TECHNICAL TRANSACTIONS | **CZASOPISMO TECHNICZNE**
FUNDAMENTAL SCIENCES | NAUKI PODSTAWOWE

3-NP/2014

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A NEW APPROACH TO BOUNDED LINEAR OPERATORS ON $C(\omega^*)$

DOMKNIĘTE OPERATORY PRZESTRZENI $C(\omega^*)$ Z NOWEJ PERSPEKTYWY

A b s t r a c t

We discuss recent results on the connection between properties of a given bounded linear operator of $C(\omega^*)$ and topological properties of some subset of ω^* which the operator determines. A family of closed subsets of ω^*, which codes some properties of the operator is defined. An example of application of the method is presented.

Keywords: retraction, projection, ultrafilter, Cech-Stone compactification

S t r e s z c z e n i e

Artykuł przedstawia metodę badania własności ograniczonego operatora liniowego na $C(\omega^*)$ poprzez badanie własności pewnej rodziny domkniętych pozbiorów ω^* wyznaczonej przez ten operator. Przedstawiony został przykład zastosowania tej metody w przypadku projekcji.

Słowa kluczowe: retrakcja, projekcja, ultrafiltr, Cech-Stone compactification

The author is responsible for the language in all paper.

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The Greek letter ω denotes the set of all natural numbers. We use the symbol *fin* for the ideal of finite subsets of ω. For $A, B \subseteq \omega$, the expression $A \subseteq {}^*B$ denotes the relation $B \setminus A \in$ *fin*; similarly $A = {}^*B$ if and olny if $A \div B \in$ *fin*. The space $\omega^* = \beta[\omega]\setminus\omega$ is the growth (Čech-Stone compactification) of the discrete topological space ω. If $A \in P(\omega)/fin$, A^* is the set $A^{\beta[\omega]} \setminus A$. The space ω^* can be viewed as the space of all non-principal ultrafilters on ω. It is well known that $\boldsymbol{B}(\omega^*)$, the algebra of all clopen subsets of ω^*, is isomorphic to $P(\omega)/fin$ (cf. [1]). Thus, for $A, B \in P(\omega)$, the condition $A = {}^*B$ is equivalent to $A^* = B^*$. ***An antichain*** in $\boldsymbol{B}(\omega^*)$ is a family of pairwise disjoint subsets of ω^*. Recall that a set $A \subseteq \omega^*$ is said to have ***ccc (countable chain condition)*** if for every antichain $\{U_\alpha\colon \alpha \in \boldsymbol{I}\} \subseteq \boldsymbol{B}(\omega^*)$, there exists a finite or countable set $\boldsymbol{I}_0 \subseteq \boldsymbol{I}$ such that $A \cap U_\alpha = \varnothing$ for all $\alpha \in \boldsymbol{I}\setminus\boldsymbol{I}_0$.

The space $C(\omega^*)$ consists of all continuous real-valued. functions on ω^* and it can be regarded as l_∞/c_0 i.e. the quotient space of l_∞ by the following equivalence relation:

$$\text{for } f_1, f_2 \in l_\infty,\ \ f_1 \approx^* f_2 \quad \text{iff } \lim_{n\to\infty} (f_1(n) - f_2(n)) = 0$$

Let f^* denote the equivalence class determined by f. Note that for $f_1, f_2 \in l_\infty$, we have $f_1 \approx^* f_2$ iff $f_1|\omega^* \approx^* f_2|\omega^*$, where $f_i\colon \beta[\omega] \to \mathbb{R}$ is a continuous extension of f_i ($i = 1, 2$). Thus, $f^* = f|\omega^*$. An equivalent definition of the (classical) norm on l_∞/c_0 is following:

$$\|f^*\|_* = \sup\ \{\lim_p |f|\colon p \in \omega^*\}$$

where the symbol $\lim_p |f|$ denotes, for an ultrafilter p, the limit to which a sequence $\{|f(n)|\colon n \in \omega\}$ converges with respect to the ultrafilter p. Thus, $C(\omega^*)$, equipped with the supremum norm, is isometric to $(l_\infty/c_0, \|\ .\ \|_*)$.

The domain of function f is denoted by dom f, the range by ran f; supp f is the closure of the set of all elements $p \in$ dom f, such that $f(p) \neq 0$.

The space C(ω^*). It is appropriate to recap on some elemetary properties of functions in space $C(\omega^*)$. Let $f\colon \omega^* ® \mathbb{R}$. (To simplify notation, the sign * will be omitted):

- For every $r \in \mathbb{R}$, the preimage $f^{-1}(r)$ is a closed G_δ set,
- If $f^{-1}(r) \neq \varnothing$, then int $f^{-1}(r) \neq \varnothing$,
- For arbitrary $\varepsilon > 0$, there exist clopen sets $U_1, U_2, \ldots, U_n \in \boldsymbol{B}(\omega^*)$ and reals $r_1, r_2, \ldots, r_n$ such that;

$$\|f - \Sigma_{i \in \{1, \ldots, n\}}\, r_i\, \chi\, U_i\| < \varepsilon,$$

where $\chi\, U_i$ denotes the characteristic function of U_i.

Bounded linear operators on C(ω^*). Assume that $T\colon C(\omega^*) \to C(\omega^*)$ is linear and bounded, and its norm is equal to M.

Fix an ultrafilter $q \in \omega^*$ and define:

$$\mathcal{N}_q = \{U \in \boldsymbol{B}(\omega^*)\colon V \forall \in \boldsymbol{B}(\omega^*)\ V \subseteq U \Rightarrow T(\chi_V)(q) = 0\},\ \mathcal{S}_q = \omega^* \setminus \mathcal{N}_q.$$

$\mathcal{N}_q$ is an open set. Consider $\mathcal{S}_q$. It is closed (by definition) and nowhere dense. To show this suppose that $\mathrm{int}(\mathcal{S}_q) \neq \varnothing$ and argue to a contradiction.

Let $U \in \boldsymbol{B}(\omega^*)$ and $U \subseteq \mathrm{int}(\mathcal{S}_q)$. Consider a family of pairwise disjoint sets $V_\alpha \subseteq U$, $\alpha < \omega_1$. By definition of $\mathcal{S}_q$, for every $\alpha < \omega_1$ there exists $W_\alpha \subseteq V_\alpha$, $W_\alpha \in \boldsymbol{B}(\omega^*)$ such that $T(\chi_{W\alpha})(q) \neq 0$. Thus, for some $\varepsilon > 0$ there exists an uncountable set $\Gamma \subseteq \omega_1$ with:

$$\forall \alpha \in \Gamma\ |T(\chi_{W\alpha})(q)| > 0.$$

Moreover, we may assume that all the $T(\chi_{W\alpha})(q)$ are positive (or negative). Fix $k \in \omega$ such that $k > (M/\varepsilon) + 1$ and a finite set $\Gamma_0 \subseteq \Gamma$ which contains at least k elements. Since T is linear, it follows that:

$$|T(\Sigma_{\alpha\in\Gamma 0}\, \chi_{W\alpha}\})(q)| = |\Sigma_{\alpha\in\Gamma 0}\, T(\chi_{W\alpha})(q)| \geq k\varepsilon > \varepsilon\, [(M/\varepsilon) + 1] > M,$$

this contradicts the assumption that M is the norm of T.

In a similar way we show that $\mathcal{S}_q$ has the c.c.c.

Lemma 1 *Suppose that* $f \in \mathbf{C}(\omega^*)$ *and* $\operatorname{supp} f \cap \mathcal{S}_q = \varnothing$. *Then* $T(f)(q) = 0$.

Proof. Suppose that this is not true. Then, since T is continuous, there exist clopen sets U_1, U_2, ..., $U_n \subseteq$ supp f and reals $r_1, r_2, ..., r_n$ such that:

$$\|f - \Sigma_{i\in\{1,\ldots,n\}}\, r_i\, \chi_{Ui}\| < \varepsilon,$$

for $\varepsilon < |T(f)|(q)/(2M)$. It follows that;

$$|T\,(f - \Sigma_{\ i\in\{1,\ldots,n\}}\, r_i\, \chi_{Ui})| < |T(f)|(q)/2$$

thus $|T(\Sigma_{i\in\{1,\ldots,n\}}\, r_i\, \chi_{Ui})| > |T(f)|(q)/2$. So, there exists $i \leq n$ such that $|T(\chi_{Ui})\, q)| > 0$. Therefore $U_i \setminus \mathcal{N}_q \neq \varnothing$. But it implies that $\varnothing \neq \mathcal{S}_q \cap U_i \subseteq \mathcal{S}_q \cap \operatorname{supp} f = \varnothing$, a contradiction.

Note that the condition $T(f)(q) = 0$ does not imply that $\mathcal{S}_q \cap \operatorname{supp} f = \varnothing$. Now an example of application of the notion $\mathcal{S}_q$ is presented.

Projections of $C(\omega^*)$ and retractions of ω^*. Assume that r: ω^*® $F \subseteq \omega^*$ is a retraction (i.e. r is continuous and $\mathbf{r} \circ \mathbf{r} = \mathbf{r}$). Recall how to define a projection P: $\mathbf{C}(\omega^*) \rightarrow V$ (i.e. a bounded linear operator such that $\mathbf{P} \circ \mathbf{P} = \mathbf{P}$) by using $\mathbf{r}$ (cf. [2]). For $f \in \mathbf{C}(\omega^*)$, $q \in \omega^*$ put:

$$\mathbf{P}(f)(q) = f(\mathrm{r}(q)).$$

$\mathbf{P}$ is linear and for every $f \neq \mathbf{C}(\omega^*)$, $\|\mathbf{P}(f)\| \leq$ \leq $\|f\|$, thus $\mathbf{P}$ is bounded. Moreover:

$$\mathbf{P}(\mathbf{P}(f))(q) = \mathbf{P}(f)(\mathrm{r}(q)) = f(\mathbf{r}(\mathbf{r}(q))) = f(\mathbf{r}(q)) = \mathbf{P}(f)(q).$$

A retraction of ω^* induces a projection of $\mathbf{C}(\omega^*)$. One can ask if a projection determines a retraction. In order to (partially) answer this question, an equivalence relation on ω^* can be defined:

$$p, q \in \omega^*, p \approx q \text{ iff for all } U \in \mathbf{B}(\omega^*),\ \mathbf{P}(\chi_U)(q) = \mathbf{P}(\chi_U)(p).$$

Note that:

- if $p \approx q$ then $\mathcal{S}_p = \mathcal{S}_q$,
- the equivalence class $[p] = \bigcup_{U\in \boldsymbol{B}(\omega^*)} (\mathbf{P}(\chi_U))^{-1}\, (\{P(\chi_U)(p)\})$ is a closed subset of ω^*.

Theorem 1 *Assume that* $\mathbf{P}$: $\mathbf{C}(\omega^*) \rightarrow V$ *is a projection and the following assertion is satisfied:*

$$\textit{for each } p \in \omega^* \textit{ there exists } q_p \in [p] \textit{ such that } \mathcal{S}_p = \{q_p\}.$$

Then $\mathbf{r}$: $\omega^* \ni p \rightarrow q_p \in \bigcup_{p\in\omega^*} \mathcal{S}_p$ *is a retraction.*

Proof. Since $q_p \approx p$, $\mathcal{S}_q = \mathcal{S}_p = \{q_p\}$ and $\mathrm{r}(q_p) = q_p$. Therefore $\mathbf{r} \circ \mathbf{r} = \mathbf{r}$.

We shall show that r is continuous. Let $\tilde{U}$ be an open subset of $\bigcup_{p\in\omega^*} \mathcal{S}_p$. Fix $q_p \in \tilde{U}$. Thus, there exists a U open subset of ω^* and $V \in \mathbf{B}(\omega^*)$ such that $U \cap \bigcup_{p\in\omega^*} \mathcal{S}_p$ and $q_p \in V \subseteq U$.

Since $S_{qp} = \{q_p\}$, it follows that $\mathbf{P}(\chi_V)(q_p) = x_p \neq 0$. Assume that for some $s \in \omega$, $P(\chi_V)(q_p) = x_p \neq 0$. Thus, $\{q_p\} \cap V = S_{qp} \cap V$, which implies that $q_s \in V$.

We showed that $q_s \in V \Rightarrow P(\chi_V)(q_s) \neq 0$. Put $W = (P(f))^{-1}(\mathbb{R}\setminus\{0\})$. W is open and $r(W) \subseteq V \cap \bigcup_{p\in\omega^*} S_p$. This finishes the proof.

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