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CONCEPTS OF A NUMBER OF C. MÉRAY, E. HEINE, G. CANTOR, R. DEDEKIND AND K. WEIERSTRASS

KONCEPCJE LICZBY CH. MÉRAYA, E. HEINEGO, G. CANTORA, R. DEDEKINDA I K. WEIERSTRASSA

A b s t r a c t

The article is devoted to the evolution of concept of a number in XVIII–XIX c. Ch. Méray's, H. Heine's, R. Dedekind`s, G. Cantor's and K. Weierstrass`s constructions of a number are considered. Only original sources were used.

Keywords: *concept of a number*, *Cantor*, *Dedekind*, *Méray*, *Heine*, *Weierstrass*

S t r e s z c z e n i e

Artykuł poświęcony jest rozwojowi koncepcji liczb w XVIII i XIX wieku. Rozważane są konstrukcje liczb Ch. Meraya, E. Heinego, G. Cantora, R. Dedekinda i K. Weierstrassa. W rozważaniach wykorzystano wyłącznie oryginalne źródła.

Słowa kluczowe: *koncepcje liczb*, *Cantor*, *Dedekind*, *Méray*, *Heine*, *Weierstrass*

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Let us review how the concept of a number developed in the 18th and 19th century.

1707. I. Newton (1642–1727)

"We mean by number not an aggregate of units, but rather a dimensionless ratio of a value to another value of the same nature taken as a unit. There may be three types of a number: an integer, a fraction, and a surd. An integer is something that is measured by a unit; a fraction is a multiple of a part of unit; and a surd is incommensurable with a unit". [1, p. 8].

1758. A. G. Kaestner (1719–1800)

"Fractions are whole numbers, a unit of which is a part of the initially chosen unit; irrational values are fractions, a unit of which is variable and represents an ever reducing part of a whole. Irrational numbers are non-extracted roots. Any such number may be put between two arbitrary close rational approximations. It is a priori assumed that the root $\sqrt[n]{a}$ exists, where a is not an *n-th* power of a rational number, and that arithmetical operations with them are possible" [2].

1821. A. Cauchy (1789–1857)

"If variables keep approximating a certain value, so that finally there is an arbitrary small difference between the variables and this value, the latter value is called a limit. Thus, e.g. the area of a circle is a limit which is approximated by areas of regular inscribed polygons: the greater is the number of their sides, the closer the approximation.

Note that measurements of a line or an arc may represent a numerical value which precisely corresponds to this length, or were obtained as a numerical result of gradual approximations from either side to a fixed point (let's call it an initial point), increasing or decreasing in length as they approach it" [3, p. 349].

Cauchy did not set any rules of procedure for irrational numbers.

1869, 1872. Ch. Méray (1835–1911)

In 1869, Méray laid down two principles of the theory of irrational numbers (immeasurable, incommensurable numbers): "1. Variable v, which sequentially takes value $v_1, v_2, \ldots, v_n, \ldots$, tending to a certain limit, if its components keep growing or decreasing, however, remaining, in the first case, less, and, in the second case, greater than a certain fixed numerical value. 2. An additional property of variable ν is that difference $v_{n+p} - v_n$ tends to zero at n increasing without limit, whatever the relation of n to p may be" [4].

Méray named irrational numbers (whether algebraic or transcendental) as immeasurable numbers.

His reasoning of 1872 was as follows ([5], Méray's italics):

"Let us call numerical value $v_{m,n}$ (whether a whole number or a fraction, positive or negative) the amount whereof depends on the value of integers m, n, …, taken in whatsoever combination of values and numbered with these indices, a variant, e.g.: $v_m = 1 + \frac{1}{2} + \ldots + \frac{1}{m-1} + \frac{1}{m}$, $v_{m,n} = \frac{1}{mn}$ is a variant of two indices.

1. If there is a V for which at sufficiently large m, n, …, difference $V - v_{m,n,\ldots}$ is arbitrary small in its absolute value for sufficiently large values of the indices, variant $v_{m,n,\ldots}$ is said to tend or converge to limit V.

If $V = 0$, variant $v_{m,n,\dots}$ is called infinitely small, as, for example, the difference between the variant and its limit.

Among variants that have no limits, one should mention those the absolute value whereof can become greater than any prescribed number; they are called *infinite* values, while those the numerical value whereof is less than a finite number are called *finite* values.

2. It is easy to assert as follows:

I. A sum and product (or product of powers) of a certain number of finite variants and constant values will be a finite value. This applies to the relation of two similar values if the denominator is not infinitely small.

II. A product of an infinitely small and constant or finite value, a sum of a certain number of such products (positive powers) and an infinitely small value which is opposite to the infinitely large value, will be an infinitely small variant.

III. A power with infinite positive index of a certain constant value or variant will be infinitely large or infinitely small, depending on the final absolute value thereof, i.e. if it exceeds an amount > 1 or is less than < 1.

IV. A sum and product (or product of powers) of a certain number of variants which have limits and a constant value have at the limit a result which would be obtained if the limit of these values is inserted in this calculation. The same applies to the ratio of two similar values, if the denominator is not infinitely small.

Immeasurable numbers

3. Let us call variant $v_{m+n,\dots}$, for which the difference between $v_{m+p,n+q,\dots}$ and $v_{m+n,\dots}$ for arbitrary p and q is less than any infinitely small variant with indices m and n, that is to say, this difference tends to zero for m, n which are infinite regardless of p and q, a convergent variant.

4. Two variants $v_{m,n,\dots}$ and $v'_{m',n',\dots}$ are equivalent if their difference $v_{m,n,\dots} - v'_{m',n',\dots}$ considered as a single variant with indices $m, n, \dots, m', n', \dots,$ is infinitely small.

Having ascertained the above, we will easily prove that:

A sum and a product (or product of powers) of a certain number of convergent variants and fixed values will be a convergent variant equivalent to a variant that would be a replacement of respective equivalents. The same applies to a ratio as well, if the denominator is not infinitely small.

5. This assertion is trivial if limits of the variants are certain numbers. However, if any of them do not converge to any *numerical* limit, this assertion is also true.

Nevertheless, let us admit that, in a figurative sense, this means that an invariant converges to a fictitious *immeasurable* limit if it converges to a point which cannot be accurately determined. If incommensurable limits of two converging variants are equal, such variants will be equivalent; a sum, product, etc. of variants converging to a certain limit, whether real or fictitious, as case may be, is a sum or product, etc. of their real or fictitious limits. If we supplement these conditions, the statements we set forth above are true and correct, as well as the cited theorems.

6. *Converging variant which is not infinitely small is finite, given the certain sign is retained*. According to our hypothesis, there is an infinite number of combinations of $m, n, \ldots$ values, corresponding to $v_{m,n,\ldots}$, the absolute value of which exceeds the fixed number δ. Let us attach sufficiently large values to $m, n, \ldots$, so that $v_{m+p,n+q,\ldots} - v_{m,n,\ldots}$ would be numerically less than δ, whatever $p, q, \ldots$ might be. Whereas $v_{m+p,n+q,\ldots}$ equals $v_{m,n,\ldots} + (v_{m+p,n+q,\ldots} - v_{m,n,\ldots})$, this equality is correct for all $p, q, \ldots$, that is to say, for all indices equal to or exceeding $v_{m,n,\ldots}$.

   Moreover, if two variants $v_{m,n,\ldots}$ and $v'_{m',n',\ldots}$ converge to incommensurable limits and are not equivalent, their difference $v_{m,n,\ldots} - v'_{m',n',\ldots}$ is finite and retains the certain sign. Depending on whether it is + or –, we would say that the immeasurable limit of the first one is greater or less than that of the second one.

   In the same way, a measurable number a is said to be greater or less than the immeasurable finite number for variant $v_{m,n,\ldots}$, depending on whether $a - v_{m,n,\ldots}$ is > or < 0.

If the absolute value of this finite difference remains less than ε, we will call it the *value* of an immeasurable number converged in accordance with ε with an excess in the first case and deficiency in the second case.

We will determine all immeasurable numbers, approximating their values with the help of a δ*, however small it might seem*" [5].

1872. H. E. Heine (1821–1881)

"The theory of functions is for the most part developed using elementary fundamental theorems, although insightful research casts some doubt on certain results, as research results are not always well argued. I can explain it by the fact that, although Mr. Weierstrass" principles are set forth directly in his lectures and indirect verbal communications, and in manuscript copies of his lectures, and are quite widely spread, they have not been published as worded by the author, under the author's control, which hampers the uniformity of perception. His statements are based on an incomplete definition of irrational numbers, and the geometric interpretation, where a line is understood as motion, is often misleading. Theorems must be based on the new understanding of real irrational numbers, which have been rightfully founded and do exist, however little they may differ from rational numbers, and the function has been uniquely determined for each value of the variable, whether it is rational or irrational.

Not that I am publishing this work unhesitatingly long since its first and more significant part *About Numbers* has been finished. Apart from complexity of presentation of such a topic, I was hesitant about publishing results of the verbal exchange of ideas which contain earlier ideas of other people and those of Mr. Weierstrass in the first place, so, all that is left to do is to implement these results, which is extremely important so as not to leave any vague issues in my narrative. I am especially thankful to Mr. Cantor from Halle for the discussion which significantly affected my work, as I borrowed his idea of general numbers which form series.

Let us call a numerical sequence a sequence consisting of numbers $a_1, a_2, a_3, \ldots, a_n, \ldots,$ when for each arbitrarily small non-vanishing number η such n number can be found that $a_n - a_{n+v} < \eta$ can be achieved for all whole positive v[1].

Let us assume that for the structure of (rational) numbers $a_1, a_2, \ldots,$ there is such a (rational) number U that $U - a_n$ decreases as n grows. In this case, U is the limit of a.

We will call general numbers, which in particular cases become rational numbers, as first-order irrational numbers. As irrational numbers are formed from first-order rational numbers A, so, in the same way, second-order numbers A' can be obtained from limits of irrational numbers, whereupon, third-order irrational numbers A'' can be obtained from them, and so on. We will let $A^{(m)}$ denote irrational numbers of order $m + 1$" [6].

1872. G. Cantor (1845–1918)

Cantor constructs a set of numerical values currently known as real numbers, supplementing a set of rational values with irrational numbers using sequences of rational numbers he called fundamental, i.e. sequences that meet the Cauchy criteria. Relations of equality, greater, and less are determined for them.

In the same way, it can be asserted, says Cantor, that a sequence can be in one of the three relations to rational number a, which results in $b = a$, $b > a$, $b < a$. Consequently, if b is a limit of the sequence, then $b - a_n$ becomes infinitely small with growing n. Cantor calls the totality of rational numbers domain A and the totality of all numerical b-values domain B. Numerical operations common for rational numbers (addition, subtraction, multiplication, and division, where the divisor is non-vanishing) which are applied a finite number of times can be extended to domain A and B. In this process, the domain A (that of rational numbers) is obtained from the domain B (that of irrational numbers) and together with the latter forms a new domain C. That is to say, if you set a numerical sequence of numbers $b_1, b_2, \ldots, b_n, \ldots$ with numerical values A and B not all of which belong to domain A, if this sequence has such a property that $b_{n+m} - b_n$ becomes infinitely small with growing n and any m, such sequence is said to have a certain limit c. Numerical values c form domain C. Relations of equality, of being greater than, less than, and elementary operations are determined as described above. However, even a recognized equality of two values b and b' from B does not imply their equivalence, but only expresses a certain relation between sequences to which they are compared.

Domain D is similarly obtained from domain C and preceding ranges, and domain E is obtained from all above domains, etc.; having completed λ of such transformations, domain L is obtained. The concept of a number as developed herein comes equipped with a seed of the necessary and absolutely infinite extension. Cantor uses *numerical amount, value*, and *limit* as equivalent.

Further, Cantor considers points on a line, defining the distance between them as a limit of a sequence and introducing relations of being "greater than", "less than", and "equal". He introduces an axiom that, a point on a line corresponds to each numerical value (and vice versa), the coordinate of such point being equal to this numerical value, and moreover, equal in the sense explained in this paragraph. Cantor calls this assertion as axiom, as it is not

[1] Sic [6, p. 174].

provable in its very nature. Thanks to this axiom, numerical values additionally gain definite objectivity, on which, however, they do not depend at all.

In accordance with the above, Cantor considers a point on a line as definite, if its distance from 0 considered with a definite sign is set as an λ-type numerical amount, value, or limit.

Further, Cantor defines multitude of points or point sets and introduces a concept of an accumulation point of the point set. A neighborhood is understood as any interval which contains this point. Thus, together with a set of points an ensemble of its accumulation points is defined. This set is known as the first derivative point set. If it consists of an infinite number of points, a second derivative point set may be formed of it, and so on [7].

The introduction of the concept of an accumulation point (condensation point) was fruitful. Other mathematicians like H. Schwarz and U. Dini started using it right away.

1872. R. Dedekind (1831–1916)

Dedekind reviews properties of equality, order, density of a multitude of rational numbers *R* (numerical field, a term introduced by Dedekind in appendices to Dirichlet's lectures he published). However, he tries to avoid geometric representations. Having defined the relation "larger" (or "smaller"), Dedekind confirms its transitivity; existence of an infinite multitude of other numbers between two numbers; and, for any number, breaking down a multitude of rational numbers into two infinite classes, so that numbers of one of them are smaller than this number and another one whose numbers are greater than this number; and the number which breaks down the numbers as described above may be assigned either to one class or to the other, in which event it will be either the greatest for the first class or the smallest for the second one.

Further, Dedekind reviews points on a line and sets properties for them in the same way as he has just set for rational numbers, stating that a point on the line corresponds to each rational number.

However, there are infinitely many points on a line which do not correspond to any rational number, e.g., the size of a diagonal line of a square with a unit side. This implies that the multitude of rational numbers needs to be supplemented arithmetically, so that the range of new numbers could become as complete and continuous as a line. Formerly, the concept of irrational numbers was associated with measurement of extended values, i.e. with geometrical representation. Dedekind tends to introduce a new concept by purely arithmetic means, that is, to define irrational numbers through rational numbers:

If the system of all real numbers is split into two classes, so that each number of the first class is less than each number of the second class, there is one, and only one, number which makes this split.

There are infinitely many sections which cannot be made by a rational number. For example, if D is a square-free integer, there is a whole positive number λ, so $\lambda^2 < D < (\lambda + 1)^2$. Therefore, it appears that one class has no greatest, and the other class has no smallest number to make a section, which makes the set of rational numbers incomplete or discontinuous. If that's the case and the section cannot be made by a rational number, let us create a new, irrational, number which will create the section. There is one, and only one, rational or irrational number which corresponds to each fixed section. Two numbers are unequal if they correspond to different sections. Relations "larger than" or "less than" may be found between them.

He defines calculations with real numbers. Herewith, he proves the theorem on continuity of arithmetic operations: "If number λ is a result of calculations which involve numbers α, β, γ, …, and if λ lies in interval L, one can specify such intervals A, B, C (in which numbers α, β, γ, … lie) that the result of a similar calculation in which, however, numbers α, β, γ,… are replaced with numbers of respective intervals A, B, C,…, will always be a number which lies in interval L" [8].

1886. K. Weierstrass (1815–1897)

Weierstrass delivered his first lecture circuit devoted to immeasurable numbers in the academic year 1861/1862. Records of his lectures from 1878 are also available. In summer term of 1886, in response to reproaches of L. Kronecker to the effect of insufficient justifiability of lectures on theory of analytic functions, Weierstrass read additional chapters devoted to foundations of the theory of functions [9]. By that time, concepts of a number of Cantor, Heine, and Dedekind already appeared. Weierstrass attempts to critically summarize them and align them with the classical concept of a number as a ratio.

Weierstrass notes incompleteness of the field of rational numbers, gives consideration to the difference between concepts of a number and a numerical value. According to Weierstrass, a number is a collection, a finite aggregate, e.g. in the form of a decimal notation. A point on a line corresponds to each number, however, it is not obvious that a number corresponds to each point. Unlike his contemporaries, he defines a real number as a limit of partial sums of absolutely convergent series, noting the need for arithmetization of the concept of a limit. He introduces order and completeness with respect to arithmetic operations.

Weierstrass created his reasoning of the theory of analytic functions. The concepts he introduces are not global in their nature – they are necessary for his constructions only. He introduces his own concepts of a continuum and connectivity which differ from those of Cantor; for analytic continuation, he simultaneously builds up a chain of open discs, which is equivalent to Heine covering lemma. Weierstrass defines a number so that it would be sufficient to define continuous changes in arithmetical values in their mutual dependence, "that is to say, an arithmetic expression is calculated in such detail that for any accuracy requirement for any amount t a function may be represented with any approximation. It is always possible to find a mathematical expression for a strictly defined continuous function as well." However, if a function represents series, this does not narrow down, this rather expands, opportunities for study of this function, but the series must have a uniform convergence. "For any value of x for which a function has been determined, it can in fact be represented".

1886. "There is an arbitrary large number of numerical values in an arbitrary vicinity of each immeasurable number which tend to be arbitrary close to it. Therefore, each immeasurable numerical value is a landmark of measurable (numerical) values defined in this case above. So what kind of a purely arithmetic method of definition of the difference between measurable and immeasurable numerical values should be? If measurable numerical values are assumed to exist, there is no sense in defining immeasurable numbers as exact bounds, as in advance it cannot be clear at all, except for, maybe, measurable and some other numerical values".

This is an expression of criticism of the Méray-Heine-Cantor design of real numbers, although he did not mention any names during his lecture. Further during this lecture, Weierstrass gave his reasons as follows:

"But it was not the numerical value which used to be definite, as a matter of fact, it was understood as a measurable number, however, it also contains other as well. Let us consider number e as an example, this number being represented by order elements $1, \frac{1}{2}, \frac{1}{6}, \dots, \frac{1}{n!}, \dots$ which form well determined series. These series unequivocally determine a numerical value which equals them; it can be said that there is no measurable numerical value which equals the represented numerical value (the so-called number e). We therefore conclude that the field of (all) values goes beyond measurable numbers".

"Using the introduced descriptive tool, it is easy to prove that each numerical value corresponds to a certain geometrical length. That is to say, a numerical value can be presented in an arithmetic form, e.g., in a decimal system, as $a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \dots,$ where $0 \le a_k < 10,\ k \ge 1,$ which means that we can present all our (positive) numerical values as segments (of length)" [9].

Cantor. About comparing various methods of introduction of a concept of a number and continuity

Having received Dedekind's work "Continuity and irrational numbers" on 28 April 1872, Cantor wrote to him: "Thank you very much indeed for your work on continuity and irrational numbers. As I could now satisfy myself, the conclusion I came to a couple of years ago proceeding from arithmetic studies, in fact, complies with your viewpoint; the difference is only in the method of introducing the numerical value. I am absolutely sure that you properly defined the essence of continuity".

However, their further correspondence contains polemics regarding the method of defining continuity, and in 1882, Cantor wrote to Dedekind: "I tried to summarize your concept of a section and use it to define the concept of a continuum, but in vain. On the contrary, my point of departure are countable "fundamental sequences" (i.e. sequences whose elements infinitely converge with one another) which seem to suit this attempt".

By 1878, from analysis of point ranges, Cantor proceeds to the concept of power of set, hypothesizes continuum, reviews continuous mappings between multitudes of various dimensions. The more acutely he feels the insufficiency of defining continuity through section. In 1879, he tries to use the Bolzano–Cauchy theorem on roots of a continuous function in an interval to prove that a continuous one-to-one mapping between two different manifolds of different orders is impossible.

In 1883, analyzing various forms of introduction of a number in his cycle of works [10] Cantor wrote: "I would like to briefly and more strictly outline the three basic forms of strictly arithmetic statement of the theory of general real numbers which are known to me and are essentially similar. They, *in the first place,* include the method of introduction Professor Weierstrass used for some years in his lectures on analytical functions and certain resemblance of which can be found in Mr. Kossak's program work (Die Elemente der Arithmetik. Berlin 1872). *In the second place*, in his work "Stetigkeit und irrationale Zahlen" (Braunschweig, 1872), Mr. Dedekind published a kind of a form of a definition. *In the third place*, in 1871, I suggested (Math. Ann. 1872, Bd. 5, S. 123) a form of a definition which has formal resemblance with that of Mr. Weierstrass … I believe this *third one*… is the simplest and most natural of all, and its another advantage is that it is most fit for analytical calculations".

"A definition of any irrational real number would always correspond to the strictly defined set of first-order rational number. This is a common feature for all forms of definitions. The difference is in the point of generation when a set will unite the number it defines and in conditions the set must meet to make a suitable basis for the respective definition of the number.

The *first* form of definition is based on a set of positive numbers a_v which will be denoted as (a_v) and which meets the condition that, whatever is the number and type of these a_v summed up in the finite number, this amount will always remain less than a certain preset threshold. Now, if we have two similar sum-totals (a_v) and (a'_v), it can be rigorously proven that three options may be in place: either each part of $1/n$ unit is always equally frequent in both populations, provided that their elements are summed up in a sufficient amount which can be increased; or $1/n$, starting from a known n, is always more frequent in the first sum-total than in the second one; or, finally, $1/n$, starting from a known n, is always more frequent in the second sum-total than in the first one. Based on these options, denoting the numbers defined by these two sum-totals (a_v) and (a'_v) by b and b', we assume that in the first case, $b = b'$, in the second case, $b > b'$, and in the third case, $b < b'$. If we merge both sum-totals into one new sum-total $(a_v + a'_v)$, this will provide basis for determination of $b + b'$. If a new sum-total $(a_v a'_v)$ is formed out of two sum-totals (a_v) and (a'_v), the elements whereof are products of all (a_v) multiplied by all (a'_v), this new sum-total will be taken as basis for definition of bb'.

We can see here that the point of generation which links a set with the number it incepts constitutes the *generation of sums*. However, it is important to note that only summing of an always *finite* number of rational elements is handled here, and it is *not* assumed in advance, for example, that number b being defined equals the sum Σa_v of infinite series (a_v). This would have been a *logical mistake*, as sum Σa_v can rather be defined only by setting it equal to a predetermined final number b. I believe this logical mistake first avoided by Weierstrass was made nearly by everybody and was not noticed only because it is the rare kind of a mistake which actually cannot do much harm to calculation. Nevertheless, I believe that all those difficulties which lie in the concept of an irrational are associated with the above mistake, while, if this mistake is avoided, an irrational number will lie in place in our soul as definitely, clearly and distinctively, as a rational number.

The form of Mr. Dedekind's definition is based on a *totality of all* rational numbers, however, divided into two groups, so that we will denote numbers of one group U_v, and numbers of the other group through B_μ, and it will always be that $U_v < B_\mu$. Mr. Dedekind calls such division of a multitude of rational numbers "section", denotes it through $(U_v \big| B_\infty)$, and puts in correspondence with number *b*. If you compare these two sections $(U_v \big| B_\infty)$ and $(U'_v \big| B'_\mu)$, as with the *first* form of determination, only *three* options are possible, according to which numbers *b* and *b′* present in sections accordingly are either equated with each other or it is assumed that $b > b'$ or $b < b'$. The first case occurs – if you abstract from certain easily regulated exceptions which arise if the numbers being defined are rational – only where sections are completely identical. This is the definitive and absolute advantage of this form of definition compared to others, that is to say, the advantage is in the fact that there is *only one* section that corresponds to the number *b*. However, this form has a large shortcoming – numbers in the analysis are *never* represented by "sections", and they have to be inserted in this form in a quite artificial and complex way.

And here follow definitions in the form of a sum of $b + b'$ and product of bb' based on new sections obtained from the two preset ones.

A shortcoming associated with the *first* and *third* forms of definition, that is to say, the same, i.e. equal numbers are presented infinitely often and, therefore, all real numbers cannot be directly unequivocally viewed, may be quite easily eliminated by way of specialization of underlying multitudes (a_v), if one of the well-known single-valued systems like the decimal system or simple continued fraction decomposition is considered.

Now, let us proceed to the *third* form of definition of real numbers. This form is based on an infinite set of rational numbers (a_v) of first potency as well, however, now, a different property is attributed to it, not like in Weierstrass' theory. Namely, I demand that, having taken an arbitrary small rational number ε, the finite number of elements of a set could be deleted, so that each two of the remaining ones could have a difference the absolute value where of would be less than ε. I call any such multitude (a_v) which can be characterized as equality $\lim\limits_{v=\infty}(a_{v+\mu} - a_v) = 0$ (with arbitrary μ) a *fundamental sequence* and put it in correspondence with a number *b* it determines, for which it would be advisable even to use the same notation (a_v) as Mr. Heine did, who, after numerous oral discussions rallied to my opinion in these matters (See Crelle's Journal, v. 74, p. 172). Such *fundamental sequence* as may be strictly developed from its concept leads to three options: either its members a_v for sufficiently great values of *v* the absolute value whereof is less than any present number; or they are, starting from a *v*, greater than a definitely predetermined positive rational number ρ, or they are, starting from a known *v*, less than a definitely predetermined negative rational value –ρ. In the first case, I say that *b* is equal to zero; in the second case, that *b* is greater than zero, or positive; and in the third case, that *b* is less than zero, or negative.

Thereafter, we proceed to elementary operations (sum, product, ratio), including those involving rational *a* and irrational number.

And it is only now that we proceed to definition of an equality and both cases of inequality of two numbers b and b' (where b' may also equal a), saying that $b = b'$, $b > b'$, or $b < b'$ – depending on whether the difference $b – b'$ equals zero, is more than zero, or less than zero.

Given these preparatory reasoning, we proceed with the first *strictly provable* theorem which says that if b is a number defined by a fundamental sequence (a_v), then the absolute value of $b - a_v$ with growing v becomes less than any conceivable rational number, or, in other words, $\lim_{v=\infty} a_v = b$.

It should be noted that the following depends on something whose essence can be easily missed: in the case of the third form of definition, the number b is not at all defined as a "limit" of elements a_v of fundamental sequence (a_v). If we accepted this, it would mean to make the same logical mistake as the one we talked about when we considered the *first* form of definition because in that case it is assumed in advance that $\lim_{v=\infty} a_v = b$ exists. However, the situation is rather reversed, that is, thanks to our previous definitions, the concept of the number b is said to have such properties and relationships to rational numbers that it can be with logical clearness concluded as follows: $\lim_{v=\infty} a_v$ exists and equals b. Forgive me all these details. They are justified by the fact that most people miss these indiscernible details and thereafter easily come across contradictions in irrational numbers and doubt them, while, had they observed the above precautions, this would easily prevent such things. In fact, if they observed these precautions, they would clearly understand that due to the *properties assigned to it by our definition*, an irrational number is as real for our spirit as a rational one, even as a whole rational number, and that it need not at all be *obtained* through a limit process. It is rather vice versa, *possessing* these properties, one can generally ascertain the soundness and clearness of limit processes. In fact, the above theorem can be easily summarized as follows: if (b_v) is a multitude of rational or irrational numbers in which $\lim_{v=\infty}(b_{v+\mu} - b_v) = 0$ (whatever μ may be), then there is a number b defined by fundamental sequence (a_v), and $\lim_{v=\infty} b_v = b$.

It therefore turns out that *those* numbers b which were determined on the basis of fundamental sequences (a_v) (I call these fundamental sequences "first-order sequences") so that they turn to be limits a_v, may be set out in different ways and as limits of sequences (b_v), where each b_v is defined with the help of first-order fundamental sequence $(a_\infty^{(v)})$ (at fixed v).

Therefore, if any such multitude (b_v) possesses such property that $\lim_{v=\infty}(b_{v+\mu} - b_v) = 0$ (with arbitrary μ), I use to call it a "*second*-order fundamental sequence".

Similarly, one can form fundamental sequences of the *third*, *fourth*,…, n^{th} order, and fundamental sequences of order α, where α is any number of the second number class.

All fundamental sequences provide the same thing for definition of any real number b as the first-order fundamental sequences. The only difference is but in a more complex extended form of assignment.

Now I use the following way of expressing it: numerical value b is given by a fundamental sequence of the n^{th}, therefore, α, order. If we dare do this, we will thus obtain a remarkably simple and, at the same time, straightforward language to describe the full abundance of diverse, often so complex, constructions of analysis in a most simple and prominent way. This, I believe, will materially contribute to the clearness and transparency of narrative. This way I protest against concerns voiced by Mr. Dedekind in the foreword to his work "Continuity and irrational numbers". It never occurred to me to introduce new numbers with the help of fundamental sequences of the second, third, etc. orders which would not have been already determined with the help of fundamental sequences of the first order: I meant only conceptually different form of an assignment. This is clearly apparent from various parts of my work.

Here I would like to address one remarkable circumstance that orders of fundamental sequences I distinguish with the help of numbers of the first and second number classes absolutely exhaust all forms of regular types of sequences, already found or not yet found, which one can imagine in analysis – exhaust in the meaning that there are no fundamental sequences at all (as I am going to strictly prove in other circumstances) the ordinal whereof could be denoted by any other number, e.g. of third number class".

Conclusions. Change in the type of mathematical definitions

Weierstrass defined a real number as a limit of partial sums of absolutely covergent series, noting the need for arithmetization of the concept of a limit. A point on a line corresponds to each number. However, it is not obvious that a number corresponds to each point. Cantor considered points on a line, defining the distance between them as a limit of a sequence and introducing relations of greater than, less than, and equal to. He introduced an axiom that, vice versa, a point on a line corresponds to each numerical value, the coordinate of such point being equal to this numerical value – equal in such meaning as set forth in this paragraph. Cantor called this statement an axiom, as it is unprovable due to its very nature. Thanks to it, numerical values additionally gain definite objectivity, on which, however, they do not depend at all. Dedekind believed that numbers are subjects of the "world of our thoughts", and it was our right to believe they were related to points. Unlike the above authors, Weierstrass defined a real number as a limit of partial sums of absolutely uniformly convergent series, noting the need for arithmetization of the concept of a limit.

Cantor was developing a perceptual theory of point sets, truly believing that applications were subsidiary issues. Years later, his theory of point sets devised as a summary of contemporary analysis formed the basis of mathematics.

Dedekind developed and arithmetical concept of a number as an algebraist, not being inclined to problems of analysis. Fifteen years later, his design led to creation of Dedekind–Peano arithmetic axiomatics.

Heine pursued educational goals. His narrative on limit and continuity was included in modern courses of analysis. Simultaneously, he set forth a number of fundamental principles: on disregarding of a certain number of points; a covering lemma; the concept of uniform convergence.

Creation of Charles Méray was recognized by his fellow countrymen a century later and is now called a "Méray–Heine" or "Méray–Cantor concept of a number".

After Cantor created the set theory, the language and internal structure of mathematics changed. It did not need anymore a geometrical or physical interpretation and gained a material descriptive component. Language and descriptive forms became the creating tool. The set theory was created as a continuation of arithmetics. However, already ten years later it formed the basis of the theory of a real number. It provided the opportunity to analyze the finest shades of designing mathematical objects and links between them. Many definitions and statements were formed verbally, retaining a high degree of abstraction. This caused discussions among mathematicians devoted to contradictions, many of which were linguistic in their nature. However, a new theory was created as a result, descriptive set theory, the key results of which belong to mathematicians of Warsaw and Moscow schools.

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