *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales* (University of Jan Kazimierz, Lvov, 1920) contained fundamental results in functional analysis.

He is recognized mostly because of his fundamental results in functional analysis. However, some of Banach's achievements are closely connected to set-theoretic topology and topological algebra. As examples, we mention the open mapping theorem, Banach-Alaoglu theorem, and Banach-Stone theorem. The fixed point theorem, although it belongs to metric geometry rather than topology, still finds great use even in solving purely topological problems.

We should also mention the method of proving the existence of categories of objects with specified properties. It became one of the hallmarks of the Lvov mathematical school.

S. Banach died in Lvov, in 1945.

Kazimierz Kuratowski

Born in Warsaw, in 1896. His Lvov period lasted for 6 years, until he left Lvov for Warsaw in 1934. The total number of works written by Kuratowski in the Lvov period is more than thirty, so we will focus only on some of them.

The famous Knaster-Kuratowski-Mazurkiewicz lemma [22] is a result which is known to be equivalent to the Brouwer fixed point theorem. However, its formulation turns out to be convenient for the use in mathematical economics, in particular, in the market equilibrium theory.

The paper [25] contains the following result: for every Peano continuum, the fixed point property implies unicoherence. As a consequence, one can derive some of Janiszewski's results mentioned above. A consequence is the unicoherence property of the plane established earlier by Mazurkiewicz.

In the paper [26] Kuratowski provided a topological characterization of the 2-dimensional sphere. To this aim he introduced the class of Janiszewski spaces, namely the spaces X satisfying the property that every continuum cutting X is unicoherent.

In his paper [19] published in Studia Mathematica, Kuratowski demonstrated a purely topological nature of the following result of S. Banach [0]: in a topological group, every subgroup satisfying the Baire property is simultaneously open and closed.

Kuratowski made important contributions to the dimension theory. In [27] he developed some ideas of Witold Hurewicz and proved a characterization theorem for dimension of perfect subsets in compact spaces in terms of mappings.

It is important also to emphasize that the first volume of Kuratowski's fundamental monograph *Topologie* [31] was published during the author's Lvov years.

Kuratowski died in Warsaw, in 1980.

Juliusz Schauder

Born in Lvov, in 1899. He was awarded his PhD in 1923 with the thesis *The theory of surface measure*.

One of the most important of Schauder's topological achievements is his fixed point theorem for the convex compact sets in linear topological spaces [44]. This is a generalization of the classical Brouwer fixed point theorem to infinite-dimensional case. However, soon after this publication it turned out that the proof worked smoothly only in the locally convex case (see also [50]). The general case was formulated as an open problem in the Scottish book. This gave a start to numerous publications in this direction.

The general case was considered by R. Cauty in 2001. The very first proof by Cauty of Schauder conjecture contained a gap. T. Dobrowolski [9] elaborated on Cauty's proof in order to make it more accessible. In this way he discovered an essential mistake.

Later, Cauty developed the theory of algebraic neighborhood retracts that allowed for obtaining fairly general results on fixed points (e.g., for the so-called locally equiconnected spaces).

Also, Schauder proved some results on domain invariance in infinite-dimensional linear spaces [43]. In collaboration with the French mathematician Jean Leray, Schauder developed the theory of degree of some nonlinear maps of Banach spaces [36]. The degree is a homotopy invariant of maps.

Juliusz Schauder died in Lvov, in 1943.

Miron Zarycki

Born in 1889, in Tarnopol region. Under the influence of Sierpiński, Zarycki's interests in mathematics turned to the set theory and real analysis.

Zarycki was awarded PhD (University of Lvov, 1930) for the thesis *Quelques notions fondamentales de l'Analysis Situs au point de vue de l'Algèbre de la Logique* published in Fund. Math. [53]. This paper is, in a sense, a continuation of Kuratowski's research on the closure operator in topological spaces. In particular, Zarycki formulated axioms for the operations of boundary of sets in topological spaces. It turned out that the system of axioms obtained is equivalent to that formulated by Kuratowski [23] and therefore the notion of the boundary can be equivalently used for defining topological spaces. This result by Zarycki was cited not only in topology but also, because of the importance of the notion of boundary in ontology, in some philosophical papers [52].

Note that Zarycki was a vice-Dean while Stefan Banach was the Dean of Department of Physics and Mathematics, and served as Dean after Banach's death. Zarycki died in Lvov, in 1961.

Stanisław Ulam

Born in Lvov, in 1909. Published his first paper in Fundamenta mathematicae when he was 20 (the paper was written two years before its publication). PhD in 1933 at the Lvov Polytechnical School.

One of the most important of Ulam's contributions to topology is the famous Borsuk--Ulam theorem [3]. It states that for any continuous map of an *n*-dimensional sphere into an *n*-dimensional euclidean space there are two antipodal points with the same image. The result was conjectured by Ulam and proved by Borsuk. (More information on Ulam's activity can be found in the article by L. Bazylevych, I. Guran and M. Zarichnyi in this volume).

Ulam died in Santa Fe (USA), in 1984.

The table below shows the equivalence of some of the mentioned results to their combinatorial and covering versions. It is notable that two results in this table belong to mathematicians from Lvov.

Algebraic topology	Combinatorics	Set covering
Brouwer fixed-point theorem	Sperner's lemma	KKM-lemma
Borsuk–Ulam theorem	Tucker's lemma	Lusternik–Schnirelmann theorem

Fig. 3. Equivalence of several topological, combinatorial, and covering results (taken from open sources)

Concluding remarks

Given the richness and heterogeneity of topological results obtained by the Lvov mathematical school, we cannot claim our brief survey to be complete. It demonstrates, however, that the achievements of mathematicians from the Lvov mathematical school in topology are in some aspects comparable with those in functional analysis.

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THE EMERGENCE OF NATIONAL MATHEMATICAL RESEARCH COMMUNITIES IN CENTRAL-EASTERN EUROPE

TWORZENIE SIĘ KRAJOWYCH ŚRODOWISK MATEMATYCZNYCH W KRAJACH EUROPY ŚRODKOWO-WSCHODNIEJ

A b s t r a c t

Since medieval times mathematics was being developed mainly in Central and Western Europe but in the 19th century it greatly expanded both to the United States and to some countries of Central-Eastern Europe. This expansion, accompanied by the rapid growth of mathematics as a whole, has recently become an object of interest and investigation. The interest, however, reflects the Western viewpoint, cf. [17], with the bibliography covering Germany, Spain, Italy, France, Moscow. The aim of this paper is to outline the history of emerging national mathematical research communities in Russia, Poland, Bohemia, Lithuania and some other countries of Central-Eastern Europe.

Keywords: *history of mathematics in countries of Central-Eastern Europe in the 19th c.*

S t r e s z c z e n i e

Od czasów średniowiecza matematyka była rozwijana głównie w Europie Środkowej i Zachodniej, ale już w XIX wieku jej badanie zostało znacznie rozszerzone w Stanach Zjednoczonych i niektórych krajach Europy Środkowo-Wschodniej. Ekspansja ta, wraz z szybkim rozwojem matematyki jako całości, stała się ostatnio obiektem zainteresowania i analizy. Zainteresowanie jednak odzwierciedla punkt widzenia Zachodu (por. [17]) obejmujący Niemcy, Hiszpanię, Włochy, Francję, Moskwę. Celem niniejszej pracy jest przedstawienie historii wschodzących krajowych matematycznych środowisk naukowych w Rosji, Polsce, Czechach, na Litwie i innych krajach Europy Środkowo-Wschodniej.

Słowa kluczowe: *historia matematyki w krajach Europy Środkowo-Wschodniej w XIX w.*

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1. A mathematician cannot work alone. Necessary to him are people to communicate with, people able to understand his work, to criticize its weak points, to appreciate and take into account its achievements. Without such people his pains are pointless and his efforts are useless.

   People able to communicate with the working mathematician form a mathematical **community**. In the antiquity and medieval times the mathematical community was very sparse and letters in Greek or Latin were a prevailing mean of communication. At the beginning of modern times, however, mathematics was gaining momentum and national languages have grown in maturity and importance. These two factors, expanding mathematics and growing importance of national languages, contributed to the emergence of **national** research communities. As the long isolation of British mathematics after Newton-Leibniz controversy over priority or the French--German enmity after the war 1870 show, the phenomenon could split one common world-wide community into separate parts. The danger was real, but **globalization** took an upper hand over the risk of separation. Eventually national communities do communicate with one another and presently form a common world-wide mathematical community.

   The aim of the present paper is to describe the emergence of some national research communities in Central-Eastern Europe and their later growth, both in size and importance, with the idea of uncovering contributing factors and finding common patterns in their rise and development.
2. Depending on the national venue, there were different causes favoring the emergence of a local mathematical community. Nevertheless, there were also some common features. Among the latter particularly important were **educational reforms** on a state scale. After the activity of Commission of National Education in Poland and changes introduced by the Revolution in France (both in the 18th century), there came reforms in Prussia, which begun after her loss in the Napoleonic wars and became widely influential in Central and Central-Eastern Europe. There were two main Prussian reformers, Wilhelm von Humboldt (1767–1835) and Friedrich Schleiermacher (1768–1834). Both deserve the credit for the establishing clear distinction between the levels of secondary schools and of higher education and the interplay between them, and for the founding Berlin university in 1810 which soon became the highly influential model of a modern university.

   In their vision the main task of secondary schools was general education based upon common curriculum imposed by the state and controlled by it. In contrast, in universities the emphasis was placed upon research, accompanied and complemented by academic freedom, in German *Lehr- und Lernfreiheit*, that is, the freedom to teach and to do research without any interference. Its implementation in the Berlin university has turned that university into a model one. The main type of a secondary school was *gymnasium* (usually consisting of eight grades); its final exam called *matura* conditioned the university immatriculation. In turn, only university graduates could apply for a teacher's position in gymnasium. In that way a positive feedback between secondary and higher education appeared. As a result, universities were getting well prepared students eager to develop their intellectual interests.

3. Prussian reforms produced a strong stimulus towards **professionalization**: mathematics became an attractive profession offering a high social status, either in the form of a well-paid job in the expanding secondary school system or an academic career within similarily expanding higher education system. There appeared well-trained professional mathematicians whose main job was to develop mathematics and/or to teach it.

   The German system has also brought with it a new vision of the role of a university professor. It consisted since then both in teaching and in doing research and was highly esteemed. In the specific case of mathematics, the new research ethic ultimately brought with it a greater **specialization** in the field, as mathematicians and mathematicians-to-be tended to focus their studies more narrowly in an effort to make their own personal contributions. Developing along these lines, the system contributed to the exponentially rapid growth of the volume of mathematics and to specialization within the field. These aspects of the development of mathematics in Germany could be clearly seen at the University of Berlin under Dirichlet, Kummer, Weierstrass, and Kronecker and at the (formerly Hannoverian) university in Göttingen [17].
4. One of the main tasks of a university professor was to prepare future researchers. The principal vehicle for the active training of young researchers became a **seminar** which offered a collaborative study of recent research papers under the leadership of a senior researcher and a joint criticism of research attempts of young attendants. Seminars proved to be highly effective and remain to this day an irreplaceable and indispensable tool in producing new researchers.

   In support of the rising mathematical community came **journals** and **associations**. The oldest scientific journals, comprising mathematics, were academy proceedings which appeared in the 17th century and were followed later by special mathematical journals publishing articles and reviews, the first of which appeared at the beginning of 19th century. One of the first was (still existing) *Journal für die reine und angewandte Mathematik*, founded 1826 and called "Crelles journal" after its long-time editor. The number of such journals, together with the number of published articles, rose so fast that in 1871 there appeared *Jahrbuch über die Fortschritte der Mathematik*, the first review journal. It ceased to exist but there are some followers.

   The growth of community has been accompanied and supported by professional associations. Among the first were *Moscow Mathematical Society* (founded 1864) and *London Mathematical Society* (founded 1865), followed soon by national ones: *Société Mathématique de France* (since 1872), *Deutsche Mathematiker-Vereinigung* (founded 1890), *Societa Italiana de Matematica* (founded 1908) and others. Poland joined the general development in 19th century but founding of the *Polish Mathematical Society* was delayed until 1919, when political changes allowed it.

   "These institutions – the graduate seminar, the specialized journal, the specialized society – together with the twin values of research and teaching largely defined the profession and, in subtler ways, the discipline of mathematics as it had developed in Germany by the end of the nineteenth century. These same institutions and values informed the emergence of mathematical research communities in a number of other countries as well and thereby served to build a common foundation for the subsequent internationalization of the field" [17, p. 1583].

In fact, the triple pattern – seminar, journal, society – could be observed in a number of countries in Central-Eastern Europe, sometimes with slight changes. For example, in partitioned Poland (period 1795–1918) national organizations of any sort were forbidden and a greater role was played by leaders enjoying authority, analogous to that of a president of a society. Professional mathematical associations in Poland and in some other countries of Central-Eastern Europe appeared only after World War I.

5. Educational reforms implemented in Germany were soon imitated, with some rather inessential changes, in the empires of Austria and Russia. They brought new life into old universities and produced some new ones. The basic principle of *Lehr- und Lernfreiheit* was preserved, thus accelerating the process of professionalization and strengthening the awareness of the significance of mathematics and of mathematicians, but also raising national ambitions.
6. Modern mathematics in Russia began with the founding of the Academy of Sciences in Petersburg by Peter the Great in 1724. The first mathematician in the Academy was Leonhard Euler (1707–1783), who worked there for most of his adult life (1727–1741 and 1766–1783). Yet there was no Russian mathematical community around. Euler worked alone, writing and communicating in Latin, French or German, and he had no students of his own. Nevertheless, his value for the Russian science was enormous. He showed that it was possible to do mathematics in a remote country and how to do it.

   In the 19th century Russia underwent elaborate educational reforms, originally patterned after those of the Polish Commission of National Education and soon thereafter following the German model. Several state universities were revived or newly founded (originally there were six – Vilnius, Dorpat, Petersburg, Moscow, Kazan, Tomsk – but later their number slowly rose). In the lack of native mathematicians a governmental policy was to send talented youngsters abroad with the objective to learn mathematics there and to come back with that knowledge. In that way first Russian mathematicians of world renown emerged: Victor Jakovlevič Bunjakowski (1804–1889) and Michail Vasil'evič Ostrogradskij (1801–1862), both had the merit of the initiation of the early mathematical community in Petersburg.
7. Mathematical communities appeared also in Dorpat, Kazan and Moscow. The last one grew to a larger significance with the figures of Nikolai Vasil'evič Bugaev (1837–1903), who had studied both in Berlin and in Paris for two-and-a half years beginning in 1863 and then returned to Moscow to influence colleagues and students through extensive teaching and vigorous support of the *Moscow Mathematical Society* (founded 1864) and of its journal *Matematičeskij Sbornik* (founded 1866). Bugaev taught a wide range of courses in, for example, number theory, the theory of elliptic functions, the calculus of variations, the theory of analytic functions etc. Moreover and most importantly, he wished to train students capable of contributing to further development of these subjects at a research level. "He also fostered and contributed to a philosophical atmosphere in which mathematics was interpreted essentially as a theory of functions and where the theory of discontinuous functions played a key role. This conception not only proved conducive to the acceptance of Georg Cantor's novel set-theoretic ideas but also

served as the foundation of the Moscow school of function theory, spearheaded in the early decades of the 20th century by Bugaev's student, D.F. Egorov, and perpetuated by Egorov's disciple, N.N. Luzin" ([17, p. 1587]; see also [20]).

The case of Moscow University "drives home the obvious point that the success of the mathematical endeavor in a given national context depends crucially on the process of training talented students in areas rich in interesting, open questions. At its core, mathematics undeniably involves proving theorems, and these students not only learned how to carry out that creative process successfully but also embraced the belief that they should pass on their insights to a subsequent generation. As they had been trained, so should they train – this philosophy came to characterize the mathematical mission internationally in the latter quarter of the 19th century. Moreover, in concert with the other factors examined above, it encouraged the formation of self-sustaining mathematical communities, that is, interacting groups of people linked by common interests" [17, p. 1587].

8. The case of Poland was different. There were old universities in Cracow (founded 1364), Vilnius (founded 1578) and Lvov (founded 1661), there were reforms introduced and executed by the Commission of National Education, which existed 1773–1794 in then still-independent country, with the noticeable progress and good prospects for further development [11]. However, in 1795 the state was partitioned among the three neighbors (Russia, Prussia, and Austria) and a century of national oppression began. The universities in Cracow and Lvov (which turned to be in the Austrian empire) fell into decline, while the old university in Vilnius, and the newly founded (in 1816) university in Warsaw (both in the Russian empire) were closed in 1832 after the unsuccessful November uprising against Russia.

The revival came only half a century later with the autonomy granted in the 1870s to the province of Galicia (embracing Cracow and Lvov) which led to a subsequent re-polonization of universities in Cracow and Lvov, followed by their rapid growth and supported by some organizational and editorial initiatives in Warsaw.

In Cracow Franciszek Mertens (1840–1927) became a professor in 1865–1884; an excellent mathematician himself, he had no disciples of his own. However, two other mathematicians came to Cracow: in 1895 Kazimierz Żorawski (1866–1953) and in 1900 Stanisław Zaremba (1863–1942), and their influence was much wider.

An important development took place also in Lvov [6]. At the university there was Józef Puzyna (1856–1919) who taught extensively before World War I, giving nearly 30 courses in different branches of mathematics and writing an original monograph on analytic functions. With the view to forming a prospective mathematical community in Lvov, he invited Wacław Sierpiński (1882–1969) in 1908 and Hugo Steinhaus (1887–1972) in 1917. Sierpiński taught there set theory and led a seminar on "applications" of that theory to topology and to the theory of real functions; among the participants were Zygmunt Janiszewski (1888–1920) and Stefan Mazurkiewicz (1888–1945), who formed a core of what later became known as the "Warsaw school of topology". At the end of World War I Janiszewski happened to be in Warsaw and then proposed (in 1918) a program how "to win an independence for Polish mathematics". The essence of the program consisted in joining efforts to work in a chosen one new field of mathematics (which eventually became "the set theory and its applications"), to work collectively and in a friendly manner (in a seminar), and to found a journal devoted to the chosen field (*Fundamenta Mathematicae*, 1920–) [7, 15]. After his untimely death the program was continued by Sierpiński, who then just came to Warsaw after the internment in Russia, and by Mazurkiewicz who also happened to be in Warsaw in that time, and the two men became leaders of the *Warsaw school of mathematics* [13, 14].

Similar development took then place in Lvov where Steinhaus invited Stefan Banach (1893–1945) to join him. Banach's Ph.D. thesis from 1920 in which he introduced "Banach spaces" became the cornerstone of functional analysis. Both Steinhaus and Banach became leaders of the Lvov school of mathematics, soon supported by the journal *Studia Mathematica* (founded 1929) devoted to the "theory of operators" [10]. The two new Polish journals, *Fundamenta Mathematicae* and *Studia Mathematica*, were the very first specialized mathematical journals in the world. The role of a research seminar in Lvov, however, was taken over by sessions of the Lvov branch of the Polish Mathematical Society and by the famous Scottish Café together with its equally famous *Scottish Book* [16, 18].

The two schools, Warsaw and Lvov alike, became soon known as the *Polish school of mathematics* [12, 13].

9. In Bohemia there was the old university in Prague (founded 1348) with the long mathematical tradition behind it. However, after the Thirty Years War (1618–1648) and under the Habsburg rule the country was being steadily Germanized to the effect that in the first half of 19th century the dominant language of its culture and science was German and up to the end of 1850s the language of instruction in secondary schools and universities, including Prague university, was solely German. But then came the Czech national revival and in 1871 there appeared first Czech mathematical courses at the Prague university; Czech departments were founded, including mathematical ones. A relentless work began on the modernization of Czech mathematical terminology, accompanied by translations into Czech of some old books and writing new original schoolbooks in Czech . The leading role in that movement, aimed at the establishment of Czech national mathematical community, was played by the *Czech Union of Mathematicians* (*Jednota českých mathematiků*), founded in 1869, and its *Journal for Cultivation of Mathematics and Physics* (*Časopis pro pěstování mathematiky a fisiky*), founded in 1872. In that way, already at the turn of 20th century, the Czech mathematical community appeared with a number of mathematicians publishing in Czech and in foreign languages (predominantly German or French) and aiming at attaining the world level.

   The Czech case is an outstanding example of a mathematical community which emerged slowly from the grass-root level and without any single leader, due to a common effort of the whole generation.

10. One of the main aims of teaching mathematics at the Prague university was to prepare teachers for secondary schools. In view of the "overproduction" of mathematical departments many of the would-be Czech teachers were leaving Bohemia going to other parts of the Austrian-Hungarian empire, predominantly to Balkans.

"After their arrival, they learned the respective foreign language and begun to create curricula for the teaching mathematics and descriptive geometry at the secondary schools and at universities. For they colleague-teachers, they wrote the first methodological manuals about the teaching mathematical subjects in their mother tongues. For their pupils they created the first brief teaching manuals and collections of mathematical exercises (…). During the few first years, they translated Czech textbooks of mathematics and of descriptive geometry to other languages (…). They set a form for the first generations of students educated in their mother tongues. In the second phase of their "mission" – usually at the end of the first decade of their stay – still inspired by Czech models, they wrote new textbooks for secondary schools and universities (…). These textbooks were widespread and used until the end of the World War I. Thanks to their quality education, high professional standard and all around activities they contributed to the creation of the mathematical terminology that has been used – except for a few modifications – until today (…). On the basis of their good experience from Bohemia they led local mathematical communities to the unification of professional associations (…) and initiated publishing professional, educational and popularization periodicals (…). In addition, they participated in the international promotion of the results of professional and pedagogical research (…)" [3, p. 43-45].

The early leading person in Croatia was Jan Pexider (1831–1873) but his premature death did not allow him to exert a deeper influence. Karel Zahradnik (1846–1916) turned out to be more influential: he became a professor of a newly founded university in Zagreb in 1875 and for many years was the only professor of mathematics there, teaching algebra, geometry, analysis, number theory, probability. His "mathematical seminar", which he led since 1886, became a breeding ground for the emerging Croatian mathematics.

To Sarajevo in Bosnia and Herzegovina came Alois Studnička (1842–1927), who worked there 1893–1907 in a technical school, significantly influencing the development of the Serbian educational system by, among others, teaching, promoting Serbian terminology, elaborating curricula etc.

An interesting development took place in Bulgaria. After the country get rid of the Turkish hegemony in 1878, it began to build its own educational system. The first "Bulgarian" university professor in mathematics in Sofia was Theodor Monin (1858–1893). He came to Bulgaria from Prague at the age of 23 to teach at a grammar school in Liven in 1881–1886, so when he got in 1887 his university appointment, he was already well versed in Bulgarian and was treated as a Bulgarian. Monin had ambitious plans to write several mathematical textbooks in Bulgarian but in 1891 he fell ill and died soon thereafter. Another Czech, Antonín Václav Šourek (1858–1926) was more lucky and eventually became more influential; he also taught first in Bulgarian schools (since 1880) but later took over the chair from Monin in 1893 and stayed in Sofia until 1914. During the World War I he worked in Bulgarian diplomacy abroad and in 1921 he resumed his university post. He succeeded in covering with his school texts in Bulgarian several branches of mathematics including plane trigonometry, solid geometry, analytic geometry, spherical geometry, and descriptive geometry. He also published several university texts on analysis, analytic geometry, synthetic geometry, descriptive geometry, and algebra. He also rendered significant services in founding *Physical and Mathematicl Society in Sofia* (1896) and its *Journal of Physical and Mathematical Society* (1904). Being also one of the founders of the Bulgarian mathematical terminology, Šourek is among the most renown Bulgarian mathematicians of the time.

Among Czech mathematicians active in Bulgaria there were also František Vítězslav Splítek (1855–1943) and Vladislav Švak (1860–1941).

11. The independent state of Lithuania was established only after World War I but Vilnius, its ancient capital, became a part of Poland. Vilnius preserved the memory of the old university there (founded 1578 and closed by Russians in 1832) and so, referring to that old tradition, Poland has re-established it in 1919 under the name of Stefan Batory University (the king of the Polish-Lithuanian state who founded the university in 1578).

    Since Vilnius became part of Poland, Lithuanians have founded their own university in Kaunas, the capital of Lithuania at that time, but there was no mathematical tradition behind it. The Faculty of Mathematics and Natural Sciences has decided to appoint staff by competition, addressed to some European universities. Among the applicants there was a German mathematician Otto Volk (1892–1989) and the Faculty decided to nominate him.

    Otto Volk completed his studies and passed teacher's examination in 1917 in Stuttgart. In 1919 he got doctor's degree in engineering (for a mathematical work in potential theory) and moved to Munich, where he received Ph.D. degree in 1920 for a work in pure mathematics (on elliptic functions) and in 1922 he completed his habilitation in mathematics, thus acquiring the rank of Associate Professor. Volk came to Kaunas in 1923 and stayed there until 1930, when he became a professor at Würzburg university (he remained there until his death). However, the years 1923–1930, which he spent in Kaunas, were very productive. Giving many courses, he supervised 31 M.A. degree in mathematics and 3 of his students (Petras Katillius, Paulius Slavenas (1901–1991) and Otanas Stanaitis (1905–1988)) were admitted to doctoral studies at Heidelberg, Yale and Würzburg, respectively. Thus due to his commitment and devotion a Lithuanian mathematical community arose.

12. The Ukrainian national revival came late in the 19th century, shaping itself in opposition to both the Russian and the Polish cultural domination. The very first mathematical paper in the Ukrainian language has appeared only in 1894 (translated from the Polish original version) and from that time on the elaboration of the Ukrainian mathematical terminology proceeded; the terminology was introduced subsequently into school textbooks in Ukrainian schools. There was no Ukrainian mathematics at the university level as Poles did not permit the founding of Ukrainian university in Lvov and the Soviets battled against Ukrainian "national deviation" in their geopolitical part. The slow progress came after World War II but the Russian language still dominated and relations with the West were few. The whole situation changed after 1990, when Ukraine became independent.

13. The emergence of national mathematical communities in remote (from a Western viewpoint) Central-Eastern European countries is a fascinating story. In general,

it followed the Western triple pattern but it has also a local flavor. Nevertheless, the phenomenon testified to the vitality of newly risen old nations whose striving for independence also expressed itself in the tendency to attain high level of development in the most noble science of mathematics. Those national ambitions were largely supported by educational reforms which took place in the 19th century in all countries considered and which resulted in creating many posts and allowing many people to make mathematics their life mission.

In Russia the story began with Euler, who set a pattern for doing mathematics but had no immediate followers. The first native Russian mathematicians appeared in the first half of 19th century (Buniakovski, Ostrogradskij) but the more influential community arose in Moscow a few decades later with the leaders who founded a professional association, a specialized journal, and who began to teach modern mathematics and to train future researchers. Great names in that movement were those of Nikolai Vasilievič Bugaev (1837–1903), his disciple Dimitrij Fedorovič Egorov (1869–1931) and his student Nikolai Nikolaevič Lusin (1883–1950), the last one together with his group called Lusitania. Thus there appeared the great Moscow school of mathematics, the leader of Russian and Soviet mathematics [20].

The old universities of Cracow and Vilnius in the Polish-Lithuanian Commonwealth were reformed at the end of 18th century with good prospects for the future, but as a consequence of partitions in the 19th century there came the time of national oppression on the part of Russia, Prussia and Austria, and several national uprisings in turn. In the shadow of the movement towards regaining independence there was, however, a slow progress, augmented by the autonomy granted to Galicia (including Cracow and Lvov) and by some relaxation of the Russian policy in central Poland (with Warsaw), towards establishing a material basis for further development. The basis included re-polonized universities in Cracow and Lvov, a newly founded (in 1872) academy in Cracow, journals founded in Warsaw etc., which after War World I gave support to the Polish school of mathematics with its two branches in Warsaw and Lvov. The school – initiated by Janiszewski in 1918, led by Sierpiński, Mazurkiewicz, Steinhaus, and Banach, and supported by the journals *Fundamenta Mathematicae* and *Studia Mathematica* – soon gained the international recognition [12–14].

For centuries Bohemia was a part of the Habsburg empire but in the middle of 19th century there came a national revival with the tendency to restore the Czech language in culture and science. In the shadow of German political and cultural domination there began a slow but steady grass-root movement which started with the introduction of Czech primary and then secondary schools, textbooks in Czech, first Czech mathematical courses in the Prague university, founding a mathematico--physical association with its specialized journal etc. The movement resulted in the emergence of Czech national community but with no genuine leaders the community did not become then a truly creative mathematical center. First Czech mathematician of international rank came only after World War I.

A great merit of Czech mathematical community consists in influencing the emergence of national mathematical communities in the Balkan states of Slovenia, Bosnia and Herzegovina, and Bulgaria. In that last country Czech mathematicians deserve particular credit for initiating Bulgarian mathematical terminology, founding its mathematico-physical professional association and its journal.

A still different development took place in Lithuania. There was a national revival in the 19th century but when Lithuania has got its independence in 1918 there was no single Lithuanian mathematician of significance. Lithuanian mathematics began with the German Otto Volk, who taught in Kaunas for 7 years and educated some good students who later became university professors. From those rather modest beginnings a wider Lithuanian mathematical community with its own journal *Litovskij Matematičeskij Sbornik* arose after World War II under the auspices of Soviet mathematics (Lithuania regained its independence only in 1990). The Ukrainian situation is delicate as the country is still strongly influenced by Russian mathematics, although its contacts with the West became more vivid after 1990.

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2-NP/2015

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LUDWIK SILBERSTEIN AND THE OPERATOR CALCULUS

LUDWIK SILBERSTEIN I RACHUNEK OPERATOROWY

A b s t r a c t

In the article we outline the life of Ludwik Silberstein (1872–1948). We present his approach to the matrix calculus and its application to the operator form of relativity. We also give the list of books on the subject written by him as well as translated by him.

Keywords: *Ludwik Silberstein*, *matrix calculus*, *theory of relativity*

S t r e s z c z e n i e

W artykukle przedstawiamy zarys życiorysu Ludwika Silbersteina (1872–1948). Omawiamy podejście do rachunku macierzowego i jego zastosowanie do podania operatorowej postaci teorii względności. Prezentujemy również listę książek i tłumaczeń z różnych języków dotyczących tej tematyki.

Słowa kluczowe: *Ludwik Silberstein*, *rachunek macierzowy*, *teoria względności*

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1. Historical background

Ludwik Silberstein is regarded by historians of science as a Polish-American mathematical physicist.

Silberstein was born on May 15th, 1872 in Warsaw, Poland, and died in Rochester, USA on January 17th, 1948. His parents were Samuel Silberstein and Emilia, nee Steinkalk. The father took great care of the education of the three children. Henryk (1858–1890) obtained his PhD with Lothar Meyer 1882 in Tubingen in chemistry [4], while Ludwik's sister Adela (1874–?) obtained a doctorate in philosophy in Zurich [3].

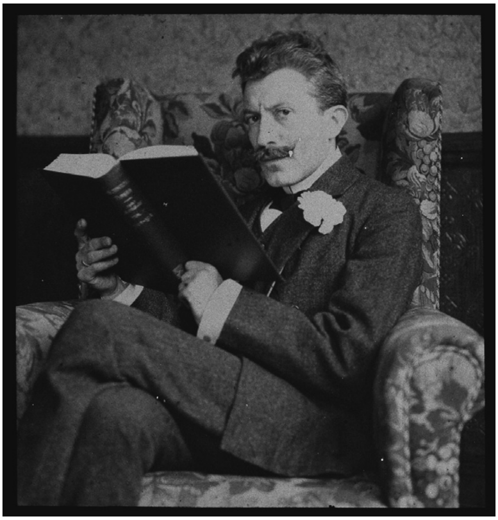
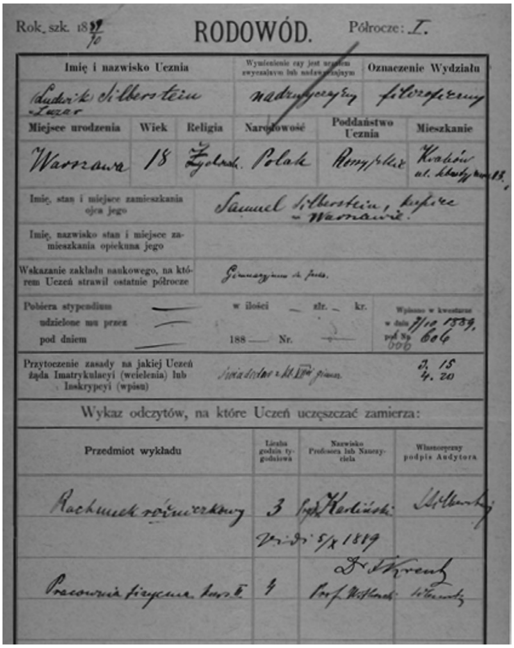


Fig. 1. Ludwik Silberstein in his home about 1915 (source: Courtesy of George Eastman House, International Museum of Photography and Film, Rochester, USA)

Ludwik went to a high school in Warsaw, and later passed the examination to attend the sixth class of the St. Hyacinth Imperial-Royal Gymnasium (c.k. Gimnazjum św. Jacka) in Cracow. His given name was Lazar, which he used until the end of high school. In 1890, he studied at Cracow's Jagiellonian University, even before completing his secondary education. He was attending calculus and laboratory classes conducted by professors August Witkowski (1854–1913) and Franciszek Karliński (1830–1906) respectively, and later always listied them among his most important teachers. There, at the Jagiellonian University, the name `Ludwik' appeared for the first time. The name 'Lazar' is seen in the matriculation document, because all school documents contained this name; however, afterwards the name 'Lazar' never appeared. From the document it follows that he regarded himself as a Pole (Fig. 2).

After a short period of study in Cracow, Silberstein spent one semester in Heidelberg and several years in Berlin, where he obtained his doctorate at the faculty of philosophy (where Max Planck was the dean) at Berlin's Friedrich-Wilhelms-University. Von Helmholtz wrote an opinion of Silberstein's dissertation *Ueber die mechanische Auffassung*


Rok szk. 1889/90 RODOWÓD. Półrocze: I.

Imię i nazwisko Ucznia	Wymienienie czy jest uczniem zwyczajnym lub nadzwyczajnym	Oznaczenie Wydziału
Ludwik Silberstein – Lazar	nadzwyczajny	filozoficzny

Miejsce urodzenia	Wiek	Religia	Narodowość	Poddaństwo Ucznia	Mieszkanie
Warszawa	18	Żydowsk.	Polak	Rosyjskie	Kraków

Imię, stan i miejsce zamieszkania ojca jego: Samuel Silberstein, kupiec w Warszawie.

Imię, nazwisko stan i miejsce zamieszkania opiekuna jego

Wskazanie zakładu naukowego, na którem Uczeń strawił ostatnie półrocze

Pobiera stypendium — w ilości — złr. — kr. udzielone mu przez — pod dniem — 188— Nr. — Wpisano w kwesturze w dniu 7/10 1889, pod Nr. 606

Przytoczenie zasady na jakiej Uczeń żąda Imatrykulacyi (wcielenia) lub Inskrypcyi (wpisu)

Wykaz odczytów, na które Uczeń uczęszczać zamierza:

Przedmiot wykładu	Liczba godzin tygodniowa	Nazwisko Profesora lub Nauczyciela	Własnoręczny podpis Audytora
Rachunek różniczkowy	3		
Pracownia fizyczna	4		

Fig. 2. Matriculation document of Ludwik Silberstein Wpis Ludwika Silbersteina na studia w Uniwersytecie Jagiellońskim (source: Jagiellonian University Archive, Cracow, Poland)

elektromagnetischer Erscheinungen in Isolatoren und Halbleitern [5], which circulated in the faculty from December 18, 1893.

Von Helmholtz wrote: “Mr Silberstein’s work shows clear understanding of difficult and abstract problems and aptness for mathematical treatment. I propose its assessment as: “sollertiae et ingenii specimen laudabile”. Below this document is a short note by the dean: “I agree in all points” and the signature of Max Planck. On July 5th, the PhD examination took place – the examiners were Max Planck in physics, Karl Hermann Schwarz in mathematics, Hans Landolt in chemistry, and Carl Stumpf in philosophy.

From 1895–1897 Silberstein was an assistant at the Politechnika Lwowska (Lviv Technical School) where Prof. Olearski was the head of the physics department. The scientific work and collaboration with Prof. Olearski were fruitful, but it was impossible for Silberstein to continue his work in Lviv. At that time Poland did not exist, having been partitioned by three empires: Russian, Prussian and Austro-Hungarian. As he was born in Warsaw, Silberstein was a Russian subject, while both Cracow and Lvov (Lviv, Lemberg) were situated in the Austro-Hungarian Empire. The authorities did not want to employ foreigners. His best-known paper from that period was probably *O tworzeniu się wirów w płynie doskonałym* [7], which allowed Bjerknes to formulate his theorem on turbulent motion, which formed the foundation of meteorological investigations and weather forecasting based on numerical calculations (1904) (application to eddy currents in the atmosphere).

In 1899 Silberstein took the position of lecturer (*libero docente*) in mathematical physics in Bologna, in 1904 moving to La Sapienza University in Rome. He was affiliated with Rome University until 1920, even though in the period 1912–1913 he was a lecturer in general relativity at London's University College, where the textbook *General Relativity* was written and published (1914). In 1920 he left Europe and went to the USA, where he remained until the end of his life, living in Rochester and collaborating with the Eastman Kodak Laboratory. He published several books, with the phrase "former lecturer at Rome University" appearing below his name, which shows he was proud of this work.

On 29 June 1905 he married Rose Eisenman from Warsaw, and they had three children (Georg P., Hedwig and Hannah).

He maintained close connections with Poland practically until the end of his stay in Europe. He corresponded with several physicists, was the organizer of the summer school of theoretical physics in Zakopane in 1904, and was very helpful in the organization of Polish science. In a letter to Banachiewicz, Sierpiński (1917) advised him to ask Silberstein in order to obtain books published in Europe.

2. Silberstein and matrix calculus

Silberstein translated several books dealing with methodology and methods of scientific investigations, written in several languages: Polish, German, French and English (see appendix A). Also, throughout his life he wrote textbooks on vectors and operators (see appendix B).

Matrix calculus was his most prominent subject of study, and he was one of the first to notice the advantages of using this calculus in physics. There are several papers written as early as 1901 and 1902. Interestingly, he proposed the quaternion form of general relativity [10]. As we know, almost twenty years later this calculus found its main application in quantum mechanics. The novelty and importance of these papers was noted by as great a specialist as Jammer [1].

In his book The conceptual development of quantum mechanics [1] Max Jammer wrote:

> "In concluding our brief survey on the early development of operators in mathematical physics, we should like to draw the reader's attention to a rather unknown paper by Ludwik Silberstein [9], which anticipated to some extend the formal aspects of the operational approach in modern quantum mechanics. The study of symbolic integrals of the equations of the electromagnetic field suggested to Silberstein [8], who is known mostly for his writings on the theory of relativity, a theory of "physical operators", in terms of which he attempted to give a unified representation of such disparate phenomena as mechanical oscillations, heat conduction, and electrodynamic process. Defining the "state" of a physical system by a time-dependent function $\psi(t)$, Silberstein introduced what ha called "chrono-operators" $\begin{Bmatrix} H \\ t \end{Bmatrix}$ by means of

which the state at time $t = t$ can be determined from the knowledge of the state at time $t = 0$,in accordance with the equation $\psi(t) = \begin{Bmatrix} H \\ t \end{Bmatrix} \psi(0)$. He defined the inverse operator $\begin{Bmatrix} H \\ t \end{Bmatrix}^{-1}$ by $\begin{Bmatrix} H \\ -t \end{Bmatrix}$, showed that $\begin{Bmatrix} H \\ nt \end{Bmatrix} = \begin{Bmatrix} H \\ t \end{Bmatrix}^n$ and that $\begin{Bmatrix} H \\ t \end{Bmatrix} = \{e^{Ft}\}$, where F is another operator, connected with H in a definite manner, but independent of t. Silberstein even spoke of the superposition of states and mentioned numerous other details which, including the notation, made their appearance in the operational formalism of quantum mechanics some twenty-five years later. “Every class of physical phenomena”, declared Silberstein in 1901, “or at least those amenable to quantitative treatment, will be characterized by corresponding physical operators; and the scientific study of natural phenomena will proceed by detailed examinations of the properties of these operators, based on observation and nurtured by experiment”.

3. Quaternion form of general relativity

Quaternions are quadrupels of real numbers. Quaternions can be written in different ways:

1. The matrix form

$$\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} = \begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix}$$

where $z = a + bi$, $w = c + di$.

2. The algebraic form

$$i = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad k = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

with a unit matrix 2 × 2 for real r. $r = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$. Each quaternion can be explicity write as $q = a + bi + cj + dk$, where a, b, c, d are real.

Quaternions satisfy the following relations:

$$i\hat{}2 = j\hat{}2 = k\hat{}2 = ijk = -1$$

$$ij = -ji = k$$

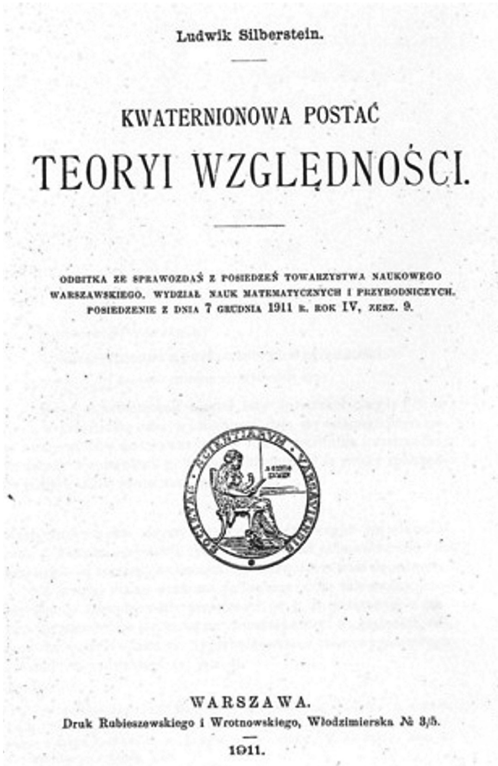
$$jk = -kj = i$$

$$ki = -ik = j$$

$$1q = q1 = q \text{ for each } q$$

$$rq = qr \text{ if } r \in \mathbb{R},\ q \text{ a quaternion}$$

$$q\overline{q} = |q|^2$$



Ludwik Silberstein.

KWATERNIONOWA POSTAĆ

TEORYI WZGLĘDNOŚCI.

ODBITKA ZE SPRAWOZDAŃ Z POSIEDZEŃ TOWARZYSTWA NAUKOWEGO WARSZAWSKIEGO. WYDZIAŁ NAUK MATEMATYCZNYCH I PRZYRODNICZYCH. POSIEDZENIE Z DNIA 7 GRUDNIA 1911 R. ROK IV, ZESZ. 9.

WARSZAWA.
Druk Rubieszewskiego i Wrotnowskiego, Włodzimierska № 3/5.

1911.

Fig. 3. The cover of the book *Kwaternionowa Postać Teoryi Względności*

We thank several archives for giving us access to the documents. This work started with the collaboration of dr Hilmar Duerbeck (1948–2014), whose absence is very sad and unfortunate.

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Appendix A

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2-NP/2015

KAROLINA KARPIŃSKA*

TEACHING THINKING IN TERMS OF FUNCTIONS – FULFILLING THE FUNDAMENTAL IDEA OF THE MERANO PROGRAMME AT TORUN CLASSICAL GRAMMAR SCHOOL IN THE EARLY TWENTIETH CENTURY

WYRABIANIE NAWYKÓW MYŚLENIA FUNKCYJNEGO – REALIZACJA POSTULATU PROGRAMU MERAŃSKIEGO W TORUŃSKIM GIMNAZJUM KLASYCZNYM W PIERWSZYCH LATACH DWUDZIESTEGO WIEKU

A b s t r a c t

In this article, one of the main postulates of the Merano Programme for teaching mathematics will be analysed, namely: teaching thinking in terms of functions. This postulate will be discussed in the context of its implementation at Torun Classical Grammar School. Through the detailed analysis of mathematics curricula implemented in the Torun lassical Grammar School at the beginning of the twentieth century, the mathematical textbooks then used and school-leaving examinations problems from the years 1905–1911, the degree of fulfillment of the said postulate of the Merano Programme at the Torun school will be assessed.

Keywords: *mathematics*, *functions*, *dependent variables*, *Merano Programme*, *Torun Grammar School*, *Torun Classical Grammar School*, *Torun Real School*, *school-leaving examinations in the early 20th c.*

S t r e s z c z e n i e

W niniejszym artykule analizie zostanie poddany jeden z głównych postulatów Programu Merańskiego dotyczących nauczania matematyki, czyli wyrabiania nawyków myślenia funkcyjnego. Postulat ten omówimy w kontekście jego realizacji w toruńskim Gimnazjum Klasycznym. Szczegółowa analiza programów nauczania matematyki realizowanych w toruńskim Gimnazjum Klasycznym na początku XX wieku, stosowanych wówczas podręczników do matematyki oraz zestawów zadań maturalnych z lat 1905–1911, pozwoli ocenić stopień realizacji wspomnianego postulatu Programu Merańskiego w szkole toruńskiej.

Słowa kluczowe: *matematyka*, *funkcje*, *zmienne zależne*, *Program Merański*, *Gimnazjum Toruńskie*, *toruńskie Gimnazjum Klasyczne*, *toruńska Szkoła Realna*, *egzaminy maturalne w pierwszych latach XX wieku*

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1. Introduction

The Merano Programme was the reform of teaching mathematics and natural sciences, which has been prepared for secondary schools in Prussia: Classical Grammar Schools, Real Grammar Schools and Higher Real Schools. The Merano Programme was announced in 1905 at the conference of the Society of German Natural Scientists and Physicians (dt.: *Gesellschaft Deutscher Naturforscher und Ärzte*) in Merano[1]. Its main author was Felix Klein – a mathematician of Göttingen.

2. Postulates of the Merano Programme

The Merano Proramme changed the overall organization of secondary education. It stated that since 1905 three types of schools in Prussia: Classical Grammar Schools, Real Grammar Schools and Higher Real Schools, had the same rights. Since 1905, each of these schools had to put equal emphasis on education in terms of both mathematics and natural sciences, as well as philological and historical sciences [17, p. 17]. Thus, Grammar Schools lost their one-sided humanistic direction, and Real School – its mathematical and natural sciences direction. Each school was giving a general education and their graduates would join a university on equal terms[2].

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1 Merano is a city located in the southern part of Tyrol, northern Italy. Tyrol in the years 1814–1919 was a part of Austria [19, p. 117]. After World War I, under the treaty of St. Germain (20 September 1919), its southern part, along with Merano, was annexed to Italy [13, pp. 69-70].

2 Up to 1905, the school-leaving certificate obtained in Grammar Schools, Real Grammar Schools and Higher Real Schools, enabled the commencement of study at university or institute of technology under certain conditions [17, p. 11]:

- if a graduate of Grammar School wanted to enter an institute of technology, he had to pass an exam in mathematics and natural sciences; this obligation did not concern Real Grammar Schools and Higher Real Schools graduates,
- if a graduate of Real Grammar School or Higher Real School wanted to start studying at a faculty different from mathematics or natural sciences, he had to provide a certificate of sufficient knowledge of ancient languages,
- medical and theological studies were open only to people who obtained the school-leaving certificate in Grammar Schools.

Additional exams aroused a fear in Grammar School students of studying mathematics and natural sciences [17, p. 11].

The aim of the Merano reform was to provide equal level of education at Grammar Schools, Real Grammar Schools and Higher Real Schools, and thus the elimination of additional entry exams at university and institute of technology.

In the case of mathematics lessons, the Merano Programme recommended the same curricula and timetables (four hours per week) in the corresponding classes of Grammar Schools and Real Grammar Schools. Higher Real Schools had the same curriculum framework as Grammar Schools and Real Grammar Schools, but they discussed certain issues in more detail. The hourly schedule of mathematics classes at Higher Real Schools was as follows (from the lowest class): 5, 5, 6, 6, 5, 5, 5, 5, 5 [11].

A compendium of guidelines of the Merano Programme connected with teaching of mathematics was as follows [17]:

General learning objectives:
- logical thinking,
- ability to think independently,
- ability to model natural phenomena mathematically,
- awareness that mathematics plays an important role in all areas of life and is essential for human and industrial society.

Specific learning objectives:
- development of spatial imagination,
- **teaching thinking in terms of functions**,
- combining various mathematical problems,
- paying attention to the application of mathematics,
- the balance between applications and theory in mathematics,
- a common approach to plane geometry and stereometry,
- emphasis on the history of mathematics.

Teaching methods:
- the genetic method – 'one should connect ideas, put new knowledge in an inseparable relation with the knowledge already gained, eventually associate the knowledge with the rest of the school curriculum, more and more, so that combination of knowledge would grow and students become more aware',
- the psychological principle – the material should be adapted to the course of the intellectual development of students,
- the principle of utility – showing that mathematics is important for everyday life.

Teaching thinking in terms of functions became a symbol of the mathematical part of the Merano Programme [17, p. 14]. In the curricula recommended by the Merano Programme it was clearly stated that up to the Lower Tertia this should be done by educating the student's intuition of variability – students should observe that the change of some values affects changes of others, they should observe the way of changing and notice that each result is in a relationship with changing some values. Ability to distinguish independent and dependent values, understanding the relationship between them and appealing to student's intuition, allowed for making in the Upper Tertia first graphical representations of such relationships (between two variables: the independent variable and the dependent variable) and for simultaneous introduction of the concept of a function and its graph. The students of the highest classes were to learn functions such as the quadratic function, trigonometric functions and the logarithmic function, check their properties and use them, for example, to solve equations. The only issue in which the Merano Programme left freedom to teachers was whether to discuss differential and integral calculus or not [17, p. 20].

Let us see if the curricula implemented at Torun Classical Grammar School after 1905 contained issues related to teaching thinking in term of functions and, if so, whether they were consistent with the guidelines of the Merano Programme.

3. Teaching thinking in terms of functions at Torun Classical Grammar School

3.1. The curricula of mathematics

Analysis of the curricula contained in the school reports of Torun Grammar School leads to the conclusion that in the years 1905–1911 the curricula of mathematics at Torun Classical Grammar School were identical [8] from year to year. Namely:

Sexta:	Arithmetic operations on integer numbers (absolute and denominate numbers). The German measures, weights and coins along with practicing decimal notation and easy calculations in the decimal system. Preparation for making calculations on fractions (4 hours per week).
Quinta:	Divisibility of numbers. Factorization. Common fractions. Four arithmetic operations on numbers written in the decimal system. Simple tasks using the Rule of Three (4 hours per week).
Quarta:	Calculations on decimal numbers. The Rule of Three (simple and composed) with integers and fractions. Tasks related to civic life, the simplest cases of calculating percents, interests and discounts. – Preparation for geometry. Exercises in using the compass and ruler. Learning about straight lines, angles and triangles (4 hours per week).
Lower Tertia:	The introduction of positive and negative numbers. Tasks that are based on solving equations of the first degree with one unknown. – Extending the learning on triangles. Parallelograms. Chords and angles in a circle. Construction problems (3 hours per week).
Upper Tertia:	Review of operations on fractions which are denoted with the use of letters. Proportions. Equations of the first degree with one or more unknowns. Powers whose exponents are positive integers. – Learning about the circle. Equality of figures in terms of the surface area. Calculating the area of straight-line figures. Construction problems (3 hours per week).
Lower Secunda:	Powers, roots, logarithms. Calculations using four-digit logarithmic tables. Simple quadratic equations with one and two unknowns. – The similarity, proportionality of straight lines in a circle, golden section. Regular polygons. Circumference and area of a circle, tasks associated with these problems. Construction problems (4 hours per week).
Upper Secunda:	Equations, in particular quadratic equations with more than one unknown. Harmonic points, radii of circles, secants. Application of algebra in geometry[3].

[3] In the textbooks of the nineteenth and early twentieth century, part of geometry dealing with applications of algebra in geometry was called algebraic geometry. The main task of algebraic

Construction problems, especially with using algebraic analysis[4]. Goniometry. Simple calculations related to solving triangles (4 hours per week).

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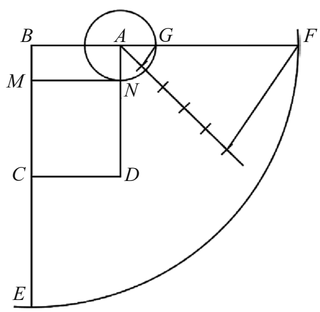
geometry was to find, through algebraic calculations, specific information about a given geometric figure (solid), taking into consideration the specified information relating to this figure (solid), for example: the length of some of its sides, heights, diagonals, measures of certain angles, or a relationship between some elements of this figure (solid) [1, p. 219].
Example of algebraic geometry problem:
Exercise [1, p. 247, ex. 22]. A triangle with area equal to a^2 is given. One of the sides of this triangle is divided into two parts, which are to each other as 3:7. Then through the dividing point of this side we draw a line parallel to the side of the triangle which is adjacent to the shorter part of the division. How big is each of the two parts of the triangle which are defined by this parallel line?
It was believed that of all the applications of algebra, the most important was its use in geometry: to formulate propositions and solve construction problems, using a method called algebraic analysis – this method was a part of algebraic geometry [1, p. 219].

4 The following scheme of reasoning was called 'the solution to the construction problem using algebraic analysis method': 'firstly, the unknown lines of the figure should be found using algebraic calculations, then the arithmetic expressions should be constructed and used to construct the whole figure' [1, p. 255].
Solution of the following exercise illustrates the way in which in the nineteenth century construction problems were solved by algebraic analysis:
Exercise [1, p. 255]. Divide a given rectangle into two parts by a line parallel to one side of this rectangle so that the circumferences of these parts are to each other as 2:3.
Solution [1, pp. 255-256]:



Analysis: Let a and b be the sides of a given rectangle, let the line dividing the rectangle be parallel to a and located at a distance of x from a; then:

$$2a + 2x : 2a + 2(b - x) = 2 : 3,$$

therefore:

$$x = \frac{2b - a}{5}.$$

Construction: Let $ABCD$ be a given rectangle; let us extend the side BC (through C) to the point E in such a way that: $BF = BE = 2b$, then $AF = 2b - a$. On AF we mark out $AG = \frac{1}{5}AF$; next, on the side AD we mark out $AN = AG$ and draw NM – a line parallel to AB. Thus, $ABCD$ is divided through NM into two parts satisfying the given condition.
Requirements: $2b > a$.

□

Besides the algebraic analysis which was a method of solving construction problems, there also existed a branch of mathematics with the same name: algebraic analysis. The main task of algebraic analysis, as a branch of mathematics, was to solve the three types of tasks ([15, pp. VI– VII).
Assumption: Let an equation $y = f(x)$ be given, where x is an independent variable, f – a function, y – a dependent variable.

Lower Prima: Arithmetic and geometric sequences; civic calculations; quadratic equations; extension of the concept of numbers on imaginary numbers. Apollonius's problems[5] according to the old methods and other construction tasks. Solving triangles using the sum and difference of their sides, radii of tangent circles, angles and heights of these triangles. The main theorems on relative position of points, lines and planes in space. Calculation of the areas and volumes of solids, such as: prism, pyramid, cylinder, cone and sphere. Revision of the material of previous classes (4 hours per week).

Upper Prima: Binomial theorem for exponents that are positive integers. The equations of higher degrees, which can be reduced to quadratic equations. Basic information about the coordinates, the equation of a straight line, a circle and conics. Construction problems. Repetition of stereometry; the introduction of certain formulas of spherical trigonometry in relation to the Earth and Sky[6].

Problem 1. Knowing x and f, determine y.
Problem 2. Knowing y and f, determine x.
Problem 3. Knowing x and y, determine the formula of the function f.

These tasks were related to one of the fundamental concepts of mathematical analysis: that of a function, and their solution required the use of algebraic calculations, hence the name 'algebraic analysis'.

In some cases, to the three problems above the following were appended (if they could be solved algebraically) [15, p. VII].

Problem 4. Knowing the formula of the function f, determine its properties.
Problem 5. Knowing the properties of the function f, determine its formula.

In the nineteenth century many textbooks were written containing an exposition of algebraic analysis, for example [7, 14, 15, 18].

In *Principles of algebraic analysis with supplements to the 'Arithmetic to school- and self-study'* [*Anfangsgründe der algebraischen Analysis nebst Ergänzungen zur Arithmetik für den Schul- und Selbst-Unterricht*] by K. Koppe [7], the chapter devoted to the algebraic analysis included the following topics: general comments about functions and the most important theorems about equations of higher degrees (for example, cubic equations were solved there), series (discussed, for example, were: convergence of series, exponential series, logarithmic series and periodic functions), complex numbers.

5 Apollonius's problems are problems that require the construction of a circle satisfying three conditions, each of which may have one of the following forms:
- the circle has to pass through a given point;
- the circle has to be tangent to a given line;
- the circle has to be tangent to a given circle [10, p. 16].

For example, the following problem is an Apollonius's problem: Construct a circle which passes through the three given points.

6 In addition, the curriculum of physics in the Upper Prima contained mathematical geography and mathematical astronomy. A comprehensive article about teaching of mathematical astronomy can be found in the school report of Grammar School in Bydgoszcz from the year 1906/1907 [5, pp. 3-22].

Revision of the previous classes, according to the guidelines of Mehler[7] (4 hours per week).

Note that there was no theory of functions at Torun Classical Grammar School curricula in the years 1905–1911, there were also no clear signs that habits of thinking in terms of functions were developed. On the other hand, it can be seen that the Merano Programme attached a great importance to showing functional relations, discussing the functions, their graphs and applications.

Bearing in mind that the curricula of Torun Grammar School were written concisely, and also that in the curricula of the top class (Upper Prima) of the Classical Grammar School in the years 1902–1905 there were 'remarks about functions' [8], after 1905, we will explore further the implementation of this issue of the Merano Programme (namely: teaching thinking in terms of functions) at Torun Classical Grammar School by analyzing the textbooks from that time and school-leaving examinations problems.

Were habits of thinking in terms of functions formed in mathematics lessons at Torun Classical Grammar School? An answer to this question will be given on the basis of *Methodically structured collection of problems* [*Methodisch geordnete Aufgabensammlung*] by F. Bardey [3] (used in the classes from Lower Tertia to Upper Prima at Torun Classical Grammar School), textbook *Fundamental theorems of elementary mathematics* [*Hauptsätze der Elementar-Mathematik*] by F.G. Mehler [12] (used in the classes from Quarta to Upper Prima at Torun Classical Grammar School)[8] and school-leaving examinations tasks from the years 1905–1911.

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[7] That is, according to the issues contained in the textbook: *Basic theorems of elementary mathematics* [*Hauptsätze der Elementar-Mathematik*] by F. G. Mehler [12].

[8] In the late nineteenth century and the first decade of the twentieth century at Torun Grammar School the following textbooks of mathematics were used [8, 1907, p. 15]:

1. *Fundamental theorems of elementary mathematics* [*Hauptsätze der Elementar-Mathematik*] by F.G. Mehler [12]:
   - classes IV – IU (Quarta – Upper Prima), Classical Grammar School,
   - classes IV – IU (Quarta – Upper Prima), Real Grammar School.
2. *Methodically structured collection of problems* [*Methodisch geordnete Aufgabensammlung*] by F. Bardey [3]:
   - classes IIIL – IU (Lower Tertia – Upper Prima), Classical Grammar School,
   - classes IV – IU (Quarta – Upper Prima), Real Grammar School.
3. *Four-digit logarithmic tables with mathematical, physical and astronomical tables* [*Vierstellige Logarithmentafeln nebst mathematischen, physikalischen und astronomischen Tabellen*] by A. Schülke [16]:
   - classes IIL – IU (Lower Secunda – Upper Prima), Classical Grammar School,
   - classes IIL – IU (Lower Secunda – Upper Prima), Real Grammar School.
4. *Problems in elementary arithmetic* [*Aufgaben zum Ziffernrechnen*] by J. Blümel and R. E. Pflüger, parts III, IV and V:
   - classes VI – IV (Sexta – Quarta), Classical Grammar School and Real Grammar School.

3.2. Methodically structured collection of problems [Methodisch geordnete Aufgabensammlung] by F. Bardey

Bardey's *Methodically structured collection of problems* includes more than 8000 exercises. Before certain batches of exercises the author placed the theory needed to solve them. However, these cases were only individual and concerned issues that were not discussed in the *Fundamental theorems of elementary mathematics* [*Hauptsätze der Elementar--Mathematik*] by F. G. Mehler [12]. Analyzing the contents of tasks and the theory in Bardey's collection we can observe that there is no mention of the word 'function' at all. Nevertheless, in a supplement to a collection of problems, on page 326 (the whole collection of problems contains 330 pages) the author took up the explanation of what the independent variables and the dependent variables are. He noted that 'the independent variable' is the name for a variable, denoted generally by x, which may take any value, for example: 0, 1, 2, 3 and so on. 'The dependent variable' is the name for a variable, usually denoted by y, whose value depends on the value of the variable x, and this relationship is defined by a certain equation with two variables x and y.

On the basis of the definition of a function given by T. Gutkowski in *Elementary algebra*, part I (Warsaw 1918), which was evaluated by K. Wuczyńska (in [20]) as a textbook maintaining the spirit of the Merano Programme [20, p. 271, 278], it can be stated that according to Klein 'dependent variable' and 'function' were synonymous.

Thus, it appears that, despite the fact that in *Methodically structured collection of problems* [*Methodisch geordnete Aufgabensammlung*] the word 'function' does not appear, the author dealt with functions. After defining the independent variable and the dependent variable (or function), he explained in which way the relationship between them can be represented graphically. He began with a graphic representation of first-degree equations with variables x and y, then passed to the equations of the second degree, pointed out that their graphic representations are conics, and using graphs he found minimum and maximum values of both variables [3, pp. 326-330].

Although, in the collection of problems, functions were discussed in a symbolic way (Bardey dealt with graphical representations of functions, but did not use them e.g. to solve equations; he did not explore the properties of the function either), the thinking in the term of function was very often used by the author.

Bardey tried to develop the student's intuition of variability and functional dependence. He prepared tasks that allowed students to observe that the change of some values affects changes of others. Later, he went one step further and demanded that students made analysis of occurring changes. Some other time he demanded to describe mathematically the relationship between values, and finally, he passed to the tasks in which students, knowingly or not, had to write formulas of functions, calculate when the functions take the given values or find their minima and maxima.

Some of the tasks may be treated as strictly preparatory for the introduction of dependent variables. Among them are the following:

– Tasks that point out to the fact that the change in some values causes others to change, for example:

Exercise [3, p. 4, ex. 47]. What values are taken by the following expressions:

1. $a - b + c - d$	9. $a + b(c - d)$
2. $a - (b + c) - d$	10. $a - b(c - d)$
3. $a - b - c + d$	11. $(a - b)c + d$
4. $a - b - (c + d)$	12. $(a + b) : c - d$
5. $(a + b)\,(c + d)$	13. $a + b : c + d$
6. $(a - b)\,(c - d)$	14. $(a + b) : (c + d)$
7. $a + bc - d$	15. $a - b : c - d$
8. $a - b + cd$	16. $(a - b) : (c - d)$

1. for $a = 30$, $b = 12$, $c = 3$, $d = 2$;
2. for $a = 96$, $b = 36$, $c = 6$, $d = 3$;
3. for $a = 72$, $b = 12$, $c = 6$, $d = 2$?

Exercise [3, p. 22, ex. 142]. By how much a product $831 \cdot 754$ will decrease, if its first factor increases by 1, and the second factor will decrease by 1? How much this product will increase when his first factor will decrease by 1 and the other one will increase by 1?

Exercise [3, p. 22, ex. 143]. By how much a rectangle (meaning: the area of the rectangle), whose sides have lengths 793 and 137 feet, will increase if the longer side of this rectangle is extended by 5 feet and the shorter side is extended by 7 feet (give an answer without calculating the area of the rectangle)?

Exercise [3, p. 83, ex. 1]. What are the logarithms with base 2 of the numbers 2, 4, 64, 16, 128, 32, 1, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{8}$?

Exercise [3, p. 91, ex. 92]. What value does the expression $\sqrt{1-\frac{ac}{b^2}}$ take for $a = 28{,}371$, $b = 39{,}832$ and $c = 41{,}504$?

Exercise [3, p. 91, ex. 93]. The same as above for $a = 173{,}54$, $b = 375{,}42$ and $c = 280{,}19$?

– The change of some values does not always cause others to change, for example:

Exercise [3, p. 239, ex. 14]. The diagonal of a rectangle is 65 feet long. If the smaller side of this rectangle is shortened by 17 feet and the larger side is extended by 7 feet, the diagonal of the rectangle will not be changed. How long are the sides of this rectangle?

– Tasks that point to the fact that some values can depend on others by means of mathematical formulas, for example:

Exercise [3, p. 59, ex. 39]. Knowing that $\sqrt{50} = a$, calculate: $\sqrt{8}$, $\sqrt{18}$, $\sqrt{32}$, $\sqrt{72}$ and $\sqrt{98}$.

Exercise [3, p. 84, ex. 36]. Knowing that log 2 = 0,30103 and log 3 = 0,47712, calculate: log 4, log 5, log 6, log 8, log 9, log 12, log 15, log 16, log 18, log 20, log 24, log 25, log 27, log 30, log 32, log 36, log 40, where log 10 =1 (the base of the logarithm is 10).

Exercise [3, p. 84, ex. 20]. How big are: log 530, log 5300, log 53000, log 5300000, log 5,3, log 0,53, log 0,0053, when log 53 = 1,7243? (Prove!)

– Certain values are directly or indirectly proportional, for example:

Exercise [3, p. 4, ex. 49]. Calculate in the memory[9]:

1. 1 lot costs respectively: 2, 7, 10, 15 pfennig; how much does 1 gram cost?
2. 1 lot costs respectively: 4, 6, 9, 14 pfennig; how much does 1 pound cost?
3. 1 pound costs respectively: 65, 70, 80, 85 pfennig; how much does 1 Hundredweight cost?
4. 1 litre costs respectively: 10, 15, 20, 24 pfennig; how much does 1 hectolitre cost?
5. 1 metre costs respectively: 3, 5, $7\frac{1}{2}$ Mark, 9 Mark 30 pfennig; how much does 1 centimetre cost?
6. 1 Hundredweight costs respectively: 13, 50, 300, $715\frac{1}{2}$ Mark, 87 Mark 30 pfennig; how much does 1 pound cost?
7. 1 gram costs respectively: 4, 11, 15 Mark, 7 Mark 30 pfennig; how much does 1 lot cost?
8. 1 pound costs respectively: $1\frac{1}{2}$, $2\frac{1}{2}$, 5 Mark, 3 Mark 20 pfennig; how much does 1 lot cost?
9. 1 hectolitre costs respectively: 7, 9, $20\frac{1}{2}$ Mark, 30 Mark 40 pfennig; how much does 1 litre cost?

Exercise [3, p. 22, ex. 153]. Prove the following theorems:

1. 1 pound costs *a* pfennig, therefore *b* hundredweight costs *ab* Mark.
2. 1 lot costs *a* pfennig, therefore *b* grams costs *ab* Mark.
3. 1 lot costs *a* pfennig, therefore *b* pounds costs $\frac{1}{2}ab$ Mark.
4. 1 litre costs *a* pfennig, therefore *b* hectolitres costs *ab* Mark.

Exercise [3, p. 22, ex. 154.2]. How much does respectively: 4, $5\frac{1}{8}$, $6\frac{1}{4}$ pounds, 9 pounds 8 lot, cost, when 1 lot costs respectively: 2, 3, 4, 5 pfennig?

Other tasks, already touching upon the essence of dependent variables, for example:

Exercise [3, p. 271, ex. 85]. An equilateral triangle is given. From the height of this triangle we build another equilateral triangle, from the height of the second triangle we build a third triangle and so on. What is the sum of all these triangles (including the given triangle)?

Exercise [3, p. 92, ex. 106] What is the base of a logarithm in which:

1. log 10 = 2;
2. log 100 = 3;
3. log 3 = 4;
4. log 33 = 3;
5. log 444 = 4;
6. log 666 = 6?

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[9] 1 pound = 30 lot = 500 grams; 1 Hundredweight = 100 pounds [10, p. 127]. 1 Mark = 100 pfennig.

Exercise [3, p. 246, ex. 17]. Find one solution of a diophantine equation $13x + 5y = 444$, then determine another solutions and find the general form of solution of this equation.

There are also the tasks that require writing a formula of a function and calculating the minimum or maximum values of this function[10], such as:

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[10] In mathematics lessons at Torun Grammar School tasks relating to the calculation of minimum and maximum values of functions were solved without using differential calculus (at that time in the Torun Grammar School curricula there was no differential calculus).

In *Principles of algebraic analysis with supplements to the 'Arithmetic to school- and self-study'* [*Anfangsgründe der algebraischen Analysis nebst Ergänzungen zur Arithmetik für den Schul- und Selbst-Unterricht*] [7] Karl Koppe showed the method of solving this kind of tasks without using the derivatives. This method was based on three fundamental theorems:

Theorem 1 [7, p. 1]. Let m and x are positive numbers, then the following expressions (dependent on x):

1. $2mx - x^2$,
2. $3m^2x - x^3$,
3. $3mx^2 - 2x^3$,
4. $4m^3x - x^4$,
5. $4m^2x^2 - 2x^4$,
6. $4mx^3 - 3x^4$,

attain the greatest value when $x = m$.

Proof. 1 [7, p. 1]. The expression $2mx - x^2$ is equal to m^2 when $x = m$. To demonstrate that this is the greatest value of $2mx - x^2$, we have to add to this expression $m^2 - m^2 = 0$. As a result, we obtain the following equation: $2mx - x^2 = m^2 - (m^2 - 2mx + x^2) = m^2 - (m - x)^2$. The subtrahend $(m - x)^2$ vanishes when $x = m$, whereas in all other cases it assumes a positive value. Thus, the above expression is greatest when $x = m$.

□

Theorem 2 [7, p. 1, 3]. Let m and x be positive numbers, then the following expressions (dependent on x):

1. $\frac{m^2}{x} + x$,
2. $\frac{m^3}{x^2} + 2x$,
3. $\frac{2m^3}{x} + x^2$,
4. $\frac{m^4}{x^3} + 3x$,
5. $\frac{2m^4}{x^2} + 2x^2$,
6. $\frac{3m^4}{x} + x^3$,

attain the smallest value when $x = m$.

Proof. 1 [7, p. 3]. It is easy to see, that: $\frac{m^2}{x} + x = \frac{m^2 + x^2}{x} = 2m + \frac{(m - x)^2}{x}$. Firstly, we have to notice that $\frac{(m - x)^2}{x}$ is always non-negative. Therefore, the expression $\frac{m^2}{x} + x$ is the smallest when the second summand of $2m + \frac{(m - x)^2}{x}$ vanishes, that is, when $x = m$.

□

Exercise [3, p. 217, ex. 130]. A circle of radius r is given. Find a rectangle inscribed in this circle, which:

1. has the largest area,
2. has the greatest perimeter[11].

Theorem 3 [7, p. 4]. If an expression dependent on x (x assumes only positive values) for a specific value of x is the largest or the smallest, the same applies to expression which arises from the previous one by:

– adding, subtracting, multiplying, or dividing the previous expression by an expression independent of x,
– raising the previous expression to any power,
– taking any root of the previous expression.

[11] Solution of this problem consists of two parts. The first part is concerned with finding a rectangle with the largest area, the second one – a rectangle with the greatest perimeter. We will use: Theorem 1 and Theorem 3, which were given in the previous footnote.

Solution [7, p. 6; in original notation]:

Let x and y are the sides of the rectangle. Then, we obtain the following equations:

$$x^2 + y^2 = 4r^2 \quad \text{and}$$

1. xy = maximum.
   From the above equation we eliminate y:
   $$x\sqrt{4r^2 - x^2} = \text{maximum.}$$
   On the base of Theorem 3, placed in the previous footnote, we can raise the left hand side of this equation to the square (it does not change the solution of this equation):
   $$4r^2x^2 - x^4 = \text{maximum.}$$
   Now, we want to bring the last equation into the form:
   $$4m^2x^2 - x^4 = \text{maximum}$$
   (because we want to use Theorem 1 given in the previous footnote), so we multiply by 2 its left hand side:
   $$8r^2x^2 - 2x^4 = \text{maximum.}$$
   Hence, we obtain that:
   $$m^2 = 2r^2,$$
   or:
   $$m = r\sqrt{2}.$$
   On the basis of Theorem 1, we obtain that:
   $$x' = r\sqrt{2} = y'.$$
   Thus, of all rectangles inscribed in a circle, the square has the largest area.
2. $2x + 2y$ = maximum, or: $x + y$ = maximum.
   From the above equation we eliminate y:
   $$x + \sqrt{4r^2 - x^2} = \text{maximum.}$$

Exercise [3, p. 217, ex. 123]. Of all triangles satisfying the property: the sum of the base and the height is equal to a, find the one which has the greatest area and calculate this area.

Exercise [3, p. 217, ex. 117]. Of all rectangles of the same area, find the one with the smallest circumference.

Moreover, there were more sophisticated problems that required using the formula of velocity, cosine theorem, Pythagoras's theorem, writing a formula of a function and calculating for which arguments function takes a given value, such as:

Exercise [3, p. 215, ex. 106]. In a vertex of an equilateral triangle with a side a, there are two bodies. One of them moves along one side of the triangle at a speed of m m/s and the second body moves along another side that is adjacent to the previous one, at a speed of n m/s. When will the distance between these bodies be equal to the height of the triangle?

Exercise [3, p. 215, ex. 104]. Two circles are given. The centers of these circles are located on two perpendicular lines and move toward the point of intersection of these lines. The first circle has a radius of 100 meters, in each second it covers a distance of 3 meters and the center of this circle is located 247 meters from the intersection of the perpendicular lines. The second circle has a radius of 35 meters, in each second it covers a distance of 2 meters and the center of this circle is located 169 meters from the intersection of perpendicular lines. After how many seconds will the circles be internally tangent?

Exercise [3, p. 210, ex. 41]. Two bodies A and B are situated on different arms of an angle. The body A is located 123 meters from the vertex of the angle and starts to move away from the vertex at a speed of 239 m/s. The body B is located 239 meters from the vertex and starts to move toward to the vertex at a speed of 123 m/s. When will the distance of these bodies be equal to 850 meters?

In *Methodically structured collection of problems* [*Methodisch geordnete Aufgabensammlung*], Bardey also puts tasks whose solution requires using the 1-1 property (injectivity) of the power functions, exponential and logarithmic functions, for example:

Exercise [3, p. 193, ex. 191]. Solve the equation: $\sqrt{x+3}+\sqrt{2x-3}=6$.

Exercise [3, p. 119, ex. 7]. Solve the equation: $(a^{x-5})^{x-6}=(a^{x-8})^{x-1}$.

We raise to the square the left hand side of this equation (on the basis of Theorem 3):

$$4r^2+2x\sqrt{4r^2-x^2}=\text{maximum},$$

and remove the constants, which will not change the solution (on the basis of Theorem 3):

$$x\sqrt{4r^2-x^2}=\text{maximum}.$$

Once again, we raise to the square the left hand side of last equation:

$$4r^2x^2-x^4=\text{maximum}.$$

This equation is the same as the equation obtained at the first point of this solution. Therefore, further steps of this reasoning (point 2) are the same as steps described at point 1. Thus, of all rectangles inscribed in a circle, the square has the greatest perimeter.

Exercise [3, p. 202, ex. 156]. Solve the equation: $\log\sqrt{7x+5}+\frac{1}{2}\log(2x+7)=$ $=1+\log 4,5$.

In the collection of problems, Bardey did not include the definition of an injective function. Moreover, in the solutions, which were included in Bardey's book, there was no remark that at a certain moment author used the 1-1 property of a function. This suggests that students used the 1-1 property of functions, but they were not aware of it.

3.3. Fundamental theorems of elementary mathematics [Hauptsätze der Elementar--Mathematik] by F. G. Mehler

The textbook *Fundamental theorems of elementary mathematics* [*Hauptsätze der Elementar-Mathematik*] [12] was divided by Mehler into five main chapters: planimetry, algebra, trigonometry, series and binomial theorem, stereometry. In this textbook, as the title indicates, only theory was included, namely: definitions, theorems, proofs and certain patterns of reasoning, e.g. methods of solving equations [12, pp. 58–72] or a method of converting numbers into continued fractions [12, pp. 72–80]. Only in a few cases the author did exercises designed to apply theory in practice [12, pp. 13–15, 21, 26–27, 34, 41–45, 50, 75, 77, 78–79, 97, 100, 109–111]. Tasks to be solved independently by students were almost completely ignored by Mehler (one of them can be found: [12, p. 33]).

In Mehler's textbook, functions were discussed only in the context of trigonometric functions – the author gave definitions of trigonometric functions and he thoroughly discussed their properties: the relationship between trigonometric functions, signs of trigonometric functions, their periodicity, evenness and oddness.

In *Fundamental theorems of elementary mathematics* [*Hauptsätze der Elementar-Mathematik*] there was no definition of a dependent variable, there was also no evidence of indicating functional relationship between certain values. Furthermore, in the chapter concerning series and binomial theorem, when Mehler develops e^x in a power series, he avoids explaining that e^x is an exponential function [12, p. 104]. Therefore, it can be assumed that Mehler did not intend to make general considerations about functions. He did not apply functional approach to various mathematical problems either. He discussed only trigonometric functions, because they are the basis of planimetric and stereometric considerations (including spherical trigonometry). They are also very important to calculations on complex numbers.

It should be stressed that Mehler's textbook was published in 1869, that is, 36 years before the promulgation of the Merano reform. Therefore, we cannot criticize the author for the lack of implementation of the Merano Programme postulates. Should Torun Grammar School have definitively abandoned Mehler's textbook after 1905? It turns out that not necessarily so. Together with Bardey's collection of problems, this textbook perfectly implemented the postulate of functional approach to various mathematical problems. Definitions and tasks included in Bardey's collection of problems allowed one to analyse the contents of Mehler's textbook in terms of dependent variables and as a result, the vast majority of the material could be read in the terms of function. For example:

- **Ptolemy's Theorem** (if a quadrilateral can by inscribed in a circle, then the product of its diagonals is equal to the sum of the products of the pairs of opposite sides [12, p. 32]) can be understood as follows: the product of the diagonals of a quadrilateral inscribed in a circle is a function of its sides.
- **Exercise** [12, p. 33, §87]. Construct a fourth proportional to the three given lines: *a*, *b* and *c* (it means: the fourth component *x*, such that $\frac{b}{c} = \frac{c}{x}$).

  The solution of this exercise allowed the students to note that the fourth proportional *x* is a function of the three given lines: *a*, *b* and *c*.

  In a similar way the following theorems can be read:
- **Theorem** [12, p. 17, §47]. The sum of angles of a polygon with *n* sides is equal to $2n - 4$ right angles.
- **Theorem** [12, p. 82, §160]. Assume that: the initial capital is equal to *c*, the annual interest on one thaler is equal to *p* and *n* is the number of years after which the capital will increase to the amount of *k*. Then:

$$k = c(1 + p)^n .$$

- **Theorem** [12, p. 122, §224]. If *h* is a height of a cone, *r* a radius of its base, and *s* its lateral face, then the lateral area of this cone is equal to πrs.

3.4. School-leaving examinations

By analyzing the school-leaving examinations in mathematics which were carried out at Torun Classical Grammar School in the years 1905–1911 [8], we conclude that the teachers of mathematics at this institution had to attach a great importance to construction tasks. As a part of every written school-leaving examination in mathematics at Torun Classical Grammar School in the years 1905–1911, there was a construction task. Some of them required finding the geometric locus of points. Tasks of this type provide excellent opportunities for practicing thinking in terms of functions – students can observe how the change of some values affects changes of others, moreover, they have to look for a relationship between certain values.

Examples of school-leaving examinations problems:

Exercise [8, 1908, p. 9, ex. 1]. Construct a triangle when

1. radius of a circle circumscribed about this triangle,
2. one of angles of this triangle and
3. a square, which is the sum of squares of the sides adjacent to the given angle

are given.

Exercise [8, 1907, p. 9, ex. 1]. Construct a quadrilateral when

1. two osculate sides of the quadrilateral,
2. the ratio of the two other sides and
3. both diagonals

are given.

Exercise [8, 1910, ex. 2]. Draw a rectangle whose circumference and area are equal to the circumference and area of a given triangle[12].

[12] Solution of the exercise:
Given: a triangle *ABC* with sides: *a*, *b* and *c*.
Description of the construction (sketch):

1. We construct a rectangle with the same area as the given triangle:
   - we determine the center of the base *AB* of the given triangle *ABC*, we denote it by *D*,
   - we mark out a perpendicular line to the base *AB* passing through the vertex *A* (line *l*) and perpendicular line to the base *AB* passing through the point *D* (line *u*),
   - we mark out the parallel line to the base *AB* passing through the vertex *C* (line *n*),
   - a point of intersection of *n* with *l* we denote by *E*,
     a point of intersection of *n* with *u* we denote by *F*,
   - rectangle *ADEF* has the same area as the given triangle *ABC*.
2. We draw a straight line *m*, then select any point on this line and draw the perpendicular line to *m* passing through the selected point – a line *r*.
3. On the same side of the line *m* we draw a few rectangles, each of them satisfying the following properties:
   - the circumference of the rectangle is equal to $a + b + c$,
   - one of its sides lies on the line *m*,
   - line *r* is the axis of symmetry of the rectangle.

   Then, we can notice that the left upper vertices of these rectangles determine a straight line (denote it by *s*), right upper vertices also designate a straight line (denote it by *t*). Lines *s* and *t* are symmetrical with respect to the line *r*. A piece of line *s* restricted by *m* and *r* (denote it by s') is the geometric locus of the left upper vertices of the rectangles with one of the sides lying on *m* and with a circumference equal to $a + b + c$. The piece of line *t* restricted by *m* and *r* (denote it by t') is the geometric locus of the right upper vertices of the rectangles with one of the sides lying on *m* and with a circumference equal to $a + b + c$.
4. On the same side of the line *m* which we chose in the previous point (point 3), we draw a few more rectangles, each of them satisfying the following properties:
   - its area is equal to the area of the rectangle *ADEF*,
   - one of the sides of this rectangle lies on the line *m*,
   - line *r* is the axis of symmetry of this rectangle.

   Then, the left upper vertices of these rectangles determine a piece of a hyperbola (denote it by *p*) and so do the upper right vertices, too (denote this piece of a hyperbola by *q*).
   The pieces *p* and *q* are symmetrical with respect to the line *r*, *p* is the geometric locus of the left upper vertices of the rectangles with one of the sides lying on *m* and with area equal to the area of rectangle *ADEF*; analogous reasoning can be applied to *q*.
5. The point of intersection of s' with *p* (denote it by *X*) determines the left upper corner of the rectangle we are looking for. Then, the right upper corner of this rectangle is the point of intersection of p' with *q* and simultaneously is a reflection of the point *X* in the line *r* (denote it by *Y*). It is not difficult to find the other two vertices of this rectangle: *Z* is the projection of *X* onto *m*, *T* is the projection of *Y* onto *m*.
6. The second point of intersection of s' with *p* will generate a second rectangle satisfying the desired properties.

4. Conclusions

In conclusion, in the curricula of the various classes of Torun Classical Grammar School in the years 1905–1911 there were no functions, but it certainly can be said that functions were discussed there. Teachers of mathematics introduced a definition of a dependent variable, which, according to Klein, was a synonym for a function. In Bardey's collection of problems the definitions of an independent variable and a dependent variable were the foundation for introducing a coordinate system and discuss graphical representation of linear and quadratic equations (here conics were introduced, as a graphical representations of quadratic equations). In the years 1905–1911 in the curricula of top class Prima Classical Grammar School, there was the following issue: 'Basic information about the coordinates, the equation of a straight line, a circle and conics' [8]. Mehler's textbook did not include this issue, therefore, it had to be implemented on the basis of Bardey's collection of problems. Thus, students of top class Prima had to learn the definition of the dependent variable.

The considerations made in this article prove that not all recommendations of the Merano Programme which involved teaching of functions were implemented at Torun Grammar School. On the basis of the contents of *Methodically structured collection of problems* [*Methodisch geordnete Aufgabensammlung*] by F. Bardey [3], the textbook *Fundamental theorems of elementary mathematics* [*Hauptsätze der Elementar-Mathematik*] by F.G. Mehler [12] and school-leaving examinations tasks from the years 1905–1911, it can be assumed that at Torun Grammar School a graph of logarithmic function and monotonicity of functions were not discussed, moreover, equations were not solved by the graphical method. However, we cannot be sure about it. In support of it, let us mention here the situation at Grammar School in Bydgoszcz in 1906/1907. There, like at Torun Classical Grammar School, in the curricula there were no considerations of functions [9]. However, in the curriculum of the Upper Prima, there was an issue of 'information about coordinates and theory of conics' [9, p. 27]. At Bydgoszcz Grammar School, in mathematics lessons in Upper Prima two books were used: *Elementary mathematics* [*Die Elementar Mathematik*] by L. Kambly [6] and *Methodically structured collection of problems* [*Methodisch geordnete Aufgabensammlung*] by F. Bardey[13]. In *Elementary mathematics* neither theory of functions nor dependent variables appeared, there was also no theory of conics. Thus the issue mentioned had to be implemented on the basis of *Methodically structured collection of problems* [*Methodisch geordnete Aufgabensammlung*] by F. Bardey. Although in Bardey's collection of problems neither the word 'function' nor consideration of the logarithmic function appeared, mathematics lessons at Bydgoszcz Grammar School filled in these gaps. This is confirmed by one of the school-leaving examinations tasks:

Exercise [9, p. 27, Easter, ex. 1]. Draw a graph of a function $y = \log x$ and discuss its main properties, in particular calculate the logarithm of a given number using the mean value[14].

[13] In addition, in Bydgoszcz Grammar School: *Complete logarithmic and trigonometric tables* [*Vollständige logarithmische und trigonometrische Tafeln*] by E. F. August [2; 9, p. 27] was used.

[14] This means: calculate the logarithm of a given number using the interpolation method.

We can say with certainty that the postulate of teaching thinking in terms of functions was executed very well at Torun Classical Grammar School.

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A NOTE ON THE MATHEMATICAL PUBLICATIONS IN THE *DISSERTATIONS AND REPORTS OF MEETINGS OF THE ACADEMY OF ARTS AND SCIENCES IN CRACOW* IN THE YEARS 1874–1951

NOTKA O PUBLIKACJACH MATEMATYCZNYCH W *ROZPRAWACH I SPRAWOZDANIACH Z POSIEDZEŃ AKADEMII UMIEJĘTNOŚCI W KRAKOWIE* (1872–1894)

A b s t r a c t

This paper contains some information on the Academy of Arts and Sciences in Cracow. It gives a detailed list of mathematical publications in the *Dissertations and Reports of Meetings of the Academy of Arts and Sciences in Cracow* in the years 1874–1951.

Keywords: *Academy of Arts and Sciences in Cracow*, *Mathematical publications*, *Dissertations and Reports of Meetings of the Academy of Arts and Sciences in Cracow*

S t r e s z c z e n i e

Artykuł zawiera pewne informacje o Akademii Umiejętności w Krakowie i listę publikacji matematycznych wydrukowanych w *Rozprawach i Sprawozdaniach Akademii Umiejętności w Krakowie* w latach 1874–1951.

Słowa kluczowe: *Akademia Umiejętności w Krakowie*, *publikacje matematyczne*, *Rozprawy i Sprawozdania z Posiedzeń Akademii Umiejętności w Krakowie*

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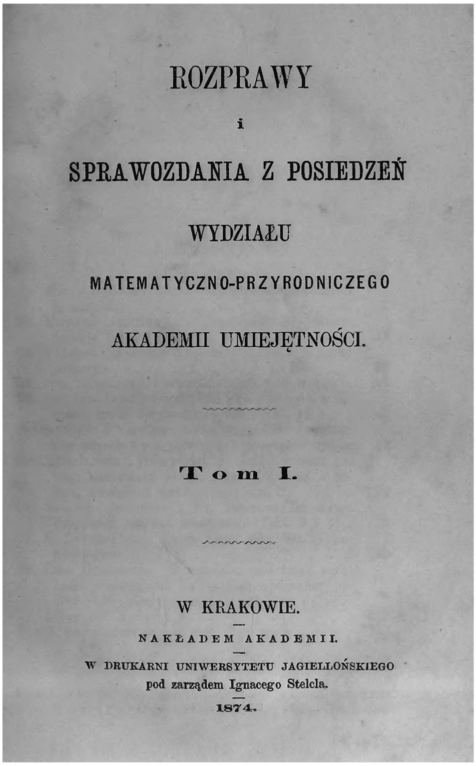
1. Some information on the Academy of Arts and Sciences in Cracow and the *Dissertations and Reports of Meetings of the Academy of Arts and Sciences in Cracow*

The Academy of Arts and Sciences was founded in 1872, as a result of the transformation of the Cracow Learned Society, which had existed since 1815. *Dissertations and Reports of Meetings of the Academy of Arts and Sciences in Cracow* was a continuation of the Memoirs of the Academy of Sciences in Cracow and the Annals of the Cracow Scientific



ROZPRAWY

i

SPRAWOZDANIA Z POSIEDZEŃ

WYDZIAŁU

MATEMATYCZNO-PRZYRODNICZEGO

AKADEMII UMIEJĘTNOŚCI.

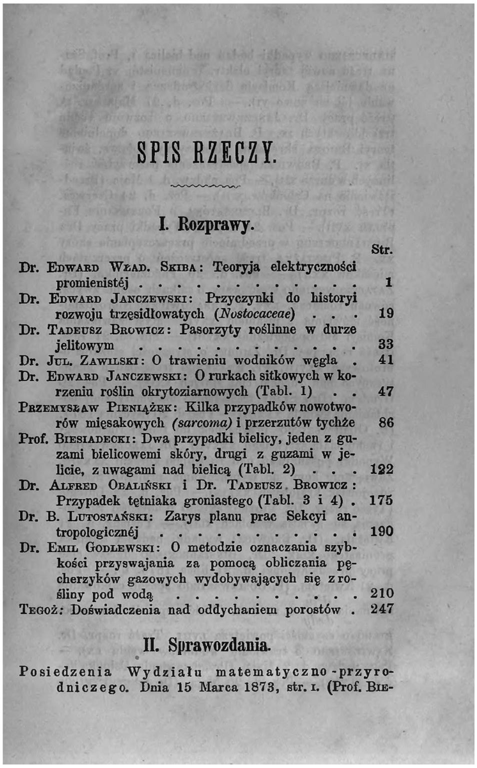
Tom I.

W KRAKOWIE.

NAKŁADEM AKADEMII.

W DRUKARNI UNIWERSYTETU JAGIELLOŃSKIEGO
pod zarządem Ignacego Stelcla.

1874.



SPIS RZECZY.

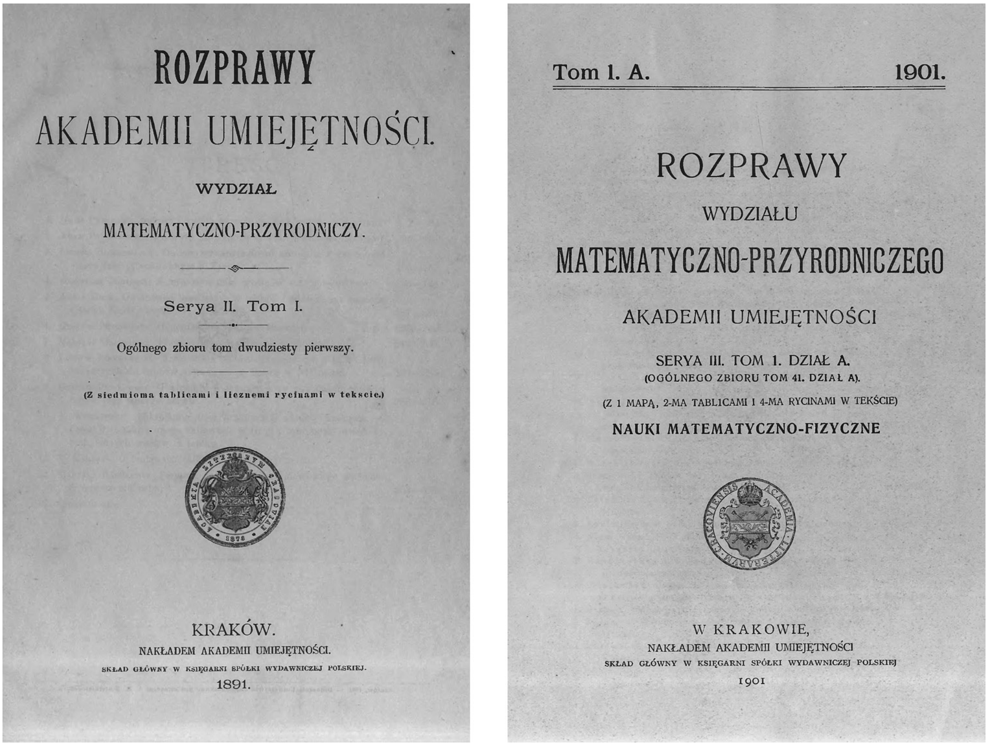
I. Rozprawy.

II. Sprawozdania.

Posiedzenia Wydziału matematyczno-przyrodniczego. Dnia 15 Marca 1873, str. I. (Prof. Bie-

Fig 1. The cover page and list of papers in first volume of *Dissertations and Reports of Meetings of the Academy of Arts and Sciences in Cracow*

Society. This journal was printed in 60 volumes from 1874 to 1921. In all volumes of the *Reports of the Meetings of the Academy of Arts and Sciences in Cracow* there were printed 869 scientific papers, including 90 mathematical works, and 776 articles in various fields of science. Among them 27 works were related to differential equations (before 1951 there were printed 74 volumes of *Reports of the Academy of Arts and Sciences in Cracow*).



ROZPRAWY
AKADEMII UMIEJĘTNOŚCI.
WYDZIAŁ
MATEMATYCZNO-PRZYRODNICZY.
Serya II. Tom I.
Ogólnego zbioru tom dwudziesty pierwszy.
(Z siedmioma tablicami i licznemi rycinami w tekście.)
KRAKÓW.
NAKŁADEM AKADEMII UMIEJĘTNOŚCI.
SKŁAD GŁÓWNY W KSIĘGARNI SPÓŁKI WYDAWNICZEJ POLSKIEJ.
1891.

Tom 1. A. 1901.
ROZPRAWY
WYDZIAŁU
MATEMATYCZNO-PRZYRODNICZEGO
AKADEMII UMIEJĘTNOŚCI
SERYA III. TOM 1. DZIAŁ A.
(OGÓLNEGO ZBIORU TOM 41. DZIAŁ A).
(Z 1 MAPĄ, 2-MA TABLICAMI I 4-MA RYCINAMI W TEKŚCIE)
NAUKI MATEMATYCZNO-FIZYCZNE
W KRAKOWIE,
NAKŁADEM AKADEMII UMIEJĘTNOŚCI
SKŁAD GŁÓWNY W KSIĘGARNI SPÓŁKI WYDAWNICZEJ POLSKIEJ
1901

Fig. 2. The cover pages of XXI (first volume of the second series) and XLI (first volume of the third series) volumes of *Dissertations of the Academy of Arts and Sciences in Cracow*

2. List of mathematical papers in the *Dissertations and Reports of Meetings of the Academy of Sciences in Cracow* [2] in the years 1874–1951 (in Polish) – (we preserve the original 19th- century spelling):

Volume V (1878)

Jan Nep. Franke i Antoni Jakubowski: *Maciéj Głoskowski, matematyk polski XVII-go wieku* (pp. 126-159).

Volume VI (1880)

Gustaw Kummer: *Rzecz o dwóch na płaszczyznach leżących krzywych rzędu drugiego. (Tablica VI i VII)* (pp. 143-194).

Józef Tetmajer: *Nowy wzór do całkowania za pomocą szeregów* (pp. 231-245).

Volume VII (1880)

Dr Władysław Zajączkowski: *O pewnéj własności pfafijanu* (pp. 67-74).

(In this volume also: F. Karliński, *Ułatwienie obliczenia współczynników wzoru Bessela używanego w meteorologii* (pp. 59-66)).

Volume X (1883)

Wł. Gosiewski: *O pewném zadaniu z teorii prawdopodobieństwa* (pp. 137-140).

Wł. Gosiewski: *O nieogólności pewnéj zasady rachunku całkowego* (pp. 141-147).

Mieczysław Łazarski: *O konstrukcyi osi w perspektywie koła. (Tablica III)* (pp. 148--154).

A. J. Stodółkiewicz: *Przyczynek do całkowania równań różniczkowych z dwiema zmiennemi* (pp. 225-228).

Volume XI (1884)

Dr Łazarski Mieczysław: *O zamianie krzywych rzędu drugiego na koła za pomocą rzutów. (Tabl. V)* (pp. 145-154).

A. J. Stodółkiewicz: *O całkowaniu pewnego równania różniczkowego linijnego rzędu drugiego* (pp. 155-159).

Volume XIII (1886)

Wł. Gosiewski: *Łatwy sposób dowodzenia twierdzenia odwrotnego do twierdzeniu Jakóba Bernoulliego* (pp. 153-159).

A. J. Stodółkiewicz: *O równaniu różniczkowém linijném Pfaffa* (pp. 160-171).

Dr M.A. Baraniecki: *O przekształceniu koła na przecięcie stożkowe* (pp. 172-182).

Dr M.A. Baraniecki: *O funkcjach Bernoulliego* (pp. 183-195).

(In this volume also: Wł. Gosiewski: *O średnich składowych odkształcenia ciała stałego, sprężystego jednorodnego, a w szczególności izotropowego*, (pp. 143-152)).

Volume XV (1887)

A. J. Stodółkiewicz: *Przyczynek do nauki o całkowaniu równań różniczkowych linijowych rzędu drugiego* (pp. 36-43).

Dr Łazarski Mieczysław: *O konstrukcji i własnościach krzywych rzędu czwartego z punktem potrójnym. (Tablica IX)* (pp. 224-249).

Dr Łazarski Mieczysław: *O wpływie punktów i stycznych szczególnych na rząd i klasę krzywych płaskich* (pp. 279-285).

Volume XIX (1889)

I. Dickstein: *O metodzie teologicznej Hoene–Wrońskiego rozwiązywania równań algebraicznych* (pp. 167-192).

F. Mertens: *O niektórych całkach oznaczonych* (pp. 204-224).

(In this volume also: Wł. Gosiewski: *Teoria zjawisk Weyhera* (pp. 193-203) and L. Birkenmajer: *O równowadze kinetycznej płynu nieściśliwego* (pp. 225-235)).

Volume XX (1890)

F. Mertens: *O wprowadzeniu nowych zmiennych do wyrażeń różniczkowych* (pp. 267--271).

J. Rajewski: *O pewnych całkach określonych* (pp. 272-281).

I. Dickstein: *Dopełnienie artykułu o metodzie teologicznej Hoene–Wrońskiego rozwiązywania równań algebraicznych* (pp. 287-291).

Volume XXI (1891)

F. Mertens: *O funkcyach całkowitych symetrycznych* (pp. 333-352).

Volume XXII (1892)

Józef Puzyna: *Kilka uwag o ogólnej teoryi krzywych algebraicznych* (pp. 1-29).

F. Mertens: *O zastosowaniu teoryi funkcyj symetrycznych do wyprowadzenia układu zupełnego utworów niezmiennikowych dla form o dwóch zmiennych* (pp. 141-171).

A. J. Stodółkiewicz: *O pewnym kształcie układów równań różniczkowych o różniczkach zupełnych* (pp. 299-303).

Volume XXIII (1893)

Kazimierz Żorawski: *O pewnem odkształceniu powierzchni* (pp. 225-291).

Volume XXIV (1893)

Kazimierz Żorawski: *Uzupełnienie ciągłych grup przekształceń* (pp. 34-40).

Kazimierz Żorawski: *Niezmienniki różniczkowe pewnej nieskończonej ciągłej grupy przekształceń* (pp. 41-55).

S. Dickstein: *Zasady teroyi liczb Wrońskiego* (pp. 73-104).

Volume XXV (1893)

Stanisław Kępiński: *O całkach rozwiązań równań różniczkowych zwyczajnych liniowych jednorodnych rzędu 2-go (*pp. 264-328).

Volume XXVI (1893)

A. J. Stodółkiewicz: *O całkowaniu pod postacią skończoną równań różniczkowych liniowych rzędu n-go* (pp. 100-104).

A. J. Stodółkiewicz: *Sposób d'Alemberta w zastosowaniu do równań różniczkowych rzędu n-go ze spółczynnikami stałymi* (pp. 105-111).

K. Olearski: *Nowy sposób całkowania pewnych równań różniczkowych pierwszego rzędu o dwu zmiennych* (pp. 131-141).

W. Kretkowski: *O funkcyach równych co do wielkości i różnych co do natury* (pp. 142--144).

A. J. Stodółkiewicz: *O kilku klasach równań różniczkowych liniowych rzędu n-go* (pp. 145--150).

W. Kretkowski: *O pewnej tożsamości* (pp. 151-154).

S. Dickstein: *O rozwiązaniu kongruencyi* $z^n - a^n = 0$ (mod$M$) (pp. 155-159).

K. Żorawski: *O zbieżności iteracyi* (z dwiema figurami w tekście) (pp. 271-288).

K. Żorawski: *Drobne przyczynki do teoryi przekształceń i jej zastosowań* (pp. 289--300).

J. Puzyna: *O wartościach funkcyi analitycznej na okręgach spółśrodkowych z kołem zbieżności jej elementu* (z 7 figurami w tekście) (pp. 311-361).

K. Żorawski: *O pochodnych nieskończenie wielkiego rzędu* (pp. 419-433).

K. Olearski: *Sprostowanie pomyłek drukarskich w rozprawie: Nowy sposób całkowania pewnych równań różniczkowych* (pp. 434-436).

Volume XXVII (1895)

A. J. Stodółkiewicz: *Kilka uwag o czynniku całkującym równań różniczkowych* (pp. 131--138).

S. Kępiński: *O związkach dwuliniowych między stałemi całek rozwiązań pewnych równań różniczkowych rzędu 2-go* (z 3-ma rycinami w tekście) (pp. 384-399).

Volume XXVIII (1895)

K. Żorawski: *O wielkościach zasadniczych ogólnej teoryi powierzchni* (pp. 1-7).

F. Mertens: *Przyczynek do rachunku całkowego* (pp. 53-66).

F. Mertens: *O zadaniu Malfattego* (pp. 67-92).

K. Żorawski: *O całkach niezmiennych ciągłych grup przekształceń* (pp. 232-273).

Volume XXIX (1895)

K. Żorawski: *Iteracye i szeregi odwracające* (pp. 240-249).

K. Żorawski: *O linii wskazującej krzywiznę powierzchni* (pp. 250-265).

Volume XXX (1896)

S. Kępiński: *O funkcyach Fuchsa dwu zmiennych zespolonych* (pp. 211-221).

Volume XXXI (1897)

J. Puzyna: *Do teoryi szeregów potęgowych* (pp. 270-289).

Volume XXXII (1896)

W. Zajączkowski: *O inwolucyi punktów na liniach tworzących powierzchni skośnej* (pp. 279--301).

Volume XXXIII (1898)

S. Dickstein: *Wiadomość o korespondencyi Kochańskiego z Leibnizem* (pp. 1-9).

K. Żorawski: *O pewnych związkach w teoryi powierzchni* (pp. 107-11).

Volume XXXIV (1899)

K. Żorawski: *O całkowaniu pewnej kategoryi równań różniczkowych zwyczajnych rzędu trzeciego* (pp. 141-205).

K. Żorawski: *Przyczynek do teoryi nieskończenie małych przekształceń* (pp. 218-232).

Volume XXXVII (1900)

S. Kępiński: *O peryodach całek hypereliptycznych* (pp. 63-80).

S. Kępiński: *O całkach równań różniczkowych z sobą sprzężonych rzędu 2-go posiadających trzy punkty osobliwe* (pp. 112-138).

K. Żorawski: *O zbieżności szeregów odwracających* (z tabl. I II i III) (pp. 139-153).

K. Żorawski: *Przyczynek do geometryi nieskończenie małych przekształceń* (pp. 154-175).

Volume XXXVIII (1901)

K. Żorawski: *O pewnych zmianach długości liniowych elementów podczas ruchu ciągłego układu materyalnych punktów. Część I-sza* (pp. 353-365).

Ł.E. Böttcher: *O własnościach pewnych wyznaczników funkcyjnych* (pp. 382-389).

Volume XXXIX (1902)

K. Żorawski: *O pewnem zagadnieniu z teoryi podobnego odwzorowania powierzchni* (pp. 218-235).

K. Żorawski: *O zachowaniu ruchu wirowego* (pp. 236-250).

Volume XLI (1901)

S. Zaremba: *O tak zwanych funkcyach zasadniczych w teoryi równań fizyki matematycznej* (pp. 241-275).

S. Kępiński: *O całkach rozwiązań równań różniczkowych z sobą sprzężonych rzędu 2-go posiadających trzy punkty osobliwe (ciąg dalszy)* (pp. 276-288).

S. Zaremba: *O teoryi równania Laplace'a i o metodach Neumanna i Robina* (pp. 350-405).

S. Zaremba: *Przyczynek do teoryi pewnego równania fizyki matematycznej* (pp. 490-504).

J. Rajewski: *O funkcyach hypergeometrycznych i ich przekształceniach* (pp. 505-552).

Volume XLII (1902)

S. Kępiński: *O całkach rozwiązań równań różniczkowych rzędu drugiego z sobą sprzężonych* (pp. 45-69).

K. Żorawski: Uwaga o pochodnych nieskończenie wielkiego rzędu (pp. 212-215).

Volume XLIII (1903)

C. Russjan: *Kilka twierdzeń z teoryi wyznaczników* (pp. 8-13).

S. Zaremba: *Uwagi o pracach prof. Natansona nad teoryą tarcia wewnętrznego* (pp. 14-21).

S. Zaremba: *O metodach średniej arytmetycznej Neumanna i Robina w przypadku gdy ograniczenie nie jest spójne* (pp. 39-70).

J. Puzyna: *O sumach nieskończenie wielu szeregów potęgowych i o twierdzeniu Mittag–Lefflera z teoryi funkcyi* (pp. 148-178).

S. Zaremba: *O pewnem uogólnieniu klasycznej teoryi tarcia wewnętrznego* (pp. 223-246).

S. Zaremba: *O pewnem zagadnieniu hydrodynamiki będącem w związku ze zjawiskiem podwójnego załamania w cieczach odkształcanych i rozbiór pracy prof. Natansona o tym przedmiocie* (pp. 247-266).

C. Russjan: *Metoda Pfaffa całkowania równań różniczkowych cząstkowych rzędu pierwszego.* Część I (pp. 351-396).

S. Zaremba: *O pewnej postaci doskonalszej teoryi relaksacyi* (pp. 492-502).

S. Zaremba: *Zasada ruchów względnych i równania mechaniki fizycznej (Odpowiedź prof. Natansonowi)* (pp. 503-510).

C. Russjan: *Metoda Pfaffa całkowania równań różniczkowych*, Część II (pp. 511-576).

Volume XLV (1906)

S. Kępiński: *Całkowanie równania* $\frac{\partial^2 j}{\partial \xi^2} - \frac{i\partial^2 j}{\xi \partial \xi^2} = 0$ (pp. 1-10).

S. Zaremba: *Ogólne rozwiązanie zagadnienia Fouriera* (pp.19-118).

Volume XLIX (1910)

W. Sierpiński: *Pewne twierdzenie o liczbach niewymiernych* (pp. 433-444).

Volume L (1911)

A. Rosenblatt: *Badania nad kształtami krzywych algebraicznych stopnia szóstego (z tabl. II–XXIII)* (pp. 317-370).

Volume LI (1911)

L. Lichtenstein: *Przyczynek do teoryi równań różniczkowych liniowych o pochodnych cząstkowych drugiego rzędu typu eliptycznego. Całki okresowe i podwójnie okresowe* (pp. 81-112).

A. Rosenblatt: *Przyczynek do klasyfikacyi powierzchni rozwijalnych algebraicznych* (pp. 113-151).

J. Puzyna: *O systemach krzywych z grupą pseudoliniowych podstawień* (pp. 201-324).

L. A. Birkenmajer: *Aforyzmy z teoryi podstawień* (pp. 379-464).

Volume LII (1912)

W. Sierpiński: *O pewnym układzie równań funkcyjnych który wyznacza funkcyę mającą pantachiczne przedziały stałości* (pp. 1-15).

W. Sierpiński: *O pewnym szeregu wielomianów którego suma przedstawiać może przy odpowiednim uporządkowaniu składników dowolną funkcyę ciągłą* (pp. 33-43).

A. Hoborski: *O pewnem zastosowaniu zasady najmniejszej wartości* (pp. 89-161).

A. Rosenblatt: *Badania nad pewnemi klasami powierzchni algebraicznych nieregularnych i nad biracyonalnemi przekształceniami nie zmieniającymi tych powierzchni* (pp. 197--294).

Volume LVI (1917)

H. Steinhaus: *Niektóre własności szeregów trygonometrycznych i szeregów Fouriera* (pp. 175-225).

Volume LVII (1918)

A. Hoborski: *Uwagi o podstawach geometryi rzutowej* (pp. 1-82).

L. A. Birkenmajer: *O związku twierdzenia Wilsona z teoryą reszt kwadratowych* (pp. 137--149).

L. A. Birkenmajer: *Wymierne trójkąty Herona i Hindów* (pp. 175-191).

Volume LX (1921)

A. Maksymowicz: *Z teorji szeregów sumowalnych metodą Cesaro–Hölder* (pp. 1-42).

A. Łomnicki: *Uogólnienie wzoru interpolacyjnego Lagrange'a* (pp. 173-179).

Volume LX VI (1928)

K. Żorawski: *Własności pewnej kategorii przekształceń punktowych na plaszczyźnie* (pp. 37-70).

K. Żorawski: *Ruchy sztywne i kompleksy linjowe* (pp. 278-340).

Volume LX VIII (1929)

K. Żorawski: *O pewnych przekształceniach przestrzeni będących w związku z własnościami funkcyj zmiennych zespolonych* (pp. 130).

In the next volumes LXIX–LXXIV (1951) of *Reports of the Academy of Arts and Sciences* in Cracow the mathematical papers are not published.

3. Conclusions

In all mathematical papers (about 90 of them) in *Dissertations and Reports of Meetings of the Academy of Arts and Sciences* in Cracow (first series) and in *Dissertations of the Academy of Arts and Sciences* in Cracow (second and third series) [2] in the years 1874–1951 there were published 27 papers in differential equations [1]. These papers were published by the following seven mathematicians: Alojzy Jan Stodółkiewicz (1856–1934) – eight works, S. Kępiński (1867–1908) – eight papers, S. Zaremba (1863–1942) – five papers, K. Olearski and C. Russjan – two works and one work by K. Żorawski (1866–1953) and Władysław Zajączkowski (1837–1898) each.

Out of remaining papers about 28 ones concerned classical mathematical analysis; 16 differential geometry; 7 algebra; 1 analytic geometry; 5 algebraic geometry. The subjects of other papers are mechanics, history of mathematics, algebra and probability. Therefore, including works in differential equations into analysis one can see that about 55 papers among 90 are devoted to mathematical analysis. Among authors of mathematical papers there are the famous mathematicians outside from Cracow such as e.g. A. Łomnicki, W. Sierpiński and H. Steinhaus. Therefore *Academy of Arts and Sciences* in Cracow played an important role in the scientific life of the Polish nation. Hence Polish mathematicians were involved in an active manner in the development of main mathematical ideas.

References

[1] Koroński J., *Prace z równań różniczkowych w Rozprawach i Sprawozdaniach z Posiedzeń Akademii Umiejętności w Krakowie w latach 1874–1951* (in preparation).

[2] *Dissertations and Reports of Meetings of the Academy of Arts and Sciences* in Cracow, Vol. 1–74, Kraków 1874–1951.

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JAN KOROŃSKI*

STANISŁAW KĘPIŃSKI (1867–1908) AND HIS PAPERS IN THE FIELD OF DIFFERENTIAL EQUATIONS

STANISŁAW KĘPIŃSKI (1867–1908) I JEGO PRACE Z RÓWNAŃ RÓŻNICZKOWYCH

A b s t r a c t

The subject of this paper is an analysis of the publications of Stanisław Kępiński in the field of ordinary and partial differential equations. In particular we present part I and part II of the monograph (textbook) of Stanisław Kępiński on the ordinary and partial differential equations.

Keywords: *ordinary differential equations*, *partial differential equations*, *publications of Stanisław Kępiński*

S t r e s z c z e n i e

Artykuł poświęcony jest prezentacji publikacji Stanisława Kępińskiego w dziedzinie zwyczajnych i cząstkowych równań różniczkowych. W pracy prezentujemy zwłaszcza dwuczęściową monografię (podręcznik) z równań różniczkowych zwyczajnych i cząstkowych.

Słowa kluczowe: *równania różniczkowe zwyczajne*, *równania różniczkowe cząstkowe*, *publikacje Stanisława Kępińskiego*

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1. Stanisław Kępiński (11.11.1867–24.03.1908)

Stanisław Kępiński was born on November 11, 1867, in Bochnia near Cracow [1, 3]. He graduated from the III gymnasium in 1885 in Cracow and passed his matura exam. In the same year he began studying mathematics, physics and astronomy at the Jagiellonian University. In 1890 he received the degree of Doctor of Philosophy on the basis on the paper: *O całkowaniu równań różniczkowych cząstkowych rzędu drugiego.* Kępiński went on to postdoctoral studies in Göttingen, from where he returned to do his habilitation in year 1893 on the basis of his work: *O całkach rozwiązań równań różniczkowych zwyczajnych liniowych jednorodnych rzędu drugiego*, Rozprawy AU, seria A, Vol. 26, 1893, p. 264-328. Next he worked at the Jagiellonian University (in the years 1893–1898) and in the years 1883–1896 he was a teacher in a high school in Cracow (Wyższa Szkoła Realna w Krakowie). He lectured on mathematics at Jagiellonian University, since the year 1896 as extraordinary professor [1]. In 1898 he was appointed a professor of mathematics in Lvov at the Polytechnic School, of which he was the rector in the academic year 1903/1904. He also lectured on mathematics at the Lvov University [3]. Kępiński was one of the greatest Polish mathematicians of the nineteenth century. His research papers dealt mainly with differential equation. In 1907 Kępiński published a monograph (textbook) on the ordinary and partial differential equations [*Podręcznik równań różniczkowyh ze szczególnem uwzględnieniem potrzeb techników i fizyków*, Cz. l, *Równania różniczkowe zwyczajne*, Z I. Związkowej Drukarni, Lwów 1907, Biblioteka Politechniczna, Vol. l8, Cz. l, VIII, 195 pp.; and 2. *Podręcznik równań różniczkowyh ze szczególnem uwzględnieniem potrzeb techników i fizyków*, Cz. 2, *Równania różniczkowe cząstkowe*, Z I. Związkowej Drukarni, Lwów 1907, Biblioteka Politechniczna, Vol. 18, Cz. 2, VI, 199 pp.].

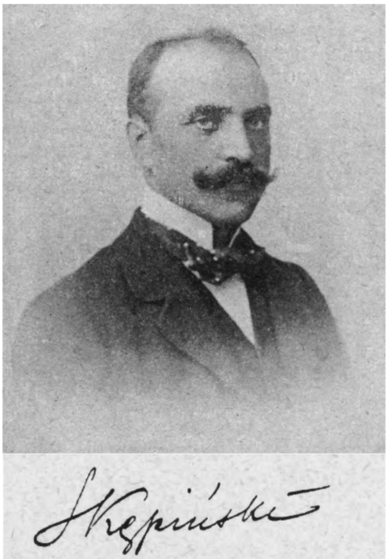

Fig. 1. Stanisław Kępiński (11.11.1867–24.03.1908) [3]

2. List of Stanisław Kępiński`s publications in the field of differential equations (We keep the original nineteenth-century spelling):

1. *Własności szczególnych trójek punktów trójkąta*, Prace Mat.-Fiz., Vol. 2, 1890, pp. 169--219.
2. *O całkowaniu równań różniczkowych cząstkowych rzędu drugiego*, Dissertation, 1891.
3. *O całkach rozwiązań równań różniczkowych zwyczajnych liniowych jednorodnych rzędu 2-go*, Rozprawy Akad. Um. Wydz. Mat.-Przyr. w Krakowie, Vol. 25, 1893, pp. 264-328.
4. *Z teoryi nieciągłych grup podstawień liniowych, posiadających spółczynniki rzeczywiste*, Rozprawy Akad. Um. Wydz. Mat.-Przyr. w Krakowie, Vol. 26, 1893, pp. 37-66.
5. *O związkach dwuliniowych między stałemi całek rozwiązań pewnych równań różniczkowych rzędu 2-go*, Rozprawy Akad. Um. Wydz. Mat.-Przyr. w Krakowie, Vol. 27, 1895, pp. 384-399.
6. *O funkcyach Fuchsa dwu zmiennych zespolonych*, Akad. Um. Wydz. Mat.-Przyr. w Krakowie, Vol. 30, 1896, pp. 211-221.
7. *Sur les fonctions de Fuchs à deux variables complexes,* Bull. Intern. Acad., 1895, pp. 288-289.
8. *Ueber Fuchs'sche Functionen zweier Variabein*, Math. Annalen., 1896. Bd. 47, pp. 573--578.
9. *O peryodach całek hypereliptycznych rodzaju p = 2*, Prace Mat.-Fiz., Vol. 9, 1898, pp. 139-163.
10. *Sur les intégrales des solutions des équations du second ordre, équivalentes â leur adjointe, avec trois points singuliers*, Bull. Intern. Acad., 1898, pp. 67-75.
11. *Ueber die Periodicitätsmoduln der hyperelliptischen Integrale*, Bull. Intern. Acad., 1898, pp. 270-272.
12. *Przemówienie na uroczystej inauguracyi roku szkolnego w Lwowskiej Szkole Politechnicznej*, Czasop. Techn. 1900, Rocz. 18, Nr 21, pp. 265-266; Nr 22, pp. 273-275 .
13. *O krzywej normalnej* φ *rodzaju* ρ = 3, Prace Mat.-Fiz., 1900, Vol. 11, pp. 1-22.
14. *O peryodach całek hypereliptycznych*, Rozprawy Akad. Um. Wydz. Mat.-Przyr. w Krakowie, Vol. 37, 1900, pp. 63-80.
15. *O całkach rozwiązań równań różniczkowych, z sobą sprzężonych, rzędu 2-go, posiadających trzy punkty osobliwe*, Rozprawy Akad. Um. Wydz. Mat.-Przyr. w Krakowie, Vol. 37, 1900, pp. 112-138.
16. *Geometrya analityczna*, Lwów 1901, pp. 258 (Litografia).
17. *O całkach rozwiązań równań różniczkowych, z sobą sprzężonych, rzędu 2-go, posiadających trzy punkty osobliwe*, Rozprawy Akad. Um. Wydz. Mat.-Przyr. w Krakowie, Vol. 41, 1901, pp. 276-288. Idem, Buli. Intern. Acad., 1901, pp. 134-141.
18. *Wykłady matematyki. Cz. 1. Rachunek różniczkowy i zastosowania*, Lwów 1902–1903, VIII, pp. 419 (Litografia).

19. *O całkach rozwiązań równań różniczkowych, rzędu drugiego, z sobą sprzężonych*, Rozprawy Akad. Um. Wydz. Mat.-Przyr. w Krakowie, Vol. 42, 1902, pp. 45-69.
20. *Ueber Integrale der Lösungen der gewöhnlichen linearen sich selbst adjungierten Differentialgleichungen zweiter Ordnung*, Buli. Intern. Acad., 1902, pp. 65-88.
21. *Geometrya analityczna na płaszczyźnie*, Lwów 1903, pp. 238 (Litografia).
22. *Integration de Differentialgleichung* $\frac{\partial^2 j}{\partial \xi^2} - \frac{i \partial^2 j}{\xi \partial \xi^2} = 0$, Buli. Intern. Acad., 1905, pp. 198-205.
23. *Über die Differentialgleichung* $\frac{d^2 z}{dx^2} + \frac{m+1}{x}\frac{dz}{dx} - \frac{n}{x}\frac{dz}{dt} = 0$, Math. Annalen., 1905, Bd. 61, pp. 397-405.
24. *O drganiach poprzecznych prętów sprężystych*, Prace Mat.-Fiz., 1905, Vol. 16, pp. 71--107.
25. *Całkowanie równania* $\frac{\partial^2 j}{\partial \xi^2} - \frac{i \partial^2 j}{\xi \partial \xi^2} = 0$, Rozprawy Akad. Um. Wydz. Mat.-Przyr. w Krakowie, Vol. 45, 1906, pp. 1-10.
26. *Podręcznik równań różniczkowyh ze szczególnem uwzględnieniem potrzeb techników i fizyków. Cz. l. Równania różniczkowe zwyczajne*, Z I. Związkowej Drukarni, Lwów 1907, Biblioteka Politechniczna, Vol. l8, Cz. l, VIII, pp. 195.
27. *Podręcznik równań różniczkowyh ze szczególnem uwzględnieniem potrzeb techników i fizyków. Cz. 2. Równania różniczkowe cząstkowe*, Z I. Związkowej Drukarni, Lwów 1907, Biblioteka Politechniczna, Vol. l8, VI, pp. 199.
28. *Wykłady matematyki: równania różniczkowe*, Wyd. 2, W. Kutyłowski-Sokół, Lwów 1933, VII, pp. 283.: il, err.

3. Monograph (textbook) of Satnisław Kępiński on the ordinary and partial differential equations

Kępiński published about 30 works [2]. More than 20 of his publications are devoted to ordinary and partial differential equation. He was also the author of a monograph (a textbook) on the ordinary and partial differential equation.

This monograph was entitled *Podręcznik równań różniczkowyh ze szczególnem uwzględnieniem potrzeb techników i fizyków. Cz. 1, Równania różniczkowe zwyczajne,* Z I. Związkowej Drukarni, Lwów 1907, Biblioteka Politechniczna, Vol. l8, Cz. l., VIII, pp. 195 and *Podręcznik równań różniczkowyh ze szczególnem uwzględnieniem potrzeb techników i fizyków. Cz. 2. Równania różniczkowe cząstkowe*, Z I. Związkowej Drukarni, Lwów 1907, Biblioteka Politechniczna, Vol. 18, VI, pp. 199. The subject of first part of the above monograph are ordinary differential equations. In the second part Kępiński considered partial differential equations.

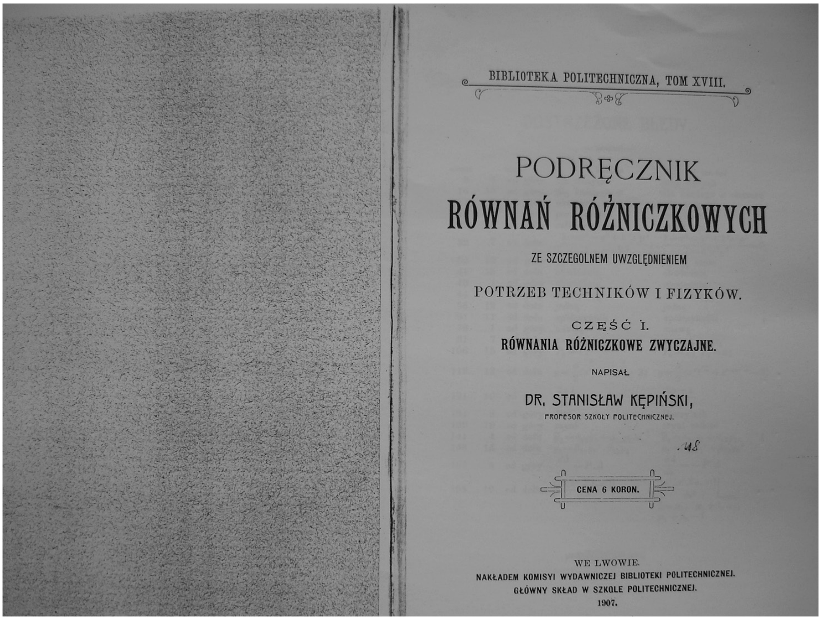

Fig. 2. The cover page of the first part of the monograph of S. Kępiński on ordinary differential equation

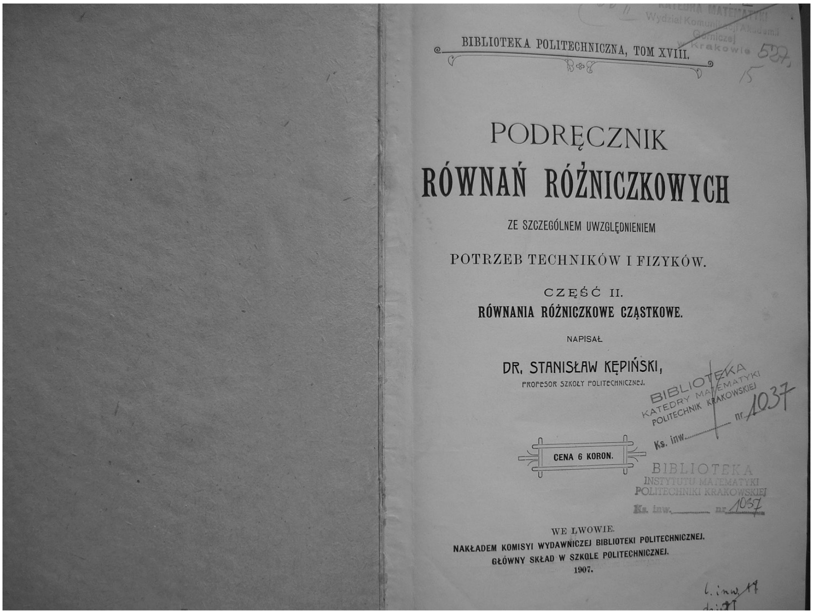

Fig. 3. The cover page of the second part of the monograph of S. Kępiński on partial differential equation

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[1] Archiwum Uniwersytetu Jagiellońskiego, sygn. WF. II 124, Kępiński Stanisław (teczka osobowa).

[2] Kępiński S., The papers presented at the above *List of Stanisław Kępiński's publications in the field of differential equations* (see part 2 of this paper).

[3] Żorawski K., *Stanisław Kępiński. Wspomnienie pośmiertne*, Wiadomości Matematyczne, Vol. XII, z. 5–6, Warszawa 1908, 161-167.

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JAN KOROŃSKI*

STANISŁAW ZAREMBA (1863–1942) AND HIS RESULTS IN THE FIELD OF DIFFERENTIAL EQUATIONS

STANISŁAW ZAREMBA (1863–1942) I JEGO WYNIKI W DZIEDZINIE RÓWNAŃ RÓŻNICZKOWYCH

A b s t r a c t

The subject of the paper is presentation of the publications of Stanisław Zaremba in the field of partial differential equations. We present selected results in detail. Zaremba published about 120 works. Among them more than 60 were devoted to partial differential equations.

Keywords: *differential equations*, *partial differential equations*, *publications of Stanisław Zaremba*

S t r e s z c z e n i e

Artykuł poświęcony jest prezentacji publikacji Stanisława Zaremby w dziedzinie równań różniczkowych cząstkowych. Wybrane rezultaty zaprezentowano bliżej. Zaremba opublikował około 120 prac, z których ponad 60 jest poświęconych równaniom różniczkowym cząstkowym.

Słowa kluczowe: *równania różniczkowe zwyczajne*, *równania różniczkowe cząstkowe*, *publikacje Stanisława Zaremby*

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1. Stanisław Zaremba (03.12.1863–23.11.1942)

Stanisław Zaremba was born on 3th December 1863 in a village Romanówka [1]. In 1881 he finished high school (the German gymnasium in St. Petersburg) and next studied engineering at the Institute of Technology in St. Petersburg (getting an engineering diploma in 1886). Then in 1887 he went to Paris, where he studied mathematics for his doctorate at the Sorbonne in 1899, advised by Darboux and Picard.

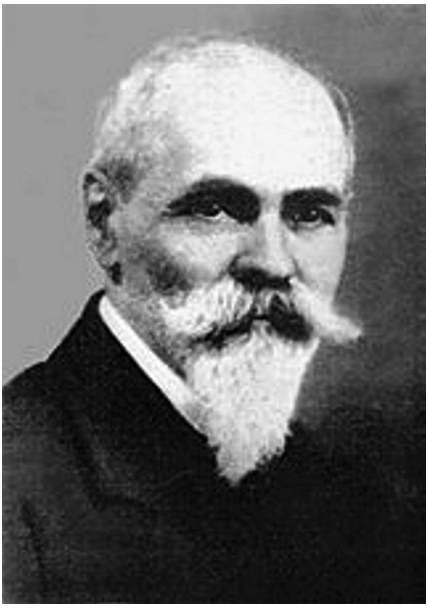

Fig. 1. Stanisław Zaremba (03.12.1863–23.11.1942)

As a topic for his dissertation Zaremba chose the ideas introduced by Riemann in 1861. His doctoral thesis *Sur un problème concernant l'état calorifique d'un corp homogène indéfini* was presented in 1889. At that time Zaremba got in touch with many mathematicians of the French school. He maintained these ties, engaging in a wide international cooperation after returning to Poland. In particular he collaborated with Painlevé and Goursat. Before 1900 Zaremba taught in secondary schools in France. At that time he concentrated hard on his research. He published his results in French mathematical journals, which resulted in his work becoming well known and highly respected by leading French mathematicians such as Poincaré and Hadamard. Zaremba's publications concerned mainly partial differential equations. These publications played a very important role in the world development of mathematical sciences. In 1900 he came back to Cracow being nominated *extraordinary professor* and in 1905 the *ordinary professor* (*full professor*) at the Jagiellonian University. He worked in Jagiellonian University until 1935 (when he retired). In that year he obtained the title of the *honorary professor of the Jagiellonian University*, a very unusual titular dignity. Zaremba died (at the age of 79) on 23th November 1942.

2. List of Stanisław Zaremba's publications in the field of differential equations

(On the basis of [2], where about 118 publications of S. Zaremba are mentioned)

1. *Sur un problème concernant l'état calorifique d'un corps solide homogène idéfini (*Dissertation in Sorbone – advisored by Darboux and Picard) Paris, chez Gauttier--Villars, 1899.
2. *Note concernant l'intégration d'une équation aux dérivées partialles*, C. R. d l' Ac. de Paris, 1890.
3. *Note concernant l'intégration d'une équation aux dérivées partialles,* Annales de l' École Normale, 3ieme Série 7, fasc. 3, mai 1890.
4. *Sur la réduction du nombree des périodes d'une fonction périodique*, Bull. de la Société Mathématique de France, 1894.
5. *Contribution a la théorie de la fonction de Green*, Bull. de la Société Mathématique de France, 1896.
6. *Note sur la methode des approximations successives de M. Picard*, C. R. d *l'*Ac. de Paris, 1897.
7. *Sur la méthode des approximations successives de M. Picard*, Journal de Mathématiques Pures et Appliquées 5ieme Série 3, fasc. 3, 1897.
8. *Sur la méthode des approximations successives de M. Picard,* Journal de Liouville, 1897.
9. *Note sur le problème de Dirichlet*, C. R. d *l'* Ac. de Paris, 1897.
10. *Note sur le problème de Dirichlet*, Annales de l' École Normale Supérieure, 1897.
11. *Sur le problème de Dirichlet*, Annales de l' École Normale Supérieure, 1897, 3ieme Série 14, juillet, 1897.
12. *Zastosowanie metody Picarda do równań różniczkowych trzech zmiennych*, Prace Mat.--Fiz., T. IX,1898.
13. *O zasadzie Dirichleta*, Prace Mat.-Fiz., T. IX, 1898.
14. *Sur l'équation aux dérivées partialles* $\Delta u + \xi u + f = 0$, Bull. de la Socciété Mathématique de France, 1898.
15. *Note sur un theoreme de M. Poincaré*, C. R. d *l'* Ac. de Paris,1899.
16. *Sur l'équation aux dérivées partialles* $\Delta u + \xi u + f = 0$ *et sur les fonctions harmoniques*, Annales de l' École Normale, 3ieme Série 16, octobre, 1899.
17. *Note sur le développement d'une fonction arbitraire en une série procedant suivant les fonctions harmoniques*, C. R. d *l'* Ac. de Paris, 1899.
18. *Sur le développement d'une fonction arbitraire en une série procédant suivant les fonctions harmoniques*, Journal de Liouville, 1890.
19. *O równaniu o pochodnych cząstkowych* $\Delta u + \xi u + f = 0$ *i o funkcjach harmonicznych*, Prace Mat.-Fiz., T. XI,1900.
20. *O rozwinięciu dowolnej funkcji w szereg (funkcji) harmonicznych*, Prace Mat.-Fiz., T. XI, 1900.

21. *Sur le développement d'une fonction arbitraire en une sérié procédant suivant les founctions harmoniques*, Journal de Mathématiques Pures et Appliquées, 1900.
22. *Contribution a la théorie de l'équation aux dérivées partialles* $\Delta v + \xi v = 0$, Annales de la Faculté des Sciences de l' Université de Toulouse, 1900.
23. *Contribution a la théorie des'équations de la Physique*, C. R. d *l'* Ac. de Paris, 1901.
24. *O tak zwanych funkcjach zasadniczych w teoryi równań fizyki matematycznej*, Rozprawy Wydz. Mat.-Przyrodniczego Akademii Umiejętności, Seria III, T. I, 1901.
25. *Sur les fonctions dites fondamentales la théorie des équations de la physique*, Bulletin International de l'Academie des Sciences de Cracovie, 1901.
26. *Note sur l' integration de l'équation aux dérivées partialles* $\Delta u + \xi u = 0$, C. R. d *l'* Ac. de Paris, 1901.
27. *O teoryi równania Laplace'a i o metodach Neumanna i Robina*, Rozprawy Wydz. Mat.--Przyrodniczego Akademii Umiejętności, Seria III, T. I, 1901.
28. *Sur la théorie de l'équation de Laplace at les méthodes de Neumann et de Robin*, Bulletin International de l'Academie des Sciences de Cracovie, 1901.
29. *Przyczynek do teoryi pewnego równania fizyki matematycznej*, Rozprawy Wydz. Mat.--Przyrodniczego Akademii Umiejętności, Seria III, T. I, 1901.
30. *Contribution a la théorie d' une équation de la physique*, Bulletin International de l'Academie des Sciences de Cracovie, 1901.
31. *Sur les méthodes de la moyenne arithmétique de Neumann et de Robin de la cas d'une frontiére non connexe*, Bulletin International de l'Academie des Sciences de Cracovie, 1902.
32. *Sur l' intégration de l'équation* $\Delta u + \xi u = 0$, Journal de Mathématiques Pures et Appliquées, 1902.
33. *Sur l' intégration de l'équation* $\Delta u + \xi u = 0$, Journal de Liouville, 1902.
34. *Détermination du cas ou les fonctions fondamentales de M. Poincaré sont déductibles de celles de M. Roy et de celles de M. Stekloff*, Bulletin International de l'Academie des Sciences de Cracovie, 1902.
35. *Sur une généralisation de la théorie classique de la viscosité*, Bulletin International de l'Academie des Sciences de Cracovie, 1903.
36. *Contribution a la théorie des fonctions fondamentales*, Annales de l' École Normale, 1903.
37. *O metodach średniej arytmetycznej Neumanna i Robina w przypadku, gdy ograniczenie nie jest spójne*, Rozprawy Wydz. Mat.-Przyrodniczego Akademii Umiejętności, 1903.
38. *Les fonctions fondamentales de M. Poincaré et la méthode de Neumann pour frontière compasèe des polygenes curvilignes*, Journal de Mathématiques Pures et Appliquées, 1904.
39. *Contribution à la théorie d' une équation fonctionelle de la physique*, Rendiconti del Circolo Matematico di Palermo, 1904.

40. *Solution generale du problème de Fourier*, Bulletin International de l'Academie des Sciences de Cracovie, 1905.
41. *Ogólne rozwiązanie zagadnienia Fouriera*, Rozprawy Wydz. Mat.-Przyrodniczego Akademii Umiejętności, 1905.
42. *Sur les fonction de Green et quelques-unes de ses applications*, Bulletin International de l'Academie des Sciences de Cracovie, 1906.
43. *L'equation biharmonique et une classe remarquable de fonctions fondamentales harmoniques*, Bulletin International de l'Academie des Sciences de Cracovie, 1907.
44. *Równania biharmonijne i pewien szczególny rodzaj funkcyj harmonijnych zasadniczych*, Wiadomości Matematyczne, XI, 1907 (cz. II sprawozdanie).
45. *Nowa metoda uzasadniania podstawowych własności funkcji Greena*, Wiadomości Matematyczne, XI, 1907.
46. *Sur l' intégration de l'équation biharmonique*, Bulletin International de l'Academie des Sciences de Cracovie, 1908.
47. *Sur le principe de Dirichlet*, Atti del IV Congresso Internazionale dei Matematici, Roma, 1908.
48. *Sur le princile de minimum*, Bulletin International de l'Academie des Sciences de Cracovie, 1909.
49. *Sur l' unicité de la solution du problème de Dirichlet*, Bulletin International de l'Academie des Sciences de Cracovie, 1909.
50. *Sur le calcul numérique des fonctions demandées dans le problème de Dirichlet et le problème hydrodynamique*, Bulletin International de l'Academie des Sciences de Cracovie, 1909.
51. *Le problème biharmonique restreint*, Annales de l'Ecole Normale, 1909.
52. *Sur le principe de Dirichlet*, Acta Mathematica 34, Stockholm, 1910.
53. *Sur la problème mixte relatif à équation de Laplace*, Bulletin International de l'Academie des Sciences de Cracovie, 1910.
54. *Sur un théoréme fondamental relatif à l'équation de Fourier*, Comptes Rendus du Congres International des Mathématiciens, Strassburg, 1920.
55. *Les fonctions réel es non analytiques et les solutions singulariéres des équations differentialles du premier ordre*, Annales de la Société Polonaise de Mathématiques, 5, 1922.
56. *Sur une forme remarquable de l' intégrale de l'équation des cordes vibrantes*, Nouvelles Annales de Mathématiques, Paris, 1923.
57. *Sur un problème toujours possible comprenant à titre de cas particuliers, le problème de Dirichlet et celui de Neumann*, Journal de Mathématiques Pures et Appliquées, 1927.
58. *Pogląd na współczesny stan teoryi potencjału*, Mathesis Polska, 1931.
59. *Un théoréme général relatif aux équation aux dérivées partialles du second ordre lineaires et du type hyperbolique*, Bulletin International de l'Academie des Sciences de Cracovie, 1934.

3. Selected scientific results of Stanisław Zaremba in the field of partial differential equations

This presentation is based on the papers mentioned in [7] and [5]: T. Ważewski, J. Szarski, *Stanisław Zaremba*, *Studia z dziejów katedr Wydziału Matematyki, Fizyki, Chemii Uniwersytetu Jagiellońskiego* (ed. S. Gołąb), Wydawnictwa Jubileuszowe – Tom XV, Uniwersytet Jagielloński, Kraków, 1964, pp. 103-117; the A. Pelczar elaboration [4] entitled: *Stanisław Zaremba* (*100th anniversary of taking up a chair at the Jagiellonian University*). *Prepared for the International Conference 90 years of the reproducing Kernel Property, Kraków, April 16–21, 2000 organised by the Chair of Functional Analysis of the Jagiellonian University.* The last one is also based on the the paper [5]. Because Ważewski, Szarski and Pelczar (world-famous specialists in the field of differential equations) had written about Zaremba's scientific results and characterised Zaremba's mathematical achievements, below we are going to cite *in extenso* a part of Pelczar's elaboration [4] on Zaremba's works connected with the subject of differential equations.

"... Among his important results there are those concerning the elliptic equation

$$\Delta u + \xi u + f = 0 \tag{1}$$

with boundary Dirichlet conditions as well as Neumann and Fourier type conditions. Some of these results were included into the canon of the fundamental knowledge on the theory of partial differential equations. Before talking about some details let us quote a sentence from the book [J. Mawhin, *Metody wariacyjne dla nieliniowych problemów Dirichleta,* (Polish version of the book *Problèmes de Dirichlet variationneles non linéaires*; translated by D.P. Idziak, A. Nowakowski, S. Walczak), Warszawa 1995] of Jean Mawhin: *According to Bouligand Zaremba's contribution to the development of the theory of the Dirichlet problem is the same as that of Poincaré and Lebesgue.*

In the paper [S. Zaremba, Sur le problème de Dirichlet, *Annales de l'École Normale* (3), 14, 1897, 251-258] properties of the Green function G for a Dirichlet problem in the three dimensional space is considered and it is shown that the function

$$u = \int_S \frac{\partial G}{\partial n} \sigma ds$$

is a solution to a given Dirichlet problem with the boundary condition described by a continuous function σ, a discussion of properties of u in the case of non-continuos σ is included as well.

In the paper [S. Zaremba, Sur l'équation aux dérivées partielles $\Delta u + \xi u + f = 0$ et sur les fonctions harmoniques, *Annales de l'École Normale,*(3)16, 1899, 427-463]. Zaremba discussed the equation (1) for $f = 0$ with the condition

$$\frac{\partial G}{\partial n} = hu, \tag{2}$$

where h is a non-negative constant and $\dfrac{\partial G}{\partial n}$ is the interior normal derivative. He proved that there exist a sequence of eigenvalues and corresponding sequence $\{U^k\}$ of orthonormal eigenfunctions. He proved also that if ξ is not an eigenvalue then the problem (1)–(2) (with f = 0) has exactly one solution. Moreover, every function satisfying the boundary condition (2) can be represented as a Fourier series with respect to the sequence $\{U^k\}$ of eigenfunctions. Zaremba developed some idea of Poincaré and used *generalised potentials* defined by replacing in the classical definition of Newtonian potential the function $\dfrac{1}{r}$ by the function $\dfrac{\exp(-\eta r)}{r}$, where η is a complex number such that $re\,\eta > 0$ and $\eta^2 + \xi = 1$. This notion of generalised potentials introduced by Zaremba turned out to be very useful in several other problems.

Analogous results for the homogenous problem: (1) with f = 0 and the boundary problem u = 0 are given in the paper [S. Zaremba, Sur le développement d'une fonction arbitraire en un série procédant suivant les fonctions harmoniques, *Journal de Mathématiques pures et appliquées,*(5),6, 1900, 47-72].

In [S. Zaremba, Contribution à la théorie de l'équation aux dérivées partielles $\Delta u + \xi u = 0$, *Annales de la Faculté des Sciences d l'Université de Toulouse,* (32), 3, 1900, 5-12]. Zaremba gave some conditions sufficient for derivatives of arbitrary order of solutions of Dirichlet problems for the homogenous equation (1) (with f = 0) in a domain D to be continuous in the closure of D. Importance of this result is underlined by Jean Mawhin in the preface to the Polish version of his book [J. Mawhin, *Metody wariacyjne dla nieliniowych problemów Dirichleta,* (Polish version of the book *Problèmes de Dirichlet variationneles non linéaires*; translated by D.P. Idziak, A. Nowakowski, S. Walczak), Warszawa 1995].

The paper [S. Zaremba, Sur l'intégration de l'équation Δu + ξu = 0, *Journal de Mathématiques pures et appliqueés*, (5), 8, 1902, 59-117] deals with the following problem. Let D be a bounded domain in the three dimensional real space, S be the intersections of the boundaries of D and $-D$, n the normal unit vector directed into the domain D. For a function u of three real variables and a given point $x^0 \in S$ we put

$$\left(\frac{\partial u}{\partial n}\right)_i = \lim \frac{1}{t}(u(x^0 + tn) - u(x^0) \quad \text{as} \quad t \to 0+,$$

$$\left(\frac{\partial u}{\partial n}\right)_i = \lim \frac{1}{t}(u(x^0 + tn) - u(x^0) \quad \text{as} \quad t \to 0-,$$

and for a function v and $x^0 \in S$, $(\nu)_i = \lim \nu(x)$ $x \to x^0$, $x \in D$ and $(\nu)_e = \lim \nu(x)$ $x \to x^0$, $x \notin D$.

We look for two solutions u and v of the homogenous equation (1) (that is with f = 0), which are generalised potentials of single layer and of double layer respectively, such that

$$\left(\frac{\partial u}{\partial n}\right)_e - \left(\frac{\partial u}{\partial n}\right)_i = \lambda\left[\left(\frac{\partial u}{\partial n}\right)_e + \left(\frac{\partial u}{\partial n}\right)_i\right] + 2\varphi,$$

$$(\nu)_e - (\nu)_i = \lambda[(\nu)_e + (\nu)_i] + 2\varphi,$$

Where λ is a parameter and φ is a given function defined on S. Zaremba proved that this problem has (under general, relatively weak regularity assumptions) a solution which is an analytic function of λ and has at most one essential singularity (at infinity) and single poles at points belonging to sequence of real numbers independent on the function φ. This permits to deal with the Neumann method assuming weak regularity conditions. An analogous problem on the plane is considered in the paper [S. Zaremba, Les fonctions fondamentales de M. Poincaré et la méthode de Naumann pour une frontière composée des polygones curvilignes, *Journal de Mathématiques pures et appliquées,* (5),10, 1904, 395-444] (without the assumption of the continuity of $\frac{\partial u}{\partial n}$).

In an earlier paper [S. Zaremba, Sur la méthode d'approximations successives de M. Picard, *Journal de Mathématiques pures et appliquées,* (5), 3, 1897, 311-329] Zaremba deals with successive approximations for solutions of a non-linear equation

$$\Delta u = f\left(x, y, z, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right).$$

This paper, as well as the paper [S. Zaremba, Contribution a la théorie de la fonction de Green, *Bulletin de la Société Mathématique de France,* 54, 1896, 19-24], is cited in [A. Sommerfeld, Randwertaufgaben in der Theorie der partiellen Differentialgleichungen, [in:] *Ecyklopädie der Mathematischen Wissenschaften,band II-1,* Leipzig 1907, 505-570] (p. 528) where a canon of the theory of elliptic partial differential equations is presented.

The paper [S. Zaremba, Le problème biharmonique restreint, *Annales de l'École Normale*, (3), 26, 1909, 337-404] is an extension of an unpublished note presented to the Paris Academy of Sciences and characterised by that Academy as *extrémement honorable.* A biharmonic problem considered there is such that we are looking for a solution *u* of the equation

$$\Delta^2 u = 0$$

considered in a domain *D* such that

$$u = \varphi \quad \text{and} \quad \frac{\partial u}{\partial x_i} = \frac{\partial \varphi}{\partial x_i}$$

on the boundary ∂D of the domain *D*,
where φ is a sufficiently regular function defined on ∂D (in particular the second power of the Laplacian of φ is assumed to be integrable) requesting certain natural regularity condition to be fulfilled by *u.* Zaremba proved that in order to solve that problem it is sufficient to find a function *v* harmonic in *D*, such that v^2 is integrable and for every harmonic function *h* such that h^2 is integrable on *D* the following equality is satisfied

$$u = \int_D \Delta\varphi h d\tau = \int_D v h d\tau.$$

Zaremba proved theorems on existence and uniqueness of solutions to that problem. He proved also that this problem can be solved by determining the minimum of an integral on *D*.

Basing on some result of the paper [S. Zaremba, Contribution à la théorie d'une équation fonctionelle de la physique, *Rendiconti del Circolo Matematico di Palermo,* 19, 1904, 140--150] Witold Wilkosz (1891–1941) proved a theorem on analyticity of harmonic functions (see [W. Wilkosz, Sur un point fondamental de la théorie du potentiel, *Comptes Rendus de l'Académie des Sciences de Cracovie,* 174, 1922, 435-437]).

The Dirichlet problem with non-continuous boundary conditions was treated in the paper [S.Zaremba, Sur l'unicité de la solution du problème de Dirichlet, *Bulletin International de l'Académie des Sciences de Cracovie, Classe des Sciences Mathématiques et Naturelles,* 1909, 561–564]; the main result of this paper is the first one of that type.

Several other papers were devoted to the theory of the Dirichlet problem (see for instance [S. Zaremba, Sur le calcul numérique des fonctions demandées dans le problème de Dirichlet et le problème hydrodynamique, *Bulletin International de l'Académie des Sciences de Cracovie, Classe des Sciences Mathématiques et Naturelles,* 1909, (2), 125-195], [S. Zaremba, Sur le principe du minimum, *Bulletin Internationale de l'Académie des Sciences de Cracovie, Classe des Sciences Mathématiques et Naturelles,* 1909, (7), 197-264], [S. Zaremba, Sur le principe de Dirichlet, *Acta Mathematica,* 34, 1911, 293-316], [S. Zaremba, Sur un problème toujours possible comprenant à titre de cas particuliers, le problème de Dirichlet et celui de Neumann, *Journal de Mathématiques pures et appliquées,*(9), 6, 1927, 127-163]). In papers [S. Zaremba, Sur le principe du minimum, *Bulletin Internationale de l'Académie des Sciences de Cracovie, Classe des Sciences Mathématiques et Naturelles,* 1909, (7), 197-264], [S. Zaremba, Sur le principe de Dirichlet, *Acta Mathematica*, 34, 1911, 293-316], [S. Zaremba, Sur un problème toujours possible comprenant à titre de cas particuliers, le problème de Dirichlet et celui de Neumann, *Journal de Mathématiques pures et appliquées,*(9), 6, 1927, 127-163] Zaremba developed his beautiful and fruitful idea of solving instead of the original Dirichlet problem some other problem which has always solutions and which can be reduced to the Dirichlet problem if the last one has a solution. Paper [S. Zaremba, Sur le calcul numérique des fonctions demandées dans le problème de Dirichlet et le problème hydrodynamique, *Bulletin International de l'Académie des Sciences de Cracovie, Classe des Sciences Mathématiques et Naturelles,* 1909, (2), 125-195] gives some numerical method of solving Dirichlet problems and – in a sense – extends the idea of the paper [S. Zaremba, L'équation biharmonique et une classe remarquable de fonctions fonda-mentales harmoniques, *Bulletin International de l'Académie des Sciences de Cracovie, Classe des Sciences Mathématiques et Naturelles,* 1907, (3), 147-196] which will be referred to later on in another context.

The Dirichlet problem was also the subject of Zaremba presentation [S. Zaremba, Sur le principe de Dirichlet, *Atti del IV Congresso Internazionale dei Matematici (Roma, 6–11 Aprile 1908),* Vol. II, Communicazioni delle sezioni I e II, Roa 1909, 194-199] during the IV-th International Congress of Mathematicians in Rome in 1908. In papers [S. Zaremba, Sur le principe du minimum, *Bulletin Internationale de l'Académie des Sciences de Cracovie, Classe des Sciences Mathématiques et Naturelles,* 1909, (7), 197-264] and [S. Zaremba, Sur le principe de Dirichlet, *Atti del IV Congresso Internazionale dei Matematici (Roma, 6–11*

Aprile 1908), Vol. II, Communicazioni delle sezioni I e II, Roma 1909, 194-199] Zaremba introduced generalised solutions into the direct method of variational calculus built up by Hilbert (see [8]). In [S. Zaremba, Sur le principe du minimum, *Bulletin Internationale de l'Académie des Sciences de Cracovie, Classe des Sciences Mathématiques et Naturelles,* 1909, (7), 197-264] an example of a domain in which there is no solution of a linear Dirichlet problem; it was the first such example in the literature, as it is pointed out by Jean Mawhin in [J. Mawhin, *Metody wariacyjne dla nieliniowych problemów Dirichleta,* (Polish version of the book *Problèmes de Dirichlet variationneles non linéaires*; translated by D.P. Idziak, A. Nowakowski, S. Walczak] and by Pierre Dugac, Beno Eckman, Jean Mawhin and Jean-Paul Pier in the section "Guidelines 1900–1950" (see [*Development of Mathematics 1900–1950*, edited by Jean-Paul Pier, Birkhäuser Verlag, Basel-Boston-Berlin, 1994], p. 6) presenting a list of the most important results obtained in this period the paper [42] is cited in the bibliography. In the same place [*Development of Mathematics 1900–1950*, edited by Jean-Paul Pier, Birkhäuser Verlag, Basel-Boston-Berlin 1994] Zaremba is mentioned as the author of a *method of orthogonal projection in Dirichlet problem.*

In the paper [S. Zaremba, Sopra un teorema d'unicita relativo alla equazione della onde sferiche, *R.C. della Accademia dei Lincei,* (5), 24, 1915, 904-908] an equation of so-called spherical wave is considered. There is given a method of estimation of

$$u = \int \mathrm{grad}^2 u \, d\tau,$$

where u is a solution of that equation. The idea of Zaremba used in his method was applied later on by Friedrichs and Levy in order to get known (now) integral inequalities satisfied by general solution of hyperbolic equations. These inequalities have been generalised by Juliusz Schauder (and became some fundamental elements in the survey of the theory of hyperbolic equations).

Zaremba considered also several other problems. He discussed for example, as it has been mentioned already, problems of Neumann and Fourier. An important contribution to the theory of Fourier problem was presented in [S. Zaremba, Solution générale du problème de Fourier, *Bulletin International de l'Académie des Sciences, Classe des Sciences Mathématiques et Naturelles,* 1905, 69-168].

The Fourier equation

$$\Delta_{x,t} - \frac{\partial u}{\partial t} = 0, \quad u = u(x,t),$$

was the subject of Zaremba's presentation (see [S. Zaremba, Sur un théoreme fondamental relatif à l'èquation de Fourier, *Compte Rendus du Congrés International des Mathématiciens (Strasbourg 22–30 Septembre 1920)*, Toulouse 1921, 343-350]) during the International Congress of Mathematicians in Strasbourg in 1920.

Let us now present some special part of Zaremba's contribution to the development of the theory *of reproducing kernels* (see for instance [F. H. Szafraniec, The reproducing kernel Hilbert space and its multiplication operators, *Operator Theory: Advances and Applications*, 114, 2000, 253-263]). The best and probably the shortest way to do it is by referring to the Aronszajn paper [N. Aronszajn, Theory of reproducing kernels, *Trans. Amer. Mat. Soc.*, 68, 1950, 337-404]. He wrote: "Examples of kernels of the type in which we are interested have been known for a long time, since all the Green's functions of self-adjoint ordinary differential equations (as also some Green's functions – the bounded ones – of partial differential equations) belong to this type (…) There have been and continue to be two trends in the consideration of these kernels (…). The second trend was initiated during the first decade of the century in the work of S.Zaremba" [1. S. Zaremba, *L'équation biharmonique et une classe remarquable de fonctions fonda-mentales harmoniques, Bulletin International de l'Académie des Sciences de Cracovie, Classe des Sciences Mathématiques et Naturelles,* 1907, (3), 147–196)*,* 2. S. Zaremba*, Sur le calcul numérique des fonctions demandées dans le problème de Dirichlet et le problème hydrodynamique, Bulletin International de l'Académie des Sciences de Cracovie, Classe des Sciences Mathématiques et Naturelles, 1909, (2), 125-195*)] *on boundary value problems for harmonic and biharmonic functions. Zaremba was the first to introduce, in a particular case, the kernel corresponding to a class of functions, and to state its reproducing property* (…). *However, he did not develop any general theory, nor did give any particular name to the kernels he introduced.* In that way one links certain results of Zaremba with some important part of the modern theory of operators which shows how deep were those result being now more than ninety years old...".

Most of Zaremba's scientific results were obtained in connection with particular questions belonging to theoretical physics [3–6]. Therefore they are therefore not the subject of these considerations.

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A FEW FACTS CONCERNING THE OUTSTANDING POLISH MATHEMATICIAN FRANCISZEK LEJA

KILKA FAKTÓW DOTYCZĄCYCH WYBITNEGO POLSKIEGO MATEMATYKA FRANCISZKA LEJI

Abstract

We present a collection of facts given by the outstanding Polish mathematician Franciszek Leja (1885–1979) in his unpublished memoirs, add some other information about him and list some institutions bearing his name.

Keywords: *20th century European mathematicians*, *Jagiellonian University*, *biography*, *Franciszek Leja*

Streszczenie

Przedstawiamy pewne fakty z nieopublikowanych wspomnień wybitnego matematyka Franciszka Leji[1] (1885–1979), garść innych informacji oraz listę instytucji noszących jego imię.

Słowa kluczowe: *europejscy matematycy dwudziestego wieku*, *Uniwersytet Jagielloński*, *biografia*, *Franciszek Leja*

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[1] Franciszek Leja preferred this way of declining his name.

1. Foreword

Franciszek Leja was born on 27th January 1885 in Grodzisko and died on 11th October 1979 in Kraków. He is buried in Grodzisko Górne because he had wished it.

In the late 1970s his students asked him to write down his memoirs [40]. He admitted in the introduction that it was a difficult task since he had to rely almost entirely on his memory of the period of about 90 years. The memoirs end with the year 1958. They were handwritten, and Janina Poznańska, the secretary of the Institute of Mathematics of the Jagiellonian University, typed them. Józef Siciak had one copy. The Jagiellonian University Students' Math Society (Koło Matematyków Studentów UJ) obtained 4 copies for its library. The state school in Grodzisko Górne has also a copy.

The author of this article is not a historian, thus this note is not a strictly historical one. Chapters 2, 3, 5 and 6 of this article are almost entirely based on [40]. They may therefore contain some errors if Leja remembered some facts incorrectly. The biographical articles about Franciszek Leja written by his students Józef Siciak ([46] and [47]) and Witold Kleiner ([10]) were also mainly based on [40]. We should emphasize that the authors of these articles are not historians either. The biographical entry on Leja in [12] seems to be based on the same materials. For more recent notes having more historical character we refer the reader to [3] and [6].

A few years ago Krzysztof Ciesielski (Institute of Mathematics, Jagiellonian University) researched the archives of the Jagiellonian University to record all doctorates in mathematics obtained at the Jagiellonian University. The list, which he prepared, is given in [50]. We used it in chapter 7. Another source of information was [51], where the promotions in the Institute of Mathematics of the Polish Academy of Sciences are listed.

2. Life [40]

Franciszek Leja was born in a small village of Grodzisko Górne near Leżajsk. His parents had a small holding of about 4 ha, and two sons and four daughters (cf. [48]). As a little boy Franciszek often played with three boys who were only a year or two older: his brother Józef, his uncle (the youngest brother of his father) Wojciech and a neighbour Jan. Of course the children had their duties too. Franciszek was herding cows and babysitting his younger sisters, he also had to look after the hens and bring pinecones and firewood from the forest.

He finished a three-class folk school in the village and then the fourth class in Leżajsk. He attended gymnasium in Jarosław in 1896–1904. He passed his secondary school-leaving examination in Jarosław in 1904.

Leja enrolled in the Faculty of Philosophy of the Lwów University. He studied until 1909, when he passed an exam which allowed him to become a teacher of mathematics and physics in secondary schools.

In 1910 Leja started to work at the Fourth Gymnasium in Kraków and in 1911 was transferred to Bochnia. He published his first article on non-Euclidean geometry in the annuals of the gymnasium in Kraków. He spent 9 months from November 1912 till June

1913 in Paris. In September 1913 he started to work in the Fifth Gymnasium in Kraków, but he was also appointed to a part-time assistant job at the Jagiellonian University.

When the First World War broke out, Leja joined the Legion of Lwów. Nine other men from Grodzisko Górne went with him. However, he resigned when the Austrian regiment demanded a pledge of allegiance to Austria.

Leja obtained his Ph.D. at the Jagiellonian University in Kraków in June 1916. The title of his thesis was *Własność niezmiennicza równań różniczkowych zwyczajnych ze względu na przekształcenia stycznościowe* (Polish) [Invariant property of ordinary differential equations with respect to contiguous transformations]. According to [6], the paper [14] contains this thesis.

In 1919 he was among the 16 mathematicians who founded Mathematical Society in Kraków and he became its secretary for the period of 1919–1921 (cf. [39]). This organization was transformed into the Polish Mathematical Society in 1920. In 1922 Leja passed his examination to obtain the title of the docent (the associate professor) of mathematics. [15] was his habilitation thesis (cf. [6]).

In 1923 he received two offers: he could obtain the chair of mathematics either at the Poznań University or at the Warsaw Technical University. He chose Warsaw and obtained the chair of mathematics in the Faculty of Chemistry. In 1924 he married Janina Mizerska. In the same year his habilitation from the Jagiellonian University was accepted by the University of Warsaw. Thus, even though he had the chair at the Warsaw Technical University, he gave also lectures at the University of Warsaw.

Fig. 1. Photo from 1930 by Jadwiga Lanczewska (source: Library of the Institute of Mathematics, Polish Academy of Sciences)

Professor Leja planned to stay in Warsaw for good and even started to build a house there. But when he was the Dean of the Faculty of Chemistry, two professors from the Faculty intercepted a letter addressed to the Dean and answered it in his name without telling him about it. When Leja found out about it, he asked the Rector to call them to account and to punish them. However, the authorities of the Technical University decided to hush the situation up. Leja did not like it and decided that he did not want to stay in Warsaw any longer.

In 1936 he returned to Kraków. He obtained a chair of mathematics at the Jagiellonian University, namely the one vacated by Stanisław Zaremba (1863–1942), who had retired. In 1937–1938 two well known mathematicians visited Kraków: Henri Lebesgue (1875–1941) from France and Mauro Picone (1885–1977) from Italy. They gave lectures at the Jagiellonian University. Franciszek Leja was invited to go to France and Italy to give talks in his turn. Together with his wife he went to Paris in the first half of 1939. After his lecture, he was told that nobody expected him to arrive. They were sure that the war would start soon. The travel to Italy turned out to be impossible.

On the 6th November 1939 he was among the 183 academics arrested by gestapo and taken to Sachsenhausen concentration camp. After the international interventions the older professors were freed in Spring 1940. Although Leja was not in the first group freed in March 1940, he was back in Kraków in May 1940. He was allowed then to go to Grodzisko Górne. He worked in his garden and prepared his textbook on Calculus during the remainder of the Second World War.

In 1947 his first textbook for students [27], written during the war, was published. It was a very popular book and its multiple editions always disappeared quickly from the bookshops. Later other textbooks: [32, 36, 38] were published.

In 1957 Leja was invited to a conference on analytic functions in Helsinki. He participated in the International Congress of Mathematicians held in Edinburgh from 14 to 21 August 1958 and gave a lecture entitled *Sur les moyennes arithmétiques, géométriques et harmoniques des distances mutuelles des points d'un ensemble* there.

Józef Siciak recalls that professor Leja was very active even after his retirement in 1960 (when he was 75 years old). The Institute of Mathematics was located on the fifth floor of the university building on Reymonta 4 and Franciszek Leja regularly went there never using the lift, even when he was 80 or 90 years old. In 1963 the University of Łódź awarded him its honorary doctorate.

3. Scholarships and studies ([40]; for a wider background on the education in that area and that time see [2])

Now it is quite easy to obtain a secondary or even higher education. It was not so for Franciszek Leja. He could attend the three-class folk school in Grodzisko Górne, but when he finished it, it seemed that he would not be able to go any further. His parents could not afford longer education for all of their children. They decided to send only their oldest son Józef to the secondary school. But Józef did not want to leave the village and especially the horses he loved and suggested that Franciszek should go instead. Franciszek went willingly.

He finished the fourth class in Leżajsk first and then the classical gymnasium in Jarosław thanks to a private scholarship founded by the late Rev. Czesław Kaczorowski and to money he earned by tutoring. During those years he was constantly undernourished. For a long time he was the only person from Grodzisko Górne, who attended a secondary school. In the second class he had trouble with the subject of proportions. The teacher seemed not to be able to explain it clearly. Later, however, Franciszek had no more problems with mathematics. While he was in the sixth class, a pupil from the eighth class gave him mathematical tasks which Franciszek solved. During his secondary school-leaving exam he wrote the solutions of the test in mathematics for half of his class. The other half obtained the solutions from another boy. The whole class passed the exam.

His parents wanted him to become a priest and were disappointed when he refused. He intended to study at the Lwów University, but his family could not help him financially. Fortunately, their parson Feliks Świerczyński helped Franciszek. He talked with his parents and succeeded in soothing them. He also provided a further scholarship (from the same fund left by Rev. Kaczorowski): 20 koronas per month. Franciszek needed at least 80 for his boarding, living and studies. He earned the missing amount accepting various jobs: he was a tutor, a land surveyor, an accountant, an assistant manager in a small cigarette tube factory and he was also employed at collecting documents in libraries about the history of the Church (his employer then was Rev. Adam Sapieha (1867–1951) – later a bishop and a cardinal).

Fig. 2. The credit book of Franciszek Leja (provided by Józef Siciak)

Leja wrote later about his studies at the Faculty of Philosophy of Lwów University in his memoirs [40]: "I was interested not only in mathematics and physics, but in philosophy and psychology too. The lectures in psychology were really good, but unfortunately I could listen to only a few of them because of my jobs which I needed to earn my living. On the other hand, the "metaphysics" disappointed me, even though I attended every lecture. When I finished this course, I began to have much more respect for mathematics and its deductive methods than for philosophy".

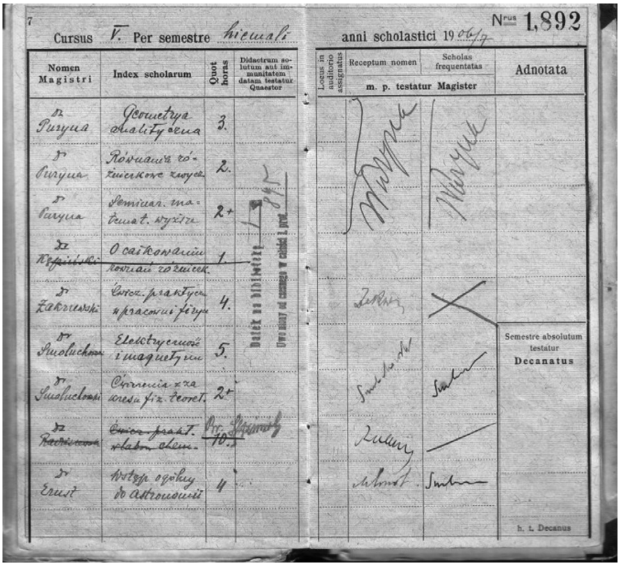


Fig. 3. From the credit book of Franciszek Leja. Some courses given by Puzyna, others by Smoluchowski

During his studies Leja attended lectures given e.g. by Józef Puzyna (1856–1919) in mathematics and by Marian Smoluchowski (1872–1917) in physics.

Fig. 4. From the credit book of Franciszek Leja Some courses given by Rajewski are deleted in the second year

Unfortunately, Puzyna was the only mathematician at the university apart from Jan Rajewski (1857–1906), whose classes Leja could attend only in his first year of studies. Thus there were not enough mathematical courses (cf. [6]). That is why Leja said in [40]:

"I was not satisfied with my studies in Lwów and I told my friends who congratulated me on my final exam: Alas, I do not know this mathematics.

At the university, I had only a chance to learn that there is a huge building of higher mathematics, but I had no opportunity to get to know anything from this building because there were not enough lectures and courses. That is why I decided to try to continue my studies while teaching at a high school".

Leja planned to study further and that is why he refused the first assignment to a post in a school in Drohobycz. He wanted to work in an academic center, close to a university. Fortunately, he obtained another post in Kraków.

In 1911 Kazimierz Żorawski (1866–1953), a professor of mathematics at the Jagiellonian University in Kraków, noticed the first article published by Leja. He liked the paper so much that he sought out its author and offered him a subsidy from the fund left by late Władysław Kretkowski (1840–1910) (cf. the CV of Leja from 1916 in [2]). This money sufficed for one year of studies abroad. Leja accepted with pleasure. He could choose the Sorbonne in Paris or Göttingen in Germany. He spoke German better than French, so he decided to choose France to have an opportunity to learn its language better. And thus he went to Paris for the academic year 1912/1913. He learnt French, studied mathematics at the Sorbonne and attended high teas at Władysław Mickiewicz' house on Thursdays there. He was also able to visit London.

Kazimierz Żorawski was later the advisor of Leja's Ph.D. thesis. And when Leja started to work part-time at the Jagiellonian University in 1913, his wages were paid from some private funds distributed by the Academy of Learning.

4. Adoption [4]

The only child of Franciszek Leja and his wife died in infancy. On the other hand, Józef Leja, the oldest brother of Franciszek, who married Aniela Pawlik in 1908, had six children. He was not able to educate all of them, he was too poor, just like his parents were before. Franciszek saw that and offered in 1927 that he might adopt Józef's son Jan Leja (1918–2009) and give him home and education. Hence, starting from 1927, Jan lived with his uncle, first in Warsaw then in Kraków. Jan did not know why he was chosen for adoption in preference to one of his brothers. Perhaps the reason was his willingness to run errands for his uncle when Franciszek came to Grodzisko Górne.

In 1935, after a skiing trip from Zakopane to Kraków, Jan had to lie in bed for five months with rheumatic fever. His uncle provided him with a series of mathematical Calculus problems. In this way he tried to keep the boy entertained. By the time Jan recovered, he had completed a first year course in Calculus. As we already mentioned, after World War II Franciszek Leja published a very popular textbook on Calculus [27].

In 1937 Jan started to study at the Mining Academy in Kraków (which later became the AGH University of Science and Technology).

In September 1939 Jan sent his uncle to Grodzisko Górne and joined the Polish Army. His unit was intercepted by Russians. Fortunately Jan succeeded to escape and to get to Lwów where he met his older brother Stanisław Leja (1912–2000), who in 1938 had finished his studies in mathematics and since then had worked at the University of Lwów (cf. [5]). Stanisław had also joined the Polish Army, his unit had also been intercepted by Russians and he had also succeeded to flee. Both brothers wanted to join the resistance, but each of them was imprisoned by Russians, in different circumstances. Jan was sent to a slave labour camp Vorkuta Gulag, Stanisław was deported to Sverdlovsk Oblast by Ural (he had to work as a woodcutter there). Both managed to survive the terrible conditions and to join the Polish Army formed under Władysław Anders (1892–1970) in 1941. It was really a miracle that they met in Persia, when each of them was in a different unit of the Army.

The story of Jan's life is related in [4], a good account of Polish history written by the brother-in-law of Jan Leja.

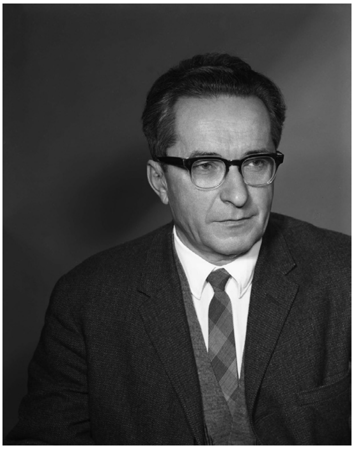


Fig. 5. Jan Leja, photo from 1966 (source: University of British Columbia Archives, [UBC 5.1/1731])

Eventually, Jan became a professor of mining and mineral process engineering at the University of British Columbia, Vancouver, in Canada, and his brother Stanisław a professor of mathematics at the Western Michigan University, Kalamazoo, in the USA. In 1976 the Maria Curie-Skłodowska University in Lublin awarded Jan Leja its honorary doctorate. Jan had six children.

Just after the Second World War Franciszek Leja took another young boy to his household – it was Roman Danak (1935–1994) (cf. also [48]), a grandson of Franciszek's sister Katarzyna Danak. The father of the boy was killed by Germans. In the late 1950s Roman Danak studied oriental philology at the Jagiellonian University and later became a journalist and a science fiction writer [7].

In [40] Leja referred only once to Jan, saying that while they lived in Warsaw, he and his wife took care of his nephew who attended gymnasium. He said nothing more about the adoption or any other help he gave other young people.

5. Benefits to Grodzisko Górne [40]

Franciszek Leja never forgot his family home. He usually spent his holidays in Grodzisko Górne, his birthplace. He was also a benefactor of this village.

Already in his early youth, when he attended gymnasium, he organized, together with his friends, lectures, theater performances and fêtes in the village. The collected money was used for buying books for the library of the Association of the Folk School.

In 1910 he and his uncle Wojciech helped to found the Dairy Cooperative. In 1912 he was one of the founders of the Peasant Orchestra in Grodzisko Górne (cf. [4]).

6. Hunger [40]

When Franciszek Leja was 90 years old, he said "I am wondering why I have lived so long. Probably because I have often been hungry" (see [10]).

When he was a child, the villages in Galicia were really poor and everybody was there quite often undernourished. Later it was so too. Jan Leja said that as a child in Grodzisko Górne he had enough food only on Christmas Eve during the traditional celebratory meal (cf. [4]).

When Franciszek attended gymnasium, he ate in a canteen. The portions were very meagre and never satisfying. He remembers a discussion between hungry pupils there, on the best method of consumption: whether quick or slow eating made the eater more sated. An older boy finally decided that it was better to swallow very fast, because in this way the food, not arranged, took more place, so the stomach seemed to be more filled.

Franciszek Leja was hungry in Paris too, at least at the beginning of his stay. Namely, he did not want to take all his money with himself on the journey. He took only a part of it and his uncle had the rest and was to send an amount to Paris each month. Almost just after leaving Poland, Franciszek lent most of his money to an acquaintance, who seemingly needed it only for a few days. Unfortunately, the man never contacted Franciszek again; thus Leja lost the money and had to live on very little till the next part of money arrived.

In the beginning of his period in Warsaw the conditions were difficult too. It was not easy to have enough food. And of course Leja was on the verge of starvation quite often during the Second World War.

Finally, Leja recalled another difficult situation, which happened while he was abroad after the war. He was invited, together with Mieczysław Biernacki (1891–1959) and Zygmund Charzyński (1914–2001), to a conference on analytic functions in Helsinki in 1957. All three of them should have obtained 60 U.S. dollars from the Polish authorities for their expenses. But they were given only bonds of the Polish Bank, and outside of Poland those were only worth one third of the sum which they theoretically represented. The conference lasted 8 days and the talks of the Polish mathematicians were planned for the second half of the conference. Therefore, Leja and the others could not leave earlier and had to eat very sparingly.

7. Teaching

As we said before, Franciszek Leja was the author of very popular textbooks [27, 32, 36, 38]. He prepared parts of the texts testing them on his students during the lectures (cf. [48]).

Leja's students after the Second World War had a nickname for him: they called him *Grandpa* (cf. [48]). He was always very elegant and seemed to be working all the time. The exams he gave were really difficult but he always cared for his students and was proud of their achievements.

Professor Leja was the advisor of at least 9 Ph.D. theses. His students were: Jan Leśniak (1947; later a professor at the Higher College of Teacher Training in Kraków), Jerzy Górski (1950; later a professor of the University of Silesia in Katowice), Witold Kleiner (1954; now a professor emeritus at the Jagiellonian University), Franciszek Bierski (1959; later a professor at the AGH University of Science and Technology), Andrzej Szybiak (also 1959; later e.g. a professor at the University of Tlemcen, in Algeria), Józef Siciak (1960; now a professor emeritus at the Jagiellonian University), Władysław Bach (also 1960; lived only for 35 years but succeeded in obtaining his habilitation at the Jagiellonian University in 1965), Czesław Loster (1961; later worked at the Technical University of Kraków), and finally Bolesław Szafirski (1963; now a professor emeritus at the Jagiellonian University too). Seven of the promotions took place at the Jagiellonian University ([50]) and two: those of Witold Kleiner and Andrzej Szybiak, in the Institute of Mathematics of the Polish Academy of Sciences ([51]). According to Bolesław Szafirski, the youngest on the list (who was exactly half a century younger than Leja), starting from 1950 there were no other doctorates advised by Leja than those listed here. We do not know whether Leja advised any other Ph.D. thesis (apart from that one of Jan Leśniak) before that year.

The author of this article is a scientific 'great granddaughter' of Franciszek Leja. Namely, Leja was the advisor of the Ph.D. thesis of Józef Siciak, who was the advisor of the one of Wiesław Pleśniak and he in his turn was the advisor of the Ph.D. thesis of Marta Kosek.

8. Scientific achievements. For the whole list of publications see [10]

Franciszek Leja's research interests started with differential equations (see e.g. [13–15]) and he had some results in this domain later too (see e.g. [29]).

In [42] Pierre Dugac, Beno Eckmann, Jean Mawhin and Jean-Paul Pier give a list of the main mathematical achievements in the first half of the twentieth century, so called *Guidelines 1900-1950*. There, under the year number 1925, one can find: *Leja F.; Schreier O., Topological groups*. Namely, Leja and Schreier introduced the notion of an abstract topological group independently. In the references of the book [42] two papers by Leja: [19, 20] and two papers by Schreier are listed. Let us cite here one more article on this subject, namely [21].

Leja worked on (power) series (see e.g. [18, 22, 23]) and interpolation theory (see e.g [25, 26, 28]). His research on the theory of analytic functions (in particular those of several variables) (see e.g. [16, 17, 30]) and extremal problems (see e.g. [31, 33, 34]) was taken up by his students and their students and is now continued by a strong group at the Jagiellonian University.

The importance of the work of Leja and his students was underlined in the Preface of [44] (as Józef Siciak noticed in [47]): "The external field problem has its origins in the work of C.F. Gauss, and is sometimes referred to as the" Gauss variation problem. "O. Frostman investigated the problem and the Polish school headed by F. Leja made important contributions during the period 1935–1960 that have greatly influenced the present work. A rebirth of interest in the Gauss variational problem occured in the 1980's when E.A. Rakhmanov and, independently, collaborators H. N. Mhaskar and E. B. Saff used potentials with external fields to study orthogonal polynomials with respect to exponential weights on the real line".

Paragraph III.5 of this book is entitled *The function of Leja and Siciak* and it starts with the following sentence: "In this section we introduce and investigate a function – due to F. Leja and J. Siciak – that gives the smallest upper bound for polynomials majorized by a weight on a set Σ". In this way the authors introduce the Leja-Siciak extremal function. They refer to 13 papers and one book by Leja (i.a. [24, 33, 36]) and many other articles by his students: Jerzy Górski (10 papers), Witold Kleiner (10), Józef Siciak (8).

Let us also mention another very important book, cited in [44] too. Maciej Klimek, a scientific 'grandson' of Franciszek Leja, namely a student of Józef Siciak, now a professor in Uppsala University, wrote the monograph [11] in 1991. The book gives a comprehensive study of plurisubharmonic functions. In Kraków this subject started with analytic functions of multiple variables studied by Leja. Many results obtained by the group working at the Jagiellonian University (built of the students of Leja and their students in turn) are discussed in this book, even though of course they do not exhaust the whole material. The author cites his own results but also refers to papers written by Leja (to [24, 37] and to book [36]), by his student Józef Siciak, by students of Siciak: Wiesław Pleśniak, Marek Jarnicki, Sławomir Kołodziej and finally to a few ones written by one scientific 'great grandson' of Leja, namely Mirosław Baran, a student of Pleśniak.

It is not the aim of this paper to discuss in detail the scientific achievements of Leja and their impact on the mathematics. For a whole account we refer the reader to [46, 10, 47] (cf. also [6]). We chose only to mention here two objects which are named after Leja. Let us start with a famous strong result, namely the Polynomial Lemma of Leja introduced in [24]. We will present only its simplest form (cf. [36]).

Polynomial Lemma *Let E be a connected compact set (consisting of at least 2 points) in the complex plane, M be a positive number. For every $\varepsilon > 0$ there exist positive numbers δ and N such that:*

if (P_n) is a sequence of polynomials where P_n is of degree at most n and $|P_n(z)| \le M$ for all $z \in E$, $n \in \{1, 2, 3, \dots\}$, then for every $w \in E$

$$|P_n(z)| \le M(1+\varepsilon)^n, \quad \textit{whenever} \quad |z-w| < \delta,\ n > N.$$

For a generalization of this lemma and some information on its impact see e.g. [41, 43, 45] (cf. Leja polynomial condition in [11] too).

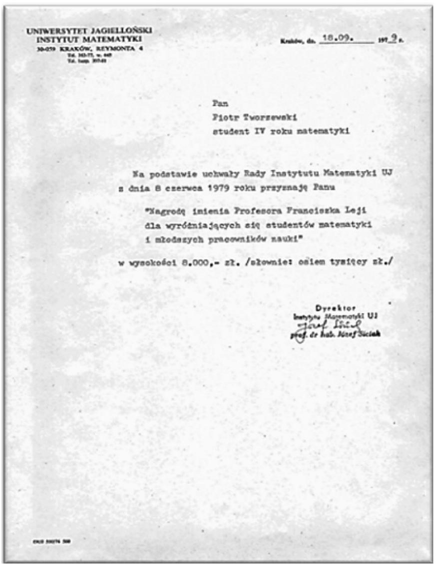
An important notion is that of a Leja sequence (also known under the name of Leja points), defined by Leja in [35], independently from Edrei [8]. Let us recall the definition. Let E be a nonempty compact set in the complex plane and (a_n) be a sequence of points in E. We say that (a_n) is a *Leja sequence* if

$$|(a_n - a_0)\dots(a_n - a_{n-1})| = \max_{z\in E} |(z - a_0)\dots(z - a_{n-1})|, \quad n \in \{1,2,3,\dots\}.$$

For some generalizations of this notion and results on it see e.g. [9, 44, 49] and the references given there. Recently, pseudo Leja sequences were defined and turned out to be a useful tool in the numerical optimization (see [1]).

9. The foundation, the school, streets and a lecture hall

Leja was thankful for the scholarship which had enabled him to study. In his turn he tried to help other people (as we had seen already). In 1977 he gave 200 000 złoty (it was a professor's yearly salary then) to the Institute of Mathematics of the Jagiellonian University for scholarships (awards) for oustanding students and young mathematicians. The first three of the beneficiaries were Piotr Jakóbczak in 1978, Piotr Tworzewski in 1979 and Maciej Klimek in 1980. Later the Leja Foundation was established and thus the scholarships are still granted. The whole list of the beneficiaries is given in [52] together with other informations about the scholarship. We present here photos of two diplomas.



UNIWERSYTET JAGIELLOŃSKI
INSTYTUT MATEMATYKI
30-059 KRAKÓW, REYMONTA 4

Kraków, dn. 18.09. 1979 r.

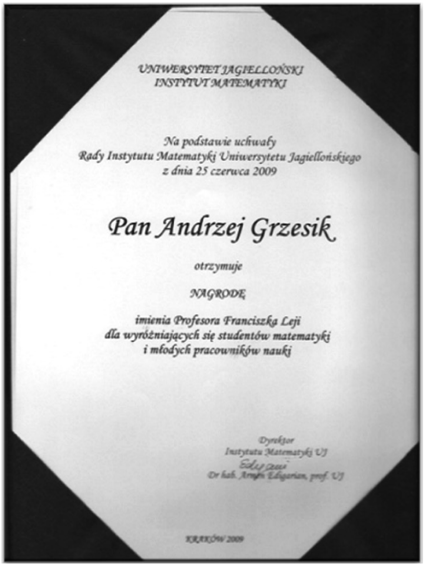
Pan
Piotr Tworzewski
student IV roku matematyki

Na podstawie uchwały Rady Instytutu Matematyki UJ z dnia 8 czerwca 1979 roku przyznaję Panu

"Nagrodę imienia Profesora Franciszka Leji dla wyróżniających się studentów matematyki i młodszych pracowników nauki"

w wysokości 8.000,- zł. /słownie: osiem tysięcy zł./

Dyrektor
Instytutu Matematyki UJ
prof. dr hab. Józef Siciak



UNIWERSYTET JAGIELLOŃSKI
INSTYTUT MATEMATYKI

Na podstawie uchwały
Rady Instytutu Matematyki Uniwersytetu Jagiellońskiego
z dnia 25 czerwca 2009

Pan Andrzej Grzesik

otrzymuje

NAGRODĘ

imienia Profesora Franciszka Leji
dla wyróżniających się studentów matematyki
i młodych pracowników nauki

Dyrektor
Instytutu Matematyki UJ
Dr hab. Armen Edigarian, prof. UJ

KRAKÓW 2009

Fig. 6. The beneficiaries allowed the use of their diplomas

In 1998 the primary school in Grodzisko Górne celebrated its hundredth anniversary. There was a big fête and the school was then named after Franciszek Leja.

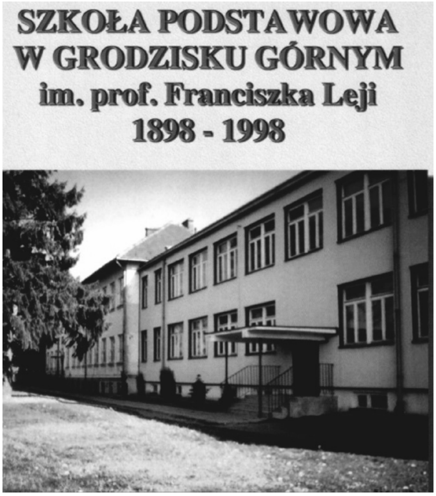


Fig. 7. The cover of [48]

In 2002 the gymnasium in Grodzisko Górne joined the primary school. Now it is a Complex of Schools named after Franciszek Leja: the primary school and the gymnasium together (cf. [53]). The schools make the name of Franciszek Leja known in some circles other than those of mathematicians. In [54] the competition scores from the Global Conference on Educational Robotics in Albuquerque in 2015 are given. One can read there that Franciszek Leja State School in Grodzisko Górne, Poland, won the first place in the International Botball Double Elimination, the fourth place in the International Botball Seeding and the Overall Judges' Choice Award there. More than 50 teams participated in the competitions and other awarded teams came from the USA (most of them), Austria, China and Kuwait. Polish television related some information about earlier achievements of the school in this area ([55, 56]). Unfortunately, the reporters spoke only about the school in Grodzisko Górne and did not mention its patron.

At least two streets are named after Franciszek Leja: one in Kraków and one in Jarosław.

Fig. 8. Streets in Kraków and Jarosław (photos by Marta Kosek and Jerzy Szczepański)

The biggest lecture theater in the new building of the Faculty of Mathematics and Informatics of the Jagiellonian University is now called the Franciszek Leja lecture hall.

Fig. 9. Lecture theater in the new building of the Faculty of Mathematics and Informatics of the Jagiellonian University (photos by Marta Kosek)

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TECHNICAL TRANSACTIONS | **CZASOPISMO TECHNICZNE**
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2-NP/2015

LIDIA OBOJSKA*

SOME REMARKS OF JAN ŚLESZYŃSKI REGARDING FOUNDATIONS OF MATHEMATICS OF STANISŁAW LEŚNIEWSKI

KILKA UWAG JANA ŚLESZYŃSKIEGO DOTYCZĄCYCH PODSTAW MATEMATYKI STANISŁAWA LEŚNIEWSKIEGO

A b s t r a c t

Jan Śleszyński, a great mathematician, is considered a pioneer of Polish logic; however, he was not connected with the famous Warsaw School of Logic (WSL). He believed that his mission was a critical evaluation of work of other logicians in the field of foundations of mathematics and proof theory. Among his writings we find several notes regarding the work of Stanisław Leśniewski (the co-founder of the WSL) and his collective set theory. These remarks are the subject of investigation of the presented paper.

Keywords: *foundations of mathematics*, *set theory*, *element*, *set*

S t r e s z c z e n i e

Jan Śleszyński, wielki matematyk, uważany jest za pioniera polskiej logiki, chociaż nie był związany ze słynną Warszawską Szkołą Logiki (WSL). Śleszyński uważał, że jego misją była krytyczna ocena prac dotyczących podstaw matematyki oraz teorii dowodu autorstwa innych logików. Wśród jego zapisków znajdujemy między innymi uwagi dotyczące teorii zbiorów kolektywnych Stanisława Leśniewskiego, współzałożyciela WSL. Uwagi te są przedmiotem analizy niniejszego artykułu.

Słowa kluczowe: *podstawy matematyki*, *teoria zbiorów*, *element*, *zbiór*

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1. Introduction

The beginning of the twentieth century was incredibly rich in significant discoveries in exact sciences. Well known all over the world are the achievements of Polish scholars of the Lvov School of Mathematics, Warsaw School of Logic or Warsaw School of Mathematics. Jan Śleszyński is one of few mathematicians living in this period who studied logic and foundations of mathematics, and who does not belong to any of these schools. Even though he is considered a pioneer of Polish logic, Śleszyński is barely present in the consciousness of Poles. He is well known for his famous two-volume work: *The Proof Theory*, but almost nobody knows that he left a fairly rich heritage: 4 thousand pages of manuscripts (some edited by Stanisław Krystian Zaremba) containing analysis, lectures, speeches and comments. Among them one can find remarks dealing with the early work of Stanisław Leśniewski dedicated to the foundations of set theory, called *mereology*, which is still the subject of scientific investigations of logicians and philosophers[1]. Therefore, in this paper, we would like to focus on Leśniewski's foundations of mathematics and Śleszyński's remarks dealing with this work.

2. Jan Śleszyński (1854–1931) and Stanisław Leśniewski (1886–1939)

Jan Śleszyński, the Polish mathematician and logician, was born in 1854 in Łysianka (Ukraine). In 1871–1875 he studied mathematics at the University of Odessa. After finishing his studies, he moved from Odessa to Kiev, where he taught mathematics in secondary schools. In 1880 he received a scholarship from the Russian government and went to Berlin, where he attended the lectures of L. Kronecker, E.E. Kummer and K. Weierstrass. In Berlin he prepared his thesis on the calculus of variations and in 1882 he came back to Odessa to work at the University. In his master's thesis he gave a precise proof of the restricted form of the Central Limit Theorem. In 1898 he became a full professor and retired in 1909.

At that time, the Polish Academy of Arts and Sciences received a donation from Władysław Kretkowski, designed, among other things, to increase the number of lectures in mathematics at the Jagiellonian University. The Foundation invited Śleszyński to come as a professor to Krakow. He accepted the invitation to take the chair and in October 1911 moved permanently to Krakow.

Stanisław Leśniewski was much younger than Śleszyński. He was born in Serpukhov (near Moscow) in Russia, 30 March 1886. Because his father was a railway engineer, they often changed their place of residence. After graduating from a high school in 1904, he went to Germany to study philosophy. He studied in Leipzig and Heidelberg. In 1909 he appeared in Munich, and in 1910 went to Lvov to write his PhD dissertation under the supervision of Kazimierz Twardowski. In Lvov he attended the lectures of Wacław Sierpinski and Józef Puzyna on set theory and there he met Jan Łukasiewicz. After obtaining the PhD, he travelled a little: he went to France, Italy and St. Petersburg (1912/1913) and in the middle of 1913 he moved to Warsaw.

[1] For ex. D. Mieville – Switzerland, A.J. Cotnoir, B – UK, R. Poli; L. Biacino – Italy, A. Varzi – USA, A. Betti – NL, P. Forrest – Australia, etc.

In Lvov Leśniewski was influenced by the philosophy of Twardowski, especially by the theory of subjects of Husserl, which he applied to his set theory.

A very important event for Leśniewski was the reading of the work of Łukasiewicz *On Aristotle's principle of contradiction* (in 1911). For the first time he learned of the existence of symbolic logic and of Russell's famous antinomy. However, the reading of this work filled Leśniewski with a long-lasting aversion to logic, which he broke only in 1918. That time, since 1914, he had devoted to a detailed analysis of the work of English logicians – *Principia Mathematica*.

A very fruitful discussion around the work of Łukasiewicz resulted in various works published within the *Review of Philosophy* (*Przegląd Filozoficzny*) in the period 1911–1913[2], one of which is especially important for us. In 1913 Leśniewski wrote a paper: *Is the class of classes not subordinated to themselves, subordinated to itself?*, which was a prelude to the work on general set theory (*Podstawy ogólnej teorii mnogości)* called by Leśniewski: mereology, and published in Moscow in 1916. This work closed the so-called grammatical and philosophical period of Leśniewski's creativity and opened a new chapter in his life dedicated to the foundations of mathematics[3]. These two papers were subjected to investigations of Jan Śleszyński and they will be discussed in detail later in the text.

In that period, around 1915, Śleszyński actively participated in Polish academic life, giving lectures on mathematical logic, proof theory, probability. He participated in meetings of the Polish Mathematical Society, Polish Philosophical Society, etc. In the Krakow period he did not publish a lot. Beside *The Proof Theory* (already mentioned), in 1926 *The Theory of Determinants* was published. Both works are based on Śleszyński's lectures. They were written up by his students and not by the author himself. In addition, in 1923 in volume 3 of the *Guide for Self-Taught* (*Poradnik dla samouków*) two papers of Śleszyński appeared: *The importance of logic to mathematics* and *On the first stages of development of infinitary concepts*. In 1919 K. Żorawski took the position at the Warsaw University of Technology, and Śleszyński became a professor of mathematics and mathematical logic at the Jagiellonian University. In 1924 he completely retired from the academic life.

As A. Hoborski wrote, it was characteristic of Śleszyński to explain any doubts of logical and mathematical nature in a way that any mathematical reasoning could become complete, i.e. there shouldn't be any hidden rules applied in deductive reasoning. For this reason Śleszyński studied the work of British philosophers, Bertrand Russel and Alfred N. Whitehead "Principia Mathematica" (PM) very deeply and applied their ideograms to rewrite all doubtful proofs he found in literature. He verified theories of other authors and pointed out all shortcomings and mistakes, because, as he believed, his mission was a critical evaluation of work in the foundations of mathematics. Therefore, we find in his notes detailed remarks, written by him or by S. K. Zaremba, regarding not only the early

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[2] One can find a detailed discussion in [7]. We have to add here that many very important ideas, developed later, have their roots in Leśniewski's intuitions from that period: e.g., many-valued logics of Łukasiewicz; the distinction between language and metalanguage – the research that later on was continued by A. Tarski, etc.

[3] The complete presentation of his general set theory took place only in years 1927–1931, when it was published in the periodical *Philosophical Reviews* (*Przegląd Filozoficzny*).

mathematical works of Leśniewski, but also the works of Łukasiewicz, and perhaps other authors.

Leśniewski, after four years spent in Moscow, returned to Warsaw in 1919 and became the Chair at the Department of Philosophy of Mathematics. Since that time he had been teaching at the University of Warsaw. During this period he only worked on the foundations of mathematics; he constructed his own symbolic language applied to the construction of his systems: Protothetics, Ontology and Mereology[4], and created his own theory of semantic types applied broadly in the Warsaw School of Logic for a certain period of time. In 1936 he became a full professor.

3. Stanisław Leśniewski and his foundations of mathematics

The work of S. Leśniewski *On the foundations of general set theory,* published in Moscow in 1916, is the first attempt to formalize the general set theory also known as a collective set theory or mereology; a full formalization followed much later, in the years 1927–1931. This work is valuable because it removes the problem of Russell's antinomy, but retains the original intuition of the term set introduced by Georg Cantor.

The paradox of set theory (i.e. the antinomy of irreflexive classes) is conceptually the simplest theory within the set theory. It was discovered by Russell (in 1901) while he studied the work of G. Frege *Grundegesetze der Arithmetic* (1893). The antinomy of irreflexive classes constitutes that there is no set composed of only those sets, which are not elements of themselves:

$$Z = \{X : X \notin X\}.$$

If we assume that R is such a set, and ask whether R is its own element, we obtain that:

$$R \in R \to R \notin R.$$

Hence, if we assume that $R \in R$, pursuant to the rules of inference, we obtain that:

$$R \notin R,$$

and vice versa. Thus, we get two contradictory expressions. Leśniewski suspected that the essence of the paradox lies in the mistaken notion of the term "set", hence the desire to consolidate properly the foundations of mathematics was born, and the mereology was created.

Leśniewski uses only one primitive term: part. This term is reserved only for proper parts, i.e. those which are different from the whole. The relation of being a part is irreflexive and transitive. In contrast to the relation of ingrediens (i.e. improper part) which partially orders the set of objects on which it is determined, i.e. it is reflexive, antisymmetric, and transitive. For Leśniewski being an ingrediens of a set is equivalent to being an element of this set. The essence of the concepts of "element" and "set" is explained by Borkowski very well:

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[4] The complete collection of Leśniewski's works may be found in [10].

> "Terms "set", "element of a set" are used in double meaning. One meaning is that the term "set" signifies objects made of parts, collectives, that is conglomerates of various types. Elements of such a set are understood as its parts, and the term "part" is understood in its common meaning, e.g. a leg of a table constitutes part of the table. A pile of stones in this meaning is a set of those stones. Particular stones and various parts of those stones, that is molecules or atoms are equally elements of this set. According to this meaning the set of given stones is identical with e.g. a set of all atoms constructing those stones. Elements of a set understood in this manner, e.g. a set of all tables, are not only particular tables, but e.g. legs of tables and other parts of tables. (…). The second meaning of the term "set" and "element of a set" is used e.g. when we talk about a set of European countries and we recognize particular European countries such as e.g. Poland, France, Italy etc. to be elements of the set, and we do not use various parts of those countries as elements of this set. In this meaning e.g. the Tatra Mountains or the Małopolska Upland are not elements of the set of European countries, despite the fact that they are parts of certain European countries. In this meaning we often use those terms when we talk e.g. about a set of Polish cities and we recognize particular cities e.g. Wroclaw, Warsaw etc. as elements of this set and we do not recognize particular streets, squares and other parts of those cities (…) as elements of this set. In this meaning we cannot identify the notion of an element of a set with the common notion of part [1]".

Mereology, hence, is the theory in which the whole can be seen in different ways[5]: let us consider, for example, a chessboard, and a class of its own fields. Each field of the chessboard is its element (because it is its part); it is an element of the class of its fields (which has 64 elements), but the same chessboard can be considered as a class of eight--fields rectangular bands (hence it has 8 elements). This perspective is totally unacceptable in classical set theory, where the set is uniquely determined by its elements.

To define the concept of a whole Leśniewski uses the terms "class" and "mereological sum". A mereological class is a concrete object which consists of all its parts, while the sum of any non-empty set of objects is a concrete set of such parts that overlap with some elements of that set. Thus, one can immediately see the difference between the classical set theory and the theory of collective sets: the latter requires that components are connected to each other; we take into account the relationship between the individual elements, in the former this relationship is not taken into consideration. Of course it causes some problems for the theory itself, but also opens new possibilities for the understanding of the concept of continuum, where the whole cannot be considered only as a sum of its parts. In this perspective, perhaps a new definition of mereological continuum might be offered in future.

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[5] A detailed description of the essence of mereology presented in the language of the first order logic is presented in [7] and [8].

4. Śleszyński's notes

In one of Śleszyński's manuscript dating from the period 1919–1921, stored in the archive of the University of Warsaw [9], we find a detailed analysis of the first two works of Leśniewski [2, 3] dedicated to the foundations of set theory[6]. Let us begin with some basic concepts given by Leśniewski:

DEFINITIONS (p. 22):

I. *„I apply the term: 'ingrediens of the object P' to denote the object P itself or each of its part".*

II. *The term: 'set of objects m' I apply to denote any object P, which fulfills the following condition:*
   - *if J is an ingrediens of the object P, then some ingrediens of an object J is an ingrediens of some m, which is an ingrediens of the object P.*

III. *The term: 'class of objects m' – I apply to denote each object P, which fulfills the following conditions:*
   - *Each m is an ingrediens of the object P,*
   - *If J is an ingrediens of the object P, then some ingrediens of the object J is an ingrediens of some m.*

IV. *„I apply the term: 'element of the object P', to denote any object P1, if at some denotation of the term x, the following conditions are fulfilled:*
   - *P is class of objects x,*
   - *P1 is x.*

AXIOMS (p. 20):

I. *If an object P is part of the object P1, then the object P1 is not part of the object P.*

II. *If an object P is part of an object P1, the object P1 is part of an object P2, then the object P is part of the object P2.*

III. *If an object is m, then some object is a class of m.*

IV. *If P is a class of objects m, and P1 is a class of objects m, then P is P1.*

The first definition given by Leśniewski defines the notion of a proper part, and the second and third define the concept of a collection (a set and a class). The concept of a class (a set) given above differs from the classical understanding, that of Cantor: in mereological collections (classes), as we mentioned before, the relation between the elements of sets (classes) is taken into consideration, while in Cantorian concept it is ignored. The fourth definition of Leśniewski defines the concept of an element of a class. While the first two axioms determine the properties of the relation of being a part, which is irreflexive and transitive, Axioms 3 and 4 are related to the existence and uniqueness of classes.

We find these definitions, axioms and some statements in the manuscript of Śleszyński, with accompanying notes, in the first twenty-five pages.

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6 A PhD thesis written by P. Borowik under the supervision of J. Woleński was dedicated to the same topic, but the presented analysis was made independently.

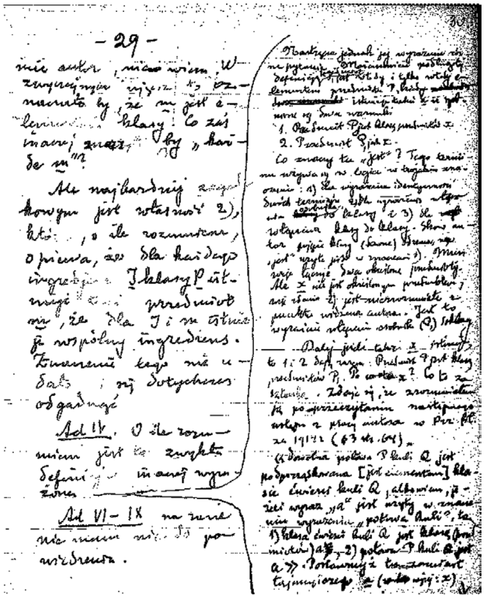

Fig. 1. Page 29 of Śleszyński's Manuscript [9]

Regarding the first two axioms of Leśniewski (AI, AII), Śleszyński formalizes them using the language from *Principia Mathematica* as follows:

Let *apb* means that *a* is a part of an object *b*. Then the Axioms I and II can be written as follows:

(I) $apb \cdot \supset \cdot \sim (bpa)$

(II) $apb.\, bpc \cdot \supset \cdot apc$

Here we find nothing special. Śleszyński compares Leśniewski's axioms to the axiomatization of line made by Vailati, where the predicate "*p*" means that a single point precedes the other. Hence, the system of axioms (I)**,** (II) can be replaced by the system (Ia) and (II):

(Ia) $\sim (apa)$

(II) $apb.\, bpc \cdot \supset \cdot apc$

Then Śleszyński refers to Leśniewski's proofs, and – for example – the proof (Ia) can be done in the following way:

(1) $(\mathrm{I})\, [b \setminus a] \rightarrow apa \cdot \supset \cdot \sim (apa)$

$(1).L \rightarrow \sim (apa)$ [7]

[7] Here " · " means conjunction; the sign: " · ⊃ · " – implication; "(1).*L* →" the result following from the conjunction of the assumption (1) and the logical rules signed by *L*; "(I) [*b**a*]" – we substitute "*b*" for "*a*" in assumption (I).

Instead, if we begin with Axioms (Ia) and (II), then the proof would be as follows (applying of course the classic rules of inference, which Śleszyński symbolically denotes by the letter: *L*):

(1) $(\text{II})\,[c \setminus a] \rightarrow apb.\, bpa \cdot \supset \cdot\, apa$

(2) $(\text{Ia}) \rightarrow \sim (apa)$

(3) $(1).(2).L \rightarrow \sim (apb.\, bpa)$

(4) $(3).L \rightarrow\ apb \cdot \supset \cdot \sim (bpa)$, which proves (I).

We must add here that Leśniewski deliberately does not use the language of classical logic, because he believes that it is imprecise; he distinguishes between the concept of existence and that of being: every object that exists – by Leśniewski – is, but not everyone which is , exists.

Now, let us take into consideration the definition of ingrediens (and being an element) (DF.I and DF.IV). Śleszyński compares this notion to the concept of inclusion of one class in another class, and distinguishes three different meanings of the word “is” (jest):

(1) It expresses the identity of two objects.
(2) It expresses the inclusion of a single object into a class.
(3) It expresses the inclusion of one class into another class.

He writes:

> “Since the author (Leśniewski) distinguishes the concept of class, so “is” has the meaning as described in point (1). It must therefore combine two determined objects. But *x* is not a determined object, so the sentence (number 2 in DF. IV) is incomprehensible from the point of view of the author. It expresses the inclusion of a single object (*P*1) into a class.
>
> Furthermore, if such x exists, then by the 1st and the 2nd condition it follows that: an object *P* is a class of objects *P*1. Hence, what about this *x*? What’s the trick? It seems that I have understood it after reading the paragraph [given below] from the work of the author published in the Review of Philosophy (Przegląd Filozoficzny) in 1914, vol. 3, p. 64.
>
> <Any half *P* of a ball *Q* is subordinated [= is an element] of a class of quarters of the ball *Q*, since if the term “*a*” is used in a sense of being “half of a ball”, then:
>
> (1) A class of quarters of the ball *Q* is a class of objects “*a*”,
> (2) A half of the ball *Q* is “*a*”>.
>
> Let’s put here instead of the mysterious “*a*” (previously *x*), what it really means. Then, the condition (2) [in DF. IV] is eliminated since it is obvious, while the condition (1) would be as follows:
>
> A class of quarters of the ball *Q* is a class of halves of the ball Q, and the author does not hesitate to declare it explicitly in the article in the Review of Philosophy (p. 64): <a class of halves of the ball *Q* is also a class of quarters of the ball *Q*>. The same expression I cannot find in this work, hence the comprehension becomes very difficult”.

In fact, the Definition III underlines that a class (a collection) can be viewed in different ways (not all objects have to be **m** if a class is defined as a collection of objects **m**): the ball can be seen as a class of its halves and as a class of its quarters. Moreover, the Definition II is redundant in comparison with Definition III.

Also subjected to the criticism is the term “objects **m**”. If there was only one item **m**, them **m** means the name of the object – by Śleszyński. Conversely, if the object is not an individual, but there are many **m’s**, what are these **m’s**? (p. 28). Addressing this issue, Śleszyński states that: “**m** is the common name for the elements of a class”(p. 28) and could not even suspect that Leśniewski uses this term unconsciously.

Next, Śleszyński investigates the correctness of Leśniewski’s proofs applying the language of *Principia Mathematica* (p. 48) but he does not make any essential changes. He notes the use of three terms: ingrediens, element and sub-multitude which are synonyms. It seems that Śleszyński did not fully understand the concept of a class applied by Leśniewski, which totally differs from the classical concept. Hence he criticizes Leśniewski, but he does not reject his ideas (p. 55). Additionally, analyzing Leśniewski’s proofs of theorems and performing them again in a classical way, he greatly appreciates Leśniewski’s accuracy and precision (p. 73).

At the end of the manuscript, we also find the analysis of Leśniewski’s paper entitled: *Is the class of classes which are not subordinated to themselves, subordinated to itself*?, where a formulation of the antinomy of irreflexive classes is given. Śleszyński criticizes Leśniewski’s use of the term “subordinated” (p. 80) and does not understand why he uses it. At this point we find the example of a class of halves of a ball and a class of quarters of a ball given by Śleszyński, who concludes that the term “class” used by Leśniewski does not denote “a collection of objects, in which there is no internal relationship between objects [intuitively that’s how we interpret the collections in the sense of G. Cantor]” (p. 84).

On the page 94 of Śleszyński’s comments we find a beautiful summary of the whole theory of Leśniewski written in the language of PM. The first part consists of two axioms (for the relation of being a part and its property: irreflexivity and transitivity), followed by 19 claims. The second part consists of the definition of ingrediens and 5 claims. The third part contains the definitions of multitude and class together with 2 axioms and 4 theorems (p. 95). The fourth part (the most important contribution of Śleszyński) contains the definition of an element and 17 claims (p. 96), in which it is shown that the term ‘element’ is synonymous with the term ‘ingrediens’; that every object is its own element, and that the ratio epsilon (Peano’s “is”) is transitive, and “part of the multitude of objects **m** may not be any of these objects” (p. 99) – which is an absolute novelty in relation to the classical set theory. This shows the impossibility of construction of the antinomy of Russell within mereology.

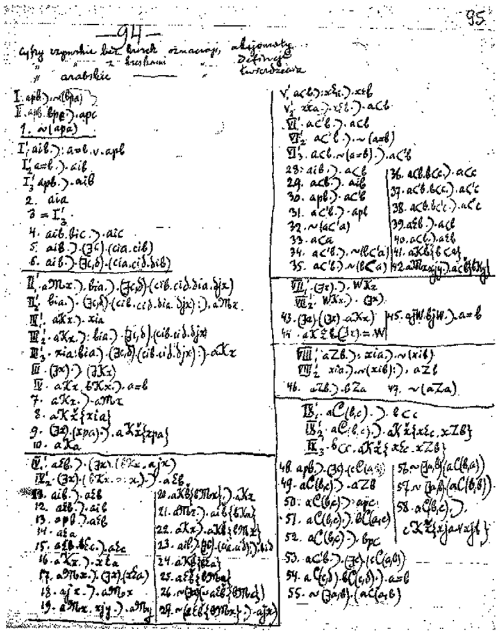

Fig. 2. Page 94 of Śleszyński's Manuscript [9]

5. Conclusions

Summarizing, in general, the criticism of Śleszyński can be considered very positive. It emphasizes Leśniewski's accuracy and precision, and the work itself contains neither logical nor formal errors. As for the lack of understanding of certain terms, one can always have doubts, but it is not a formal shortcoming of this work.

It is a pity that Śleszyński's notes were not published during his life. Perhaps the reception of Leśniewski's ideas could have been easier. Leśniewski's systems are not currently used as foundations of mathematics; maybe the reason lies in the language applied by the author. However, his work can be considered a masterpiece of mathematical precision and accuracy.

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2-NP/2015

ROMAN SZNAJDER*

KACZMARZ ALGORITHM REVISITED

JESZCZE O ALGORYTMIE KACZMARZA

A b s t r a c t

In 1937, Stefan Kaczmarz proposed a simple method, called the *Kaczmarz algorithm*, to solve iteratively systems of linear equations $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ in Euclidean spaces. This procedure employs cyclic orthogonal projections onto the hyperplanes associated with such a system. In the case of a nonsingular matrix **A**, Kaczmarz showed that his method guarantees convergence to the solution of $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$. The Kaczmarz algorithm was rediscovered in 1948 and became an important tool in medical engineering. We briefly discuss generalizations of this method and its ramifications, including applications in computer tomography, image processing and contemporary harmonic analysis.

Keywords: *Kaczmarz method*, *systems of linear equations*, *computer tomography*, *image reconstruction*

S t r e s z c z e n i e

W 1937 roku Stefan Kaczmarz zaproponował prostą metodę [*KA*], zwaną obecnie *algorytmem Kaczmarza*, za pomocą której można rozwiązywać iteracyjnie układy równań liniowych $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ w przestrzeniach euklidesowych. Metoda ta używa cyklicznego ciągu rzutów ortogonalnych na hiperpłaszczyzny związane z tym układem. W przypadku macierzy odwracalnej **A** Kaczmarz pokazał, że jego metoda gwarantuje zbieżność do rozwiązania układu równań $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$. Metoda ta została ponownie odkryta w 1948 roku i stała się ważnym narzędziem w inżynierii medycznej. Omawiamy tutaj pokrótce uogólnienia tej metody i ich zastosowania w tomografii komputerowej, przetwarzaniu obrazów i we współczesnej analizie harmonicznej.

Słowa kluczowe: *metoda Kaczmarza*, *układy równań liniowych*, *tomografia komputerowa*, *rekonstrukcja obrazu*

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1. The origins of the Kaczmarz algorithm

Stefan Kaczmarz has been primarily known as a member of the Lvov School of Mathematics and a collaborator of Stefan Banach and Hugo Steinhaus (see a celebrated monograph [20])[1]. His professional interests were orthogonal series, theory of real functions, and applications of mathematics. The aim of this article is to highlight the importance of Kaczmarz's pioneering work [21], which has found numerous applications across a number of fields, including image processing, computer tomography, and image compression. The paper appeared in 1937 in German and is hardly known to the majority of the Polish mathematical community, although it became widely recognized in the Western hemisphere. For years, researchers have been using the German original, as the English version appeared only in 1993, translated by professor P.C. Parks [22].

Fig. 1. Stefan Kaczmarz (1895–1939)

It seems that the only note on Kaczmarz algorithm in Polish literature was by Cegielski [7]. In this paper, the author, a noted expert on modern iterative computational methods, refers to an extensive list [8] of English-language publications on the Kaczmarz method. In his recent monograph [9], among others, Cegielski studies convergence of such a type of methods.

In its original formulation, the Kaczmarz algorithm (*KA*) states the following: for a given $m \times n$ matrix $\mathbf{A}$ and a vector $\boldsymbol{b} \in R^m$, we wish to find a solution to the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$. Let $\boldsymbol{x}^0 \in R^n$. Define the sequence of vectors

$$\boldsymbol{x}^{k+1} := \mathbf{x}^k - \frac{<\boldsymbol{a}^i, \boldsymbol{x}^k> - b_i}{\left\|\boldsymbol{a}^i\right\|^2}\boldsymbol{a}^i, \quad \text{where } k+1 \equiv i \ (\mathrm{mod}\ n)$$

[1] For a superb and exhaustive monograph on Kaczmarz's research and private life, see [25].

and $\boldsymbol{a}^i$ is the i-th row of matrix $\mathbf{A}$ (where $\|\cdot\|$ stands for the Euclidean norm in R^n). Kaczmarz originally considered systems with a square matrix and showed that for a nonsingular matrix $\mathbf{A}$, the sequence $(\boldsymbol{x}^k)$ converges to the solution, regardless of the starting point $\boldsymbol{x}^0 \in R^n$.

How does the algorithm work? For any $1 \leq k \leq n$, the above formula presents the orthogonal projection of the point $\boldsymbol{x}^k$ onto the affine hyperplane

$$H_i = \{\boldsymbol{x} \in R^n : <\boldsymbol{a}^i, \boldsymbol{x}> = b_i\} \quad (i = 1, \ldots, n).$$

The point $\boldsymbol{x}^{n+1}$ is projected again onto the hyperplane H_1, $\boldsymbol{x}^{n+2}$ is projected onto H_2, and so on. It is also called a *cyclic projection method*. In general, when A is of full rank, one gets $\boldsymbol{x}^k \to \boldsymbol{x}$ for some solution $\boldsymbol{x}$ to the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.

The *KA* is historically the first numerical method exploring sequences of orthogonal projections onto hyperplanes. This method, while completely elementary – high school students should be able to grasp it – is quite powerful. On the other hand, certain issues such as the speed of convergence, were hard to settle.

In the simplest case of two intersecting lines l_1 and l_2 on the plane with normal (linearly independent) vectors $\boldsymbol{a}^1$ and $\boldsymbol{a}^2$, for a given seed point $\boldsymbol{x}^0$, $(\boldsymbol{x}^k)$ is the sequence of alternating orthogonal projections on these lines. The sequence obviously converges to the common point $\boldsymbol{x}$ of the given lines.

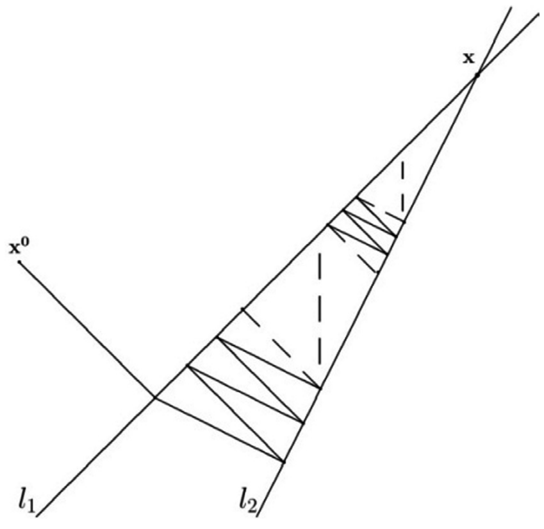

Fig. 2. The Kaczmarz method in the case of intersecting lines on the plane

Soon after the Kaczmarz's discovery, in 1938, Gianfranco Cimmino [10] proposed a similar iterative method: one reflects a given point $\boldsymbol{x}^k \in R^n$ about all hyperplanes H_i and averages these reflections with respect to a fixed probability vector $\boldsymbol{w} = (w_1, \ldots, w_n)$:

$$\boldsymbol{x}^{k+1} = \boldsymbol{x}^k + 2\sum_{i=1}^{n} w_i (P_{H_i}\boldsymbol{x}^k - \boldsymbol{x}^k).$$

Cimmino showed that his method of simultaneous reflections converges to a solution. Both Kaczmarz and Cimmino algorithms were established ahead of their time and their invention passed in math community virtually unnoticed. These methods evolved from original forms and their generalized versions are currently widely used to solve large systems of linear equations. Although Cimmino's algorithm is slightly slower than the *KA*, the advantage of this method is the opportunity of using parallel processors. We should point out that Cimmino's method was extended to simultaneous projections onto closed convex sets.

2. Reemergence of the Kaczmarz algorithm

For more than a decade the *KA* was in oblivion. It resurfaced in separate publications of Bodewig [2] (1948), Forsythe [14] (1953), and Tompkins [32] (1949). The methods of projections were already known to these authors and the *KA* was performing remarkably well, despite slow computers in the early 1950s.

Systematic studies and applications of the *KA* started in 1970 with the paper by Gordon, Bender and Herman [15]. The *KA* was rediscovered by these authors and is known as the *Algebraic Reconstruction Technique* in computer tomography. For matrix $\mathbf{A} \in R^{m \times n}$, solvability of the matrix equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ indeed means finding (reconstructing) $\boldsymbol{x}$ from the data: $< \boldsymbol{a}^i, \boldsymbol{x} > = b_i$, $i = 1, 2, \ldots, m$; we assume here that $m \gg n$ (overdetermined system) and $\mathbf{A}$ is of full rank.

In 1971, Tanabe [31], undertook the effort of generalizing the *KA* and providing a deeper insight into the theoretical aspects of this method. He showed that the sequence $(\boldsymbol{x}^k)$ of iterates always converges, regardless of the consistency of the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$. He also noticed that the *KA* can be used to approximate the Moore-Penrose pseudoinverse $\mathbf{A}^{\dagger}$ of a matrix $\mathbf{A}$.

The *KA* was implemented in the first medical scanner in 1972. It became clear that there were a lot of potential applications of the *KA*. During the second half of the past century, the computational mathematics community witnessed an outburst of various iterative techniques, including simultaneous projections, relaxation and averaging techniques. Faster hardware and efficient software contributed to a rapid increase in applications of these techniques in medical sciences, e.g., in reconstruction of 3*D* images through 2*D* projections (computer tomography), see Brooks [4]. The Kaczmarz method can be considered as a special case of the POSC (*Projection onto Convex Sets*) method, see [9] and references therein, including Kiwiel [23], and the important survey paper by Bauschke and Borwein [1]. This technique plays a prominent role in signal and image processing, particularly in medical image processing, and in such disciplines as operations research and game theory. Byrne's monograph [6] emphasizes the importance of the so-called block Kaczmarz method; see also [5].

3. Evolution of the Kaczmarz method

Over the time, the *KA* evolved to become faster and more efficient. One of important modifications of the original *KA* is the so-called *randomized Kaczmarz algorithm*:

$$x^{(k+1)} := x^k + \frac{b_{p(i)} - \langle a^{p(i)}, x^k \rangle}{\left\| a^{p(i)} \right\|^2} a^{p(i)},$$

where, with $p(i) \in \{1, \ldots, m\}$, the row $a^{p(i)}$ is chosen at random with probability $\frac{\left\| a^{p(i)} \right\|^2}{\| \mathbf{A} \|_F^2}$; here, $\| \mathbf{A} \|_F$ is the Frobenius norm of matrix **A**, i.e., $\| \mathbf{A} \|_F^2 = \sum_{i=1}^{m} \left\| a^i \right\|^2 =$ $= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2$. The magnitude of rows a^i of **A** cannot be ignored. Theoretical results on exponential convergence in expectation were obtained by Strohmer and Vershynin in their seminal work [30]. In fact, it is the first paper where such an estimate was obtained for the randomized *KA*. An interesting follow-up was done by Deanna Needell in [27]; see also her other work [18], [28], and [29]. In [13] the authors presented a modified version of the randomized *KA*, which in most cases significantly improved the convergence rate. They utilized the Johnson-Lindenstrauss dimension reduction technique to keep the runtime at the same order as that of original randomized version. The Johnson-Lindenstrauss lemma is a very interesting fact in its own right: it is one of the most quoted results in analysis, and deserves a separate exposition:

Let $\delta > 0$ *and S be a finite set of points in* R^n*. Then for any d satisfying*

$$d \geq C \frac{\log |S|}{\delta^2},$$

there exists a Lipschitz mapping $F : R^n \to R^d$ *such that*

$$(1-\delta) \left\| s_i - s_j \right\|^2 \leq \left\| F(s_i) - F(s_j) \right\|^2 \leq (1+\delta) \left\| s_i - s_j \right\|^2,$$

for all $s_i, s_j \in S$ [19].

In recent times, several master's and doctoral theses on spectral tomography and the block *KA* appeared, see [3, 4] (mentioned earlier), and [12]. An earlier dissertation by Grangeat [16] addressed the 3*D* image reconstruction from 2*D* X-ray pictures, however it did not directly refer to the *KA*. It is worthwhile to mention [33], which presented a practical application of the *KA*.

4. Extension of the Kaczmarz method to Hilbert spaces

In 1977, McCormick [26] was the first to investigate the *KA* in Hilbert spaces. In 2001, Kwapień and Mycielski [24] proposed an efficient generalization of the *KA* to infinite-dimensional Hilbert space. Here is their generalization: Let *H* be a Hilbert space and let $(e^n)_{n=0}^{\infty}$ be a sequence of unit vectors in *H*. Given $x \in H$, the Kaczmarz algorithm is defined as:

$$\boldsymbol{x}^0 := <\boldsymbol{x},\boldsymbol{e}^0>\boldsymbol{e}^0, \text{ and } \forall n \in N,\ \boldsymbol{x}^n := \boldsymbol{x}^{n-1} + <\boldsymbol{x}-\boldsymbol{x}^{n-1},\boldsymbol{e}^n>\boldsymbol{e}^n.$$

It is important to notice that in a finite-dimensional case, the iterative sequence generated by the *KA* is always convergent, while the situation in the infinite-dimensional setup may differ. We say that the sequence $(\boldsymbol{e}^n)_{n=0}^{\infty}$ is *effective* if and only if $\forall \boldsymbol{x} \in H \ \ \lim_{n\to\infty} \boldsymbol{x}^n = \boldsymbol{x}$. Kwapień and Mycielski introduced the following sequence $(\boldsymbol{g}^n)_{n=0}^{\infty}$:

$$\boldsymbol{g}^0 := \boldsymbol{e}^0, \text{ and } \forall n \in N,\ \boldsymbol{g}^n := \boldsymbol{e}^n - \sum_{i=0}^{n-1} <\boldsymbol{e}^n,\boldsymbol{e}^i>\boldsymbol{g}^i.$$

Thus, we have $\boldsymbol{x}^n = \sum_{i=0}^{n} <\boldsymbol{x},\boldsymbol{g}^i>\boldsymbol{e}^i$. They showed that *the sequence* $(\boldsymbol{e}^n)_{n=0}^{\infty}$ *is effective if and only if* $(\boldsymbol{g}^n)_{n=0}^{\infty}$ *is a tight frame with constant 1 for H*. Let us recall that the sequence $(\boldsymbol{g}^n)_{n=0}^{\infty}$ is a *tight frame* with constant 1 if $\|\boldsymbol{v}\|^2 = \sum_{n\in N} \left| <\boldsymbol{v},\boldsymbol{g}^n> \right|^2$ for each $\boldsymbol{v} \in H$.

In 2005, Haller and Szwarc [17] made a follow up and connected Kwapień-Mycielski results with construction of frames and tight frames in harmonic analysis. They showed that a sequence $(\boldsymbol{e}^n)_{n=0}^{\infty}$ *is an effective sequence if and only if it is linearly dense in H and for a certain matrix* **C** *associated with the Gram matrix of the sequence* $(\boldsymbol{e}^n)_{n=0}^{\infty}$, $\mathbf{C}^*\mathbf{C}$ *is an orthogonal projection, i.e.,* **C** *is partial isometry*.

Recently, Czaja and Tanis pursued further studies [11] concerning the nature of the *KA*. Their starting point was the observation that if a sequence $(\boldsymbol{e}^n)_{n=0}^{\infty}$ is an orthonormal basis in *H*, then $\boldsymbol{g}^n = \boldsymbol{e}^n$ and by the Kwapień-Mycielski theorem, $(\boldsymbol{e}^n)_{n=0}^{\infty}$ is an effective sequence. They introduced the concept of an *almost effective sequence* and characterized such sequences (under the assumption of being a Bessel sequence) in terms of frames. At the beginning of this section, we mentioned that McCormick extended the classical *KA* to solve $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, where **A** is a bounded linear transformation on $l^2(N)$ and $\boldsymbol{b} \in \mathrm{Ran}(A)$. McCormick considered a finite-dimensional approximation of such a problem by a sequence of increasing finite-dimensional subspaces. After imposing certain frame conditions on rows of the matrix operator **A**, one can get a better convergence estimate of the process. Thus, assume that an infinite dimensional matrix **A** has rows $\boldsymbol{a}^i \in l^2(N)$ that form a linearly dense system in $l^2(N)$ and choose the initial guess $\boldsymbol{x}^0 := \boldsymbol{b}_0 \dfrac{\boldsymbol{a}^0}{\|\boldsymbol{a}^0\|^2}$. Czaja and Tanis showed the following: *Let* $\mathbf{A}: l^2(N) \to l^2(N)$ *be a bounded (matrix) transformation. Then, for the initial guess* $\boldsymbol{x}^0$, *the KA algorithm always converges to a solution if and only if* **A** *is surjective with rows that form an orthogonal basis for* $l^2(N)$.

This note covers only selected contributors to the development of the Kaczmarz method. Other names of particular note include M. Benzi, Y. Censor, P.L. Combettes, S.D. Flåm, F. Natterer, and C. Popa.

5. Epilogue

Professor W. Orlicz once mentioned: "It seems that Cracovian Calculus of Tadeusz Banachiewicz and Kaczmarz method are the most important Polish achievements in numerical analysis between the wars". In the era of modern computing, *Cracovian Calculus* became obsolete and serves merely as a historical artifact and provides an important example of a non-associative algebra, while the Kaczmarz method is an efficient technique with many prosperous years ahead.

I thank Dr. Danuta Ciesielska from Pedagogical University of Cracow for her help in supplying a copy of the bibliographical item [8]. My thanks also go to Professor Wojtek Czaja from the Norbert Wiener Center, University of Maryland, College Park, USA, for turning my attention to his recent work [11] and discussions on the Kaczmarz method. I am also grateful to an anonymous referee for comments, which improved precision of this paper.

Dedication: I dedicate this paper to my wife Joanna.

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JAN WOLEŃSKI*

PHILOSOPHY OF EXACT SCIENCES (LOGIC AND MATHEMATICS) IN POLAND IN 1918–1939

FILOZOFIA NAUK ŚCISŁYCH (LOGIKI I MATEMATYKI) W POLSCE W LATACH 1918–1939

A b s t r a c t

This paper describes the philosophy of logic and mathematics in Poland in the years 1918–1939. The special attention is attributed to the views developed in the Polish Mathematical School and the Warsaw School of Logic. The paper indicates various differences between mathematical circles in Warszawa, Lvov and Kraków.

Keywords: *set theory*, *the foundations of mathematics*, *pluralism*

S t r e s z c z e n i e

Artykuł opisuje filozofię logiki i matematyki w Polsce w latach 1918–1939. Szczególną uwagę zwrócono na poglądy rozwinięte w Polskiej Szkole Matematycznej oraz Warszawskiej Szkole Logicznej. Artykuł wskazuje na rozmaite różnice pomiędzy środowiskami matematycznymi w Warszawie, Lwowie i Krakowie.

Słowa kluczowe: *teoria mnogości*, *podstawy matematyki*, *pluralizm*

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Poland recovered its independence in 1918 and the building of education and science became one of the main tasks of new Polish state. Due to partition of the country into three zones, Russian, German (formerly, Prussian) and Austro-Hungarian (formerly, Austrian) a the end of 18^{th} century, it was a very challenging aim. The policy of particular states occupying Poland toward cultural ambitions of Poles was different. Restricting attention to the period after 1864, relatively best situation occurred in Galicia, that is, Polish territory belonging to the Habsburg empire. A considerable liberalization of Austro-Hungary in the 1870s resulted in a partial autonomy of Galicia and considerable freedom in cultivating Polish culture. As far as the issue concerns science and higher education, Galicia had two universities and the Polytechnic School.The Jagiellonian University in Kraków continued its character as fully Polish university after a short period (after 1849) in which German language served as official teaching. The University of Lvov became polonized around 1870; the same concerns the Lvov Polytechnic School. Academy of Arts and Sciences was established in Kraków in 1872. These institutions managed normal scientific and/or educational activities in Polish intellectual circles.

The situation in the Russian zone was complex and essentially dependent on the actual policy of Tsarist authorities, which was sometimes relatively liberal or sometimes took a completely opposite course. In 1862, the Main School, practically a university was established in Warsaw. However, it was closed in 1869 and replaced by the Imperial University with Russian as the official language. On the other hand, Poles developed in the Russian zone, mostly in Warsaw, a half-official, but principally tolerated by Russians, system of education, which played an important role. In 1907, the authorities agreed for the rise of the Warsaw Scientific Society. The Mianowski Kasa, the Foundation for Supporting Polish Science, established in 1881, was another significant institution. The worst situation occurred in the German zone. The policy of Germanization, called *Kulturkampf*, was consequently executed by German authorities. To exclude Polish scientific and educational life (on higher level than secondary school) was one of the principles of *Kulturkampf.* The Poznań Society of the Friends of Science existed since 1857, but its real role was not particularly great. Although both Galician universities attracted many students from German and Russian zones, many Poles decided to study abroad, mostly in Germany, Austria, Russia or France.

The above (very) brief historical account explains why the unification of Polish educational system linked with creation of new universities became one of priorities after 1918. In the years 1915–1920, four universities (I omit other types of high schools – note that the term “high school” refers, according to Polish tradition, to any school at the university level) were established or renewed, namely in Warszawa (renewed), Poznań (new), Vilna (renewed) and Lublin (new; it was a catholic university and it did not play any role in exact sciences). In 1916, the Mianowski Kasa invited leading Polish scientists working in Poland and abroad to answer the question “What should be done in independent Poland in order to develop national science?” The very numerous answers outlined the state of art and needs of Polish science. This material was included in the two first volumes of the journal “Nauka Polska. Jej potrzeby, organizacja i rozwój” (Polish Science. Its Needs, Organization and Development)), published in 1918–1919 (presumably, it was the first professional journal devoted to the science of science). This enterprise shows that the building the foundations of science in Poland was considered very seriously by Polish scholars. Zygmunt Janiszewski's famous programmatic essay on the needs of mathematics in Poland was one of papers published in this volume.

Speaking about remarkable development of mathematics and logic in Poland in 1918–1939, one should not ignore Polish (or Warsaw) positivism. This movement, influenced by French and English positivism, was closely connected with the mentioned Main School. Polish positivism was a reaction to romanticism and its unsuccessful attempts to recover Polish independence by military fighting, directed mostly against Russia considered in Poland as the main invader. Two uprisings, namely in 1830–1931 (the November Uprising) and in 1963–1964 (the January Uprising) ended with total defeats as well as with subsequent political and cultural repressions. Polish positivists proposed another solution, namely so-called organic (or basic) work consisting in improving society (particularly, its lower classes) by increasing its economic prosperity and education. This second point was especially important for the rise of interests in the philosophy of science in Poland. Since Polish tradition was not very strong in this field, many translations of works of foreign leading philosophers and scientists appeared in 1865–1918. The Mianowski Kasa was very instrumental in this enterprise. Books and papers on the foundations of exact sciences written by Dedekind, Enriques, Helmholtz, Jevons, Maxwell, Mill, Bain, Poincaré, Riemann, Pieri, Russell, Young and Whitehead were published in Polish. They allowed Polish intellectuals to follow the world development of science and its methodology. In this environment, first Polish scientists, like mathematicians Samuel Dickstein, Władysław Gosiewski and Edward Stamm (he was also a philosopher) began their activity as professional scholars.

Looking at the academic level, the philosophy of exact sciences in the interwar period did not start from the scratch around 1918. In Lvov, Kazimierz Twardowski, professor of philosophy since 1895, created a very good atmosphere for the development of logic. His metaphilosophical program of scientific philosophy, insisting on clarity of thinking and expressing thoughts in a language, demanding that philosophers should correctly justify the proposed statements and opposing metaphysical speculations, considered logic as a natural instrument of achieving cognitive goals. Twardowski did not work in logic, except a very few contributions to general logic, mostly semantics and the methodology of sciences. However, he recommended mathematical logic s worthy to a serious study to his students. Alfred Tarski summarized the role of Twardowski in the development of logic in Poland in the following words [18, p. 20].

Almost all researchers, who pursue the philosophy of exact sciences in Poland, are indirectly or directly the disciples of Twardowski, although his own works could be hardly be counted within this domain.

Twardowski trained a number of students who began to work in formal logic and the methology of sciences. This group included Jan Łukasiewicz, Stanisław Leśniewski, Kazimierz Ajdukiewicz, Tadeusz Czeżowski Tadeusz Kotarbiński and Zygmunt Zawirski. The mentioned scholars formed the first generation of the so-called Lvov–Warsaw School, a powerful group of analytic philosophy (in fact, it functioned as the Lvov School until 1915). Logic and the foundations of mathematics became popular among Lvov mathematicians still before 1914. Wacław Sierpiński lectured on set theory at the end of the first decade pf the 20th century. He was joined by Zygmunt Janiszewski. Both, Sierpiński and Janiszewski, decided to devote their habilitation lectures to the problems of foundations of mathematics (the former spoke about the concept of mathematical correspondence and the latter – about realism and idealism in mathematics). However, Krakow, not Lvov, became the first serious centre of mathematical logic in Poland. Stanisław Zaremba a distinguished Polish mathematician, had strong interests in logic and the foundations of mathematics. He considered logic as a peripheral branch of mathematics, having only a secondary importance, mainly in teaching mathematicians. He was influenced in this attitude by the French style of doing mathematics. Zaremba's views prevailed among mathematicians in Krakow. It is interesting that the Jagiellonian University had a special professorship in mathematical logic, occupied by Jan Śleszyński. Other logicians working in Krakow included Leon Chwistek and Witold Wilkosz. Warszawa appeared on the stage of logic in 1915, when German authorities (the city was occupied by Germans since the summer end of 1915) allowed to re-open the (Polish) university. Łukasiewicz was appointed as the professor of philosophy and began systematic lectures in logic and the foundations of mathematics which attracted many young mathematicians. He was joined by Leśniewski, who became the professor of the philosophy of mathematics in 1919. Also Kotarbiński moved to Warsaw as professor of philosophy in 1918.

According to the mentioned Janiszewski program, Polish mathematicians should concentrate on carefully chosen mathematical fields which could be promising from the point of view achieving new results. Set theory, topology and their applications to classical mathematics were identified as such domains. This choice made mathematical logic and the foundations of mathematics as located in the very centre of mathematics. Janiszewski also postulated that Poland should have a special mathematical journal published in international languages. This idea found its realization in *Fundamenta Mathematicae* (the first volume appeared in 1920). The first idea of *Fundamenta Mathematicae* particularly stressed the significance of logic and the foundations of mathematics by projecting two series of the journal, namely one devoted to set theory, topology and their applications, and second to logic and the foundations. However, this idea was abandoned and the *Fundamenta* was published as the unified journal, but with a considerable amount of papers on foundational problems. Lvov became the second main centre of Polish mathematical school. Krakow remained more traditional in the spirit of Zaremba. Sierpiński, Janiszewski and Stefan Mazurkiewicz (all move to Warsaw after 1918) played the main role in Warszawa, while Stefan Banach and Hugo Steinhaus became the leaders in Lvov. Yet one important difference between the two centres of modern mathematics in Poland must be noted. Although mathematicians in Lvov worked mainly on functional analysis, the operator theory and similar problems, eventually with using of set theory and topology, the mathematical circle in Warszawa focused more on abstract matters.

The University of Warsaw organized the separate Faculty of Mathematics and Natural Sciences. Łukasiewicz and Leśniewski, two philosophers with rather a limited mathematical education were appointed as professors just at this specialized faculty. Incidentally, it was a brave sociological move, not practised in other countries. Łukasiewicz and Leśniewski, supported by leading mathematicians working at the Warsaw Univwersity, particularly Sierpiński, Stefan Mazurkiewicz and Kazimierz Kuratowski (Janiszewski prematurely died in 1920) began very intensive teaching in mathematical logic and very soon found many gifted students. Due to these activities, the Warsaw School of Logic was established. It included such logicians like Tarski, Adolf Lindenbaum, Mordchaj Wajsberg, Moses Presburger, Stanisław Jaśkowski, Bolesław Sobociński, Jerzy Słupecki and Andrzej Mostowski. This school had two parents, mathematics and philosophy, more explicitly the Warsaw School of Logic can be regarded as a part of Polish Mathematical School and the group originated with Twardowski since 1895, extended to the Lvov-Warsaw School after 1918. Last but not least, Twardowski idea of the development of Polish philosophy and the Janiszewski program shared similar points. Both great organizers of Polish science were convinced that Polish scholars have to have a very close contacts with novelties and tendencies executed in leading scientific investigations. By a coincidence, mathematical logic became a common focus for Twardowski and Janiszewski. On the other hand, although we have no written historical evidence, it is quite possible that Twardowski, Sierpiński and Janiszewski discussed the organization of science in Poland after recovering independence and reached similar conclusions, also those articulated by Janiszewski in his program. To complete personal issues, Ajdukiewicz stayed in Lvov (with exception of the years 1926-1928, when he taught in Warszawa) and lectured on logic for philosophers and mathematicians; he was appointed as professor of philosophy. The University of Lvov established the chair for mathematical logic in 1928. Chwistek won the competition for this postand he created a small school working in logic.

The double, mathematical and philosophical genesis, of the Warsaw School of Logic, immediately provokes the question about the philosophy of exact sciences in this circle. A similar problem concerns the entire Polish Mathematical School. Due to the mentioned differences between Warsaw and Lvov mathematicians, the first group treated the foundational the the second. Mathematicians from Kraków are not counted among members of Polish Mathematical School. As I already noticed, Zaremba had strong interests in logic and the foundations of mathematics, but he considered logic as a fairly marginal branch of mathematics, having only a secondary and auxiliary importance as a preparation for doing hard mathematics. The already indicated French influence on Zaremba resulted in his even hostile attitude to logic, similar to Poincaré's view. Zaremba's opinions prevailed among mathematicians in Krakow. Śleszyński's position was cancelled when he became retired. More importantly, there was a great controversy between Zaremba and Polish Mathematical School concerning the foundations of mathematics. Zaremba entirely rejected the view that set theory constitutes the fundament of mathematics. In fact, Wilkosz was the only exception and conducted investigations in abstract (or pure) set theory. Chwistek, as I already noticed, moved to Lvov. The above explanations justify the further schematization of my report about the philosophy of exact sciences in Poland in the period 1918–1939. It is reasonable to concentrate on the Warsaw branch of Polish Mathematical School, particularly the Warsaw School of Logic and on Chwistek who developed own approach to logic and the foundations of mathematics.

It is convenient to start with Leśniewski and Chwistek. Both developed general schemes, *grand logics*, so to speak, for grounding logic and mathematics best on very explicit philosophical premises. Leśniewski proposed a comprehensive system consisting of three parts, protothetic (a generalized propositional calculus), ontology (a logic of terms) and mereology (the theory of parts and wholes). Two first parts constitute pure logic, but the third one, mereological theory of classes, functions as a substitute of set theory, although

Leśniewski himself considered mereology as a part of logic. Leśniewski hoped to base the entire mathematics on his system. Leśniewski's formal systems are radically nominalistic (no abstract objects are admitted), fully formalized (he accepted so-called intuitionistic formalism consisting in the view that formal languages are always interpreted; note that this view has nothing in common with intuitionism as a program in the foundations of mathematics) and realistic (logic and mathematics describe reality; the term "ontology" was chosen by Leśniewski, because he considered the logic of terms as the general theory of objects. Chwistek began his logical investigations by attempts to improve Russell's ramified theory of types. More specifically, Chwistek tried to combine Poincaré's constructivism (predicativism) and Russell's approach by eliminating the axiom of reducibility. As a result he obtained a version of the simple theory of types based on strong nominalistic presuppositions. However, limitations of this solution in capturing the entire mathematics by the modified simple theory of type pushed Chwistek to a different conception consisting in constructing the hierarchy of semantic systems. Semantics in Chwistek's sense is a general formal theory of expressions, but not a theory of relations between languages and what they refer to. Thus, semantics is rather similar to syntax in it standard understanding. Chwistek in his new logical construction also preserved nominalism. The foundational proposals of Leśniewski and Chwistek, although different in essential points, can be considered as versions of logicism, that is, grounding mathematics as a part of logic, provided that mereology and semantics (in Chwistek's sense) are considered as parts of logic. Both, Leśniewski and Chwistek took this position.

Chwistek and Leśniewski were exceptions in Poland, because other Polish logicians and mathematicians did not develop general conceptual schemes as capturing mathematics and mathematical knowledge. Consequently, the Polish Mathematical School and the Warsaw Logical School had no official philosophy of exact sciences. More specifically, Polish logicians and mathematicians did not ascribed themselves to one of the dominant foundational projects of the first half of the 20th century, namely logicism, formalism or intuitionism. Using a label, popular nowadays (see [7]) the philosophy of logic and mathematics was considered in Poland as the second philosophy, according to the slogan (it is a paraphrase of the saying beginning with *primum vivere*) *primum mathematicari*, *deinde philosophari.* Putting this otherwise, mathematical practise is first, but mathematical philosophy – the second one. This background allowed to employ simultaneously various influences stemming from reading books and papers written by foreign scholars and having personal contacts with them. Due to the fame of *Principia Mathematica*, Russell was extremely popular in Poland, but nobody, even Chwistek or Leśniewski, accepted logicism as the only correct foundational scheme. In fact, Łukasiewicz began his serious logical investigations after reading Russell's, *The Principles of Mathematics*, pubhlished in 1903. Traces of formalism are clearly present in Chwistek, Ajdukiewicz, Leśniewski and Tarski; two first studied in Göttingen, but the rest knew formalism from the literature only. I already noticed, Poincaré's influence on Chwistek, but the latter was inspired by constructivism (predicativism) of the former, but not by his conventionalism. It interesting that even in the case of Polish authors (Chwistek, Leśniewski) proposing own foundation al schemes, we easily recognize influences coming from different, often mutually rival, philosophical programs. Thus, pluralism as a general standpoint in the philosophy of logic and mathematics can be regarded as one of the most characteristic features of the worldview of the Polish school of logic and mathematics. Incidentally, pluralism was also a characteristic view in the entire philosophical Lvov–Warsaw School.

On the other hand, some Polish authors strongly influenced the rise of a quite new foundational paradigm. The programs of logicism, formalism and intuitionism were replaced in the 1930s by new projects, namely set theoretical, constructivism and proof-theoretical (see [10]). Due to the role of set theory in the reconstructing the whole edifice of classical mathematics, stressed by the Polish Mathematical School, it was quite natural that Polish mathematicians and logicians contributed to the set theoretical paradigm. Excluding set theory from logic, contrary to logicism, resulting in favouring first-order logic as the logic by most Polish logicians; higher-order logic was considered as a part of set theory. Although formalism did not attract Poles as a general foundational programs, several Tarski's works on consequence operations and general deductive systems can be viewed as contributions to the proof-theoretical foundational scheme, although Tarski himself was never committed to the view that formalism is the best proposal in grounding the foundations of logic and mathematics. In fact, most Polish mathematicians and logicians abstain from explicit declarations what is the best in science.

The attitude of Polish logicians and mathematicians to intuitionism and constructivism as grand projects in the foundations of mathematics provides perhaps a particularly instructive illustration of Polish pluralism in doing the philosophy of logic and mathematics. As it is well-known, constructivism (intuitionism is its very radical version) considers non--constructive mathematical methods as defective in a way which cannot be improved. This radical verdict also concerns infinitistic procedures exceeding those mathematical methods which can be formalized within arithmetic of natural numbers (in this respect, constructivism is somehow more liberal than Hilbert's original formalism admitting finitely performable constructions as the only admissible methods in the so called real mathematics). Tarski [16, p. 713] said once:

> "As an essential contribution of the Polish school to the development of metamathematics one can regard the fact that from the very beginning it admitted into metamathematical research all fruitful methods, whether finitary or not".

The message coming from this fragment is a good example of application the dictum *primum mathematicari*, *deinde philosophari.* Tarski's view can be characterized as methodological Platonism. Paradoxically, Tarski himself frequently expressed his sympathy to nominalism. However, if someone sharply distinguishes mathematics and its philosophical interpretations, such a discrepancy unnecessarily must be regarded as a lack of coherence.

Sierpiński commenting the axiom of choice, controversial as it is commonly admitted, applying the same ideology ([13, p. 95]; this opinion appeared in Sierpiński's writings much earlier, in fact, since early 1920s) as Tarski did in the following words:

> Still, apart from our personal inclinations to accept the axiom of choice, we must take into consideration, in any case, its role in the Set Theory and in the

> Calculus. On the other hand, since the axiom of choice has been questioned by some mathematicians, it is important to know which theorems are poved with its aid, and to realize the exact point at which the proof been based on ther axiom of choice, for it has frequently happened that various authors have made use of the axiom of choice in their proofs without being aware of it. And after all, even if no one questioned the axiom of choice, it would bnot be without interest to investigate which proofs are based on it and which theorems are proved without its aid – this as we know , id also done with regard to other axioms.

This attitude produced the entire program of investigations concerning the status of the axiom of choice and the continuum hypothesis in the entire body of mathematics. The deep results obtained by Polish mathematicians working on this issue (Sierpiński, Kuratowski, Tarski, Lindenbaum, Mostowski) belong to the most remarkable achievements of the Polish Mathematical School. They very essentially contributed to the development of the set-theoretical project of the foundations of mathematics.

Although Polish logicians did not belong to the constructivism school in the foundations of mathematics, they worked on topics suggested by intuitionism. Let me mention resuts of Łukasiewicz (axiomatization of intuitionistic propositional calculus, investigations of relations between classical and intutionistic logic), Tarski (intuitionistic logic and topology), Wajsberg (the separation theorem for intutionistic propositional calculus) and Jaśkowski (the construction of adequate matrix for intuitionistic propositional calculus). Mostowski remarked [9, p. VII]:

> "I am inclined to think that a satisfactory solution of the problem of the foundations of mathematics will follow the path pointed out by constructivism or in a direction close to it. However, it would be impossible to write a textbook of logic on this base at the moment".

He preceded these words by remarking that a deeper discussion of the philosophical foundations of logic does not belong to the scope of formal logic.

Generally speaking, the principles of extensionality and compositionally were axiomatically adopted as regulative ideas governing logic. In propositional calculus, according to this principle, every operation must be extensional, that acting as a truth-function of its constituents. Consequently, many-valued logic has to be equally extensional as classical (bivalent) logic. This view motivated Łukasiewicz to interpreted modalities within many-valued logic. The modal extensions of classical logic, like Lewis-style systems, were considered as somehow defective as not fully extensional. The principle of extensionality was extended to interpretations of quantifiers and semantic constructions. In particular, Tarski's famous semantic definition of truth was extensional in this sense, that he defined the set of true sentences, not the intensional concept of truth. Of course, one can, and Tarski did that, consider the definition in question as explaining the meaning of the term "true', but this move requires the assumption that extension and intension strictly coincide in this case.

Polish logicians generally accepted realism as a view concerning the relation of logic and reality. Roughly speaking, logic was conceived as a description of reality. This view was, as I already noticed, advanced by Leśniewski, was also shared by Łukasiewicz in the interwar period (he took a more conventionalist position after 1945). However, both leaders of the Warsaw School of Logic had considerably different answers to the question which logic is correct. Leśniewski decisively preferred classical logic and considered many-valued logics as formal algebraic constructions. On the contrary, Łukasiewicz maintained that one of the variety of rival systems is satisfied in the real world (he hoped that infinitely many-valued logic is correct). Tarski also preferred classical logic. He did not justify this view by ontological arguments, but appealed to simplicity, universality and productivity of rthe classical system. A consequence of realism (Tarski presumably shared this philosophy of logic) consisted in the rejection of the distinction between logical and extra-logical concepts as well as between logical and empirical, at least if both discriminations are taken as absolute. Tarski tried to explain the essence of logical notions by regarding them as invariant under all one-to-one transformations of the universe into itself. It was a generalization of the Erlangen program in the foundations geometry, proposed by Felix Klein.

Although Polish logicians had no official philosophy of logic, several important investigations were strongly motivated by philosophical considerations. Perhaps the most important examples of how philosophy influenced logic are provided by many-valued logic and the semantic definition of truth. Łukasiewicz's main (initial, because he changed his mind later) motivation in proposing many-valued system of logic explicitly focused on attempts to reject determinism for its inconsistency with postulates of freedom, creativity and responsibility. Tarski in his semantic truth-definition explicitly followed Aristotle's tradition seeing truth as saying as things are. On the other hand, one should definitely observe that motivation does not mean justification. In particular, Łukasiwicz regarded the principle of bivalence as metalogical, not purely logical and Tarski opted for the classical truth-definition for its intuitive plausibility and being subjected to a rigorous mathematical treatment. Consequently, we must try to justify bivalence or its denial by separate investigations, ontological or scientific, because pure logic is insufficient in this respect.

Finally, let me say something about the social significance of logic according to the Polish school. This goes back to Twardowski. He wrote [19, p. 71]:

> "The lack of logical education not only decreases the intellectual level from the theoretical point of view, but also brings ignorance and obscurity in practical applications of our thoughts. And the whole our life is this practical application".

Similar thoughts are to be found in Łukasiewicz [5, p. 615; 6, p. 5]:

> "I believe that only just mathematical logic will teach us strict thinking. In my opinion, it is its greatest significance and task, even its social mission.
>
> My entire scientific activity was guided by the remote thought that we will come at one time to more correct views about the world and life by improving logical thinking".

And Tarski said [15, p. XV]:

> "I shall be happy if this book contributes to the wider diffusion of logical knowledge. The course of historical events has assembled in this country [USA – J.W.] the most eminent representatives of contemporary logic, and has thus created here especially favourable conditions for the development of logical thought. These favourable conditions can, of course, be easily overbalanced by other and more powerful factors. It is obvious that the future of logic, as well as of all theoretical science, depends essentially upon normalizing the political and social relations of mankind, and thus upon a factor which is beyond the control of professional scholars. I have no illusions that the development of logical thought, in particular, will have an essential effect upon the process of normalization of human relationships; but I believe that the wider diffusion of the knowledge of logic may contribute positively to the acceleration of this process. For, one the one hand, by making the meaning of concepts precise and uniform in its own field and by stressing the necessity of such a precision and uniformiazation in any other domain, logic leads to the possibility of better understanding between those who have the will to do so. And, on the other hand, by perfecting and sharpening the tools of thought, it makes men more critical – and thus makes less likely their being misled by all the pseudo-reasonings to which they are in various parts of the world incessantly exposed today".

Perhaps we can sum up the last quotations by repeating Łukasiewicz's and Tarski's sayings (unpublished, but preserved in the oral Polish tradition): "Logic is morality of speech and thought"; "Religion (you can say "ideology") divides people, logic brings them together".

Bibliographical Appendix

For general accounts of Polish Mathematical School see [3, 4]. The Lvov branch of Polish Mathematical School is extensively presented in [2]. For account of the Lvov–Warsaw School, see [2, 14, 20]. The book [12] contains a detailed exposition of the philosophy of logic and mathematics in the interwar period as well as provides an extensive bibliography of sources.

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