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THE DEGREE OF APPROXIMATION OF FUNCTIONS
FROM EXPONENTIAL WEIGHT SPACESRZĄD APROKSYMACJI FUNKCJI Z WYKŁADNICZYCH
PRZESTRZENI WAGOWYCH

Abstract

This paper presents a study of the approximation properties of modified Szász-Mirakyan operators for functions from exponential weight spaces. We present theorems giving the degree of approximation by these operators using a modulus of continuity.

Keywords: linear positive operators, Bessel function, modulus of continuity, degree of approximation

Streszczenie

W artykule badamy aproksymacyjne własności zmodyfikowanych operatorów typu Szász-Mirakjana dla funkcji z wykładniczych przestrzeni wagowych. Przedstawiamy twierdzenia podające rząd aproksymacji funkcji przez operatory tego typu, wykorzystując moduł ciągłości.

Słowa kluczowe: dodatni operator liniowy, funkcja Bessela, moduł ciągłości, rząd aproksymacji

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1. Introduction

Let us denote the set of all real-valued functions continuous on $\mathbb{R}_0 = [0; \infty)$ by $C(\mathbb{R}_0)$. In paper [6] we investigated Szász-Mirakyan type operators defined as follows

$$L_n^\nu(f; x) = \begin{cases} \sum_{k=0}^{\infty} q_{n,k}^\nu(x) f\left(\frac{2k}{n+p}\right), & x > 0; \\ f(0), & x = 0 \end{cases} \quad (1)$$

and

$$q_{n,k}^\nu(x) = \frac{1}{I_\nu(nx)} \frac{(nx)^{2k+\nu}}{2^{2k+\nu} \Gamma(k+1) \Gamma(k+\nu+1)},$$

where Γ is the gamma function and I_ν the modified Bessel function defined by

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{z^{2k+\nu}}{2^{2k+\nu} \Gamma(k+1) \Gamma(k+\nu+1)}.$$

Approximation properties of these operators in exponential weight spaces were studied. Such spaces were denoted by

$$E_p = \{f \in C(\mathbb{R}_0): w_p f \text{ is uniformly continuous and bounded on } \mathbb{R}_0\},$$

where w_p is the exponential weight function defined as follows

$$w_p(x) = e^{-px}, \quad p \in \mathbb{R}_0 \quad (2)$$

for $x \in \mathbb{R}_0$.

In the spaces we introduced the norm

$$\|f\|_p = \sup\{w_p(x)|f(x)|: x \in \mathbb{R}_0\} \quad (3)$$

and we established ([6], Theorem 2.1) that operators L_n^ν are linear, positive, bounded and transform the space E_p into E_p .

In the present paper, we shall prove theorems giving a degree of approximation of functions from E_p by these operators. We use the weighted modulus of continuity of the first and the second order defined as follows,

$$\omega(f, E_p; t) = \sup\{\|\Delta_h f\|_p : h \in [0, t]\} \quad (4)$$

and

$$\omega^2(f, E_p; t) = \sup\{\|\Delta_h^2 f\|_p : h \in [0, t]\} \quad (5)$$

respectively, where

$$\Delta_h f(x) = f(x+h) - f(x), \quad \Delta_h^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$$

for $x, h \in \mathbb{R}_0$.

The note was inspired by the results of [8, 9] which investigate approximation problems for integral operators defined in weighted spaces. The considered method of proving the main theorems is also found in papers [1-4, 10].

2. Auxiliary results

The preliminary results, which we immediately obtained from papers [5-7] and definition (1), are recalled below.

Lemma 2.1 ([5], Lemma 8)

For all $v \in \mathbb{R}_0$ there exists a positive constant $M(v)$ such that for all $n \in \mathbb{N}$ and $x \in \mathbb{R}_0$, we have

$$\left| \frac{I_{v+1}(nx)}{I_v(nx)} \right| \leq M(v), \quad nx \left| \frac{I_{v+1}(nx)}{I_v(nx)} - 1 \right| \leq M(v).$$

Through elementary calculations we get

Lemma 2.2 ([6], Lemma 2.2)

For all $n \in \mathbb{N}$, $v, p \in \mathbb{R}_0$ and $x \in \mathbb{R}_0$

$$L_n^v(1, x) = 1, \quad L_n^v(t, x) = \frac{nx}{n+p} \frac{I_{v+1}(nx)}{I_v(nx)},$$

$$L_n^v(t^2, x) = \left(\frac{nx}{n+p} \right)^2 \frac{I_{v+2}(nx)}{I_v(nx)} + \frac{2nx}{(n+p)^2} \frac{I_{v+1}(nx)}{I_v(nx)},$$

$$\begin{aligned}
L_n^v(t-x, x) &= x \left(\frac{n}{n+p} \frac{I_{v+1}(nx)}{I_v(nx)} - 1 \right), \\
L_n^v((t-x)^2; x) &= x^2 \left(\left(\frac{n}{n+p} \right)^2 \frac{I_{v+2}(nx)}{I_v(nx)} - \frac{2n}{n+p} \frac{I_{v+1}(nx)}{I_v(nx)} + 1 \right) \\
&\quad + \frac{2nx}{(n+p)^2} \frac{I_{v+1}(nx)}{I_v(nx)}.
\end{aligned}$$

Lemma 2.3 ([6], Lemma 2.5)

For all $v, p \in \mathbb{R}_0$ there exists a positive constant $M(v, p)$ such that for all $n \in \mathbb{N}$, we have

$$\|L_n^v(1/w_p; \cdot)\|_p \leq M(v, p).$$

An obvious consequence of the above lemma and definition (3) is

Theorem 2.4 ([6], Theorem 2.1)

For all $v, p \in \mathbb{R}_0$ there exists a positive constant $M(v, p)$ such that for all $n \in \mathbb{N}$ and $f \in E_p$, we have

$$\|L_n^v(f; \cdot)\|_p \leq M(v, p) \|f\|_p.$$

Applying Lemma 2.1 and Lemma 2.2, we obtain

Lemma 2.5

For all $v, p \in \mathbb{R}_0$ there exists a positive constant $M(v, p)$ such that for all $n \in \mathbb{N}$ and $x \in \mathbb{R}_0$, we have

$$|L_n^v((t-x)^2; x)| \leq M(v, p) \frac{x(x+1)}{n}.$$

Lemma 2.6 ([6], Lemma 2.6)

For all $v, p \in \mathbb{R}_0$ there exists a positive constant $M(v, p)$ such that for all $n \in \mathbb{N}$ and $x \in \mathbb{R}_0$, we have

$$w_p(x) \left| L_n^v \left(\frac{(t-x)^2}{w_p(t)}; x \right) \right| \leq M(v, p) \frac{x(x+1)}{n}.$$

3. Approximation theorems

The following theorems estimate the weighted error of approximation for functions belonging to the spaces $E_p^k = \{f \in E_p: f', f'', \dots, f^{(k)} \in E_p\}$, where $f^{(i)}$ is denoted the i -th derivative of f .

Theorem 3.1

For all $\nu, p \in \mathbb{R}_0$ there exists a positive constant $M(\nu, p)$ such that for all $n \in \mathbb{N}$, $x \in \mathbb{R}_0$ and $g \in E_p^1$, we have

$$w_p(x)|L_n^\nu(g; x) - g(x)| \leq M(\nu, p)\|g'\|_p \sqrt{\frac{x(x+1)}{n}}.$$

Theorem 3.2

For all $\nu, p \in \mathbb{R}_0$ there exists a positive constant $M(\nu, p)$ such that for all $n \in \mathbb{N}$, $x \in \mathbb{R}_0$ and $f \in E_p$, we have

$$w_p(x)|L_n^\nu(f; x) - f(x)| \leq M(\nu, p)\omega\left(f, E_p; \sqrt{\frac{x(x+1)}{n}}\right).$$

The proof for the above theorems is analogous to the proof of Theorem 4 and Theorem 5 which are detailed in paper [5].

Theorem 3.2 implies the following corollary.

Corollary 3.3

If $\nu, p \in \mathbb{R}_0$ and $f \in E_p$, then for all $x \in \mathbb{R}_0$

$$\lim_{n \rightarrow \infty} \{L_n^\nu(f; x) - f(x)\} = 0.$$

Moreover, the above convergence is uniform on every set $[x_1, x_2]$ with $0 \leq x_1 < x_2$.

Remark 3.4

The above result can be achieved in a different way; see [7] for more details.

Analogous with papers [8, 9], we define operators H_n^ν to estimate the error of approximation by the second moduli of continuity (5).

$$H_n^\nu(f; x) = L_n^\nu(f; x) - f(L_n^\nu(t; x)) + f(x) \tag{6}$$

for $\nu, p \in \mathbb{R}_0$, $f \in E_p$ and $x \in \mathbb{R}_0$. By using Lemma 2.2 we obtain

$$H_n^\nu(f; x) = L_n^\nu(f; x) - f\left(\frac{nx}{n+p} \frac{I_{\nu+1}(nx)}{I_\nu(nx)}\right) + f(x).$$

Observe that the operators are linear. Moreover, Lemma 2.2 allows us to write

$$H_n^\nu(1; x) = 1, \quad H_n^\nu(t - x; x) = 0. \quad (7)$$

Lemma 3.5

For all $\nu, p \in \mathbb{R}_0$ there exists a positive constant $M(\nu, p)$ such that for all $n \in \mathbb{N}$, $x \in \mathbb{R}_0$ and $g \in E_p^2$, we have

$$w_p |H_n^\nu(g; x) - g(x)| \leq M(\nu, p) \|g''\|_p \frac{x(x+1)}{n}.$$

Proof. Let $x \in \mathbb{R}_0$ and $g \in E_p^2$ be fixed. Through the use of the Taylor formula we can write

$$g(t) - g(x) = (t - x)g'(x) + \int_x^t (t - u)g''(u) du$$

for $t > 0$. By applying the linearity of H_n^ν and (7) we derive

$$|H_n^\nu(g; x) - g(x)| = |H_n^\nu(g(t) - g(x); x)| = \left| H_n^\nu\left(\int_x^t (t - u)g''(u)du; x\right) \right|. \quad (8)$$

Furthermore, the definition of H_n^ν implies

$$\begin{aligned} H_n^\nu\left(\int_x^t (t - u)g''(u)du; x\right) \\ = L_n^\nu\left(\int_x^t (t - u)g''(u)du; x\right) - \int_x^{L_n^\nu(t; x)} (L_n^\nu(t; x) - u)g''(u)du \end{aligned}$$

Estimating (8), we have

$$\begin{aligned} |H_n^\nu(g; x) - g(x)| \\ \leq L_n^\nu\left(\left|\int_x^t (t - u)g''(u)du\right|; x\right) + \left|\int_x^{L_n^\nu(t; x)} (L_n^\nu(t; x) - u)g''(u)du\right|. \end{aligned}$$

Since

$$\left| \int_x^t (t-u)g''(u)du \right| \leq \frac{1}{2} \|g''\|_p (t-x)^2 (e^{px} + e^{pt})$$

and

$$\begin{aligned} \left| \int_x^{L_n^v(t;x)} (L_n^v(t;x) - u)g''(u)du \right| &\leq \frac{1}{2} \|g''\|_p (L_n^v(t;x) - x)^2 (e^{px} + e^{pL_n^v(t;x)}) \\ &\leq \frac{1}{2} \|g''\|_p (L_n^v(t-x;x))^2 e^{px} (1 + e^{pL_n^v(t-x;x)}) \\ &\leq \frac{1}{2} M(v,p) \|g''\|_p (L_n^v(t-x;x))^2 e^{px} \end{aligned}$$

we get

$$\begin{aligned} w_p(x) |H_n^v(g;x) - g(x)| &\leq \frac{1}{2} \|g''\|_p L_n^v((t-x)^2;x) + \frac{1}{2} \|g''\|_p w_p(x) L_n^v\left(\frac{(t-x)^2}{w_p(t)};x\right) \\ &\quad + \frac{1}{2} M(v,p) \|g''\|_p (L_n^v(t-x;x))^2 \end{aligned}$$

Applying Hölder's inequality to the term $L_n^v(t-x;x)$ and Lemmas 2.5, 2.6, we obtain the desired estimation.

Theorem 3.6

For all $v, p \in \mathbb{R}_0$ there exists a positive constant $M(v,p)$ such that for all $n \in \mathbb{N}$, $x \in \mathbb{R}_0$ and $f \in E_p$, we have

$$\begin{aligned} w_p |L_n^v(f;x) - f(x)| &\leq M(v,p) \omega^2\left(f, E_p; \sqrt{\frac{x(x+1)}{n}}\right) + \omega(f, E_p; |L_n^v(t-x;x)|). \end{aligned}$$

Proof. Let $x \in \mathbb{R}_0$ and f_h be the second order Steklov mean of $f \in E_p$, i.e.

$$f_h(x) = \frac{4}{h^2} \int_0^{\frac{h}{2}} \int_0^{\frac{h}{2}} \{2f(x+s+t) - f(x+2(s+t))\} ds dt, \quad x \in \mathbb{R}_0, h > 0.$$

Notice that

$$f(x) - f_h(x) = \frac{4}{h^2} \int_0^{\frac{h}{2}} \int_0^{\frac{h}{2}} \Delta_{s+t}^2 f(x) ds dt.$$

By definitions (3) and (5), we get the following estimations

$$\|f - f_h\|_p \leq \omega^2(f, E_p; h)$$

and since

$$f_h''(x) = \frac{1}{h^2} \left(8\Delta_{h/2}^2 f(x) - \Delta_h^2 f(x) \right)$$

we can write

$$\|f_h''\|_p \leq \frac{9}{h^2} \omega^2(f, E_p; h).$$

The above inequalities imply that the Steklov mean f_h and f_h'' belong to E_p . Moreover, by linearity of L_n^v and connection (6), we have

$$\begin{aligned} & |L_n^v(f; x) - f(x)| \\ & \leq H_n^v(|f - f_h|; x) + |f(x) - f_h(x)| + |H_n^v(f_h; x) - f_h(x)| \\ & \quad + |f(L_n^v(t; x)) - f(x)|. \end{aligned}$$

Applying the above estimation, Theorem 2.1 and Lemma 3.5, we conclude that

$$\begin{aligned} & w_p(x) |L_n^v(f; x) - f(x)| \\ & \leq w_p(x) H_n^v(|f - f_h|; x) + w_p(x) |f(x) - f_h(x)| \\ & \quad + w_p(x) |H_n^v(f_h; x) - f_h(x)| + w_p(x) |f(L_n^v(t; x)) - f(x)| \\ & \leq M(v, p) \|f - f_h\|_p + M(v, p) \|f_h''\|_p \frac{x(x+1)}{n} \\ & \quad + w_p(x) |f(L_n^v(t; x)) - f(x)| \\ & \leq M(v, p) \omega^2(f, E_p; h) \left(1 + \frac{1}{h^2} \frac{x(x+1)}{n} \right) \\ & \quad + \omega(f, E_p; |L_n^v(t-x; x)|), \end{aligned}$$

where $L_n^v(t-x; x) = x \frac{I_{v+1}(nx)}{I_v(nx)} - x$. Substituting $h = \sqrt{\frac{x(x+1)}{n}}$, we get the assertion of our theorem.

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