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FIXED POINTS OF α-NONEXPANSIVE MAPPINGS

PUNKTY STAŁE ODWZOROWAŃ α-NIEODDALAJĄCYCH

A b s t r a c t

This paper is connected with the theory of α-nonexpansive mappings, which were introduced by K. Goebel and M. A. J. Pineda in 2007. These mappings are a natural generalisation of nonexpansive mappings from the point of view of the fixed point theory. In particular, they proved that in Banach spaces all $\alpha = (\alpha_1, \ldots, \alpha_n)$ -nonexpansive mappings with α_1 big enough, namely $\alpha_1 \geq 2^{\frac{1}{1-n}}$, have minimal displacement equal to zero. This paper introduces some new results connected with this problem.

Keywords: *α-nonexpansive mappings*, *minimal displacement*, *fixed point*

S t r e s z c z e n i e

Niniejszy artykuł jest związany z odwzorowaniami α-nieoddalającymi, które zostały wprowadzone przez K. Goebla i M. A. J. Pinedę w 2007 r. Odwzorowania te są naturalnym uogólnieniem odwzorowań nieoddalających z punktu widzenia teorii punktu stałego. Wyżej wspomniani autorzy wykazali, że w przestrzeniach Banacha odwzorowania $\alpha = (\alpha_1, \ldots, \alpha_n)$ -nieoddalające, mające odpowiednio duże α_1, a dokładniej $\alpha_1 \geq 2^{\frac{1}{1-n}}$, posiadają minimalne przesunięcie równe zeru. W artykule przedstawiono pewne nowe wyniki z związane z tym problemem.

Słowa kluczowe: *odwzorowania α-nieoddalające*, *minimalne przesunięcie*, *punkt stały*

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1. Introduction and preliminaries

Let (X, d) be a metric space, and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index satisfying $\alpha_1 > 0,\ \alpha_n > 0,\ \alpha_i \geq 0,\ i = 2, \ldots, n-1$ and $\sum_{i=1}^{n} 1$. In [2], the following notions were introduced:

The mapping $T : X \to X$ *is said to be* α*-Lipschitzian with constant* $k \geq 0$, *if*

$$\sum_{i=1}^{n} \alpha_i d(T^i x, T^i y) \leq k d(x, y) \text{ for all } x, y \in X.$$

The mapping $T : X \to X$ *is said to be* α*-nonexpansive* (α*-contraction), if T is* α*-Lipschitzian with constant* $k = 1$ ($k < 1$ *resp.).*

Denote the Lipschitz constant with $k(T)$ *and the* α*-Lipschitz constant of T with* $k(\alpha, T)$.

Define also $d(T) := \inf\{d(x, Tx),\ x \in X\}$, *which we will call the minimal displacement of T. Sometimes it is also called the approximate fixed point of T.*

These notions are natural generalisations of Lipschitzian mappings, nonexpansive mappings and contractions from the point of view of the fixed point theory. For more information concerning α-nonexpansive mappings and other Lipschitzian mappings connected with the fixed point theory, we refer to [4].

In [2], the authors proved the following:

Theorem 1.1. (see also [4], chapter 3) *Let X be a Banach space, let C be a nonempty, closed, convex and bounded subset of X. Let* $T : C \to C$ *be an* $\alpha = (\alpha_1, \ldots, \alpha_n)$-*-nonexpansive mapping where* $\alpha_1 \geq 2^{\frac{1}{1-n}}$. *Then* $d(T) = 0$.

Notice that the problem of determining the set of multi-indices α for which each α-nonexpansive mapping T has $d(T) = 0$ is still open.

The aim of this paper is to prove two results (Theorem 2.1, Theorem 2.2) which give a partial answer to the above open problem (see [4]).

Before proceeding further, let us recall the generalised Banach contraction principle (abbreviated to GBCP), which is formulated as follows:

Theorem 1.2. ([1], [5]) *In complete metric space X if for some* $N \geq 1$ *and* $0 < \mathrm{M} < 1$ *the mapping* $T : X \to X$ *satisfies* $\min\{d(T^j x, T^j y),\ 1 \leq j \leq N\} \leq M d(x, y)$ *for any* $x, y \in X$, *then T has the unique fixed point.*

In author's PhD thesis [6] the more general version of the above theorem was presented. Let us recall it without proof.

Theorem 1.3. *Let* (X, d) *be a complete metric space,* $N \geq 1$. *Assume that* $\phi : [0, \infty) \to [0, 1]$ *is a continuous, non-increasing function satisfying* $\phi(t) = 1$ *if, and only if,* $t = 0$. *Let* $T : X \to X$ *be such that* $\min\{d(T^j x, T^j y),\ 1 \leq j \leq N\} \leq \phi(d(x, y)) \cdot d(x, y)$ *for all* $x, y \in X$. *Then T has the unique fixed point.*

2. Main results

Firstly, let us note a simple fact, there exist some α-Lipschitzian mappings which are not α-nonexpansive; however, their minimal displacement is equal to zero; moreover, they may have the unique fixed point.

This is illustrated by:

Example 2.1. Let $T : l_\infty \cap \{x \in l_\infty : x_i \geq 0,\ i \in \mathbb{N}\} \to l_\infty \cap \{x \in l_\infty : x_i \geq 0,\ i \in \mathbb{N}\}$ be defined in the following way: $T : x = (x_1, x_2, \ldots) \to Tx := \left(1, \frac{2x_3}{1+x_3}, \frac{\frac{1}{2}x_2}{1+x_2}, \frac{2x_5}{1+x_5}, \frac{\frac{1}{2}x_4}{1+x_4}, \ldots\right)$. Then T is not α-nonexpansive for any α; however, for properly chosen $\alpha = (\alpha_1, \alpha_2)$ the mapping T is α-Lipschitzian with constant k arbitrarily close to 1. Moreover, T has the unique fixed point.

Obviously, the mapping T has the unique fixed point (1, 0, 0, ...).

Also, we have $\|Tx - Ty\| \leq 2\|x - y\|$ and $\left\|T^i x - T^i y\right\| \leq \|x - y\|,\ i \geq 2$ for any $x, y \in l_\infty \cap \{x \in l_\infty : x_i \geq 0,\ i \in \mathbb{N}\}$. On the other hand, taking $x^n = \left(0, 0, \frac{1}{n}, 0, 0, \ldots\right)$ and $y^n = (0, 0, \ldots)$ we have $\frac{\left\|Tx^n - Ty^n\right\|}{\left\|x^n - y^n\right\|} = \frac{2 \cdot \frac{1}{n}}{1 + \frac{1}{n}} n = \frac{2}{n+1} \to 2,\ n \to \infty$; therefore, $k(T) = 2$. Similarly, $k(T^2) = 1$ and $k(T^i) \geq 1,\ i \geq 3$.

In l_∞, it is not possible to choose α such that $\alpha_1 > 0$ and T is α-nonexpansive; however, $\frac{1}{n}\|Tx - Ty\| + \frac{n-1}{n}\left\|T^2 x - T^2 y\right\| \leq \frac{n+1}{n}\|x - y\|$; therefore, assuming n to be big enough, the mapping T is $\alpha = \left(\frac{1}{n}, \frac{n-1}{n}\right)$-Lipschitzian with constant $k(\alpha, T)$ arbitrarily close to 1.

It is worth mentioning that the existence and uniqueness of the fixed point of T also follows from Theorem 1.3. Indeed, we have $T^2 x = \left(1, \frac{x_2}{1 + \frac{3}{2}x_2}, \frac{x_3}{1 + 3x_3},\right.$

$$\frac{x_4}{1+\frac{3}{2}x_4}, \frac{x_5}{1+3x_5}, \ldots\Bigg) \quad \text{and} \quad \left|\frac{x_i}{1+\frac{3}{2}x_i} - \frac{y_i}{1+\frac{3}{2}y_i}\right| \le \left|\frac{x_i - y_i}{1+\frac{3}{2}(x_i+y_i)+\frac{9}{4}x_i y_i}\right| \le \frac{|x_i - y_i|}{1+|x_i+y_i|} \le$$

$\le \dfrac{|x_i - y_i|}{1+|x_i+y_i|} \le \dfrac{\|x-y\|}{1+\|x-y\|}$, $i \in \mathbb{N}$. The latter inequality follows from the fact that $t \to \dfrac{t}{1+t}$ is an increasing function on $[0,\infty)$. Similarly, $\left|\dfrac{x_i}{1+3x_i} - \dfrac{y_i}{1+3y_i}\right| \le \dfrac{\|x-y\|}{1+\|x-y\|}$; therefore, $\|T^2x - T^2y\| \le \dfrac{1}{1+\|x-y\|}\|x-y\|$, so T satisfies the assumptions of Theorem 1.3 with $\phi(t) := \dfrac{1}{1+t}$.

Now, let us exchange the condition $\alpha_1 \ge 2^{\frac{1}{1-n}}$ with the other regularity condition of a mapping T.

Theorem 2.1. *Let X be a Banach space, let $0 \in C \subset X$ be nonempty, closed, convex and bounded. Let $T: C \to C$ be a $\alpha = (\alpha_1, \ldots, \alpha_n)$-nonexpansive mapping such that $\|T^i(\mu x) - T^i(\mu y)\| \le \|T^i(\lambda x) - T^i(\lambda y)\|$ for any $x, y \in C$, $0 \le \mu \le \lambda$, $i \in \{1, \ldots, n\}$.*

Then $d(T) = 0$.

Proof. Fix $k \ge 1$. Define $S_k := \left(1 - \frac{1}{k}\right)T$. Obviously, $S_k x = \frac{1}{k}\cdot 0 + \left(1-\frac{1}{k}\right)Tx \in C$. Then

$$\|S_k x - S_k y\| = \left(1-\frac{1}{k}\right)\|Tx - Ty\|.$$

Next, we have:

$$S_k^2 x = S_k\left(\left(1-\frac{1}{k}\right)Tx\right) = \left(1-\frac{1}{k}\right)T\left(\left(1-\frac{1}{k}\right)Tx\right);$$

therefore, by assumptions:

$$\begin{aligned}\|S_k^2 x - S_k^2 y\| &= \left\|\left(1-\frac{1}{k}\right)T\left(\left(1-\frac{1}{k}\right)Tx\right) - \left(1-\frac{1}{k}\right)T\left(\left(1-\frac{1}{k}\right)Ty\right)\right\| \\ &= \left(1-\frac{1}{k}\right)\left\|T\left(\left(1-\frac{1}{k}\right)Tx\right) - T\left(\left(1-\frac{1}{k}\right)Ty\right)\right\| \\ &\le \left(1-\frac{1}{k}\right)\|T^2x - T^2y\|.\end{aligned}$$

Similarly, for $i \geq 3$ we have:

$$S_k^i x = T_k^{i-1}\left(\left(1-\frac{1}{k}\right)Tx\right) = \left(1-\frac{1}{k}\right)T\left(\left(1-\frac{1}{k}\right)T\left(\ldots\left(\left(1-\frac{1}{k}\right)Tx\right)\ldots\right)\right);$$

therefore:

$$\left\|S_k^i x - S_k^i y\right\| = \left\|\left(1-\frac{1}{k}\right)T\left(\ldots\left(\left(1-\frac{1}{k}\right)Tx\right)\ldots\right) - \left(1-\frac{1}{k}\right)T\left(\ldots\left(\left(1-\frac{1}{k}\right)Ty\right)\ldots\right)\right\|$$

$$= \left(1-\frac{1}{k}\right)\left\|T\left(\left(1-\frac{1}{k}\right)T\left(\ldots\left(\left(1-\frac{1}{k}\right)Tx\right)\ldots\right)\right) - T\left(\left(1-\frac{1}{k}\right)T\left(\ldots\left(\left(1-\frac{1}{k}\right)Ty\right)\ldots\right)\right)\right\|$$

$$\leq \left(1-\frac{1}{k}\right)\left\|T\left(T\left(\ldots\left(\left(1-\frac{1}{k}\right)Tx\right)\ldots\right)\right) - T\left(T\left(\ldots\left(\left(1-\frac{1}{k}\right)Ty\right)\ldots\right)\right)\right\|$$

$$\leq \left(1-\frac{1}{k}\right)\left\|T^i x - T^i y\right\|.$$

By assumptions, $\sum_{i-1}^{n} \alpha_i \left\|T^i x - T^i x\right\| \leq \|x-y\|$ for some $\alpha = (\alpha_1, \ldots, \alpha_n)$ satisfying $0 \leq \alpha_i \leq 1$ and $\sum_{i=1}^{n} \alpha_i = 1$. Therefore:

$$\min\left\{\left\|S_k^j x - S_k^j y\right\|,\ 1 \leq j \leq n\right\} \leq \sum_{i=1}^{n} \alpha_i \left\|S_k^i x - S_k^i y\right\| \leq \left(1-\frac{1}{k}\right)\|x-y\|,$$

By Theorem 1.2, S_k has the unique fixed point. Denote this fixed point by x_k. We get:

$$\|x_k - Tx_k\| = \|S_k x_k - Tx_k\| = \left\|\left(1-\frac{1}{k}\right)Tx_k - Tx_k\right\| = \frac{1}{k}\|Tx_k\| \to 0,\ k \to \infty,$$

this completes the proof.

□

For $k \geq 3$, there exists a mapping T which does not satisfy the assumptions of Theorem 1.1; however, for $k \geq 2$, it satisfies the assumptions of Theorem 2.1. This will be illustrated by the following example:

Example 2.2. Fix $k \geq 2$. Let $\tau : [-1,1] \to [-1,1]$ be a non-decreasing function, having the Lipschitz constant $k(\tau) = k$, concave on [–1,0], convex on [0,1] and such that $\tau(0) = 0$. Now define $T : B_{l_1} \ni x = (x_1, x_2, \ldots) \to Tx := (\tau(x_2), \frac{k}{k^2-1}x_3, x_4, x_5, \ldots) \in B_{l_1}$.

We will show that the assumptions of Theorem 1.1 are not satisfied for any multi- -index α of length n. Notice, that:

$$\|Tx-Ty\| = |\tau(x_2)-\tau(y_2)| + \frac{k}{k^2-1}|x_3-y_3| + \sum_{i=4}^{\infty}|x_i-y_i|$$
$$\le k|x_2-y_2| + \frac{k}{k^2-1}|x_3-y_3| + \sum_{i=4}^{\infty}|x_i-y_i|,$$
$$\|T^2x-T^2y\| = \left|\tau\left(\frac{k}{k^2-1}x_3\right)-\tau\left(\frac{k}{k^2-1}y_3\right)\right| + \frac{k}{k^2-1}|x_4-y_4| + \sum_{i=5}^{\infty}|x_i-y_i|$$
$$\le \frac{k}{k^2-1}|x_3-y_3| + \frac{k}{k^2-1}|x_4-y_4| + \sum_{i=5}^{\infty}|x_i-y_i|.$$

Therefore, $k(T)=k$ and $k(T^i)=\dfrac{k^2}{k^2-1}>1,\ i\ge 2.$

It is easy to see that for $k\ge 3$, the assumptions of Theorem 1.1 are not satisfied for any multi-index α of length n. Indeed, we would need to have $\alpha_1 \ge 2^{\frac{1}{1-n}} \ge 2^{\frac{1}{1-3}} = \dfrac{\sqrt{2}}{2} > \dfrac{1}{2}$ and thus for any such $\alpha=(\alpha_1,\ldots,\alpha_n)$, the mapping T would not be α-nonexpansive.

We will now show that T satisfies the assumptions of Theorem 2.1; therefore, $d(T)=0$.

It is enough to take n = 2. It is easy to check that $\dfrac{1}{k}\|Tx-Ty\| + \dfrac{k-1}{k}\|T^2x-T^2y\| \le$ $\le \|x-y\|$; therefore, T is $\left(\dfrac{1}{k},\dfrac{k-1}{k}\right)$-nonexpansive. We only have to show that

$$\|T(\mu x)-T(\mu y)\| \le \|T(\lambda x)-T(\lambda y)\| \text{ for any } x,y\in B_{l_1},\ 0\le\mu\le\lambda.$$

If $|\tau(\mu x_2)-\tau(\mu y_2)| \le |\tau(\lambda x_2)-\tau(\lambda y_2)|$, then obviously,

$$\|T(\mu x)-T(\mu y)\| = |\tau(\mu x_2)-\tau(\mu y_2)| + \left|\frac{k}{k^2-1}\mu x_3 - \frac{k}{k^2-1}\mu y_3\right| + \sum_{i=4}^{\infty}|\mu x_i-\mu y_i|$$
$$\le |\tau(\lambda x_2)-\tau(\lambda y_2)| + \left|\frac{k}{k^2-1}\lambda x_3 - \frac{k}{k^2-1}\lambda y_3\right| + \sum_{i=4}^{\infty}|\lambda x_i-\lambda y_i|$$
$$= \|T(\lambda x)-T(\lambda y)\|$$

It is enough to prove that $|\tau(\mu v)-\tau(\mu w)| \le |\tau(\lambda v)-\tau(\lambda w)|$ for $v\ne w$ and $0<\mu<\lambda$. Firstly, assume that $v,w>0$. Without the loss of generality, we can assume that $w<v$. Therefore, $0<\mu w<\mu v,\ \lambda w<\lambda v$.

Assume that $0<\mu w<\mu v\le\lambda w<\lambda v$. Choose $a\in(\lambda w,\lambda v]$ such that $a-\lambda w=\mu v-\mu w$. Of course, such an a exists since $\lambda(v-w)\ge u(v-w)$. Therefore, $\mu v=\frac{a-\mu v}{a-\mu w}\mu w+$ $+\frac{\mu v-\mu w}{a-\mu w}a$ and $\lambda w=\frac{a-\lambda w}{a-\mu w}\mu w+\frac{\lambda w-\mu w}{a-\mu w}a$. Due to the convexity of τ on [0,1]

$$\begin{aligned}\tau(\mu v)&=\tau\left(\frac{a-\mu v}{a-\mu w}\mu w+\frac{\mu v-\mu w}{a-\mu w}a\right)\\&\le\frac{a-\mu v}{a-\mu w}\tau(\mu y)+\frac{\mu v-\mu w}{a-\mu w}\tau(a)\end{aligned}$$

$$\begin{aligned}\tau(\lambda w)&=\tau\left(\frac{a-\lambda w}{a-\mu w}\mu y+\frac{\lambda w-\mu w}{a-\mu w}a\right)\\&\le\frac{a-\lambda w}{a-\mu w}\tau(\mu y)+\frac{\lambda w-\mu w}{a-\mu w}\tau(a)\\&=\frac{\mu v-\mu w}{a-\mu w}\tau(\mu w)+\frac{a-\mu v}{a-\mu w}\tau(a)\end{aligned}$$

Adding the above estimates side-by-side and taking into consideration the fact that τ is non-decreasing, we get:

$$\begin{aligned}\tau(\mu v)+\tau(\lambda w)&\le\left(\frac{a-\mu v}{a-\mu w}+\frac{\mu v-\mu w}{a-\mu w}\right)\tau(\mu w)+\left(\frac{\mu v-\mu w}{a-\mu w}+\frac{a-\mu v}{a-\mu w}\right)\tau(a)\\&=\tau(\mu w)+\tau(a)\le\tau(\mu w)+\tau(\lambda v),\end{aligned}$$

this implies that $|\tau(\mu v)-\tau(\mu w)|\le|\tau(\lambda v)-\tau(\lambda w)|$.

On the other hand, if $0<\mu w<\lambda w\le\mu v<\lambda v$, then let us choose $a\in(\mu v,\lambda v]$ such that $a-\mu v=\lambda w-\mu w$. Of course, such an a exists since $(\lambda-\mu)v\ge(\lambda-\mu)w$. Then $\lambda w=\frac{a-\lambda w}{a-\mu w}\mu w+\frac{\lambda w-\mu w}{a-\mu w}a$ and $\mu v=\frac{a-\mu v}{a-\mu w}\mu w+\frac{\mu v-\mu w}{a-\mu w}a$. Due to the convexity of τ on [0,1], we have:

$$\begin{aligned}\tau(\lambda w)&=\tau\left(\frac{a-\lambda w}{a-\mu w}\mu w+\frac{\lambda w-\mu w}{a-\mu w}a\right)\\&\le\frac{a-\lambda w}{a-\mu w}\tau(\mu w)+\frac{\lambda w-\mu w}{a-\mu w}\tau(a)\end{aligned}$$

$$\begin{aligned}\tau(\mu v) &= \tau\left(\frac{a-\mu v}{a-\mu w}\mu w + \frac{\mu v-\mu w}{a-\mu w}a\right)\\ &\le \frac{a-\mu v}{a-\mu w}\tau(\mu w) + \frac{\mu v-\mu w}{a-\mu w}\tau(a)\\ &= \frac{\lambda w-\mu w}{a-\mu w}\tau(\mu y) + \frac{a-\lambda w}{a-\mu w}\tau(a)\end{aligned}$$

Again, adding the above estimations side-by-side, we get:

$$\begin{aligned}\tau(\lambda w)+\tau(\mu v) &\le \left(\frac{a-\lambda w}{a-\mu w}+\frac{\lambda w-\mu w}{a-\mu w}\right)\tau(\mu w) + \left(\frac{\lambda w-\mu w}{a-\mu w}+\frac{a-\lambda w}{a-\mu w}\right)\tau(a)\\ &= \tau(\mu w)+\tau(a) \le \tau(\mu w)+\tau(\lambda v),\end{aligned}$$

this leads to $|\tau(\mu v)-\tau(\mu w)| \le |\tau(\lambda v)-\tau(\lambda w)|$.

Similarly, it is easy to check that the estimation $|\tau(\mu v)-\tau(\mu w)| \le |\tau(\lambda v)-\tau(\lambda w)|$ remains true for $v, w < 0$ and for other cases. This shows that T satisfies the assumptions of Theorem 2.1.

A set satisfying $\lambda x_0 + (1-\lambda)y \in C$ for all $y \in C$, $\lambda \in [0,1]$ we call *star-like set C with respect* to x_0.

Theorem 2.2. *Let X be a Banach space,* $x_0 \in X$, $N \in \mathbb{N}$, *let* $C \subset X$ *be a bounded, star-like set with respect to* x_0. *Let* $T : C \to C$ *be such that*

1. $\min\left\{\left\|T^j x - T^j y\right\|,\ 1 \le j \le N\right\} \le \|x-y\|$ for all $x, y \in C$,

2. there exists $0 \le b_0 \le 1$ *such that for all* $0 \le b \le b_0$, $1 \le j \le N-1$, $x, y \in C$

$$\left\|T(T_b^j x) - T(T_b^j y)\right\| \le (1+b)\left\|T^{j+1}x - T^{j+1}y\right\|,$$

where $T_b x = (1-b)Tx + bx_0$. *Then* $d(T) = 0$.

Proof. Fix arbitrary $x, y \in C$ and take $j \in \{1,\ldots,N\}$ such that $\left\|T^j x - T^j y\right\| \le \|x-y\|$. Let us note that:

$$\begin{aligned}\left\|T_b^j x - T_b^j y\right\| &= \left\|T_b(T_b^{j-1}x) - T_b(T_b^{j-1}y)\right\|\\ &= \left\|(1-b)T(T_b^{j-1}x) + bx_0 - (1-b)T(T_b^{j-1}y) - bx_0\right\|\\ &= (1-b)\left\|T(T_b^{j-1}x) - T(T_b^{j-1}y)\right\|\\ &\le (1-b)(1+b)\left\|T^j x - T^j y\right\|\\ &\le (1-b^2)\|x-y\|\end{aligned}$$

Therefore, for any $x, y \in C$ there exists $j \in \{1, \ldots, N\}$ such that $\|T_b^j x - T_b^j y\| \leq$ $\leq (1-b^2)\|x-y\|$. Theorem 1.2 ensures, that T_b has the unique fixed point.

Now, fix an arbitrary $\varepsilon > 0$ and choose $0 \leq b \leq b_0$ such that $\|T_b z - Tz\| =$ $= \|(1-b)Tz + bx_0 - Tz\| = b\|x_0 - Tz\| \leq \varepsilon$ for any $z \in C$.

Let $z_b \in C$ be such that $T_b z_b = z_b$.

Therefore, $\|z_b - Tz_b\| \leq \|z_b - T_b z_b\| + \|T_b z_b - Tz_b\| \leq 0 + \varepsilon = \varepsilon$, this proves that $d(T) = 0$.

□

Let us illustrate the possible application of Theorem 2.2.

Example 2.3. Let T be the same as in Example 2.2. Then T satisfies Theorem 2.2 (we have already shown that T does not satisfy Theorem 1.1 for $k \geq 3$).

Indeed, let us calculate

$$T(T_b x) = \left(\tau\left((1-b)\frac{k}{k^2-1} x_3 \right), \frac{k}{k^2-1}(1-b)x_4, (1-b)x_5, (1-b)x_6, \ldots \right)$$

and

$$T^2 x = \left(\tau\left(\frac{k}{k^2-1} x_3 \right), \frac{k}{k^2-1} x_4, x_5, x_6, \ldots \right)$$

We have:

$$\begin{aligned}
\|T(T_b x) - T(T_b y)\| &= \left| \tau\left((1-b)\frac{k}{k^2-1} x_3 \right) - \tau\left((1-b)\frac{k}{k^2-1} y_3 \right) \right| \\
&+ \left| \frac{k}{k^2-1}(1-b)x_4 - \frac{k}{k^2-1}(1-b)y_4 \right| + |(1-b)x_5 - (1-b)y_5| + \ldots \\
&\leq \left| \tau\left(\frac{k}{k^2-1} x_3 \right) - \tau\left(\frac{k}{k^2-1} \right) y_3 \right| + \left| \frac{k}{k^2-1} x_4 - \frac{k}{k^2-1} y_4 \right| + |x_5 - y_5| + \ldots \\
&= \left\| T^2 x - T^2 y \right\| \leq (1+b)\left\| T^2 x - T^2 y \right\|.
\end{aligned}$$

We have already taken into account the fact, that $|\tau(\mu s) - \tau(\mu t)| \leq |\tau(\lambda s) - \tau(\lambda t)|$ for any $0 \leq \mu \leq \lambda$, $s, t \in [-1, 1]$. We proved this fact in Example 2.2.

The estimate $\min\left\{ \|Tx - Ty\|, \left\| T^2 x - T^2 y \right\| \right\} \leq \frac{1}{k}\|Tx - Ty\| + \frac{k-1}{k}\left\| T^2 x - T^2 y \right\| \leq \|x - y\|$ shows that T satisfies Theorem 2.2.

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