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DIRECT DETERMINATION OF PERIODIC  
SOLUTION IN THE TIME DOMAIN  
FOR ELECTROMECHANICAL CONVERTERS

BEZPOŚREDNIE WYZNACZANIE  
ROZWIĄZAŃ OKRESOWYCH  
DLA PRZETWORNIKÓW ELEKTROMECHANICZNYCH  
W DZIEDZINIE CZASU

Abstract

The main aim of this paper is to identify relationships for direct determination in the time domain of periodic steady-state solutions for differential equations. A new discrete operator of differentiating has been defined. As a result, a set of algebraic equations has been written. Based on this, an algorithm for nonlinear differential equations has been proposed. Numerical tests have been carried out both for a new discrete operator and for steady-state analysis in a simple electromechanical converter.

*Keywords: analysis in time domain, discrete operator of differentiating*

Streszczenie

W niniejszej pracy przedstawiono rozważania prowadzące do równań umożliwiających obliczanie rozwiązań ustalonych bezpośrednio w dziedzinie czasu dla układów opisywanych równaniami różniczkowymi, o których wiadomo, że posiadają rozwiązania okresowe o znanym okresie. Zdefiniowano dyskretny operator różniczkowania określający chwilowe wartości pochodnej funkcji w wybranym zbiorze punktów na podstawie wartości funkcji w tym zbiorze oraz podano równania algebraiczne określające rozwiązania ustalone w tych punktach. Równania te uogólniono na układy nieliniowe posiadające rozwiązania okresowe z myślą o układach elektromechanicznych oraz energoelektronicznych. Dyskretny operator różniczkowania poddano testom dla najbardziej charakterystycznych funkcji. W artykule przedstawiono ponadto wyniki testowych obliczeń stanu ustalonego w prostym przetworniku elektromechanicznym.

*Słowa kluczowe: analiza w dziedzinie czasu, dyskretny operator różniczkowania*

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## 1. Introduction

Steady states in electrical circuits and in various types of electromagnetic objects are issues of great interest in electrical engineering because the technical parameters of devices are based on them. Steady states calculation methods are fundamental problems of electrical engineering and are basic tools for identifying the properties of electrical circuits, for example symbolic calculus which is used to analyse circuits with sinusoidal current waveform. This is a method which operates in the frequency domain. It allows the simple specification of parameters of set solutions on the basis of which one can clearly determine values of solutions in particular moments of time if necessary. The symbolic calculus in circuits with power electronic elements, even if it can be used, is not an effective method. Usually, simulation methods are used to determine steady states, increasing simulation time until the steady state is reached.

Specifying time performances on the basis of the Fourier spectra obtained with frequency methods can be insufficient in the case of deformed temporary solutions containing erratic changes of value, nothing that due to the Gibbs phenomenon. This paper presents an attempt to create an algorithm which allows the direct calculation of the temporary values of periodic steady waveforms where a circuit is described by a system of linear ordinary differential equation with temporary variable modulus which has periodic steady solutions.

## 2. Formulation of the problem

From a mathematical point of view, the particular solution is looking for to the system of differential equations in the form:

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t) \cdot \mathbf{x} + \mathbf{b}(t) \quad (1)$$

in which both matrix  $\mathbf{A}(t)$  and input vector  $\mathbf{b}(t)$  are periodic and can be represented in the Fourier series:

$$\mathbf{A}(t) = \mathbf{A}(t+T) = \sum_{k=-\infty}^{\infty} \mathbf{A}_k \cdot e^{jk\Omega t} \quad \mathbf{b}(t) = \mathbf{b}(t+T) = \sum_{k=-\infty}^{\infty} \mathbf{B}_k \cdot e^{jk\Omega t} \quad (2)$$

It can be proved that such a solution, known in engineering as steady-state, is also periodic and can be represented in form of the Fourier series:

$$\mathbf{x}(t) = \mathbf{x}(t+T) = \sum_{k=-\infty}^{\infty} \mathbf{X}_k \cdot e^{jk\Omega t}, \quad \Omega = 2\pi / T \quad (3)$$

The values of coefficients of this Fourier series comply with the infinite system of algebraic equations in the form [1, 2]:



$$\mathbf{X}^N = \frac{1}{2N+1} (\mathbf{C}^T)^* \cdot \mathbf{x}^N \quad (8)$$

Equations (4) and relationships (5) and (8) will be used to formulate equations directly determining instantaneous values of periodic solutions.

### 3. Equations defining the solution in the time-domain for linear equations

To form of algebraic equations defining the periodic solutions in the time domain we should limit infinite harmonic balance equations (4) to dimensions,  $(2N+1) \times (2N+1)$  from which we can calculate a Fourier spectra of the solution set to the  $N$ -th harmonic, inclusive. This system written down compactly has the form:

$$j \cdot \Omega^N \cdot \mathbf{X}^N = \mathbf{A}^N \cdot \mathbf{X}^N + \mathbf{B}^N \quad (9)$$

Relationships between the momentary values of a solution and a Fourier series modulus for a vector solution  $\mathbf{x}(t)$  can be, on the basis of (5) and (6), written in the form:

$$\mathbf{x}^N = \mathbf{C} \cdot \mathbf{X}^N; \quad \mathbf{X}^N = \frac{1}{2N+1} (\mathbf{C}^T)^* \cdot \mathbf{x}^N \quad (10)$$

Matrix  $\mathbf{C}$  takes the form identical to matrix  $\mathbf{C}$  given by formula (6), but each of its elements is constituted by a diagonal matrix of dimension of matrix  $\mathbf{A}(t)$  of the system (1), comprising elements equal to the appropriate element in matrix  $\mathbf{C}$ .

The system of equations which determines the solution set in the time domain can be obtained after conducting the following mathematical calculations:

$$j \cdot \frac{1}{2N+1} (\mathbf{C} \cdot \Omega^N \cdot (\mathbf{C}^T)^*) \cdot \mathbf{x}^N = \frac{1}{2N+1} (\mathbf{C} \cdot \mathbf{A}^N \cdot (\mathbf{C}^T)^*) \cdot \mathbf{x}^N + \mathbf{C} \cdot \mathbf{B}^N \quad (11)$$

Designating:

$$j \cdot \frac{1}{2N+1} (\mathbf{C} \cdot \Omega^N \cdot (\mathbf{C}^T)^*) = \mathbf{D}^N \quad (12)$$

$$\frac{1}{2N+1} (\mathbf{C} \cdot \mathbf{A}^N \cdot (\mathbf{C}^T)^*) = \mathbf{a}^N \quad (13)$$

$$\mathbf{b}^N = \mathbf{C} \cdot \mathbf{B}^N \quad (14)$$

we obtain a system of linear algebraic equations:

$$(\mathbf{D}^N - \mathbf{a}^N) \cdot \mathbf{x}^N = \mathbf{b}^N \quad (15)$$

Matrixes  $\mathbf{D}^N$  and  $\mathbf{a}^N$  obtained as a result of multiplying the matrix in brackets in relationships (12) and (13) are squared and have dimensions  $(2N+1) \times (2N+1)$ , and  $\mathbf{b}^N$  is a vector with  $(2N+1)$  elements. These dimensions correspond to the number of points in which values of periodic solutions are calculated. Matrix  $\mathbf{D}^N$  represents a differential operator, and matrix  $\mathbf{A}^N$  and  $\mathbf{b}^N$  represents values of matrix  $\mathbf{A}(t)$  and the inlet vector  $\mathbf{b}(t)$  at selected instances of time.

Matrix  $\mathbf{D}^N$ , which can be called a discrete differential operator for a periodic function, takes the form:

$$\mathbf{D}^N = \begin{bmatrix} \mathbf{0} & -\mathbf{d}_1 & -\mathbf{d}_2 & \cdots & -\mathbf{d}_N & \cdots & \cdots & \cdots & -\mathbf{d}_{2N} \\ \mathbf{d}_1 & \ddots & \ddots & \ddots & \vdots & \ddots & & & \vdots \\ \mathbf{d}_2 & \ddots & \mathbf{0} & -\mathbf{d}_1 & -\mathbf{d}_2 & & \ddots & & \vdots \\ \vdots & \ddots & \mathbf{d}_1 & \mathbf{0} & -\mathbf{d}_1 & -\mathbf{d}_2 & & \ddots & \vdots \\ \mathbf{d}_N & \cdots & \mathbf{d}_2 & \mathbf{d}_1 & \mathbf{0} & -\mathbf{d}_1 & -\mathbf{d}_2 & \cdots & -\mathbf{d}_N \\ \vdots & \ddots & & \mathbf{d}_2 & \mathbf{d}_1 & \mathbf{0} & -\mathbf{d}_1 & \ddots & \vdots \\ \vdots & & & & \mathbf{d}_2 & \mathbf{d}_1 & \mathbf{0} & \ddots & -\mathbf{d}_2 \\ \vdots & & & & & \vdots & \ddots & \ddots & -\mathbf{d}_1 \\ \mathbf{d}_{2N} & \cdots & \cdots & \cdots & \mathbf{d}_N & \cdots & \mathbf{d}_2 & \mathbf{d}_1 & \mathbf{0} \end{bmatrix} \quad (16)$$

whose elements are diagonal matrices  $\mathbf{d}_n$  of dimensions of the system (1) with the values  $d_n$  on the diagonal:

$$\mathbf{d}_n = \begin{bmatrix} d_n & & & & \\ & \ddots & & & \\ & & d_n & & \\ & & & \ddots & \\ & & & & d_n \end{bmatrix}$$

These values are calculated from relationship:

$$d_n = \frac{\Omega}{2N+1} \cdot \sum_{k=1}^N 2k \cdot \sin\left(k \cdot n \cdot \frac{2\pi}{2N+1}\right) \quad (17)$$

Matrix  $\mathbf{D}^N$  is singular, which is quite obvious, because we cannot reproduce the constant periodic function on the basis of its derivative.

Execution of operations provided in (13) and (14) is not necessary, because they determine values of matrix vector  $\mathbf{A}(t)$  and vector  $\mathbf{b}(t)$  in selected time instances. They can be designated directly from these matrices, and not using their distribution in a Fourier series. Therefore we can write:

$$\mathbf{a}^N = \begin{bmatrix} \mathbf{A}(N) & & & & & & & & \\ & \ddots & & & & & & & \\ & & \mathbf{A}(1) & & & & & & \\ & & & \mathbf{A}(0) & & & & & \\ & & & & \mathbf{A}(-1) & & & & \\ & & & & & \ddots & & & \\ & & & & & & \mathbf{A}(-N) & & \end{bmatrix}$$

$$\mathbf{b}^N = [\mathbf{b}(N) \quad \cdots \quad \mathbf{b}(1) \quad \mathbf{b}(0) \quad \mathbf{b}(-1) \quad \cdots \quad \mathbf{b}(-N)]^T$$

where by matrices  $\mathbf{A}(n)$  and vectors  $\mathbf{b}(n)$  are denoted the matrix  $\mathbf{A}(t)$  and the vector  $\mathbf{b}(t)$ , respectively, calculated for the time instances  $t_n$ . Then, the system of equations (15) takes the form:

$$\begin{bmatrix} -\mathbf{A}(N) & -\mathbf{d}_1 & \cdots & -\mathbf{d}_N & \cdots & -\mathbf{d}_{2N} \\ \mathbf{d}_1 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & -\mathbf{A}(1) & -\mathbf{d}_1 & -\mathbf{d}_2 & \vdots \\ \mathbf{d}_N & \cdots & \mathbf{d}_1 & -\mathbf{A}(0) & -\mathbf{d}_1 & \cdots & -\mathbf{d}_N \\ & & \mathbf{d}_2 & \mathbf{d}_1 & -\mathbf{A}(-1) & \ddots & \\ \vdots & & & \vdots & \ddots & \ddots & -\mathbf{d}_1 \\ \mathbf{d}_2 & \cdots & & \mathbf{d}_N & \mathbf{d}_1 & -\mathbf{A}(-N) & \end{bmatrix} \begin{bmatrix} \mathbf{x}_N \\ \vdots \\ \mathbf{x}_1 \\ \mathbf{x}_0 \\ \mathbf{x}_{-1} \\ \vdots \\ \mathbf{x}_{-N} \end{bmatrix} = \begin{bmatrix} \mathbf{b}(N) \\ \vdots \\ \mathbf{b}(1) \\ \mathbf{b}(0) \\ \mathbf{b}(-1) \\ \vdots \\ \mathbf{b}(-N) \end{bmatrix} \quad (18)$$

It is the system of algebraic equations which was sought, from which one can directly calculate the values of periodic solution in a selected set of  $2N + 1$  points.

The above considerations can be generalized to the non-linear system of differential equations of the form:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t) \quad (19)$$

in cases where it is known that there is a solution to the given equation and this solution is periodic  $\mathbf{x}(t) = \mathbf{x}(t + T)$ . In order to do this, we must write down the system (19) in the form:

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(\mathbf{x}, t)\mathbf{x} + \mathbf{b}(t) \quad (20)$$

For this equation, we can create an iterative algorithm for seeking a periodic solution based on equations (15) and (19). Such an algorithm requires an iterative solving of equations of the form (in a compact notation):

$$(\mathbf{D}^N - \mathbf{a}_i^N) \cdot \mathbf{x}_{i+1}^N = \mathbf{b}^N \quad (21)$$

in which  $\mathbf{x}_{i+1}^N$  is a vector of seeking solutions in iteration  $i + 1$ . Matrix  $\mathbf{a}_i^N$  has form:

$$\mathbf{a}_i^N = \begin{bmatrix} \mathbf{A}_i(N) & & & & & & \\ & \ddots & & & & & \\ & & \mathbf{A}_i(1) & & & & \\ & & & \mathbf{A}_i(0) & & & \\ & & & & \mathbf{A}_i(-1) & & \\ & & & & & \ddots & \\ & & & & & & \mathbf{A}_i(-N) \end{bmatrix}$$

and contains matrixes  $\mathbf{A}_i(n)$  calculated for the solution obtained in the  $i$ -th iteration for subsequent time instants. Such an algorithm requires the determination of the value of the starting solution.

#### 4. Study of discrete differentiation operator

In order to check the correctness of the operation of the differentiation operator  $\mathbf{D}^N$ , the following calculation were performed:

- the correctness of calculating the derivative of a constant function was checked. As a result of the derivative because the values of matrix elements  $\mathbf{D}^N$  comply to condition:

$$d_n = -d_{2N+1-n} \quad \text{for } n = (1, 2, \dots, N) \quad (22)$$

- correctness of calculating derivative of a function  $\cos x$  was checked. The calculation results for  $N = 100$  were shown in Fig. 1. Accuracy of function reconstruction  $-\sin x$  is of the order of  $10^{-12}$ .
- correctness of calculating derivative of a discontinuous function was checked. The derivative is not quite correctly reconstructed because there are effects similar to the Gibbs effect when calculating the value of discontinuous function on the basis of a Fourier series. This is illustrated in Fig. 2 with  $N = 100$ .

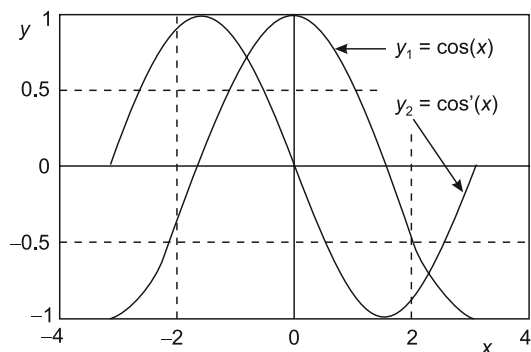


Fig. 1. Reconstruction of derivative of the function  $\cos x$  by the operator of differentiation  $\mathbf{D}^N$

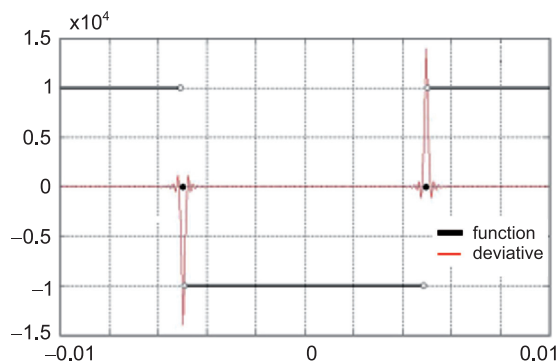


Fig. 2. Reconstruction of derivative of the discontinuous function by the operator of differentiation  $\mathbf{D}^N$

### 5. Example of designating the solution for an electromechanical transducer

In order to illustrate the proposed approach, the steady-state analysis of the simplest electromechanical converter described by the equation is presented:

$$\frac{d}{dt}(L_0 + L_1 \cdot \cos(2\varphi)) \cdot i + R \cdot i = u(t) \quad (23)$$

The steady-state conditions are supplying voltage that is mono-harmonic:

$$u(t) = \sqrt{2}U \cdot \cos(\Omega \cdot t) \quad (24)$$

at constant angular velocity  $\varphi = \Omega \cdot t + \varphi_0$ . In these conditions, it can be envisaged that the solution in steady state will be periodic. The equation of the converter has been written in the normal form, corresponding to the equation (1):

$$\frac{d\psi}{dt} = \frac{R}{L(t)} \cdot \psi + u(t) \quad (25)$$

making use of the formula  $\psi = L(t) \cdot i$ .

The calculations were performed for the following parameters:  $L_0 = 3$  H;  $L_1 = 2$  H;  $\varphi_0 = \pi/2$ ;  $U = 230$  V;  $\Omega = 2 \cdot \pi \cdot 50$  1/sec. Also, for three resistance values:  $R = 100$   $\Omega$ ;  $R = 1000$   $\Omega$ ;  $R = 2000$   $\Omega$ . It was assumed that  $N = 100$ , i.e. 201 equally distributed points were chosen in the range of voltage variation  $(-T/2, T/2)$ . On the basis of equation (21), a system of equations (18) of dimensions  $(201 \times 201)$  were set up, obtaining from its solution, the value of the linkage flux in the selected set of points. Values of current in these points were determined from the relationship  $i = \psi/L(t)$ .

The calculation results in the form of variability diagram  $i(t)$  and  $\psi(t)$  were shown in the following figures – for  $R = 100$   $\Omega$  on Fig. 3, for  $R = 1000$   $\Omega$  in Fig. 4 and  $R = 2000$   $\Omega$  for Fig. 5. In Figure 3a, the waveform of flux is practically an integral of the voltage because the value of drop in voltage across the resistance is relatively small. The current is distorted due to the variable inductance. As the resistance is being increased, the flux deforms more and more, and consequently, the current deforms as well. Waveforms were determined directly in time. A small modification also allows the direct determination of current.

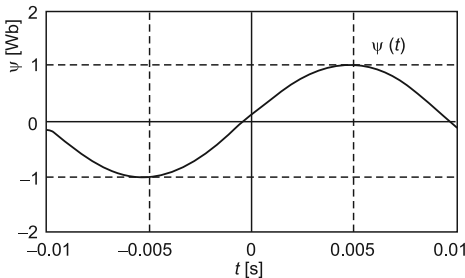


Fig. 3a. Flux waveform  $\psi(t)$  ( $R = 100$   $\Omega$ )

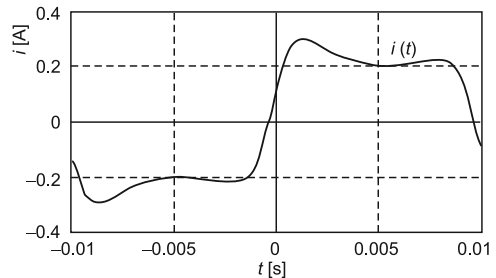
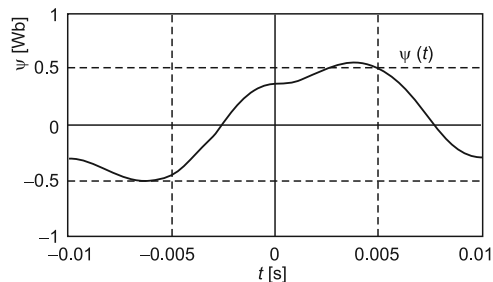
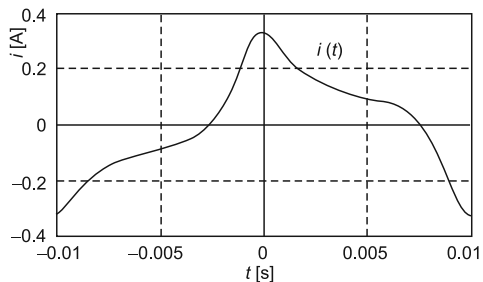
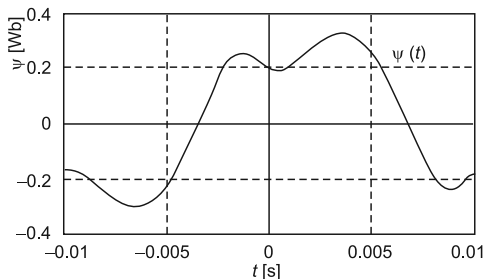
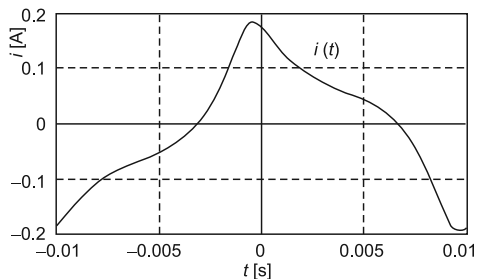


Fig. 3b. Current waveform  $i(t)$  ( $R = 100$   $\Omega$ )



Fig. 4a. Flux waveform  $\psi(t)$  ( $R = 1000 \Omega$ )Fig. 4b. Current waveform  $i(t)$  ( $R = 1000 \Omega$ )Fig. 5a. Flux waveform  $\psi(t)$  ( $R = 2000 \Omega$ )Fig. 5b. Current waveform  $i(t)$  ( $R = 1000 \Omega$ )

## 6. Conclusions

In this paper, equations for the direct determination of the instantaneous values for the periodic steady-state solution of linear differential equations with periodically variable parameters were evaluated. The obtained equations take the form of a set of linear algebraic equations and eliminate the need to use the Fourier series. The new discrete differential operator is an important element of this system.

Numerical tests of the discrete differential operator confirmed its correctness for calculation of derivatives for differentiable functions and its usefulness for the steady-state analysis of electromechanical converters.

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