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THE EXISTENCE AND UNIQUENESS OF SOLUTIONS
OF THE DIRICHLET NONLOCAL PROBLEM WITH A
NONLOCAL INITIAL CONDITION

ISTNIENIE I JEDNOZNACZNOŚĆ ROZWIĄZAŃ
NIELOKALNEGO ZAGADNIENIA DIRICHLETA Z
NIELOKALNYM WRUNKIEM POCZĄTKOWYM

Abstract

The aim of this paper is to prove the existence and uniqueness of solutions of the Dirichlet nonlocal problem with nonlocal initial condition. The considerations are extensions of results by E. Andreu-Vaillo, J. M. Mazón, J. D. Rossi and J. J. Toledo-Melero [1].

Keywords: existence and uniqueness of solutions, Dirichlet problem, nonlocal problem, nonlocal initial condition

Streszczenie

W artykule udowodniono istnienie i jednoznaczność rozwiązań nielokalnego zagadnienia Dirichleta z nielokalnym warunkiem początkowym. Rozważania są rozszerzeniami rezultatów otrzymanych przez E. Andreu-Vaillo, J. M. Mazón, J. D. Rossi i J. J. Toledo-Melero [1].

Słowa kluczowe: istnienie i jednoznaczność rozwiązań, zagadnienie Dirichleta, zagadnienie nielokalne, nielokalny warunek początkowy

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1. Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Moreover, let T be a fixed positive number and $k \in \mathbb{R} \setminus \{0\}$.

We will need the following assumption:

Assumption (H) (see: [1]). $J \in C(\mathbb{R}^n, \mathbb{R})$ is a nonnegative radial function with $J(0) > 0$ and

$$\int_{\mathbb{R}^n} J(x) dx = 1.$$

In [1], the existence and uniqueness of a solution of the following nonlocal Dirichlet boundary value problem

$$\begin{cases} u_t(x, t) = \int_{\mathbb{R}^n} J(x-y)(u(y, t) - u(x, t)) dy, & x \in \Omega, \quad t > 0, \\ u(x, t) = g(x, t), & x \notin \Omega, \quad t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega \end{cases}$$

is studied.

For this purpose the Banach fixed point theorem is applied in [1].

The existence and uniqueness of solutions of differential problems were, also, studied using the Banach fixed point theorem, by Kamont [2], Muszyński and Myszkiński [3], and Pelczar and Szarski [4].

The aim of the paper is to give a theorem on the existence and uniqueness of a solution of the following nonlocal Dirichlet boundary value problem together with the nonlocal initial condition

$$\begin{cases} u_t(x, t) = \int_{\mathbb{R}^n} J(x-y)(u(y, t) - u(x, t)) dy, & x \in \Omega, \quad t \in (0, T), \\ u(x, t) = g(x, t), & x \notin \Omega, \quad t \in (0, T), \\ u(x, 0) + kTu(x, T) = u_0(x), & x \in \Omega. \end{cases} \quad (1.1)$$

For this purpose we will also apply the Banach fixed point theorem.

We will need the assumption:

Assumption (F). $u_0 \in L^1(\Omega)$ and $g \in C((0, T); L^1(\mathbb{R}^n \setminus \Omega))$.

2. Existence and uniqueness of Solutions

Let Assumptions (H) and (F) be satisfied in this section.

Definition 2.1. A function $u \in C([0, T]; L^1(\mathbb{R}^n))$ is said to be a solution of nonlocal problem (1.1) if

$$\begin{aligned}
u(x, t) &= u_0(x) - kTu(x, T) \\
&+ \int_0^t \int_{\mathbb{R}^n} J(x-y)(u(y, s) - u(x, s)) dy ds, \quad x \in \Omega, \quad t \in (0, T),
\end{aligned}$$

and

$$u(x, t) = g(x, t) \text{ for } x \notin \Omega, \quad t \in (0, T).$$

Consider the Banach space

$$X_T = \{w \in C([0, T]; L^1(\Omega))\}$$

with the norm

$$\|w\| = \max_{0 \leq t \leq T} \|w(\cdot, t)\|_{L^1(\Omega)}.$$

The solution of problem (1.1) will be obtained as a fixed point of the operator

$$\mathcal{T}_{w_0} : X_T \longrightarrow X_T$$

defined by the formula

$$\begin{aligned}
\mathcal{T}_{w_0}(w)(x, t) &= w_0(x) - kTw(x, T) \\
&+ \int_0^t \int_{\mathbb{R}^n} J(x-y)(w(y, s) - w(x, s)) dy ds, \quad x \in \Omega, \quad t \in (0, T),
\end{aligned}$$

where

$$w(x, t) = g(x, t) \text{ for } x \notin \Omega, \quad t \in (0, T).$$

To prove the existence and uniqueness of the solution of problem (1.1), we will need the following lemma:

Lemma 2.1. *Let $w_0, z_0 \in L^1(\Omega)$. Then there is a constant*

$$C = |k| + \tilde{k}, \quad \text{where } \tilde{k} > 0, \tag{2.2}$$

depending on J and Ω such that

$$\|\mathcal{T}_{w_0}(w) - \mathcal{T}_{z_0}(z)\| \leq \|w_0 - z_0\|_{L^1(\Omega)} + CT\|w - z\|$$

for all $w, z \in X_T$.

Proof. Observe that

$$\int_{\Omega} |\mathcal{T}_{w_0}(w)(x, t) - \mathcal{T}_{z_0}(z)(x, t)| dx \leq$$

$$\begin{aligned}
& \leq \int_{\Omega} |w_0 - z_0| dx + |k| T \int_{\Omega} |w(x, T) - z(x, T)| dx \\
& + \int_{\Omega} \left| \int_0^t \int_{\mathbb{R}^n} J(x-y)[(w(y, s) - z(y, s)) - (w(x, s) - z(x, s))] dy ds \right| dx \\
& \leq \|w_0 - z_0\|_{L^1(\Omega)} + |k| T \|w - z\| + \tilde{k} T \|w - z\| \\
& = \|w_0 - z_0\|_{L^1(\Omega)} + (|k| + \tilde{k}) T \|w - z\|, \quad w, z \in X_T,
\end{aligned}$$

where \tilde{k} is a positive constant depending on J and Ω .

Consequently, since $w - z$ vanishes outside of Ω then

$$\|\mathcal{T}_{w_0}(w) - \mathcal{T}_{z_0}(z)\| \leq$$

$$\|w_0 - z_0\|_{L^1(\Omega)} + CT \|w - z\| \quad \text{for } w, z \in X_T.$$

The proof of Lemma 2.1 is complete.

Applying Lemma 2.1 we will prove the existence and uniqueness of the solution of problem (1.1).

Theorem 2.1 *Let Assumptions (H) and (F) be satisfied. Moreover, let $CT < 1$, where C is given by (2.2).*

Then there is a unique solution of problem (1.1) on the interval $[0, T]$.

Proof. Firstly, we will show that \mathcal{T}_{u_0} maps X_T into X_T . Let $z_0 \equiv 0$, $z \equiv 0$ and $w_0 \equiv u_0$ in Lemma 2.1. Then

$$\mathcal{T}_{u_0}(w) \in C([0, T]; L^1(\Omega))$$

for $w \in X_T$.

Since $CT < 1$ then taking $z_0 \equiv w_0 \equiv u_0$ in Lemma 2.1 we get that \mathcal{T}_{u_0} is a strict contraction in X_T and the existence and uniqueness of the solution of problem (1.1) follows from the Banach fixed point theorem on the interval $[0, T]$.

The proof of Theorem 2.1 is complete.

References

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