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THE POLES METHOD FOR SECOND-ORDER LINEAR TIME-VARYING SYSTEMS

METODA BIEGUNÓW DLA UKŁADÓW LINIOWYCH DRUGIEGO RZĘDU O CZASOWO ZALEŻNYCH WSPÓŁCZYNNIKACH

Abstract

In dynamic linear systems described by differential equations with constant parameters, the poles of the rational function (transfer function of the system) play an important role. This article attempts to expand the poles concept in a situation where the system is described by the second-order linear system with time-varying parameters. It then introduces the concept of characteristic equations and time-dependent poles.

Keywords: linear systems, time-varying systems

Streszczenie

W opisie liniowych systemów dynamicznych opisanych przez równania różniczkowe o stałych parametrach ważną rolę odgrywają bieguny funkcji wymiernej (transmitancji systemu). Ten artykuł rozszerza koncepcję biegunów na przypadek, gdy system jest opisany równaniem liniowym drugiego rzędu o zmiennych w czasie parametrach. Pojawiają się tu pojęcia zależnego od czasu równania charakterystycznego i zależnych od czasu biegunów transmitancji.

Słowa kluczowe: równania różniczkowe liniowe, równania różniczkowe parametryczne

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1. Introduction

In the field of linear systems analysis with constant coefficients, the well-known and functioning method is the factorisation method – this consists of transforming complex systems to *a* commutative cascade of the first-order systems.

This requires finding the poles of the transfer function – these are also the zeros of the characteristic polynomial called the eigenvalues of the system which are complex. For a stable system, they lie in the open left half-plane. It turns out that this method also works in the case of linear systems with time-variable coefficients [1–8].

Linear systems of the first order can be described by a differential equation, or by a block diagram:

$$\frac{dy}{dt} - \alpha y = x(t)$$

$$x(t) \rightarrow \boxed{\left(\frac{d}{dt} - \alpha\right)^{-1}} \rightarrow y(t)$$

The operator of the transfer function of the system can be written down using a so-called Green's function in the following form:

$$\left(\frac{d}{dt} - \alpha\right)^{-1} x(t) = \int_{-\infty}^{\infty} e^{\alpha(t-t')} 1(t-t') x(t') dt'$$

where:

$\alpha = \text{const.}$

When α is a time-depended function, its general form is:

$$\left(\frac{d}{dt} - \alpha\right)^{-1} x(t) = \int_{-\infty}^{\infty} e^{\int_{t'}^{t} \alpha(\tau) d\tau} 1(t-t') x(t') dt'$$

where:

$1(t)$ – unit step:

2. Cascade factorisation of the second-order system. The time-dependent 'Vieta's formulas'

The differential equation describing the second-order system is:

$$\frac{dy^2}{dt^2} + a(t)\frac{dy}{dt} + b(t) y(t) = x(t) \tag{1}$$

and it can be written as:

$$\frac{d}{dt}\overbrace{\left(\frac{dy}{dt}-\alpha y\right)}^{u(t)}-\beta\overbrace{\left(\frac{dy}{dt}-\alpha y\right)}^{u(t)}=x(\mathrm{t}) \tag{2}$$

where:

$\alpha(t)$, $\beta(t)$ – unknown functions,

or as a sequence of the 1st-order equations:

$$\frac{du}{dt}-\beta u=x; \qquad \frac{dy}{dt}-\alpha y=u$$

or by using the inverse operator:

$$u=\left(\frac{d}{dt}-\beta\right)^{-1}x; \qquad y=\left(\frac{d}{dt}-\alpha\right)^{-1}u$$

It corresponds to the cascade:

$$\xrightarrow{x(t)}\boxed{\left(\frac{d}{dt}-\beta\right)^{-1}}\xrightarrow{u(t)}\boxed{\left(\frac{d}{dt}-\alpha\right)^{-1}}\xrightarrow{y(t)}$$

which, however, is generally not commutative.

Differential equation (2) can be written as:

$$\frac{dy^2}{dt^2}-(\alpha+\beta)\frac{dy}{dt}+\left(\alpha\beta-\frac{d\alpha}{dt}\right)y=x \tag{3}$$

The equivalence of equations (1) and (3) requires that:

$$\begin{aligned}-(\alpha+\beta)&=a\\ \alpha\beta-\frac{d\alpha}{dt}&=b\end{aligned} \qquad \text{or} \qquad \begin{aligned}a+\alpha+\beta&=0\\ \alpha\beta-b&=\frac{d\alpha}{dt}\end{aligned} \tag{4}$$

This result could be called ‘time-dependent Vieta’s formulas’. For systems with constant coefficients, they become classic Vieta’s formulas.

From the Vieta’s formula (4), it follows the differential equation:

$$b+a\alpha+\alpha^2=-\frac{d\alpha}{dt} \tag{5}$$

which is a generalised characteristic equation of the differential equation (1) and for the constant coefficients, it becomes the classical characteristic equation. Therefore β, α coefficients, which could be called ‘time-dependent poles’ of the parametric differential equation, are determined from equations (5) and (4) [1–4].

The figures below illustrate the problem of the pole stability in the static (*a*, *b* = const) and dynamic (*a*, *b* = var) state.

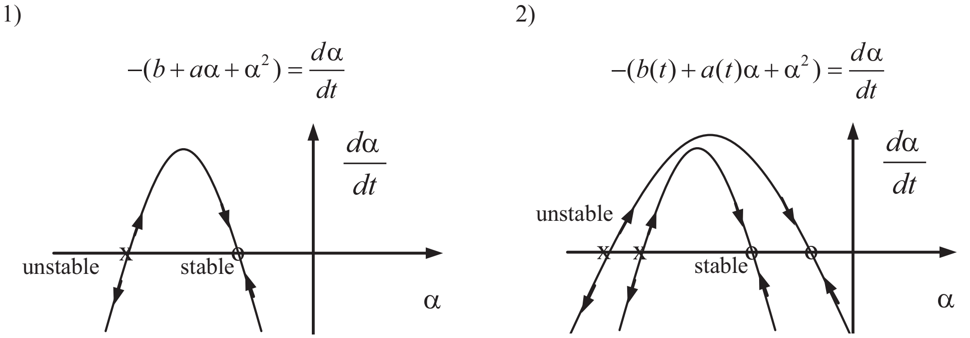

Fig. 1. The pole stability in the 1) static (*a*, *b* = const) and 2) dynamic (*a*, *b* = var) state

This is for the real poles, whereas for the complex ones the differential equation (5) can be rewritten into the form:

$$\frac{d\backslash\alpha^2}{dt} = -2\,\mathrm{Re}(\alpha)\left(b+\left(\frac{a}{\mathrm{Re}(\alpha)}+1\right)|\alpha|^2\right)\Bigg|_{\mathrm{Re}(\alpha)\to\ -\alpha/2} \to a\left(b-|\alpha|^2\right)$$

This guarantees the numerical stability of the complex root [5–10] (Fig. 2).

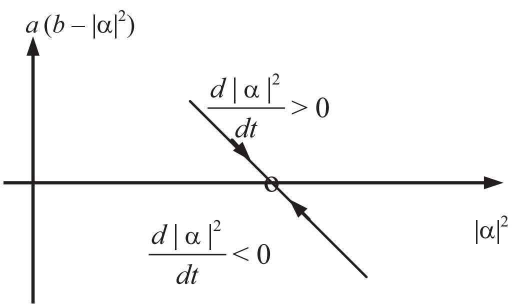

Fig. 2. The complex pole stability condition

3. The inverse problem – synthesis of poles

The inverse problem for the system of the second order is to find the coefficients $a(t)$, $b(t)$ of the differential equation at the pre-set time-variation of the poles $\alpha(t)$ i $\beta(t)$. The case of complex poles and the real equation coefficients will be considered here.

The coefficients $a(t)$, $b(t)$ are obtained from the Vieta's formulas:

$$a = -(\alpha + \beta)$$

$$b = \alpha\beta - \frac{d\alpha}{dt}$$

if they are real, they must satisfy the following equations:

$$\alpha = \alpha_R + j\alpha_I$$

$$\beta = \beta_R - j\alpha_I$$

where:

α_R, α_I, β_R – real-valued functions.

Thus, there is the relationship:

$$b = \alpha_R\beta_R + (\alpha_I)^2 - \frac{d\alpha_R}{dt} + j\left[\alpha_I(\beta_R - \alpha_R) - \frac{d\alpha_I}{dt}\right]$$

and therefore, coefficients of the differential equation are determined by the formulas:

$$a = -(\alpha_R + \beta_R)$$

$$b = \alpha_R\beta_R + (\alpha_I)^2 - \frac{d\alpha_R}{dt}$$

under the condition that:

$$\alpha_I(\beta_R - \alpha_R) - \frac{d\alpha_I}{dt} = 0 \qquad \text{or} \qquad \frac{d\alpha_I}{dt} = \alpha_I(\beta_R - \alpha_R)$$

Finally, a differential equation with separated variables can be solved analytically:

$$\alpha_I(t) = Ke^{\int(\beta_R(t) - \alpha_R(t))dt}, \quad K = \text{const}$$

The resulting formula defines the relationship between the real part and the imaginary part of the poles. In particular, for systems with constant coefficients, α and β must be adjoint to each other , thus $\beta_R = \alpha_R$ which results in $\alpha_I = \text{const.}$

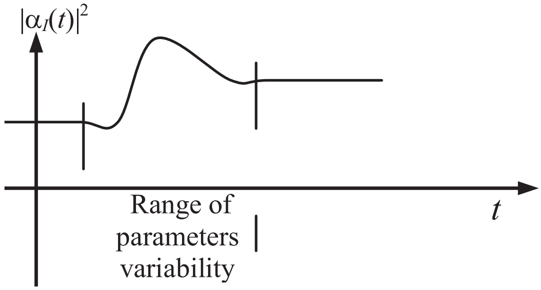

Fig. 3. The function $(\alpha_I(t))^2$ in the range of system parameters variation and outside this interval

Figure 3 shows an example of the function $(\alpha_I(t))^2$ in the range of the variation of the system parameters and outside this interval.

4. The homogeneous differential equation of the second-order and time-dependent Vieta's formulas

For the defined operators:

$$A = \frac{d}{dt} - \alpha; \qquad B = \frac{d}{dt} - \beta$$

the 1st-order homogeneous differential equations have the form:

$$Ay = 0; \qquad By = 0$$

and their solutions are:

$$y_A(t)Ke^{\int \alpha(t)dt}; \qquad y_B(t)Ke^{\int \beta(t)dt}$$

where:

α, β – time-dependent eigenvalues.

The sequences of the operators are:

$$ABy = \frac{dy}{dt}\left(\frac{dy}{dt} - \beta y\right) - \alpha\left(\frac{dy}{dt} - \beta y\right) = \frac{d^2y}{dt^2} - (\alpha + \beta)\frac{dy}{dt} + \left(\alpha\beta - \frac{d\beta}{dt}\right)y$$

and:

$$BAy = \frac{d^2 y}{dt^2} - (\alpha + \beta)\frac{dy}{dt} + \left(\alpha\beta - \frac{d\alpha}{dt}\right)y$$

This allows writing a formula for the operator commutator:

$$[A, B] = AB - BA = \frac{d}{dt}(\alpha - \beta)$$

and to express the 2nd-order homogeneous differential equation as:

$$\left(\frac{dy}{dt}\right)^2 + a(t)\frac{dy}{dt} + b(t)y(t) = \left(\frac{d}{dt} - \alpha\right)\left(\frac{d}{dt} - \beta\right) = 0 \rightarrow ABY = 0$$

where:

$\alpha + \beta + a = 0$,

$\alpha\beta - b - \frac{d\beta}{dt} = 0$ – 'Vieta's formulas'.

Theorem:
If:

$$y_A\!: \; Ay_A = 0$$

$$y_B\!: \; Ay_B = 0$$

then:

$$AB(y_A + y_B) = AB\, y_A = (BA + [A, B])y_A = [A, B]\, y_A$$

or:

$$\frac{d^2}{dt^2}(y_A + y_B) + a\frac{d}{dt}(y_A + y_B) + b(y_A + y_B) = y_A \frac{d}{dt}(\alpha - \beta)$$

where:

$\alpha + \beta + a = 0$,

$\alpha\beta - b - \frac{d\beta}{dt} = 0$,

therefore:

$$b + a\beta + \beta^2 = -\frac{d\beta}{dt}$$

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