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# A NOTE ON SINGULARITY OF FIBERS OF SINGULAR SETS

# UWAGA O SINGULARNOŚCI WŁÓKIEN ZBIORÓW SINGULARNYCH

Abstract

The paper presents a general theorem on fibers of singular sets: Let $D_1$ be a connected σ-compact Josefson manifold and let $D_2$ be a σ-compact complex manifold. Let $\Omega \subset D_1 \times D_2$ be a domain and let $\Omega \subset M$ be a singular set with respect to the family $\mathcal{F} \subset \mathcal{O}(\Omega \setminus M)$ such that the set $\{a_1 \in D_1 : \textit{the fiber } M_{(a_1,\cdot)} \textit{ is not pluripolar}\}$ is pluripolar in $D_1$. It is shown that there exists a pluripolar set $Q \subset D_1$ such that for every $a_1 \in \pi_{D_1}(\Omega) \setminus Q$, the fiber $M_{(a_1,\cdot)}$ is singular in $\Omega_{(a_1,\cdot)}$ with respect to the family $\mathcal{F}_a := \{f(a_1,\cdot) : f \in \mathcal{F}\} \subset \mathcal{O}(\Omega_{(a_1,\cdot)})$.

*Keywords*: *singular set*, *fiber of singular set*, *pluripolar set*, *complex manifold*

Streszczenie

W artykule przedstawiono dowód ogólnego twierdzenia dotyczącego własności włókien zbiorów singularnych: Niech $D_1$ będzie spójną, σ-zwartą rozmaitością Josefsona oraz niech $D_2$ będzie σ-zwartą rozmaitością zespoloną. Niech $\Omega \subset D_1 \times D_2$ będzie obszarem oraz niech $\Omega \subset M$ będzie zbiorem singularnym względem rodziny $\mathcal{F} \subset \mathcal{O}(\Omega \setminus M)$, takim, że zbiór $\{a_1 \in D_1 : \textit{włókno } M_{(a_1,\cdot)} \textit{ nie jest pluripolarne}\}$ jest pluripolarny w $D_1$. Wykazano, że istnieje wtedy zbiór pluripolarny $Q \subset D_1$ taki, że dla dowolnego $a_1 \in \pi_{D_1}(\Omega) \setminus Q$ włókno $M_{(a_1,\cdot)}$ jest singularne w $\Omega_{(a_1,\cdot)}$ względem rodziny $\mathcal{F}_a := \{f(a_1,\cdot) : f \in \mathcal{F}\} \subset \mathcal{O}(\Omega_{(a_1,\cdot)})$.

*Słowa kluczowe*: *zbiór singularny*, *włókno zbioru singularnego*, *zbiór pluripolarny*, *rozmaitość zespolona*

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## 1. Introduction and prerequisites

In [2], the authors proved a result which states that almost all sections of Riemann domains of holomorphy are regions of holomorphy.

Let $(X, p)$ be a Riemann domain over $\mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^l$, where $p = (u, v) : X \to \mathbb{C}^k \times \mathbb{C}^l$, and define $D_k := u(X)$. Let $\varnothing \neq \mathcal{F} \subset \mathcal{O}(X)$. For $a \in D_k$, define a fiber $X_a := u^{-1}(a)$, a function $p_a := v|_{X_a}$, and a family $\mathcal{F}_a := \{f|_{X_a} : f \in \mathcal{F}\}$.

**Theorem 1.1.** (Theorem 2.2 from [2]) Let $\varnothing \neq \mathcal{F} \subset \mathcal{O}(X)$ and assume that $(X, p)$ is an $\mathcal{F}$-domain of holomorphy. Then there exists a pluripolar set $S_k \subset D_k$ such that for every $a \in D_k \setminus S_k$, $(X_a, p_a)$ is an $\mathcal{F}_a$-region of holomorphy.

**Remark 1.2.** Theorem 1.1 remains true if we assume that $(X, p)$ is a countable at infinity $\mathcal{F}$-region of holomorphy (see Theorem 9.1.2 in [3]).

Now, recall the definition of a singular set. For an $n$-dimensional complex manifold $X$, let $M$ be a closed subset of $X$ such that for any domain $\Omega \subset X$, the set $\Omega \setminus M$ is connected and dense in $\Omega$ (for instance, let $M$ be a pluripolar set). Let $\mathcal{F}$ be a family of functions holomorphic on $X \setminus M$.

**Definition 1.3.** A point $a \in M$ is called *singular with respect to the family $\mathcal{F}$,* if for any open connected neighborhood $U_a$ of the point $a$, there exists a function $f \in \mathcal{F}$, that does not extend holomorphically on $U_a$. We call $M$ *singular with respect to the family $\mathcal{F}$*, if every point $a \in M$ is singular with respect to $\mathcal{F}$.

A consequence of Theorem 1.1 is a similar property of fibers of singular sets in the Riemann regions of holomorphy from [3].

**Theorem 1.4.** (Proposition 9.1.4 from [3], see also Lemma 3.3 from [2]) Let $(D, p_D)$ and $(G, p_G)$ be Riemann domains over $\mathbb{C}^k$ and $\mathbb{C}^l$, respectively. Let $\Omega \subset D \times G$ be a Riemann region of holomorphy and let $M \subset \Omega$ be a relatively closed pluripolar set that is singular with respect to a family $S \subset \mathcal{O}(\Omega \setminus M)$ There then exists a pluripolar set $P \subset D$ such that for any $a \in \pi_D(\Omega) \setminus P$ the fiber $M_{(a,\cdot)} := \{b \in G : (a, b) \in M\}$ is singular with respect to the family $S_a := \{f(a, \cdot) : f \in S\} \subset \mathcal{O}(\Omega_{(a,\cdot)} \setminus M_{(a,\cdot)})$, where $\pi_D(\Omega)$ denotes the projection of $\Omega$ to $D$.

Following the proof of Theorem 1.4, it becomes clear that we can replace the assumption of $M$ being relatively closed and pluripolar by a weaker assumption: we need only that the set $\{a \in D : \textit{the fiber } M_{(a,\cdot)} \textit{ is not pluripolar}\}$ is pluripolar.

Theorem 1.4 proved to be one of the key properties used in the theory of extensions of functions separately holomorphic on different kinds of objects called crosses. This topic has a long history in complex analysis (for the details of its evolution, see the introduction to [3]) and was started by W.F. Osgood and F. Hartogs with the famous theorem stating that every separately holomorphic function is, in fact, holomorphic ([1, 5]). One of the latest and most general and technically demanding results in the case of crosses with singularities on Riemann domains (see Theorem 3.2 in [2], Theorem 10.2.9 in [3], Main Theorem in [6]) are obtained using, among other strong tools, Theorem 1.4.

Recently, the context of cross theory has moved from Riemann domains to more general objects, such as complex manifolds or even analytic spaces (see [4]). However, the case of extensions on crosses with singularities on complex manifolds still remains open, partially because of a lack of necessary base results which were available for Riemann domains.

In this paper, we show proof of generalisation of Jarnicki and Pflug result which is one of the main tools needed to build a theory of crosses with singularities on complex manifolds. Since the proof of original Theorem 1.1 (and thus the proof of Theorem 1.4) was based on strong results, it is surprising that the proof of the main theorem presented in the next section is so elementary.

**Main Theorem.** Let $D_1$ be a connected σ-compact Josefson manifold (i.e. $D_1$ is a countable at infinity complex manifold such that every locally pluripolar set in $D_1$ is globally pluripolar) of dimension $n_1$ and let $D_2$ be a σ-compact complex manifold of dimension $n_2$. Let $\Omega \subset D_1 \times D_2$ be a domain and let $M \subset \Omega$ be a singular set with respect to the family $\mathcal{F} \subset \mathcal{O}(\Omega \setminus M)$ such that the set $\{a_1 \in D_1$: *the fiber* $M_{(a,\cdot)}$ *is not pluripolar*$\}$ is pluripolar in $D_1$. Then there exists a pluripolar set $Q \subset D_1$ such that for every $a_1 \in \pi_{D_1}(\Omega) \setminus Q$, the fiber $M_{(a_1,\cdot)}$ is singular in $\Omega_{(a_1,\cdot)}$ with respect to the family $\mathcal{F}_a := \{f(a_1,\cdot) : f \in \mathcal{F}\} \subset \mathcal{O}(\Omega_{(a_1,\cdot)})$, where $\pi_{D_1}(\Omega)$ denotes the projection of $\Omega$ to $D_1$ and for $B \subset D_1 \times D_2$ and $a_1 \in D_1$, we put $B_{(a_1,\cdot)} := \{a_2 \in D_2 : a = (a_1, a_2) \in B\}$.

## 2. Proof of Main Theorem

Fix $a = (a_1, a_2) \in M$, where $a_1 \in D_1$, $a_2 \in D_2$. Let $\Phi_j : U_j \to \widetilde{U}_j$ be a biholomorphic mapping such that $U_j$ is an open neighbourhood of $a_j$, $\widetilde{U}_j$ is an Euclidean ball in $\mathbb{C}^{n_j}$, $\Phi_j(a_j) = 0$, $j = 1, 2$, and $U_a := U_1 \times U_2 \subset \Omega$.

Define $\Phi := (\Phi_1, \Phi_2)$ and $N := \Phi(M \cap U_a)$, $\mathcal{F}_a := \{f|_{U_a} : f \in \mathcal{F}\}$, $\widetilde{\mathcal{F}}_a := \{f \circ \Phi^{-1} : f \in \mathcal{F}_a\}$. Then $N$ is a relatively closed subset of $\widetilde{U} := \widetilde{U}_1 \times \widetilde{U}_2$ and $\widetilde{\mathcal{F}}_a \subset \mathcal{O}(\widetilde{U} \setminus N)$. We show that $N$ is singular with respect to the family $\widetilde{\mathcal{F}}_a$.

Fix a $w \in N$ and define $b := \Phi^{-1}(u) \in M.$ Assume that there exists an open neighborhood $\widetilde{V}_w$ of $w$ such that every function $\tilde{f} \in \widetilde{\mathcal{F}}_a$ extends holomorphically on $\widetilde{V}_w$. Let $V_b := \Phi^{-1}(\widetilde{V}_w)$. Fix $f \in \mathcal{F}_a$ and define $\tilde{f} := f \circ \Phi^{-1} \in \widetilde{\mathcal{F}}_a$. Then $\tilde{f}$ extends to a function $\widetilde{F}$ holomorphic on $\widetilde{V}_w$. Define $F := F \circ \Phi|_{V_b} \in \mathcal{O}(V_b)$. Since $F = \widetilde{F} \circ \Phi = = \widetilde{f} \circ \Phi = f$ on the nonempty open set $V_b \setminus M$, we conclude that $F$ is a holomorphic extension of $f$ to $V_b$ – a contradiction.

Now, from Theorem 1.4, there exists a pluripolar set $\widetilde{Q}_a \subset \mathbb{C}^{n_1}$ such that for any $w_1 \in \widetilde{U}_1 \setminus \widetilde{Q}_a$ the fiber $N_{(w_1,\cdot)} := \{w_2 \in \mathbb{C}^{n_2} : (w_1, w_2) \in N\}$ is singular with respect to the family $\widetilde{\mathcal{F}}_{w_1} := \{\widetilde{f}(w_1,\cdot) : \widetilde{f} \in \widetilde{\mathcal{F}}_a\}$. Define $b_1 := (\Phi_1)^{-1}(w_1) \in U_1$, $Q_a := (\Phi_1)^{-1}(\widetilde{Q}_a)$. Then $Q_a$ is pluripolar in $D_1$ and

$$N_{(w_1,\cdot)} = \{w_2 \in \mathbb{C}^{n_2} : \exists b_2 \in U_2 : \Phi_2(b_2) = w_2, (b_1, b_2) \in M\} = \Phi_2(M_{(b_1,\cdot)}).$$

Using similar reasoning as before, we show that for any $b_1 \in U_1 \setminus Q_a$ the fiber $M_{(b_1,\cdot)}$ is singular with respect to the family $\mathcal{F}_{b_1} = \{f(b_1,\cdot) : f \in \mathcal{F}_a\}$.

From $\{U_a\}_{a \in M}$, we choose a countable covering $\{U_{a_j}\}_{j=1}^{\infty}$ of the set $M$. Define $Q := \bigcup_{j=1}^{\infty} Q_{a_j} \cup \{b_1 \in D_1 : \textit{the fiber } M_{(b_1,\cdot)} \textit{ is not pluripolar}\}$.

Because $D_1$ is a Josefson manifold, $Q$ is pluripolar in $D_1$. We show that for any $b_1 \in \pi_{D_1}(\Omega) \setminus Q$, the fiber $M_{(b_1,\cdot)}$ is singular with respect to the family $\mathcal{F}_{b_1} := \{f(b_1,\cdot) : f \in \mathcal{F}\}$.

Fix $b_1 \in \pi_{D_1}(\Omega) \setminus Q$, $b_2 \in M_{(b_1,\cdot)}$. Assume that there exists an open neighbourhood $V_{b_2}$ of $b_2$ such that any function $f(b_1,\cdot), f \in \mathcal{F}$, extends holomorphically on $V_{b_2}$. Fix $f \in \mathcal{F}$. Because $(b_1, b_2) \in M$, then there exists $a_j$, $j \in \{1, 2, \ldots\}$, such that $(b_1, b_2) \in U_{a_j} = = U_{1,j} \times U_{2,j}$. Thus $b_1 \in U_{1,j} \setminus Q_{a_j}$ and $f|_{U_{a_j}}(b_1,\cdot)$ extends holomorphically on $(V_{b_2} \cap U_{2,j})$, but we already know that the fiber $M_{(b_1,\cdot)}$ is singular with respect to the family $\{f(b_1,\cdot) : f \in \mathcal{F}_{a_j}\} = \{f|_{U_{a_j}}(b_1,\cdot) : f \in \mathcal{F}\}$ – a contradiction.

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