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# THE POLES METHOD FOR HIGHER-ORDER LINEAR TIME-VARYING SYSTEMS

# METODA BIEGUNÓW DLA UKŁADÓW LINIOWYCH WYŻSZEGO RZĘDU O CZASOWO ZALEŻNYCH WSPÓŁCZYNNIKACH

A b s t r a c t

In dynamic linear systems described by differential equations with constant parameters, the poles of the rational function (transfer function of the system) play an important role. This article attempts to expand the poles concept in a situation where the system is described by the *N*-th order linear system with time-varying parameters. It then introduces the concept of characteristic equations and time-dependent poles.

*Keywords: linear systems, time-varying systems*

S t r e s z c z e n i e

W opisie liniowych systemów dynamicznych opisanych przez równania różniczkowe o stałych parametrach ważną rolę odgrywają bieguny funkcji wymiernej (transmitancji systemu). Ten artykuł rozszerza koncepcję biegunów na przypadek, gdy system jest opisany równaniem liniowym *N*-tego rzędu o zmiennych w czasie parametrach. Pojawia się tu pojęcie zależnego od czasu równania charakterystycznego i zależnych od czasu biegunów transmitancji.

*Słowa kluczowe: równania różniczkowe liniowe, równania różniczkowe parametryczne*

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## 1. Introduction

In the field of linear systems analysis with constant coefficients, the well-known and functioning method is the factorisation method – this consists of transforming complex systems to *a* commutative cascade of the first-order systems.

This requires finding the poles of the transfer function – these are also the zeros of the characteristic polynomial called the eigenvalues of the system which are generally complex. For a stable system, they lie in the open left half-plane. It turns out that this method also works in the case of linear systems with time-variable coefficients [3, 6, 7].

Linear systems of the first and second order can be described by a differential equation, or by a block diagram.

First-order ODE:

$$\frac{dy}{dt} - \alpha y = x(t)$$

and its corresponding block diagram:

$$x(t) \rightarrow \boxed{\left(\frac{d}{dt} - \alpha\right)^{-1}} \rightarrow y(t)$$

The second-order ODE:

$$\frac{d}{dt}\overbrace{\left(\frac{dy}{dt} - \alpha y\right)}^{\leftarrow - - u(t) - - \rightarrow} - \beta\overbrace{\left(\frac{dy}{dt} - \alpha y\right)}^{\leftarrow - - u(t) - - \rightarrow} = x(\mathrm{t})$$

and its corresponding block diagram:

$$x(t) \rightarrow \boxed{\left(\frac{d}{dt} - \beta\right)^{-1}} \xrightarrow{u(t)} \boxed{\left(\frac{d}{dt} - \alpha\right)^{-1}} \rightarrow y(t)$$

Such cascade factorisation of the second-order system can be called the time-dependent 'Vieta's formulas' with time-dependent poles.

For the first-order systems, a closed-form analytical solution may be available. However, for the higher-order system with time varying parameters, the finding of poles must be carried out numerically or some special methods must be used [1, 2, 4, 5, 8–10].

## 2. Separation of time-dependent poles in the higher-order linear systems

The differential equation of the higher order:

$$\frac{d^n}{dt^n} y + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} y + \ldots + a_1\frac{d}{dt} y + a_0 y = x \tag{1}$$

is subjected to the following transformations:

$$\frac{d}{dt}\left(\overbrace{\frac{d^{n-1}}{dt^{n-1}}y+\alpha_{n-2}\frac{d^{n-2}}{dt^{n-2}}y\ +\ldots+\ \alpha_1\frac{d}{dt}y\ +\ \alpha_0 y}^{u(t)}\right)$$
$$-\alpha\left(\overbrace{\frac{d^{n-1}}{dt^{n-1}}y+\alpha_{n-2}\frac{d^{n-2}}{dt^{n-2}}y\ +\ldots+\ \alpha_1\frac{d}{dt}y\ +\ \alpha_0 y}^{u(t)}\right)=x \tag{2}$$

where:

| | | |
|---|---|---|
| $\alpha$ | – | unknown time-dependent pole, |
| $\alpha_{n-2}$, ..., $\alpha_1$, $\alpha_0$ | – | unknown coefficients of the differential equation of reduced order (also time-dependent). |

In the flowchart convention, this operation involves replacing the single block:

$$x(t)\ \rightarrow \boxed{\left(\frac{d^n}{dt^n}+a_{n-1}\frac{d^{n-1}}{dt^{n-1}}+\ldots+a_1\frac{d}{dt}+a_0\right)^{-1}} \rightarrow\ y(t)$$

by the cascade:

$$x(t)\ \rightarrow \boxed{\left(\frac{d}{dt}-\alpha\right)^{-1}} \rightarrow \boxed{\left(\frac{d^{n-1}}{dt^{n-1}}+\alpha_{n-2}\frac{d^{n-2}}{dt^{n-2}}+\ldots+\alpha_1\frac{d}{dt}+\alpha_0\right)^{-1}} \rightarrow\ y(t)$$

One can call this operation the separation of the time-dependent pole $\alpha$.
From (2), we get:

$$\frac{d^n y}{dt^n}+\alpha_{n-2}\frac{d^{n-1}y}{dt^{n-1}}+\ldots+\alpha_1\frac{d^2 y}{dt^2}+\alpha_0\frac{dy}{dt}$$
$$-\alpha\frac{d^{n-1}y}{dt^{n-1}}-\alpha\alpha_{n-2}\frac{d^{n-2}y}{dt^{n-2}}-\ldots-\alpha\alpha_1\frac{dy}{dt}-\alpha\alpha_0 y \tag{3}$$
$$+\frac{d}{dt}\alpha_{n-2}\frac{d^{n-2}}{dt^{n-2}}y+\ldots+\frac{d}{dt}\alpha_1\frac{d}{dt}y+y\frac{d}{dt}\alpha_0=x$$

equivalence of (1) and (3) gives us the system of differential equations:

$$\begin{aligned}
&\alpha_{n-2} - \alpha = a_{n-1} \\
&\alpha_{n-3} - \alpha\alpha_{n-2} + \frac{d\alpha_{n-2}}{dt} = a_{n-2} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\alpha_1 - \alpha\alpha_2 + \frac{d\alpha_2}{dt} = a_2 \\
&\alpha_0 - \alpha\alpha_1 + \frac{d\alpha_1}{dt} = a_1 \\
&-\alpha\alpha_0 + \frac{d\alpha_0}{dt} = a_0
\end{aligned}$$

or:

$$\begin{aligned}
&\alpha_{n-2} = \alpha + a_{n-1} \\
&\alpha_{n-3} = \alpha\alpha_{n-2} + a_{n-2} - \frac{d\alpha_{n-2}}{dt} \\
&\alpha_{n-4} = \alpha\alpha_{n-3} + a_{n-3} - \frac{d\alpha_{n-3}}{dt} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\alpha_1 = \alpha\alpha_2 + a_2 - \frac{d\alpha_2}{dt} \\
&\alpha_0 - \alpha\alpha_1 + a_1 - \frac{d\alpha_1}{dt} \\
&0 = \alpha\alpha_0 + a_0 - \frac{d\alpha_0}{dt}
\end{aligned} \tag{4}$$

The system of equations (4) in the dynamic state can be transformed to the normal Cauchy form:

$$\begin{aligned}
&\frac{d\alpha_{n-2}}{dt} = \alpha\alpha_{n-2} - \alpha_{n-3} + a_{n-2} \\
&\frac{d\alpha_{n-3}}{dt} = \alpha\alpha_{n-3} - \alpha_{n-4} + a_{n-3} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\frac{d\alpha_1}{dt} = \alpha\alpha_1 - \alpha_0 + a_1 \\
&\frac{d\alpha_0}{dt} = \alpha\alpha_0 + a_0 \\
&\alpha = \alpha_{n-2} - a_{n-1}
\end{aligned} \tag{5}$$

While in the static state, for systems with constant coefficients, it evolves to the following form:

$$\alpha_{n-3} = \alpha(\alpha + a_{n-1}) + a_{n-2} = \alpha^2 + a_{n-1}\alpha + a_{n-2}$$

$$\alpha_{n-4} = \alpha(\alpha^2 + a_{n-1}\alpha + a_{n-2}) + a_{n-3} = \alpha^3 + a_{n-1}\alpha^2 + a_{n-2}\alpha + a_{n-3}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (6)$$

$$\alpha_0 = \alpha^{n-1} + a_{n-1}\alpha^{n-2} + \ldots + a_3\alpha^2 + a_2\alpha + a_1$$

$$0 = \alpha^n + a_{n-1}\alpha^{n-1} + \ldots + a_2\alpha^2 + a_1\alpha + a_0$$

At the end of the process, it leads to the classical characteristic equation for the constant- coefficient equation (1). In contrast, the system of equations (4) can be called 'distributed time-dependent characteristic equation' of the variable parameter system and function $\alpha(t)$ is its 'time-dependent eigenvalue'. By solving the differential equation (5), we get the time-dependent coefficients $\alpha_{n-2}$, ..., $\alpha_1$, $\alpha_0$ of the subsystem of reduced order and time-dependent pole $\alpha$. The resulting algorithm that reduces degree of the polynomial, classically is called the Horner scheme. Therefore, it could be called a 'time-dependent Horner's scheme'.

For $n = 2$, it transforms to Vieta's formulas.

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