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RIEMANN SURFACES IN PUZYNA'S MONOGRAPH:
TEORYA FUNKCYJ ANALITYCZNYCH

POWIERZCHNIE RIEMANNA W MONOGRAFII PUZYNY:
TEORYA FUNKCYJ ANALITYCZNYCH

Abstract

In the paper, we discuss the exposition of material on the theory of surfaces in J. Puzyrna's monograph "Teoria funkcji analitycznych" [*Theory of Analytic functions*] (published at the turn of XIX and XX centuries) which is necessary for consideration of the Riemann surfaces of analytic functions. Though the monograph contains elements of the set theory, the author preferred a descriptive exposition.

Keywords: history of mathematics in Poland, Lvov university, Riemann surfaces

Streszczenie

W artykule przedstawiono treści dotyczące powierzchni Riemanna w dziele Józefa Puzyry *Teoria funkcji analitycznych* z przełomu XIX i XX w. Warto zauważyć, że chociaż monografia zawiera elementy teorii mnogości, to jednak autor preferuje opisowy sposób prezentacji materiału.

Słowa kluczowe: historia matematyki w Polsce, Uniwersytet Lwowski, powierzchnie Riemanna

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1. Introduction

In his paper *On teaching mathematics* (see [1]), the famous mathematician V. Arnold wrote that the classification theorem for surfaces is one of most important statements that the students of mathematics should necessarily know: “it is this remarkable theorem (which asserts, for example, that any compact connected oriented surface is a sphere with a number of handles) that gives the correct idea of what modern mathematics is ...”. Therefore, he asserts, this theorem should be an inevitable part of any university course of mathematics. Also, Arnold recalls that classical course of (complex) analysis by Hermite begins with the notion of a Riemannian surface and this is an important feature which makes it more *modern* than many other contemporary textbooks.

In connection with this it is interesting to look at the role and place of the Riemann surfaces, and surfaces in general, in Józef Pużyna’s monograph *Teoria funkcji analitycznych* [*Theory of Analytic functions*] published in 1898–1900.

2. Information about J. Pużyna

Józef Pużyna (1856–1919), a longtime professor of mathematics at the University of Lvov, was born on 19 March 1856 in Nowy Martynów. In 1875 he graduated from the famous Franz Joseph Gymnasium in Lvov, then he enrolled at the Faculty of Philosophy of the University of Lvov. After staying at the University of Berlin he received his doctorate in 1883 at the University of Lvov. After his habilitation in 1885 Pużyna delivered lectures in mathematics as a docent.

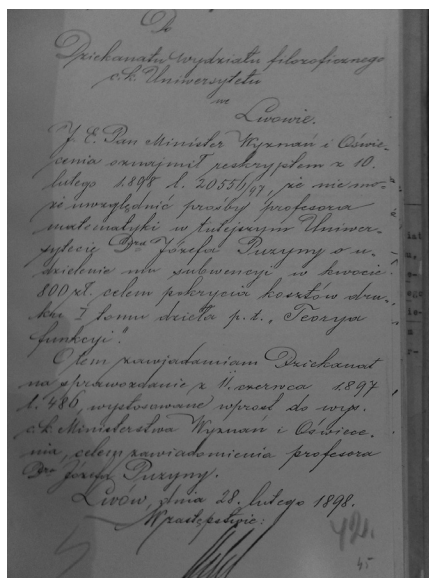


Fig. 1. Letter from the Ministry in Vienna informing on funding of the 1st volume of Pużyna’s monograph (Lvov District Archive)

He directed the Department of Mathematics at the University of Lvov as an associate professor in the years 1889-1892 and as a professor in 1892 until the end of his life, 1919. He was a very good lecturer and taught many different branches of mathematics.

In 1898, Józef Puzyna published the first volume of his famous monograph *Teorya funkcyj analitycznych*. The book was highly valued by mathematicians from Poland and by foreign mathematicians. It was a pioneering book that completely covered the modern theory of analytic functions (see: [2–5, 7, 11]).

In connection with this it is interesting to observe that in his monograph Puzyna pays a lot of attention to the notion of surface and results related to classification of surfaces.

Strictly speaking, only branched covers of complex plane are considered in the book. In his exposition of the material Puzyna follows Riemann's original approach.

Some other information concerning Puzyna's monograph can be found, e.g., in [3, 4, 7, 11]. The present note extends the material from the author's monograph [2].

3. Riemann surfaces in Puzyna's monograph

Part V of Volume II of Puzyna's monograph is entitled *Connectivity of surface (Analysis Situs). Riemann surface*. Note that Puzyna subsequently uses the term "compactness" instead of "connectedness" in his monograph. In modern terminology, the Riemann surfaces are one-dimensional complex manifolds. The notion of a Riemann surface is used in complex analysis in order to make the multi-valued analytic functions single-valued. The topological properties of the Riemann surfaces play therefore an important role in the theory of analytic functions of the one variable.

The exposition starts with the definition of closed surface and studying topological properties of surfaces by means of their sections by connected simple curves. However, the definition of surface is necessarily not rigorous as the author avoids using charts, i.e. homeomorphisms onto domains of euclidean spaces. According to Puzyna a surface is a section of a plane or an arbitrary surface (sic!) bounded by one or few closed curves.

The simple connected surfaces are introduced by means of an intuitive definition. These are the surfaces that satisfy the following properties:

- 1) Every curve connecting two points of the surface can be transformed into another one so that it does not leave the surface in the process of transformation. The endpoints of the curve either stay the same or change;
- 2) Every connected curve contained in the surface can be shrunk to an arbitrary point, while remaining on the surface in the process of shrinking;
- 3) If the surface possesses the boundary, then every simple (non-self-intersecting) curve that connects two distinct points of the boundary divides the surface into two separate parts.
- 4) Actually, if a closed surface is considered, it is supposed that its boundary consists of a chosen point in this surface.

In modern terms, the author implicitly uses the notion of homotopy (isotopy) of continuous maps in this definition.

Then n -connected surfaces are introduced. These are the surfaces in which one can make $n - 1$ cuts such that the result of cutting is a simply connected surface.

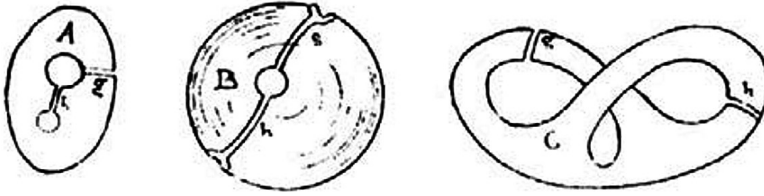


Fig. 2. Examples of 2-connected surfaces are: a planar annulus, sphere with two holes etc.
The third figure is an example of a 3-connected surface

The proofs of statements on the surfaces are based on intuitive approach as well as descriptive arguments. Therefore, the level of rigor here is necessarily strictly lower than in exposition of set-theoretic or algebraic material. Note that even simply formulated and intuitively evident statements of the planar topology can have complicated proofs, and the famous Jordan theorem is a good example supporting this statement.

Next, the notion of the genus of a surface is defined.

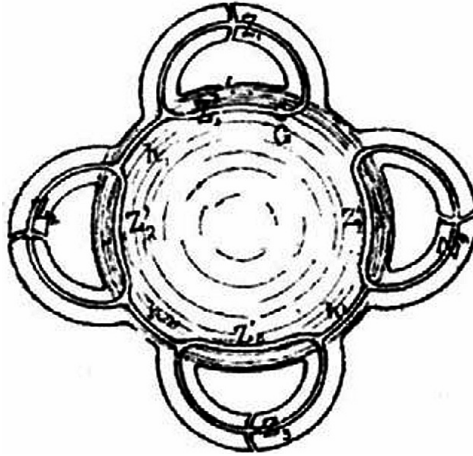


Fig. 3. A figure from Puzyna's monograph: Sphere with handles

Also, maps (transformations) of surfaces are described in the following, rather intuitive, way:

- 1) any infinitely close points remain so in the mapped surface;
- 2) any points that are finitely distant remain so in the mapped surface. Note that Puzyna does not use here language of set-theoretic topology.

The one-sided surfaces are introduced and a classification theorem (i.e. that every (oriented) surface is homeomorphic to a sphere with handles) for them is presented. The exposition of this proof is again based on an intuitive approach.

A generalization of Euler's theorem onto (triangulable) surfaces is also given (The author calls the result l'Huilier's theorem). This allows the author to consider the Euler characteristic of a surface.

The following section XIV contains a description of the construction of Riemann surfaces, first, at a neighborhood of a branching point. This construction is illustrated by the following pictures.

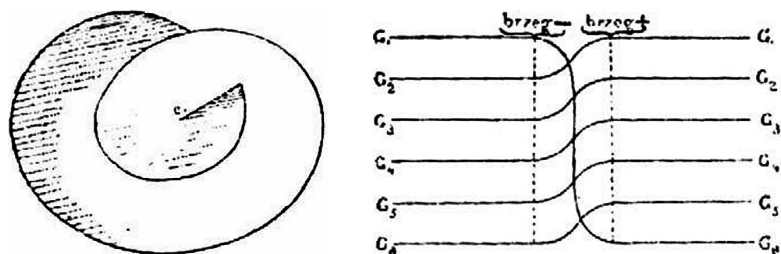


Fig. 4. Neighborhood of a branching point

It is proved that the algebraic notion of the genus of a Riemann surface can also be described in topological terms. Actually, the genus is a topological invariant of a surface.

The harmonic functions on Riemann surfaces (both open and closed) are considered in Section XX. The following Section XXI contains, in particular, a construction of a rational function on a given Riemannian surface by means of the 2nd order Abel integral.

Also, Riemannian surfaces appear in Part VIII, Section XXII devoted to maps of circular polygons in a half-plane (see the picture below).

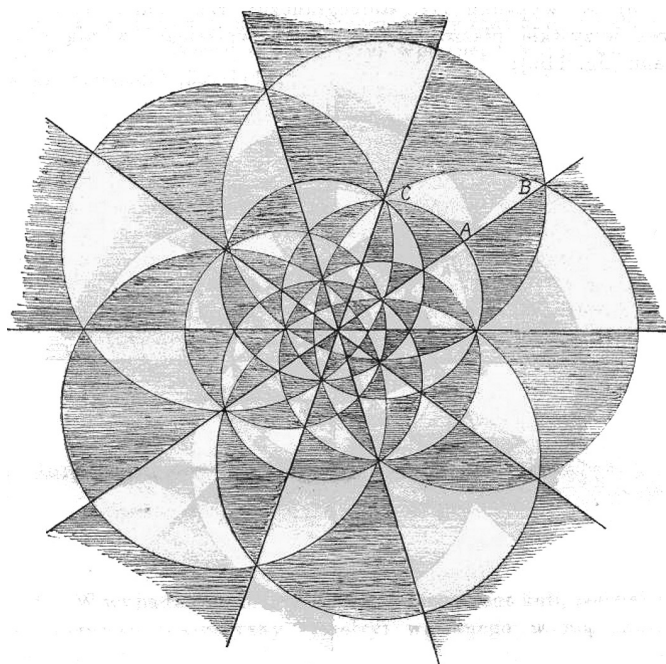


Fig. 5. Exemplary Riemannian surface

As a conclusion we emphasize the prevalence of descriptive exposition of the material concerning Riemannian surfaces despite of set-theoretic language elaborated in the initial chapters of the monograph.

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