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ANNA FORYŚ*, ANDRZEJ S. FORYŚ*

# RESONANCE PHENOMENA IN MICRO/NANOELECTROMECHANICAL SYSTEMS

# ZJAWISKA REZONANSOWE W UKŁADACH MIKRO/NANOELEKTROMECHANICZNCH

A b s t r a c t

In the paper, some aspects of nonlinearity of micro/nanoelectromechanical systems (MEMS/NEMS) are presented. Because of great values of strains of micro/nanobeams the nonlinear description is necessary. Particularly, the nonlinear inertia term is added to equation relating to motion of the beam. Numerical calculations of resonance curves and instability regions are given. Results are presented on graphs.

*Keywords*: *parametric resonance, stability, MEMS, NEMS, nonlinearities*

S t r e s z c z e n i e

W artykule przedstawiono pewne aspekty nieliniowości w mikro/nanoukładach elektromechanicznych (MEMS/NEMS). Ze względu na duże odkształcenia mikro/nanobelek nieliniowy opis jest konieczny. W szczególności do równania ruchu belki wprowadzono wyraz opisujący nieliniową bezwładność. Podano wyniki obliczeń numerycznych dla krzywych rezonansowych oraz obszarów niestateczności. Rezultaty przedstawiono na wykresach.

*Słowa kluczowe*: *rezonans parametryczny, stateczność, MEMS, NEMS, nieliniowość*

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## 1. Introduction

Microelectromechanical and nanoelectromechanical systems (MEMS and NEMS) are applied in different types of detectors and sensors e.g. in mass sensors. These systems frequently work in parametric resonance states. Great values of strain and of quality factor are characteristic features of considered mechanical systems. Therefore the nonlinear description of the systems is necessary. Response of the system depends on the types of nonlinearities.

This paper bases on references [1–4] where some mechanical micro/nanosystems connected with excitation electric systems are described. In the papers [1, 2] a parametrical excited microelectromechanical oscillator is analysed. To describe this oscillator, a Mathieu equation with nonlinearity is adopted and a perturbation method of solution is used. Two kinds of nonlinearities are taken into account: nonlinear elasticity and nonlinear excitation caused by an electric field. The system without damping and with small damping is considered. The system is used for example as mass sensor. In [3] the similar system is described but it is subject to harmonic forcing or to parametric excitation. Theoretical and experimental investigations are presented. The Duffing equation and the Mathieu equation with nonlinearity are used. In [4] a microbeam which is forced by an electric field is described. Discretization of the equation of motion of the beam and saving only the first ordinary differential equation lead to the same equation as adopted in [1–3] for lumped-mass systems.

In this paper, a nonlinear inertia force is taken into account and its effect on the motion of system is presented. The nonlinear inertia force is particularly important if we consider mass sensors. The nonlinearity changes the value of excitation frequency for which a transition between stable and unstable solutions occurs.

## 2. Equation of motion of the system with additional nonlinearity

First we consider MEMS oscillator presented in Fig. 1, [1]. The system consists of: A, B – non-interdigitated comb-drive actuators, C – flexures, D – backbone. The oscillator is excited by an electric signal. The equation of motion of the system presented in Fig. 1 has the following form (cf. [1] equation (1))

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+k_1x+k_3x^3+(r_1x+r_3x^3)V_A^2(1+\cos\theta t)=0, \tag{1}$$

where $m$ is the mass of an oscillator (a shuttle, a backbone [1, 2, 6]), $c$ is the damping coefficient, $k_1$ and $k_3$ are respectively the linear and cubic nonlinear mechanical elastic coefficients, $r_1$ and $r_3$ are respectively the linear and cubic nonlinear electrostatic stiffness of the non-interdigitated comb-fingers, finally the excitation voltage $V(t)$ applied to the system is: $V(t)=V_A\sqrt{1+\cos\theta t}$, $V_A$ and $\theta$ are positive constants.

The values of coefficients of equation (1), measured by different methods, are the following (cf. [1])

$$k_1 = 2.85\frac{\mu\text{N}}{\mu\text{m}}, \quad r_1 = 2.96\cdot 10^{-3}\,\frac{\mu\text{N}}{\text{V}^2\mu\text{m}}, \quad c = 3.88\cdot 10^{-8}\,\frac{\text{kg}}{\text{s}}, \quad m = 9.93\cdot 10^{-11}\,\text{kg},$$
$$k_3 = 0.075\frac{\mu\text{N}}{\mu\text{m}^3}, \quad r_3 = -2.1\cdot 10^{-4}\,\frac{\mu\text{N}}{\text{V}^2\mu\text{m}^3}, \quad \omega_0^2 = \frac{k_1}{m}. \tag{2}$$

Nonlinearities in equation (1) arise from the excitation (term with $r_3$) and from the property of the system (term with $k_3$).

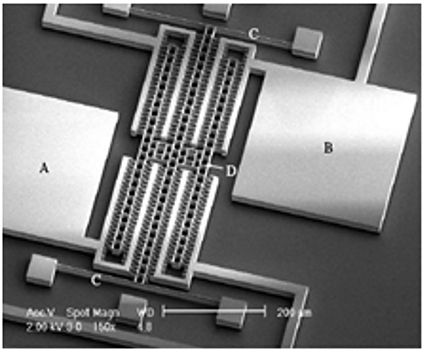


Fig. 1. Parametrically excited MEMS oscillator – electron microscope image [1]

Equation (1) is written in the form (cf. [5, 6])

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \left[\frac{k_1}{m} + \frac{r_1}{m}V_A^2(1+\cos\theta t)\right]x + \left[\frac{k_3}{m} + \frac{r_3}{m}V_A^2(1+\cos\theta t)\right]x^3 = 0. \tag{3}$$

Next we consider a microbeam presented in Fig. 2, [4]. The microbeam of length $l$ is actuated by three capacitors. Two ends of the microbeam are fixed. Equation of motion of the microbeam which is treated as Bernoulli beam is given in [4] equation (1). Limiting considerations to the first vibration mode of the microbeam one can prove (cf. [5] eq. (3), (7) and [4] eq.(21)) that the identical equation as (3) describes deflection of the center of the microbeam.

In agreement with considerations of V.V. Bolotin [7], one can add to equation (3) an additional nonlinear term – the so-called nonlinear inertia term. The nonlinear inertia term has the following form (cf. Appendix)

$$2\kappa(x^2\ddot{x} + x\dot{x}^2), \tag{4}$$

where $\kappa$ is a coefficient of nonlinear inertia; for two-articulated joint beam $2\kappa = \frac{\pi^4}{2l^2}\left(\frac{1}{3} - \frac{5}{8\pi^2}\right)$, for fixed-fixed beam $2\kappa = \frac{8\pi^4}{9l^2}\left(\frac{2}{3} + \frac{5}{16\pi^2}\right)$.

Therefore the initial equation of motion (3) is replaced with the equation

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \left[\frac{k_1}{m} + \frac{r_1}{m}V_A^2(1+\cos\theta t)\right]x + \left[\frac{k_3}{m} + \frac{r_3}{m}V_A^2(1+\cos\theta t)\right]x^3 + 2\kappa(x^2\ddot{x} + x\dot{x}^2) = 0. \tag{5}$$

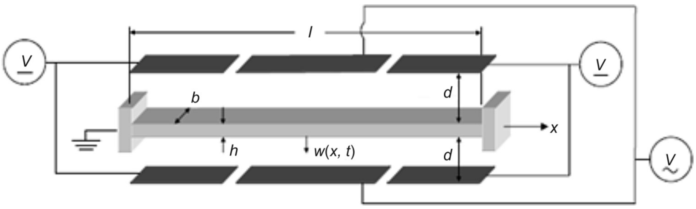


Fig. 2. Parametrically excited microbeam actuated by electric field [4]. The letter $x$ in this figure denotes a coordinate of the cross-section of the beam

Introducing the nondimensional time $\tau = \frac{\theta}{2} t$ we get the following form of equation with nonlinear inertia term

$$\frac{d^2x}{d\tau^2} + \alpha \frac{dx}{d\tau} + (\beta + 2\delta\cos 2\tau)x + (\delta_3 + \delta_3' \cos 2\tau)x^3 + 2\kappa\left[x^2 \frac{d^2x}{d\tau^2} + x\left(\frac{dx}{d\tau}\right)^2\right] = 0, \tag{6}$$

where (cf. [2, 3])

$$\alpha = \frac{2c}{\theta m},\ \beta = \left(\omega_0^2 + \frac{r_1}{m}V_A^2\right)\frac{4}{\theta^2},\ 2\delta = \frac{r_1}{m}V_A^2\frac{4}{\theta^2},\ \delta_3 = \left(\frac{k_3}{m} + \frac{r_3}{m}V_A^2\right)\frac{4}{\theta^2},\ \delta_3' = \frac{4}{\theta^2}\frac{r_3}{m}V_A^2. \tag{7}$$

## 3. Solution method, resonance curves

We look for vibration amplitudes in the steady state of the main parametric resonance on the base of Floquet theorem – at the boundaries of stable and unstable solutions, the solutions are periodic. Then solutions may be represented with Fourier series. If we confine ourselves to the first term of the series, we get

$$x = a\sin\tau + b\cos\tau. \tag{8}$$

We employ the harmonic balance method equating the coefficients at sin τ and cos τ to zero and neglecting higher harmonics. Finally we get

$$\begin{gathered}\left[-1 + \beta - \delta + \left(\frac{3}{4}\delta_3 - \kappa\right)A^2\right]a - \alpha b - \frac{1}{2}\delta_3' a^3 = 0,\\ \alpha a + \left[-1 + \beta + \delta + \left(\frac{3}{4}\delta_3 - \kappa\right)A^2\right]b + \frac{1}{2}\delta_3' b^3 = 0,\end{gathered} \tag{9}$$

where the square of vibration amplitude $A^2 = a^2 + b^2$. It is a system of two algebraic nonlinear equations of the third order for unknown $a$ and $b$. We look for non-zero solutions of (9) ($a \neq 0$, $b \neq 0$) because only in this case $A \neq 0$.

To solve the system of equations (9) we put $b = az$, where $z$ is an unknown, we get

$$a^2 = \frac{1 - \beta + \delta + \alpha z}{\left(\frac{3\delta_3}{4} - \kappa\right)(1 + z^2) - \delta_3'/2} \tag{10}$$

and next for $z$ we get the following algebraic equation of the fourth order

$$a_4 z^4 + [a_3 + a_2(1-\beta+\delta)]z^3 + 2\alpha a_1 z^2 + [a_3 + a_2(1-\beta-\delta)]z + \alpha(a_1 - a_2) = 0, \tag{11}$$

where

$$a_1 = \frac{3}{4}\delta_3 - \kappa, \quad a_2 = \frac{1}{2}\delta_3', \quad a_3 = 2\delta a_1, \quad a_4 = \alpha(a_1 + a_2). \tag{12}$$

The Ferrari method was used to solve this equation. Only real, not equal to zero roots of this equation are important, which inserted into (10) give $a^2 > 0$. If this requirement is fulfilled the amplitude of vibration is calculated as

$$A = |a|\sqrt{1+z^2}. \tag{13}$$

Results of numerical calculations are presented in the Fig. 3. The resonance curve, without taking into account the nonlinear inertia, is drawn by a dotted line. The resonance

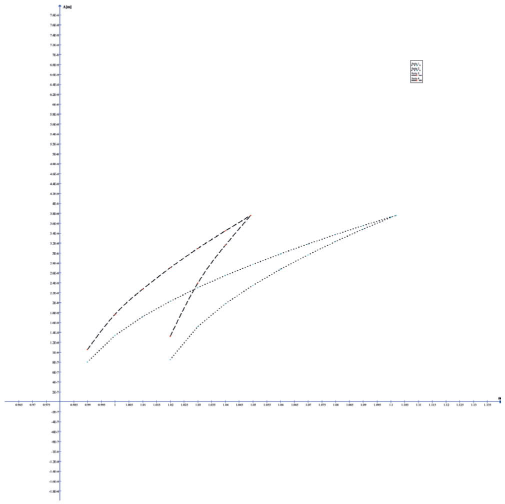

Fig. 3. Vibration amplitude $A$ versus $n = \theta/2\Omega$ for $\kappa = 0$ (series 1 and 2) and for $\kappa = 7.3 \cdot 10^9$ m$^{-2}$ (series 3 and 4)

curve, taking into account the nonlinear inertia, is drawn by a dashed line. The difference is visible. The values of amplitudes are different but the frequency of transition between zero solution region to nonzero solutions region are the same.

## 4. Regions of unstable solutions

Inserting the solution (8) to equation (6) and neglecting nonlinear terms one obtains (cf. [7, 8]) the formulae for the boundaries of the first, more dangerous region of dynamic instability in the form

$$\frac{\theta}{2\Omega} = \sqrt{1 \pm \sqrt{\mu^2 - \mu^{*2}}}, \tag{14}$$

where $\mu^*$ is the critical value of parameter of excitation, cf. [7, 8]

$$\mu^* = \frac{c}{\Omega m}, \quad \Omega = \omega_0\sqrt{1-\frac{P_t}{P_*}} = \omega_0\sqrt{1+\frac{r_1}{\omega_0^2}\frac{V_A^2}{m}}. \tag{15}$$

Therefore

$$\frac{\theta}{2\Omega} = \sqrt{1 \pm \sqrt{\left(\frac{r_1 V_A^2}{2\Omega^2 m}\right)^2 - \left(\frac{c}{\Omega m}\right)^2}}$$

or

$$\frac{\theta}{2\omega_0} = \sqrt{1+\frac{r_1}{\omega_0^2}\frac{V_A^2}{m}}\sqrt{1 \pm \sqrt{\left(\frac{r_1 V_A^2}{2\Omega^2 m}\right)^2 - \left(\frac{c}{\Omega m}\right)^2}} = \sqrt{1+\frac{r_1 V_A^2}{k_1^2} \pm \sqrt{\frac{(r_1 V_A^2)^2}{4k_1^2} - \frac{c^2}{k_1 m}\left(1+\frac{r_1 V_A^2}{k_1}\right)}}. \tag{16}$$

In the papers [1] and [5] the boundaries of the first instability region are obtained on the ground of the perturbation method and given by the formulae

$$f_1 = \frac{\omega_0}{2\pi}\left[2+\frac{r_1}{\omega_0^2}\frac{V_A^2}{m} - \sqrt{\left(\frac{r_1 V_A^2}{2\omega_0^2 m}\right)^2 - \frac{c^2}{m^2\omega_0^2}}\right],$$
$$f_2 = \frac{\omega_0}{2\pi}\left[2+\frac{r_1}{\omega_0^2}\frac{V_A^2}{m} + \sqrt{\left(\frac{r_1 V_A^2}{2\omega_0^2 m}\right)^2 - \frac{c^2}{m^2\omega_0^2}}\right], \tag{17}$$

where $f_1$ and $f_2$ are frequencies.

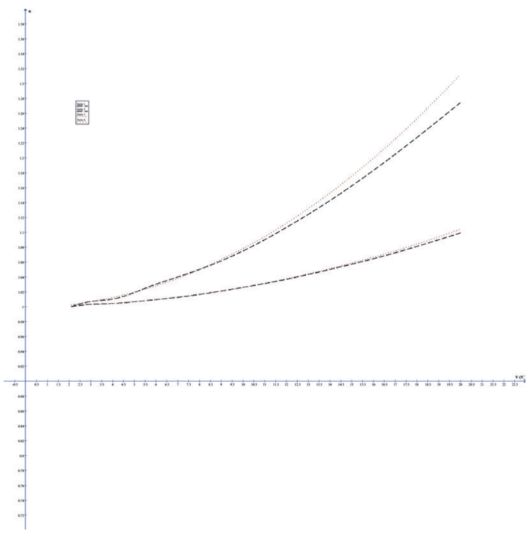

Fig. 4. Instability region (series 1 and 2; series 3 and 4 according to [1])

The results obtained by these two different methods are presented on the graphs in Fig. 4. The regions of instability are almost identical and the value of critical amplitude of voltage for which the unstable solutions occur are the same.

## 5. Conclusions

In cited papers the resonance curves are given for two special cases: 1. for nonlinear elasticity without damping or 2. for damping without nonlinear elasticity. In this paper the viscous damping together with two types of nonlinearities: the nonlinear elasticity and nonlinear inertia are taken into consideration. The nonlinear inertia term was introduced by analogy to considerations connected with beam. The solution for amplitude of vibrations is obtained in half-analytic form. The value of nonlinear inertia coefficient κ has an effect on the value of vibration amplitude.

## 6. Appendix

We quote the considerations of V.V. Bolotin [7] which concern two articulated-joint beams excited by axial force of the form $P(t) = P_0 + P_t \cos\theta t$. Limiting considerations to the first vibration mode of the beam, the time dependence of deflection $x(t)$ of the beam is described by the Mathieu equation

$$\frac{d^2x}{dt^2} + 2\varepsilon\frac{dx}{dt} + \omega_0^2\left(1 - \frac{P_0 + P_t\cos\theta t}{P_*}\right)x = 0 \tag{18}$$

or

$$\frac{d^2x}{dt^2} + 2\varepsilon\frac{dx}{dt} + \Omega^2(1 - 2\mu\cos\theta t)x = 0, \tag{19}$$

where $\varepsilon = \dfrac{c}{2m},\ \omega_0^2 = \dfrac{\pi^4}{l^3}\dfrac{EI}{m} \equiv \dfrac{k_1}{m},\ P_* = \dfrac{\pi^2}{l^2}EI = \dfrac{k_1}{\pi^2}l,\ \Omega = \omega_0\sqrt{1 - \dfrac{P_t}{P_*}},\ \mu = \dfrac{P_t}{2(P_* - P_0)};$

$\omega_0$ is natural circular frequency of the beam, $P_*$ is the value of the first Euler critical force for this beam. The replacing elastic constant $k_1$ is introduced which models the beam by a mass $m$ on the spring with elastic constant $k_1 = \dfrac{\pi^4}{l^3}EI$.

Comparing (18) and (19) with equation (3) one gets

$$P_0 = P_t = -\frac{r_1}{\omega_0^2}\frac{V_A^2}{m}P_*, \quad \mu = -\frac{r_1V_A^2}{2\Omega^2 m'}, \quad \Omega = \sqrt{\frac{k_1}{m} + \frac{r_1}{m}V_A^2}. \tag{20}$$

According to [7] one can add to equation (19) a term of nonlinear inertia (nonlinearity of geometric nature). An inertia force connected with longitudinal displacement $u$ of concentrated mass $M_L$ has the form: $-M_L\ddot{u}$ and modify the longitudinal force in eq. (18).

The longitudinal force is now $P(t) = P_0 + P_t\cos\theta t - M_L\ddot{u}$, where $u = \dfrac{1}{2}\displaystyle\int_0^l\left(\frac{\partial w}{\partial z}\right)^2 dz.$

Confining ourselves to the first vibration mode one gets $\ddot{u} = \frac{\pi^2}{2l}(x\ddot{x} + \dot{x}^2)$, where $x(t)$ is the time part of transverse displacement of the beam.

Equation (19), including (20), takes the following form

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \left[\frac{k_1}{m} + \frac{r_1}{m}V_A^2(1+\cos\theta t) + \frac{k_1}{mP_*}M_L\ddot{u}\right]x = 0. \tag{21}$$

The microbeam considered in the paper [4] has no concentrated mass on its end. Nonlinear inertia is connected only with distributed mass of the beam. In [7] is demonstrated that taking into account the distributed mass is equivalent to adding the concentrated mass $M_L = \left(\frac{1}{3} - \frac{5}{8\pi^2}\right)m$, where $m$ is the mass of the beam. Finally, the inertia term is written in the form (4).

It seems that the inertia forces cannot be neglected because of large deflections which appear in MEMS and NEMS which are taken into consideration.

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ADAM BEDNARZ*, LUDWIK BYSZEWSKI*

# ON ABSTRACT NONLOCAL CAUCHY PROBLEM

# O NIELOKALNYM ABSTRAKCYJNYM ZAGADNIENIU CAUCHY'EGO

A b s t r a c t

In this paper, we investigate the existence and uniqueness of the classical solution to an abstract nonlocal Cauchy problem. For this purpose, we apply a notion of mild solution and the Banach contraction theorem.

*Keywords*: *abstract Cauchy problem*, *nonlocal conditions*, *mild and classical solution*

S t r e s z c z e n i e

W artykule zbadano istnienie i jednoznaczność klasycznego rozwiązania abstrakcyjnego nielokalnego zagadnienia Cauchy'ego. W tym celu zastosowano rozwiązanie całkowe i twierdzenie Banacha o kontrakcji.

*Słowa kluczowe*: *abstrakcyjne zagadnienie Cauchy'ego*, *warunki nielokalne*, *rozwiązania całkowe i klasyczne*

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## 1. Introduction

We study the existence and uniqueness of the classical solution to a functional-differential abstract nonlocal Cauchy problem.

The functional-differential nonlocal problem considered in this paper, is of the form

$$u'(t) = f(t, u(t), u(a_1(t)), \ldots, u(a_r(t))), \quad t \in I, \tag{1.1}$$

$$u(t_0) + \sum_{k=1}^{p} c_k u(t_k) = x_0, \tag{1.2}$$

where $I := [t_0, t_0 + T]$, $t_0 < t_1 < \ldots < t_p \leq t_0 + T$, $T > 0$; $f : I \times E^{r+1} \to E$ and $a_j : I \to I$ ($j = 1, \ldots, r$) are given functions satisfying suitable assumptions; $E$ is a Banach space with norm $\|\cdot\|$, $x_0 \in E$, $c_k \neq 0$, ($k = 1, \ldots, p$) and $p, r \in \mathbb{N}$.

If $c_k \neq 0$, ($k = 1, \ldots, p$), then the results of the paper can be applied in kinematics to determine the evolution $t \to u(t)$ of the location of a physical object for which we do not know the positions $u(t_0)$, $u(t_1)$, …, $u(t_p)$, but we know that the nonlocal condition (1.2) holds.

The paper bases on books [3–4] and on papers [1–2].

## 2. Theorems about the existence and uniqueness of a classical solution

By $X$ we denote the Banach space $C(I, E)$, where $I = [t_0, t_0 + T]$ with the standard norm $\|\cdot\|_X$. So

$$\|w\|_X := \sup_{t \in I} \|w(t)\|, \quad w \in X.$$

Assume that $\sum_{k=1}^{p} c_k \neq -1$. A function $u \in X$, satisfying the integral equation

$$\begin{aligned} u(t) = \breve{c} \left[ x_0 - \sum_{k=1}^{p} c_k \int_{t_0}^{t_k} f(\tau, u(\tau), u(a_1(\tau)), \ldots, u(a_r(\tau)))\, d\tau \right] + \\ + \int_{t_0}^{t} f(\tau, u(\tau), u(a_1(\tau)), \ldots, u(a_r(\tau)))\, d\tau, \quad t \in I, \end{aligned} \tag{2.1}$$

where $\breve{c} \overset{df}{=} \left( 1 + \sum_{k=1}^{p} c_k \right)^{-1}$, is said to be a **mild solution** of the nonlocal problem (1.1) – (1.2).

A function $u : I \to E$ is said to be a **classical solution** of the nonlocal problem (1.1)–(1.2) if

(i) $u$ is continuous on $I$ and continuously differentiable on $I$,

(ii) $u'(t) = f(t, u(t), u(a_1(t)), \ldots, u(a_r(t)))$ for $t \in I$,

(iii) $u(t_0) + \sum_{k=1}^{p} c_k u(t_k) = x_0.$

**Theorem 2.1.** Suppose that $f : I \times E^{r+1} \to E$, $a_j : I \to I$ $(j = 1, \ldots, r)$ and $\sum_{t \in I}^{p} c_k \neq -1.$ If $u$ is a classical solution of the nonlocal problem (1.1)–(1.2), then $u$ is a mild solution of this problem.

**Proof**. Let $u$ be a classical solution of the nonlocal problem (1.1)–(1.2). Then $u$ satisfies equation (1.1) and consequently,

$$u(t) = u(t_0) + \int_{t_0}^{t} f(\tau, u(\tau), u(a_1(\tau)), \ldots, u(a_r(\tau)))\, d\tau, \quad t \in I. \tag{2.2}$$

From (2.2),

$$u(t_k) = u(t_0) + \int_{t_0}^{t_k} f(\tau, u(\tau), u(a_1(\tau)), \ldots, u(a_r(\tau)))\, d\tau, \quad (k = 1, \ldots, p). \tag{2.3}$$

By (1.2) and (2.3),

$$u(t_0) + \sum_{k=1}^{p} c_k \left[ u(t_0) + \int_{t_0}^{t_k} f(\tau, u(\tau), u(a_1(\tau)), \ldots, u(a_r(\tau)))\, d\tau \right] = x_0. \tag{2.4}$$

Since $\sum_{t \in I}^{p} c_k \neq -1,$ then (2.4) implies

$$u(t_0) = \breve{c} \left[ x_0 - \sum_{k=1}^{p} c_k \int_{t_0}^{t_k} f(\tau, u(\tau), u(a_1(\tau)), \ldots, u(a_r(\tau)))\, d\tau \right]. \tag{2.5}$$

From (2.2) and (2.5), we obtain that $u$ is a mild solution of the nonlocal problem (1.1)–(1.2). The proof of Theorem 2.1 is complete.

**Theorem 2.2.** Suppose that $f \in C(I \times E^{r+1})$, $a_j : I \to I$ $(j = 1, \ldots, r)$ and $\sum_{k=1}^{p} c_k \neq -1.$ If $u$ is a mild solution of the nonlocal problem (1.1)–(1.2), then $u$ is a classical solution of this problem.

**Proof.** Let $u$ be a mild solution of the nonlocal problem (1.1)–(1.2). Then $u$ satisfies equation (1.1) and, from the continuity of $f$, $u \in C^1(I, E)$. Now we will show that $u$ satisfies the nonlocal condition (1.2). For this purpose, observe that by (2.1),

$$u(t_0)=\breve{c}\left[x_0-\sum_{k=1}^{p}c_k\int_{t_0}^{t_k}f(\tau,u(\tau),u(a_1(\tau)),\ldots,u(a_r(\tau)))\,d\tau\right] \tag{2.6}$$

and

$$\begin{aligned} u(t_i)&=\breve{c}\left[x_0-\sum_{k=1}^{p}c_k\int_{t_0}^{t_k}f(\tau,u(\tau),u(a_1(\tau)),\ldots,u(a_r(\tau)))\,d\tau\right]+\\ &+\int_{t_0}^{t_i}f(\tau,u(\tau),u(a_1(\tau)),\ldots,u(a_r(\tau)))\,d\tau \quad (i=1,\ldots,p). \end{aligned} \tag{2.7}$$

From (2.6) and (2.7), and from some computations,

$$\begin{aligned} u(t_0)+\sum_{i=1}^{p}c_iu(t_i)&=\left(x_0-\sum_{k=1}^{p}c_k\int_{t_0}^{t_k}f(\tau,u(\tau),u(a_1(\tau)),\ldots,u(a_r(\tau)))\,d\tau\right)+\\ &+\sum_{i=1}^{p}c_i\int_{t_0}^{t_i}f(\tau,u(\tau),u(a_1(\tau)),\ldots,u(a_r(\tau)))\,d\tau=x_0. \end{aligned}$$

Therefore, the proof of Theorem 2.2 is complete.

As a consequence of Theorems 2.1 and 2.2, we obtain:

**Theorem 2.3.** Suppose that $f\in C(I\times E^{r+1}, E)$, $a_j : I\to I$ $(j = 1, \ldots, r)$ and $\sum_{k=1}^{p}c_k\neq -1$. Then $u$ is the unique classical solution to the nonlocal problem (1.1)–(1.2) if, and only if, $u$ is the unique mild solution to this problem.

Now, we will prove the main theorem of the paper.

**Theorem 2.4.** Assume that:

(i) $a_j\in C(I, I)$ $(j = 1, \ldots, r)$, $f: I\times E^{r+1}\to E$ is continuous with respect to the first variable on $I$ and there is $L > 0$ such that

$$\|f(s,z_1,\ldots,z_{r+1})-f(s,\tilde{z}_1,\ldots,\tilde{z}_{r+1})\|\le L\sum_{i=1}^{r+1}\|z_i-\tilde{z}_i\| \quad \text{for} \quad s\in I,\ z_i,\tilde{z}_i\in E\ \ (i=1,\ldots,r+1), \tag{2.8}$$

(ii) $\sum_{k=1}^{p}c_k\neq -1$

(iii) $(r+1)LT\left(1+\left|\breve{c}\sum_{k=1}^{p}c_k\right|\right)<1.$

Then the nonlocal Cauchy problem (1.1)–(1.2) has a unique classical solution $u$. Moreover, the successive approximations $u_n$ ($n = 0, 1, 2, \ldots$), defined by the formulas

$$u_0(t) = x_0 \qquad \text{for} \qquad t \in I \tag{2.9}$$

and

$$\begin{aligned} u_{n+1}(t) := \breve{c}\left[ x_0 - \sum_{k=1}^{p} c_k \int_{t_0}^{t_k} f(\tau, u_n(\tau), u_n(a_1(\tau)), \ldots, u_n(a_r(\tau)))\, d\tau \right] + \\ + \int_{t_0}^{t_i} f(\tau, u_n(\tau), u_n(a_1(\tau)), \ldots, u_n(a_r(\tau)))\, d\tau \quad \mathit{for} \quad t \in I \quad (n = 0, 1, 2, \ldots), \end{aligned} \tag{2.10}$$

converge uniformly on $I$ to the unique classical solution $u$.

**Proof.** Introduce an operator $A$ by the formula

$$\begin{aligned} (Aw)(t) := \breve{c}\left[ x_0 - \sum_{k=1}^{p} c_k \int_{t_0}^{t_i} f(\tau, w(\tau), w(a_1(\tau)), \ldots, w(a_r(\tau)))\, d\tau \right] + \\ + \int_{t_0}^{t} f(\tau, w(\tau), w(a_1(\tau)), \ldots, w(a_r(\tau)))\, d\tau, \quad w \in X, \quad t \in I. \end{aligned} \tag{2.11}$$

It is easy to see that

$$A : X \to X. \tag{2.12}$$

Now, we will show that $A$ is a contraction on $X$. For this purpose observe that

$$\begin{aligned} &(Aw)(t) - (A\tilde{w})(t) := \\ &= -\breve{c}\sum_{k=1}^{p} c_k \int_{t_0}^{t_k} [f(\tau, w(\tau), w(a_1(\tau)), \ldots, w(a_r(\tau))) - f(\tau, \tilde{w}(\tau), \tilde{w}(a_1(\tau)), \ldots, \tilde{w}(a_r(\tau)))]\, d\tau + \\ &+ \int_{t_0}^{t} [f(\tau, w(\tau), w(a_1(\tau)), \ldots, w(a_r(\tau))) - f(\tau, \tilde{w}(\tau), \tilde{w}(a_1(\tau)), \ldots, \tilde{w}(a_r(\tau)))]\, d\tau, \; w, \tilde{w} \in X, \; t \in I. \end{aligned} \tag{2.13}$$

From (2.13) and (2.8)

$$\|(Aw)(t) - (A\tilde{w})(t)\| \le (r+1)LT\left( 1 + \left| \breve{c}\sum_{k=1}^{p} c_k \right| \right) \|w - \tilde{w}\|_X, \quad w, \tilde{w} \in X, \; t \in I. \tag{2.14}$$

Let

$$q := (r+1)LT\left( 1 + \left| \breve{c}\sum_{k=1}^{p} c_k \right| \right). \tag{2.15}$$

Then, by (2.14), (2.15) and assumption (iii),

$$\left\|Aw - A\tilde{w}\right\|_X \le q\left\|w - \tilde{w}\right\|_X \quad \text{for} \quad w, \tilde{w} \in X \tag{2.16}$$

with $0 < q < 1$.

Consequently, by (2.12) and (2.16), operator $A$ satisfies all the assumptions of the Banach contraction theorem. Therefore, in space $X$ there is only one fixed point $u$ of $A$ and this point is the mild solution to the nonlocal problem (1.1)–(1.2). Consequently, from Theorem 2.3, $u$ is the unique classical solution to the nonlocal problem (1.1)–(1.2).

Now, we will prove the second part of the thesis of Theorem 2.4. To this end, observe that by (2.10) and (2.9),

$$\begin{aligned}\left\|u_1 - u_0\right\|_X &= \sup_{t\in I}\left\|u_1(t) - u_0(t)\right\| \le \left\|-\breve{c}\sum_{k=1}^{p} c_k \int_{t_0}^{t_k} f(\tau, u_0(\tau), u_0(a_1(\tau)), \ldots, u_0(a_r(\tau)))\,d\tau\right\| + \\ &+ \sup_{t\in I}\left\|\int_{t_0}^{t} f(\tau, u_0(\tau), u_0(a_1(\tau)), \ldots, u_0(a_r(\tau)))\,d\tau\right\| \le MT\left(1 + \left|\breve{c}\sum_{k=1}^{p} c_k\right|\right),\end{aligned} \tag{2.17}$$

where

$$M := \sup\left\{\left\|f(\tau, w(\tau), w(a_1(\tau)), \ldots, w(a_r(\tau)))\right\| : w \in X, \tau \in I\right\}.$$

Next, assume that

$$\left\|u_n - u_{n-1}\right\|_X \le MT\left(1 + \left|\breve{c}\sum_{k=1}^{p} c_k\right|\right) \cdot \left[(r+1)LT\left(1 + \left|\breve{c}\sum_{k=1}^{p} c_k\right|\right)\right]^{n-1} \tag{2.18}$$

for some natural $n \ge 2$.

Then, by (2.10), (2.9), (2.8) and (2.18),

$$\begin{aligned}&\left\|u_{n+1} - u_n\right\|_X = \sup_{t\in I}\left\|u_{n+1}(t) - u_n(t)\right\| = \\ &= \left\|-\breve{c}\sum_{k=1}^{p} c_k \int_{t_0}^{t_k}[f(\tau, u_n(\tau), u_n(a_1(\tau)), \ldots, u_n(a_r(\tau))) - f(\tau, u_{n-1}(\tau), u_{n-1}(a_1(\tau)), \ldots, u_{n-1}(a_r(\tau)))]\,d\tau\right\| + \\ &+ \sup_{t\in I}\left\|\int_{t_0}^{t}[f(\tau, u_n(\tau), u_n(a_1(\tau)), \ldots, u_n(a_r(\tau))) - f(\tau, u_{n-1}(\tau), u_{n-1}(a_1(\tau)), \ldots, u_{n-1}(a_r(\tau)))]\,d\tau\right\| \le \\ &\le (r+1)LT\left(1 + \left|\breve{c}\sum_{k=1}^{p} c_k\right|\right)\left\|u_n - u_{n+1}\right\|_X \le MT\left(1 + \left|\breve{c}\sum_{k=1}^{p} c_k\right|\right) \cdot \left[(r+1)LT\left(1 + \left|\breve{c}\sum_{k=1}^{p} c_k\right|\right)\right]^{n}.\end{aligned} \tag{2.19}$$

Therefore, from (2.17), (2.18), (2.19), and from induction argument,

$$\|u_n - u_{n-1}\|_X \le MT\left(1+\left|\breve{c}\sum_{k=1}^{p} c_k\right|\right)\cdot\left[(r+1)LT\left(1+\left|\breve{c}\sum_{k=1}^{p} c_k\right|\right)\right]^{n-1} \tag{2.20}$$

for all $n = 1, 2, \dots$

Inequalities (2.20) and assumption (iii) imply, by the Weierstrass theorem, the uniform convergence of the series

$$u_1 + \sum_{n=1}^{\infty}(u_{n+1} - u_n)$$

on the integral $I$ and consequently, the uniform convergence of the sequence $u_n$ on $I$.

Let

$$u_*(\mathrm{t}) := \lim_{n\to\infty} u_n(t) \quad \text{for} \quad t \in I.$$

Since $u_n$ tends uniformly to $u_*$ on $I$ then, by (2.9), (2.10) and (2.8), $u_*$ is a classical solution to the nonlocal problem (1.1)–(1.2) on $I$. But, from the first part of the thesis of Theorem 2.4, we know that there exists only one classical solution $u$ to the nonlocal problem (1.1)–(1.2) on $I$. So, $u_* = u$ on $I$.

The proof of Theorem 2.4 is complete.

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# ON NONLOCAL EVOLUTION PROBLEM FOR THE EQUATION OF THE FIRST ORDER

# O NIELOKALNYM EWOLUCYJNYM ZAGADNIENIU DLA RÓWNANIA RZĘDU PIERWSZEGO

Abstract

The aim of the paper is to prove theorems about the existence and uniqueness of mild and classical solutions of a nonlocal semilinear functional-differential evolution Cauchy problem. The method of semigroups, the Banach fixed-point theorem and theorems (see [2]) about the existence and uniqueness of the classical solutions of the first-order differential evolution problems in a not necessarily reflexive Banach space are used to prove the existence and uniqueness of the solutions of the problems considered. The results obtained are based on publications [1–6].

*Keywords*: *evolution Cauchy problem, existence and uniqueness of the solutions, nonlocal conditions*

Streszczenie

W artykule udowodniono twierdzenia o istnieniu i jednoznaczności rozwiązań całkowych i klasycznych nielokalnego semiliniowego funkcjonalno-różniczkowego ewolucyjnego zagadnienia Cauchy'ego. W tym celu zastosowano metodę półgrup, twierdzenie Banacha o punkcie stałym i twierdzenia ([2]) o istnieniu i jednoznaczności klasycznych rozwiązań ewolucyjnych zagadnień różniczkowych pierwszego rzędu w niekoniecznie refleksywnej przestrzeni Banacha. Artykuł bazuje na publikacjach [1–6].

*Słowa kluczowe*: *ewolucyjne zagadnienie Cauchy'ego, istnienie i jednoznaczność rozwiązań, warunki nielokalne*

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## 1. Introduction

Let $E$ be a Banach space with norm $\|\cdot\|$ and let $A : E \to E$ be a closed densely defined linear operator. For an operator $A$, let $\mathcal{D}(A)$, $\rho(A)$ and $A^*$ denote its domain, resolvent set and adjoint, respectively.

For a Banach space $E$, $\mathcal{C}(E)$ denotes the set of closed linear operators from $E$ into itself.

We will need the class $G(\tilde{M},\beta)$ of operators $A$ satisfying the conditions:

There exist constants $\tilde{M} > 0$ and $\beta \in \mathbb{R}$ such that

$$(C_1)\ \ A \in \mathcal{C}(E),\ \overline{\mathcal{D}(A)} = E \quad \text{and} \quad (\beta,+\infty) \subset \rho(-A),$$

$$(C_2)\ \ \left\|(A+\xi)^{-k}\right\| \leqslant \tilde{M}(\xi-\beta)^{-k} \quad \text{for each} \quad \xi > \beta \quad \text{and} \quad k = 1,2,\ldots$$

We will need the assumption:

**Assumption** $(Z)$. The adjoint operator $A^*$ is densely defined in $E^*$, i.e., $\overline{\mathcal{D}}(A^*) = E^*$.

It is known (see: [4], p. 485 and [5], p. 20) that for $A \in G(\tilde{M},\beta)$ there exists exactly one strongly continuous semigroup $T(t) : E \to E$ for $t \geqslant 0$ such that $-A$ is its infinitesimal generator and

$$\|T(t)\| \leqslant \tilde{M}e^{\beta t} \quad \text{for} \quad t \geqslant 0.$$

Throughout the paper we will assume $(C_1)$ and $(C_2)$, and assumption $(Z)$.

Moreover throughout the paper we will use the notation

$$0 \leqslant t_0 < t_1 < \ldots < t_p \leqslant t_0 + a, \quad a > 0,$$

$$J := [t_0, t_0 + a],$$

$$M := \sup\left\{\|T(t)\| \,:\, t \in [0,a]\right\}$$

and

$$X := \mathcal{C}(J,E).$$

Throughout the paper we will also assume that there exists the operator $\mathcal{B}$ with $\mathcal{D}(\mathcal{B}) = E$ given by the formula

$$\mathcal{B} := \left(I + \sum_{k=1}^{p} c_k T(t_k - t_0)\right)^{-1},$$

where $I$ is the identity operator on $E$.

The aim of the paper is to study the existence and uniqueness of mild and classical solutions to a nonlocal Cauchy problem for a functional-differential evolution equation. The nonlocal Cauchy problem considered here is of the following form:

$$u'(t) + Au(t) = f(t,u(t),u(b_1(t)),\ldots,u(b_r(t))), \quad t \in J \setminus \{t_0\}, \tag{1.1}$$

$$u(t_0) + \sum_{k=1}^{p} c_k u(t_k) = u_0, \tag{1.2}$$

where $f$ and $b_i$ ($i = 1, \ldots, r$) are given functions satisfying some assumptions, $u_0 \in E$, $c_k \neq 0$ ($k = 1, 2, \ldots, p$) and $p, r \in \mathbb{N}$.

To study problem (1.1)–(1.2) we will need the following linear problem:

$$u'(t) + Au(t) = g(t), \quad t \in J \setminus \{t_0\}, \tag{1.3}$$

$$u(t_0) = x \tag{1.4}$$

and the following definition:

A function $u : J \to E$ is said to be a classical solution to the problem (1.3)–(1.4) if

(i) $u$ is continuous on $J$ and continuously differentiable on $J \setminus \{t_0\}$,

(ii) $u'(t) + Au(t) = g(t)$ for $t \in J \setminus \{t_0\}$,

(iii) $u(t_0) = x$.

To study problem (1.1)–(1.2) we will need also the following theorem:

**Theorem 1.1.** (see [2]). *Let $g : J \to E$ be Lipschitz continuous on $J$ and $x \in \mathcal{D}(A)$.*

*Then the Cauchy problem* (1.3)–(1.4) *has exactly one classical solution $u$ given by the formula*

$$u(t) = T(t - t_0)x + \int_{t_0}^{t} T(t - s)g(s)\,ds, \quad t \in J. \tag{1.5}$$

The results obtained in the paper, are based on publications [1–6].

## 2. On mild solution

A function $u \in X$ satisfying the integral equation

$$\begin{aligned} u(t) = {} & T(t - t_0)\mathcal{B}u_0 - \\ & + \sum_{k=1}^{p} c_k T(t - t_0)\mathcal{B} \int_{t_0}^{t_k} T(t_k - s) f(s, u(s), u(b_1(s)), \ldots, u(b_r(s)))\,ds + \\ & + \int_{t_0}^{t} T(t - s) f(s, u(s), u(b_1(s)), \ldots, u(b_r(s)))\,ds, \quad t \in J, \end{aligned} \tag{2.1}$$

is said to be a *mild solution* of the functional-differential nonlocal evolution Cauchy problem (1.1)–(1.2).

Remark 2.1. *A function $u$ satisfying* (2.1) *satisfies condition* (1.2) (For the proof of Remark 2.1 see [3]).

**Theorem 2.1.** *Assume that*:

(i) *$f : J \times E^{r+1} \to E$ is continuous with respect to the first variable on $J$, $b_i : J \to J$ ($i = 1, \ldots, r$) are continuous on $J$ and there is $L > 0$ such that*

$$\| f(s, z_0, z_1, \ldots, z_r) - f(s, \tilde{z}_0, \tilde{z}_1, \ldots, \tilde{z}_r) \| \leqslant L \sum_{i=0}^{r} \| z_i - \tilde{z}_i \| \tag{2.2}$$

$$\text{for} \quad s \in J,\ z_i, \tilde{z}_i \in E \ \ (i = 0, 1, \ldots, r),$$

(ii) $(r+1)MLa\left(1+M\|\mathcal{B}\|\sum_{k=1}^{p}|c_k|\right)<1,$

(iii) $u_0 \in E.$

*Then the functional-differential nonlocal evolution Cauchy problem* (1.1)–(1.2) *has a unique mild solution.*

*Proof.* Introduce the operator $F$ on the Banach space $X$ given by the formula

$$(Fw)(t) := T(t-t_0)\mathcal{B}u_0 -$$
$$-\sum_{k=1}^{p} c_k T(t-t_0)\mathcal{B}\int_{t_0}^{t_k} T(t_k-s) f(s, w(s), w(b_1(s)), \ldots, w(b_r(s)))\, ds +$$
$$+\int_{t_0}^{t} T(t-s) f(s, w(s), w(b_1(s)), \ldots, w(b_r(s)))\, ds, \quad w \in X, \quad t \in J.$$

It is easy to see that $F$ is a mapping from $X$ into $X$ and we will show that $F$ is a contraction on $X$. For this purpose, observe that

$$\begin{aligned}
&(Fw)(t)-(F\tilde{w})(t) = \\
&-\sum_{k=1}^{p} c_k T(t-t_0)\mathcal{B}\int_{t_0}^{t_k} T(t_k-s)[f(s, w(s), w(b_1(s)), \ldots, w(b_r(s))) - \\
&\qquad - f(s, \tilde{w}(s), \tilde{w}(b_1(s)), \ldots, \tilde{w}(b_r(s)))]\, ds + \\
&+\int_{t_0}^{t} T(t_k-s)[f(s, w(s), w(b_1(s)), \ldots, w(b_r(s))) - \\
&\qquad - f(s, \tilde{w}(s), \tilde{w}(b_1(s)), \ldots, \tilde{w}(b_r(s)))]\, ds, \quad w, \tilde{w} \in X, \quad t \in J.
\end{aligned} \tag{2.3}$$

From (2.3) and (2.2)

$$\|(Fw)(t)-(F\tilde{w})(t)\| \leqslant (r+1)MLa\left(1+M\|\mathcal{B}\|\sum_{k=1}^{p}|c_k|\right)\|w-\tilde{w}\|_X, \quad w, \tilde{w} \in X, \quad t \in J. \tag{2.4}$$

Define

$$q := (r+1)MLa\left(1+M\|\mathcal{B}\|\sum_{k=1}^{p}|c_k|\right). \tag{2.5}$$

Then by (2.4), (2.5) and assumption (ii),

$$\|Fw-F\tilde{w}\|_X \leqslant q\|w-\tilde{w}\|_X \quad \text{for} \quad w, \tilde{w} \in X \tag{2.6}$$

with $0 < q < 1$.

Consequently, by (2.6), operator $F$ satisfies all the assumptions of the Banach contraction theorem. Therefore, in space $X$ there is only one fixed point of $F$ and this point is the mild solution of problem (1.1)–(1.2) So, the proof of Theorem 2.1 is complete. □

## 3. Mild and classical solutions

A function $u : J \to E$ is said to be a classical solution of the functional-differential nonlocal evolution Cauchy problem (1.1)–(1.2) if:

(i) $u$ is continuous on $J$ and continuously differentiable on $J \setminus \{t_0\}$,

(ii) $u'(t) + Au(t) = f(t, u(t), u(b_1(t)), \ldots, u(b_r(t)))$, for $t \in J \setminus \{t_0\}$,

(iii) $u(t_0) + \sum_{k=1}^{p} c_k u(t_k) = u_0$.

**Theorem 3.1.** *Assume that $f : J \times E^{r+1} \to E$ is Lipschitz continuous on $J \times E^{r+1}$. If $u$ is a classical solution to the problem* (1.1)–(1.2) *then $u$ is a mild solution of this problem.*

*Proof.* Since $u$ is a classical solution to the problem (1.1)–(1.2), $u \in X$ and $u$ satisfies the integral equation (see [2], Theorem 2)

$$u(t) = T(t-t_0)u(t_0) + \int_{t_0}^{t} T(t-s) f(s, u(s), u(b_1(s)), \ldots, u(b_r(s)))\, ds, \quad t \in J.$$

The remaining part of the proof[1] of Theorem 3.1 is as in [3]. □

**Theorem 3.2.** *Suppose that*:

(i) $f : J \times E^{r+1} \to E$, $b_i : J \to J$ $(i = 1, \ldots, r)$ *are continuous on $J$ and there is $C > 0$ such that*

$$\|f(s, z_0, z_1, \ldots, z_r) - f(\tilde{s}, \tilde{z}_0, \tilde{z}_1, \ldots, \tilde{z}_r\| \leqslant C\left(|s-\tilde{s}| + \sum_{i=0}^{r} \|z_i - \tilde{z}_i\|\right) \tag{3.1}$$

$$\textit{for} \quad s, \tilde{s} \in J, \quad z_i, \tilde{z}_i \in E \quad (i = 0, \ldots, r),$$

(ii) $(r+1)MCa\left(1 + M\|\mathcal{B}\|\sum_{k=1}^{p} |c_k|\right) < 1,$

(iii) $u_0 \in E$.

*Then the functional-differential nonlocal evolution problem* (1.1)–(1.2) *has a unique mild solution denoted by $u$. Moreover, if*

(iv) $\mathcal{B}u_0 \in \mathcal{D}(A)$ *and*

$$\mathcal{B}\int_{t_0}^{t_k} T(t_k - s) f(s, u(s), u(b_1(s)), \ldots, u(b_r(s)))\, ds \in \mathcal{D}(A) \quad (k = 1, \ldots, p)$$

*and if there is $\kappa > 0$ such that*

$$\|u(b_i(s)) - u(b_i(\tilde{s}))\| \leqslant \kappa \|u(s) - u(\tilde{s})\| \quad \textit{for} \quad s, \tilde{s} \in J$$

*then $u$ is the unique classical solution to problem* (1.1)–(1.2).

*Proof.* Since all the assumptions of Theorem 2.1 are satisfied, problem (1.1)–(1.2) possesses a unique mild solution $u$.

Now, we will show that $u$ is the unique classical solution to the problem (1.1)–(1.2). To this end, introduce

$$N := \max_{s \in J} \|f(s, u(s), u(b_1(s)), \ldots, u(b_r(s)))\| \tag{3.2}$$

---

[1] This remaining part of the proof shows why in the definition of a mild solution $u$ to the problem (1.1)–(1.2) we require that the function $u$ satisfies the integral equation (2.1).

and observe that

$$u(t+h)-u(t)=T(t-t_0)[T(h)-I]\mathcal{B}u_0-\sum_{k=1}^{p}c_kT(t-t_0)[T(h)-I]\times$$
$$\times\mathcal{B}\int_{t_0}^{t_k}T(t_k-s)f(s,u(s),u(b_1(s)),\ldots,u(b_r(s)))\,ds+$$
$$+\int_{t_0}^{t_0+h}T(t+h-s)f(s,u(s),u(b_1(s)),\ldots,u(b_r(s)))\,ds+ \tag{3.3}$$
$$+\int_{t_0}^{t}T(t-s)[f(s,u(s+h),u(b_1(s+h)),\ldots,u(b_r(s+h)))-$$
$$+f(s,u(s),u(b_1(s)),\ldots,u(b_r(s)))]\,ds \text{ for } t\in[t_0,t_0+a),\ h>0 \text{ and } t+h\in(t_0,t_0+a].$$

Consequently, by (3.3), (3.2), (3.1) and Assumption (iv),

$$\|u(t+h)-u(t)\|\leqslant$$
$$\leqslant Mh\|A\mathcal{B}u_0\|+\sum_{k=1}^{p}|c_k|Mh\left\|A\mathcal{B}\int_{t_0}^{t_k}T(t_k-s)f(s,u(s),u(b_1(s)),\ldots,u(b_r(s)))\,ds\right\|+$$
$$+hMN+MCah+MC\int_{t_0}^{t}\Big(\|u(s+h)-u(s)\|+\|u(b_1(s+h))-u(b_1(s))\|+\ldots+ \tag{3.4}$$
$$+\|u(b_r(s+h))-u(b_r(s))\|\Big)\,ds=C_*h+MC(1+r\kappa)\int_{t_0}^{t}\|u(s+h)-u(s)\|\,ds$$
$$\text{for}\quad t\in[t_0,t_0+a),\quad h>0\quad\text{and}\quad t+h\in(t_0,t_0+a],$$

where

$$C_*:=M\left[\|A\mathcal{B}u_0\|+\sum_{k=1}^{p}|c_k|\left\|A\mathcal{B}\int_{t_0}^{t_k}T(t_k-s)f(s,u(s),u(b_1(s)),\ldots,u(b_r(s)))\,ds\right\|+N+Ca\right].$$

From (3.4) and Gronwall's inequality

$$\|u(t+h)-u(t)\|\leqslant C_*e^{aMC(1+r\kappa)}h$$

for $t\in[t_0,t_0+a)$, $h>0$ and $t+h\in(t_0,t_0+a]$.
Hence $u$ is Lipschitz continuous on $J$.

The Lipschitz continuity of $u$ on $J$ combined with continuity of $f$ on $J\times E^{r+1}$ imply that $t\to f(t,u(t),u(b_1(t)),\ldots,u(b_r(t)))$ is Lipschitz continuous on $J$. This fact together with assumptions of Theorem 3.2 imply, by Theorem 1.1, that the linear Cauchy problem

$$v'(t)+Av(t)=f(t,u(t),u(b_1(t)),\ldots,u(b_r(t))),\quad t\in J\setminus\{t_0\}, \tag{3.5}$$

$$v(t_0)=u_0-\sum_{k=1}^{p}c_ku(t_k) \tag{3.6}$$

has a unique classical solution $v$ such that

$$v(t) = T(t-t_0)v(t_0) + \int_{t_0}^{t} T(t-s)f(s,u(s),u(b_1(s)),\ldots,u(b_r(\text{s})))\,ds, \quad t \in J. \tag{3.7}$$

Now, we will show that

$$u(t) = v(t) \quad \text{for} \quad t \in J. \tag{3.8}$$

To do it, observe that, by (3.6), by Remark 2.1 and by (2.1),

$$v(t_0) = u(t_0) = \mathcal{B}u_0 - \sum_{k=1}^{p} c_k \mathcal{B} \int_{t_0}^{t_k} T(t_k - s)f(s,u(s)),u(b_1(s)),\ldots,u(b_r(s)))\,ds.$$

Consequently

$$\begin{aligned} &T(t-t_0)v(t_0) = \\ &= T(t-t_0)\mathcal{B}u_0 - \sum_{k=1}^{p} c_k T(t-t_0)\mathcal{B} \int_{t_0}^{t_k} T(t_k - s)f(s,u(s),u(b_1(s)),\ldots,u(b_r(s)))\,ds, \quad t \in J. \end{aligned} \tag{3.9}$$

Next from (3.7), (3.9) and (2.1),

$$\begin{aligned} v(t) &= T(t-t_0)v(t_0) + \int_{t_0}^{t} T(t-s)f(s,u(s),u(b_1(s)),\ldots,u(b_r(s)))\,ds = \\ &= T(t-t_0)\mathcal{B}u_0 - \sum_{k=1}^{p} c_k T(t-t_0)\mathcal{B} \int_{t_0}^{t_k} T(t_k - s)f(s,u(s),u(b_1(s)),\ldots,u(b_r(s)))\,ds + \\ &+ \int_{t_0}^{t} T(t-s)f(s,u(s),u(b_1(s)),\ldots,u(b_r(s)))\,ds = u(t), \quad t \in J, \end{aligned}$$

and, therefore, (3,8) holds.

The above argument implies that $u$ is a classical solution of problem (1.1)–(1.2).

To prove that $u$ is the unique classical solution of problem (1.1)–(1.2) suppose that there is a classical solution $u_*$ of problem (1.1)–(1.2) such that $u_* \neq u$ on $J$. Then, by Theorem 3.1, $u_*$ is a mild solution of problem (1.1)–(1.2). Since, by Theorem 2.1, there exists the only one mild solution of problem (1.1)–(1.2), $u_* = u$ on $J$. Thus, the proof of Theorem 3.2 is complete.

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LUDWIK BYSZEWSKI*, TERESA WINIARSKA*

# CONTINUOUS DEPENDENCE OF MILD SOLUTIONS, ON INITIAL NONLOCAL DATA, OF THE NONLOCAL EVOLUTION CAUCHY PROBLEMS

# CIĄGŁA ZALEŻNOŚĆ CAŁKOWYCH ROZWIĄZAŃ OD NIELOKALNYCH WARUNKÓW POCZĄTKOWYCH, NIELOKALNYCH EWOLUCYJNYCH ZAGADNIEŃ CAUCHY'EGO

A b s t r a c t

The aim of the paper is to prove two theorems on continuous dependence of mild solutions, on initial nonlocal data, of the nonlocal Cauchy problems. For this purpose, the method of semigroups and the theory of cosine family in Banach spaces are applied. The paper is based on publications [1–5].

*Keywords*: *evolution Cauchy problems*, *continuous dependence of solutions*, *nonlocal conditions*

S t r e s z c z e n i e

W artykule udowodniono dwa twierdzenia o ciągłej zależności rozwiązań całkowych od nielokalnych warunków początkowych, nielokalnych zagadnień Cauchy'ego. W tym celu zastosowano metodę półgrup i teorię rodziny cosinus w przestrzeniach Banacha. Artykuł bazuje na publikacjach [1–5].

*Słowa kluczowe*: *ewolucyjne zagadnienia Cauchy'ego*, *ciągła zależność rozwiązań*, *warunki nielokalne*

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# Part I

## Continuous dependence of mild solutions, on initial nonlocal data, of the nonlocal evolution Cauchy problem of the first order

### 1. Introduction to Part I

In this part of the paper, we assume that $E$ is a Banach space with norm $\|\cdot\|$ and $-A$ is the infinitesimal generator of a $\mathcal{C}_0$ semigroup $\{T(t)\}_{t\geqslant 0}$ on $E$.

Throughout this part of the paper, we use the notation:

$$I=[0,a], \quad \text{where} \quad a>0,$$

$$M=\sup\left\{\|T(t)\|,\ t\in I\right\}$$

and

$$X=\mathcal{C}(I,E).$$

Let $p$ be a positive integer and $t_1, \ldots, t_p$ be given real numbers such that $0<t_1<\ldots<t_p$. Moreover, let $C_i$ $(i=1,\ldots,p)$ be given real numbers and

$$K:=\sum_{i=1}^{p}|C_i|.$$

Consider the evolution nonlocal Cauchy problem of the first order

$$u'(t)+Au(t)=f(t), \quad t\in I\setminus\{0\}, \tag{1.1}$$

$$u(0)+\sum_{i=1}^{p}C_iu(t_i)=x_0, \tag{1.2}$$

where $f\colon I\to E$ and $x_0\in E$.

In this part of the paper, we shall study a continuous dependence of a mild solution, on initial nonlocal data (1.2), of the nonlocal evolution Cauchy problem (1.1)–(1.2). The definition of this solution will be given in the next section.

This part of the paper is based on publications [1, 3–5]. Particularly, see Theorem 43.1 from [4].

### 2. Theorem about a mild solution of the nonlocal evolution Cauchy problem of the first order

A function $u\in X$ and satisfying the equation

$$u(t)=T(t)x_0-T(t)\left(\sum\nolimits_{i=1}^{p}C_iu(t_i)\right)+\int_0^t T(t-s)f(s)ds, \quad t\in I, \tag{2.1}$$

is said to be a mild solution of the nonlocal Cauchy problem (1.1)–(1.2).

**Theorem 2.1.** *Assume that*:

(i) $f : I \to E$ *is continuous,*

(ii) $MK < 1$,

(iii) $x_0 \in E$.

*Then the nonlocal evolution Cauchy problem* (1.1)–(1.2) *has a unique mild solution.*
*Proof.* See [1], Theorem 3.1 and page 32. □

## 3. Continuous dependence of a mild solution, on initial nonlocal data (1.2), of the nonlocal Cauchy problem (1.1)–(1.2)

**Theorem 3.1.** *Let all the assumptions of Theorem 2.1 be satisfied. Suppose that u is a mild solution (satisfying* (2.1)*) from Theorem 2.1. Moreover, let* $v \in X$, *satisfying the equation*

$$v(t) = T(t)y_0 - T(t)\left(\sum\nolimits_{i=1}^{p} C_i v(t_i)\right) + \int_0^t T(t-s)f(s)ds, \quad t \in I, \tag{3.1}$$

*be the mild solution to the nonlocal problem*

$$v'(t) + Av(t) = f(t), \quad t \in I \setminus \{0\},$$

$$v(0) + \sum_{i=1}^{p} C_i v(t_i) = y_0,$$

*where* $y_0 \in I$.

*Then for an arbitrary* $\varepsilon > 0$ *there is* $\delta > 0$ *such that if*

$$\|x_0 - y_0\| < \delta \tag{3.2}$$

*and*

$$\|u(t_i) - v(t_i)\| < \delta \quad (i = 1, \ldots, p) \tag{3.3}$$

*then*

$$\|u - v\|_X < \varepsilon. \tag{3.4}$$

*Proof.* Let $\varepsilon$ be a positive number and let

$$\delta := \min\left\{\frac{\varepsilon}{2M}, \frac{\varepsilon}{2MKp}\right\}. \tag{3.5}$$

Observe that, from (2.1) and (3.1)

$$u(t) - v(t) = T(t)(x_0 - y_0) - T(t)\left(\sum_{i=1}^{p} C_i (u(t_i) - v(t_i))\right), \quad t \in I.$$

Consequently, by (3.2), (3.3) and (3.5),

$$\|u(t)-v(t)\| \leqslant M\|x_0-y_0\|+MK\sum_{i=1}^{p}\|u(t_i)-v(t_i)\| < M\delta+MKp\delta \leqslant \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon \quad \text{for} \quad t\in I.$$

Therefore, (3.2) holds. It means that a mild solution of the nonlocal Cauchy problem (1.1)–(1.2) is continuously dependent on the initial nonlocal data (1.2).

The proof of Theorem 3.1 is complete. □

## Part II

## Continuous dependence of mild solutions, on nonlocal data, of the nonlocal evolution Cauchy problem of the second order

### 4. Introduction to Part II

In the second part of the paper, we consider the nonlocal evolution Cauchy problem of the second order

$$u''(t)=Au(t)+f(t), \quad t\in I\setminus\{0\}, \tag{4.1}$$

$$u(0)=x_0, \tag{4.2}$$

$$u'(0)+\sum_{i=1}^{p}C_iu(t_i)=x_1, \tag{4.3}$$

where $A$ is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators from the Banach space $E$ (with norm $\|\cdot\|$) into itself, $u : I \to E, f : I \to E$, $I = [0, a]$, $a > 0$, $x_0, x_1 \in E$, $C_i \in \mathbb{R}$ $(i = 1, \ldots, p)$ and $t_1, \ldots, t_p$ are as in Part I.

We will use the set

$$\tilde{E} := \{x\in E : C(t)x \text{ is of class } C^1 \text{ with respect to } t\}$$

and the sine family $\{S(t) : t \in \mathbb{R}\}$ defined by the formula

$$S(t)x := \int_0^t C(s)x\,ds, \quad x\in E,\ t\in\mathbb{R}.$$

In this part of the paper, we shall study a continuous dependence of a mild solution, on initial nonlocal data (4.2)–(4.3), of the nonlocal evolution Cauchy problem (4.1)–(4.3). The definition of this solution will be given in the next section.

The second part of the paper is based on publications [2, 4].

## 5. Theorem about a mild solution of the nonlocal Cauchy problem of the second order

A function $u \in \mathcal{C}^1(I, E)$ and satisfying the equation

$$u(t) = C(t)x_0 + S(t)x_1 - S(t)\left(\sum_{i=1}^{p} C_i u(t_i)\right) + \int_0^t S(t-s)f(s)ds, \quad t \in I, \tag{5.1}$$

is said to be a mild solution of the nonlocal Cauchy problem (4.1)–(4.3).

**Theorem 5.1.** *Assume that*:

(i) $f : I \to E$ *is continuous*,

(ii) $2CK < 1$, *where* $C := \sup\,\{\| C(t) \| + \| S(t) \| + \| S'(t) \| : t \in I\}$ *and* $K := \sum_{i=1}^{p} |C_i|$,

(iii) $x_0 \in \tilde{E}$ and $x_1 \in E$.

*Then the nonlocal evolution Cauchy problem* (4.1)–(4.3) *has a unique mild solution.*
*Proof.* See [2], Theorem 2.1. □

## 6. Continuous dependence of a mild solution on initial nonlocal data (4.2)–(4.3), of the nonlocal evolution Cauchy problem (4.1)–(4.3)

**Theorem 6.1.** *Let all the assumptions of Theorem 5.1 be satisfied. Suppose that u is a mild solution (satisfying* (5.1)*) from Theorem 5.1. Moreover, let v satisfying the equation*

$$v(t) = C(t)y_0 + S(t)y_1 - S(t)\left(\sum_{i=1}^{p} C_i v(t_i)\right) + \int_0^t S(t-s)f(s)ds, \quad t \in I, \tag{6.1}$$

*be the mild solution of the nonlocal problem*

$$v''(t) = Av(t) + f(t), \quad t \in I \setminus \{0\},$$

$$v(0) = y_0,$$

$$v'(0) + \sum_{i=1}^{p} C_i v(t_i) = y_1,$$

*where* $y_0 \in \tilde{E}$ *and* $y_1 \in E$.

*Then for any arbitrary* $\varepsilon > 0$ *there is* $\delta > 0$ *such that if*

$$\|x_0 - y_0\| < \delta, \quad \|x_1 - y_1\| < \delta \tag{6.2}$$

*and*

$$\|u(t_i) - v(t_i)\| < \delta \quad (i = 1, \ldots, p) \tag{6.3}$$

*then*

$$\|u - v\|_X < \varepsilon, \tag{6.4}$$

*where* $X = \mathcal{C}(X, E)$.

*Proof.* Let ε be a positive number and let

$$\delta := \min\left\{\frac{\varepsilon}{3C}, \frac{\varepsilon}{3CKp}\right\}. \tag{6.5}$$

Observe that, from (5.1) and (6.1),

$$u(t) - v(t) = C(t)(x_0 - y_0) + S(t)(x_1 - y_1) - S(t)\left(\sum_{i=1}^{p} C_i(u(t_i) - v(t_i))\right), \quad t \in I.$$

Consequently, by (6.2), (6.3) and (6.5),

$$\|u(t) - v(t)\| \leqslant C\|x_0 - y_0\| + C\|x_1 - y_1\| + CK\sum_{i=1}^{p}\|u(t_i) - v(t_i)\| <$$

$$< C\delta + C\delta + CKp\delta \leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \text{for} \quad t \in I.$$

Therefore, (6.4) holds. It means that a mild solution of the nonlocal Cauchy problem (4.1)–(4.3) is continuously dependent on the initial nonlocal data (4.2)–(4.3).

The proof of Theorem 6.1 is complete. □

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# THE NON-EXISTENCE OF THE FEJER–RIESZ TYPE RESULT FOR SOME WEIGHTED BERGMAN SPACES IN THE UNIT DISC

# O NIEISTNIENIU PEWNYCH OSZACOWAŃ TYPU FEJERA–RIESZA W PRZESTRZENIACH BERGMANA Z WAGĄ W KOLE JEDNOSTKOWYM

A b s t r a c t

In this note, we consider the analogues of the classical Fejer–Riesz inequality for some weighted Hilbert spaces of analytic functions in the unit disc. We prove that for some class of such spaces, the Fejer-Riesz inequality type results do not hold.

*Keywords*: *Fejer-Riesz inequality, Bergman spaces of analytic functions*

S t r e s z c z e n i e

W artykule rozważa się nierówności podobne do klasycznej nierówności Fejera–Riesza w przestrzeniach Hilberta funkcji analitycznych z wagą. Dowodzi się, że w pewnych klasach takich przestrzeni nie zachodzi odpowiednik nierówności Fejera–Riesza.

*Słowa kluczowe*: *nierówność Fejera–Riesza*, *przestrzenie Bergmana funkcji analitycznych*

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## 1. Introduction

Let $U$ be the unit disc in $\mathbb{C}$. For functions in the space $H^2(U)$ the following well-known Fejer-Riesz inequality holds (see e.g. [3], p. 46):

If $f \in H^2(U)$, and $f^*$ denotes the radial boundary values of $f$ on $\partial U$, $f^*$ being defined a.e. on $\partial U$ and $L^2$-integrable with respect to the linear Lebesgue measure on $\partial U$, then

$$\int_{-1}^{1} |f(x)|^2 dx \leq \frac{1}{2} \int_0^{2\pi} \left| f^*(e^{i\theta}) \right|^2 d\theta. \tag{1}$$

It follows from this inequality that in particular for every $f \in H^2(U)$ and every $z \in \partial U$,

$$\int_0^1 |f(tz)|^2 \, dt < +\infty. \tag{2}$$

(One should mention that the inequality (1) with 2 replaced by $p$ also holds for all $H^p$-spaces with $1 \leq p < +\infty$).

The space $H^2(U)$ can be viewed as one of some family of weighted Hilbert spaces of analytic functions in the unit disc in $\mathbb{C}$; this family can be described as follows:

Given $s > -1$, set

$$A^{2,s}(U) = \left\{ f \text{ holomorphic in } U : \int_U |f(z)|^2 (1-|z|^2)^s \, dm(z) < +\infty \right\}, \tag{3}$$

where $m$ is planar Lebesgue measure in $U$. Such spaces, also called weighted Bergman spaces, were considered by many authors; see e.g. [1, 2, 7, 8, 9].

If $f$ is holomorphic in $U$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in U$, and $s > -1$, then one can prove by integrating in polar coordinates that

$$f \in A^{2,s}(U) \text{ iff } \sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{1+s}} < +\infty. \tag{4}$$

E.g. for $s = 0$ we obtain $A^{2,0}(U) = L^2H(U)$, the so called Bergman space of all holomorphic functions in $U$ with

$$\int_U |f(z)|^2 \, dm(z) < +\infty. \tag{5}$$

The condition (4) for $s = 0$ is

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} < +\infty.$$

For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ holomorphic in $U$ we have

$$\int_U |f(z)|^2 (1-|z|^2)^s \, dm(z) = \sum_{n=0}^{\infty} |a_n|^2 \int_U |z|^{2n} (1-|z|^2)^s \, dm(z) = 2\pi \sum_{n=0}^{\infty} |a_n|^2 \int_0^1 t^{2n+1}(1-t^2)^s \, dt.$$

If $s \leq -1$, we have for $n = 0, 1, 2, \ldots$

$$\int_0^1 t^{2n+1}(1-t^2)^s\,dt = +\infty.$$

Hence for $s \leq -1$ the integral condition (3) gives the space consisting only of the zero function. But the series condition (4) yields non-zero Hilbert spaces of holomorphic functions in $U$. Therefore we set for $s \leq -1$

$$A^{2,s}(U) = \left\{ f \text{ holomorphic in } U : \sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{1+s}} < +\infty \right\}. \tag{6}$$

Such definition of the space $A^{2,s}(U)$ for $s \leq -1$ seems to be correct also by the fact that for $s = -1$ we obtain from (4) or (6) the condition $\sum_{n=0}^{\infty} |a_n|^2 < +\infty$, which is the well--known condition for $f$ to belong to the space $H^2(U)$; therefore the definition (6) gives $A^{2,-1}(U) = H^2(U)$.

Having placed the space $H^2(U)$ as $A^{2,-1}(U)$ in the above described family of spaces $A^{2,s}(U)$, $s \in \mathbb{R}$, we could ask whether the Fejer-Riesz inequality (1) valid for $H^2(U) = A^{2,-1}(U)$ also has analogues for the other spaces $A^{2,s}(U)$, $s \in \mathbb{R}$.

In [5] we have proved the result similar to (2) for the spaces $A^{2,s}(U)$ with $s > 0$:

**Proposition 1.** ([5], *Theorem 1*). *Let $s$ be a positive number. Suppose that $f \in A^{2,s-1}(U)$. Then for every $z \in \partial U$.*

$$\int_0^1 |f(tz)|^2 (1-t^2)^s\,dt < +\infty. \tag{7}$$

As was already mentioned above, for $s \leq -1$ the spaces $A^{2,s}(U)$ are defined by the series condition (6). Note that for $s > -1$ the condition (7) makes sense. Hence one can try to prove for $s$ with $-2 < s \leq -1$ the result similar to that in Proposition 1:

If $s$ is a number with $-2 < s \leq -1$, and $f \in A^{2,s}(U)$, i.e. if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic in $U$, it satisfies also according to (6)

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{1+s}} < +\infty, \tag{8}$$

then for every $z \in \partial U$,

$$\int_0^1 |f(tz)|^2 (1-t^2)^{s+1}\,dt < +\infty. \tag{9}$$

In [6] we have proved only a weakened version of the aforementioned result; it is described in [6], conditions (10) and (11). For the convenience of the reader we recall it here.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfy (8). Let $\{b_k\}_{k=0}^{\infty}$ be a new sequence which is obtained from $\{a_n\}_{n=0}^{\infty}$ in such a way that we delete all numbers $a_n$ with $a_n = 0$ and then reorder the remaining numbers $a_n$ to obtain a new sequence $\{a'_k\}_{k=0}^{\infty}$ with $|a'_0| \geq |a'_1| \geq \ldots;$ we define then $b_k = |a'_k|,\ k = 0,1,\ldots$ . We can then also prove that

$$\sum_{n=0}^{\infty} \frac{b_n^2}{(n+1)^{1+s}} < \infty.$$

The additional condition which we assume is as follows:

$$\text{The sequence } \left( \frac{b_n^2}{(n+1)^{1+s}} \right)_{n=0}^{\infty} \text{ is decreasing.} \qquad (10)$$

We have proved in [6]:

**Proposition 2.** ([6], *Proposition 2*). *Suppose that* $-2 < s \leq -1$. *Let the function f, holomorphic in U,* $f(z) = \sum_{n=0}^{\infty} a_n z^n$, *satisfy (8) (i.e.* $f \in A^{2,s}(U)$*). Suppose also that the condition (10) holds. Then for every* $z \in \partial U$

$$\int_0^1 |f(tz)|^2 (1-t^2)^{s+1} dt < \infty.$$

As mentioned above, we are still not able to prove Proposition 2 without assuming (10), although this conditions seems to be superfluous.

Consider now the spaces $A^{2,s}(U)$ with $s \leq -2$. As mentioned above, in this case $A^{2,s}(U)$ is defined by the series condition (6). Moreover, if $f \in A^{2,s}(U)$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and the coefficients $\{a_n\}_{n=0}^{\infty}$ are non-negative, then $f(t)$ is bound away from zero say for $\frac{1}{2} < t < 1$, and so the integral condition (9) holds only for $f$ equal zero. On the other hand, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $U$, $f \in A^{2,s}(U)$ with $s > -2$, and the coefficients $a_n$ are non-negative, then as explained in [6], the expression

$$\int_0^1 |f(tz)|^2 (1-t^2)^{s+1} dt$$

is estimated from below and from above by a constant time the sum of the double series

$$\sum_{k,l=0}^{\infty} a_k a_l \frac{1}{(1+k+l)^{s+2}}.$$

Hence we have assumed in [6] that for $s \leq -2$, the right analogue of the Fejer-Riesz type results, described in Propositions 1 and 2, would be the following:

If $f$ is holomorphic in $U$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and $f \in A^{2,s}(U)$, i.e.

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{1+s}} < +\infty, \tag{11}$$

then one should have that

$$\sum_{k,l=0}^{\infty} |a_k||a_l| \frac{1}{(1+k+l)^{s+2}} < +\infty. \tag{12}$$

As observed in [6], such a result is not true. Namely, we have proved the following:

**Proposition 3.** ([6], *Proposition 4*). *Let s be a real number with s ≤ –2. Then there exists a holomorphic function f such that* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *with* $a_n$ *satisfying (11), but yielding divergent series (12).*

The example of such a function, which therefore does not satisfy the Fejer-Riesz type result for $s \le -2$, is the function $f(z) = \sum_{n=2}^{\infty} a_n z^n$ with

$$a_n = \frac{1}{(n+1)^{-s/2} \log n}, \quad n = 2, 3, \ldots. \tag{13}$$

In the present note we show that the same function does not satisfy the Fejer-Riesz type results mentioned above, in some sharper sense; we prove namely.

**Proposition 4.** *Let s < –2 be given. Let the function* $f(z) = \sum_{n=2}^{\infty} a_n z^n$ *be defined by (13). Then*

$$\sum_{n=2}^{\infty} \frac{|a_n|^2}{(n+1)^{1+s}} < +\infty, \tag{14}$$

*but for every* $\sigma$ *with* $s \le \sigma \le \frac{s}{2} - 1$ *the series*

$$\sum_{k,l=2}^{\infty} |a_k||a_l| \frac{1}{(1+k+l)^{\sigma+2}} \tag{15}$$

*is divergent.*

If $s \le -2$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic in $U$, and the series $\sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{1+s}} < +\infty$, but the series $\sum_{k,l=0}^{\infty} |a_k||a_l| \frac{1}{(1+k+l)^{\sigma+2}}$ is divergent for some $\sigma \ge s$, then the value $\sigma = \frac{s}{2} - 1$ is the largest possible; we have the following result:

**Proposition 5.** *If s is a real number with* $s \le -2$, *and* $f \in A^{2,s}(U)$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, *then for every* $\sigma > \frac{s}{2} - 1$ *the series*

$$\sum_{k,l=0}^{\infty} |a_k||a_l| \frac{1}{(1+k+l)^{\sigma+2}} < +\infty. \tag{16}$$

**Proof of Proposition 4.** Fix $s < -2$. Note that we then have $s < \frac{s}{2} - 1 < -2$; therefore the condition $s \le \sigma \le \frac{s}{2} - 1$ makes sense. With $s$ as above let $f$ be defined by (13), i.e.

$$f(z) = \sum_{n=2}^{\infty} \frac{1}{(n+1)^{-s/2} \log n} z^n. \tag{17}$$

We have, for $n \ge 2$,

$$\sqrt[n]{\frac{1}{(n+1)^{-s/2} \log n}} = \frac{1}{(\sqrt[n]{n+1})^{-s/2} \sqrt[n]{\log n}}$$

and this last sequence converges to 1; so the series in the right-hand side of (17) is convergent for every $z \in U$, and hence $f$ is holomorphic in $U$.

The facts that for $s < -2$ and $a_n = \frac{1}{(n+1)^{-s/2} \log n}$ the series (14) converges, but the series (15) with $\sigma = s$ is divergent were already proved in [6], Proposition 4. Therefore, let $s < \sigma \le \frac{s}{2} - 1$. Then $\sigma = s + \varepsilon$ with some $0 < \varepsilon \le -1 - \frac{s}{2}$. We have

$$\sum_{k,l=2}^{\infty} |a_k||a_l| \frac{1}{(1+k+l)^{\sigma+2}} = \sum_{k,l=2}^{\infty} \frac{1}{(1+k+l)^{s+\varepsilon+2} (k+1)^{-s/2} \log k (l+1)^{-s/2} \log l}. \tag{18}$$

Consider the subseries of the series in the right-hand side of (18), consisting of terms for which $k = 2, 3, \ldots$ is arbitrary and $l = 2$. In this way we obtain the series

$$\sum_{k=2}^{\infty} \frac{1}{(k+3)^{s+\varepsilon+2} (k+1)^{-s/2} \log k 3^{-s/2} \log 2}.$$

We easily see that the terms in this last series are estimated from below by a constant time of

$$\sum_{k=2}^{\infty} \frac{1}{k^{s+\varepsilon+2} k^{-s/2} \log k} = \sum_{k=2}^{\infty} \frac{1}{k^{s/2+\varepsilon+2} \log k}. \tag{19}$$

Since $0 < \varepsilon \le -1 - \frac{s}{2}$, then $\frac{s}{2} + \varepsilon + 2 \le 1$. Therefore, the series in the right-hand side of (19), as well as the series in (18), i.e. the series in (15), diverge.

**Proof of Proposition 5.** We have, by Hölder's inequality,

$$\sum_{k,l=0}^{\infty} |a_k||a_l| \frac{1}{(1+k+l)^{\sigma+2}} = \sum_{k,l=0}^{\infty} \frac{|a_k||a_l|}{(k+1)^{(s+1)/2}(l+1)^{(s+1)/2}} \frac{(k+1)^{(s+1)/2}(l+1)^{(s+1)/2}}{(1+k+l)^{\sigma+2}} \le$$
$$\le \left( \sum_{k,l=0}^{\infty} \frac{|a_k|^2 |a_l|^2}{(k+1)^{s+1}(l+1)^{s+1}} \right)^{1/2} \left( \sum_{k,l=0}^{\infty} \frac{(k+1)^{s+1}(l+1)^{s+1}}{(1+k+l)^{2(\sigma+2)}} \right)^{1/2}. \tag{20}$$

Note that since $f \in A^{2,s}(U)$, then by (6)

$$\sum_{k,l=0}^{\infty} \frac{|a_k|^2 |a_l|^2}{(k+1)^{s+1}(l+1)^{s+1}} = \left( \sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{s+1}} \right)^2 < +\infty.$$

Consider the second series in the right-hand side of (20). This series has the same behavior as the series

$$\sum_{k,l=0,k+l>0}^{\infty} \frac{k^{s+1} l^{s+1}}{(k+l)^{2(\sigma+2)}}. \tag{21}$$

The terms of this last series are decreasing with respect to product order, so we can apply to the series (21) the so-called Cauchy's concentration principle (for double series). It follows from this that in order to verify that the series (21) is convergent, it is sufficient to prove that the series

$$\sum_{r,t=1}^{\infty} \frac{2^r 2^t}{(2^r + 2^t)^{2(\sigma+2)}} 2^{r(s+1)} 2^{t(s+1)} \tag{22}$$

is convergent. We have

$$\sum_{r,t=1}^{\infty} \frac{2^r 2^t}{(2^r + 2^t)^{2(\sigma+2)}} 2^{r(s+1)} 2^{t(s+1)} = \sum_{r,t=1}^{\infty} \frac{2^r 2^{r+p}}{(2^r + 2^{r+p})^{2(\sigma+2)}} 2^{r(s+1)} 2^{(r+p)(s+1)} =$$
$$= \sum_{r=1}^{\infty} \frac{2^{2r+2r(s+1)}}{(2^r)^{2(\sigma+2)}} \sum_{p=1}^{\infty} \frac{2^{p+p(s+1)}}{(1+2^p)^{2(\sigma+2)}}. \tag{23}$$

The terms of the second series in the right-hand side of (23) are estimated from above by

$$\frac{2^{p+p(s+1)}}{(2^p)^{2(\sigma+2)}}.$$

Therefore, the right-hand side of (23) is estimated from above by

$$\sum_{r=1}^{\infty} 2^{2r+2r(s+1)-2r(\sigma+2)} \sum_{p=1}^{\infty} 2^{p+p(s+1)-2p\sigma-4p} = \sum_{r=1}^{\infty} \left( \frac{1}{2^{2(\sigma-s)}} \right)^r \sum_{p=1}^{\infty} \left( \frac{1}{2^{2+2\sigma-s}} \right)^p. \tag{24}$$

Since $\sigma > \frac{s}{2} - 1 > s$, then $2 + 2\sigma - s > 0$, and $\sigma - s > 0$. Therefore both series in (24) are convergent. Hence the series in (22), as well as the second series in the right-hand side of (20) are convergent. This proves Proposition 5.

Note that if $\sigma \le \frac{s}{2} - 1$, then $2 + 2\sigma - s \le 0$, and the second series in the right-hand side of (24) diverges; therefore, we do not obtain by the above reasoning that the series in (16) is convergent for $\sigma \le \frac{s}{2} - 1$.

*The author would like to express his gratitude to the Referee for having pointed out some mistakes in the previous version of the paper.*

*The Referee also asked about the zero sets of the functions from the spaces $A^{2,s}(U)$, or more generally, $A^{p,s}(U)$, $p > 0$, for different values of s; in particular whether those zero sets depend on s. Up to now, we have not obtained the results in this direction, but it can be an interesting subject of further investigations.*

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# DISTRIBUTION-VALUED FUNCTIONS

# FUNKCJE O WARTOŚCIACH DYSTRYBUCYJNYCH

**Mathematics Subject Classification:** 46F05, 46M40.

A b s t r a c t

We study continuity and differentiability of a distribution-valued function. It is understood in a strong sense due to inductive limit topology.

*Keywords*: *distribution-valued function, inductive limit topology*

S t r e s z c z e n i e

W artykule przedstawiono zagadnienie ciągłości i różniczkowalności funkcji o wartościach dystrybucyjnych. Pojęcia te są rozumiane w sensie silnym dzięki zastosowaniu topologii induktywnej.

*Słowa kluczowe*: *funkcja o wartościach dystrybucyjnych*, *topologia induktywna*

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## 1. Introduction

Distribution-valued functions are a natural and convenient tool in constructing linear mathematical models for many physics phenomena and solving differential equations. The spaces of distributions and tempered distributions can be treated as duals to the nuclear spaces: the space of test functions $\mathcal{D}(\Omega)$ and rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n)$ respectively, which are equipped with inductive limit topology [3, 6]. Usually, continuity and differentiability of a distribution-valued function is understood in a weak sense, that is a function $T : M \ni x \mapsto T(x) \in \mathcal{D}'(\Omega, B)$ is of class $\mathcal{C}^1$ if the map $M \ni x \mapsto T(x)\varphi \in B$ is of class $\mathcal{C}^1$ for any test function $\varphi$ [1, 2, 7, 8].

In this paper, we consider distribution-valued functions which are continuous and differentiable in a strong sense due to inductive limit topology. We prove that if a distribution--valued function is differentiable in the sense of Definition 4.2, then it is differentiable in the weak sense (Corollary 4.4), the same refers to tempered distribution-valued functions (Remark 3.2, 3.5, Lemma 5.1). A similar approach was presented in [9] where summable in a strong sense distribution-valued functions were considered, and with the use of absolutely continuous distribution-valued functions, solutions to several Cauchy problems (Dirac equation, Navier-Lamé equation, biparabolic equation) were constructed. On the other hand, in [2] the parameter product of a distribution and a smooth (in a weak sense) distribution-valued function is introduced. This kind of product can be use in quantum electrodynamics, but also in modelling the vibration of a plate with piezoelectric actuators of an arbitrary shape [10].

## 2. Preliminaries

Let $\mathcal{S}$ be a locally convex space and let sn $\mathcal{S}$ denote the family of all continuouos seminorms on $\mathcal{S}$. Assume there is a decreasing sequence of convex balanced subsets of $\mathcal{S}$ that forms a local base in $\mathcal{S}$ and let $q_m$ be the Minkowski functional of the $m$-th set of some fixed base of that kind. Then $(q_m)_{m\in\mathbb{N}}$ is a seperating family of continuous seminorms on $\mathcal{S}$ and introduces the same topology on $\mathcal{S}$ as sn $\mathcal{S}$ does.

Let $B$ be a Banach space over a scalar field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) and $\mathcal{L}(\mathcal{S}, B)$ be the space of all $\mathbb{K}$-linear continuous mappings $\mathcal{S} \to B$. For every $p \in$ sn $\mathcal{S}$, let $\mathcal{L}((\mathcal{S}, p), B)$ be the space of all linear $p$-continuous mappings $\mathcal{S} \to B$. Let $T \in \mathcal{L}((\mathcal{S}, p), B)$ and denote

$$\left|T\right|_p = \sup_{p(\varphi)\le 1} \left|T(\varphi)\right|. \tag{1}$$

It is well known that $\mathcal{L}((\mathcal{S}, p), B)$ with the norm (1) is a Banach space. Moreover

(i) if $p, q \in$ sn $\mathcal{S}$ and $p \le q$ then $\left|\cdot\right|_p \ge \left|\cdot\right|_q$ and the canonical injection $\mathcal{L}((\mathcal{S}, p), B) \hookrightarrow$ $\hookrightarrow \mathcal{L}((\mathcal{S}, q), B)$ is continuous,

(ii) $\mathcal{L}(\mathcal{S}, B) = \bigcup_{p\in \mathrm{sn}\,\mathcal{S}} \mathcal{L}(\mathcal{S}, p), B) = \bigcup_{m=1}^{\infty} \mathcal{L}(\mathcal{S}, q_m), B).$

We consider the space $\mathcal{L}(\mathcal{S}, B)$ endowed with the inductive limit topology with respect to the family of canonical injections

$$\mathcal{L}((\mathcal{S}, q), B) \hookrightarrow \mathcal{L}(\mathcal{S}, B) \tag{2}$$

for all $q \in$ sn $\mathcal{S}$, that is the finest locally convex topology on $\mathcal{L}(\mathcal{S}, B)$ such that all the mappings (2) are continuous. This topology is also determined by the smaller family of inclusions,

$$\mathcal{L}((\mathcal{S}, q_m), B) \hookrightarrow \mathcal{L}(\mathcal{S}, B)$$

for $m \in \mathbb{N}$.

**Example 2.1.** *Test functions and distributions. Let* $\mathcal{D}(\Omega)$ denote the linear space of infinitely differentiable functions with compact supports which map $\Omega \in$ top $\mathbb{R}^n$ into $\mathbb{K}$ and let $\mathbb{K} \subset \Omega$ be compact. Then $\mathcal{D}_K(\Omega)$ denotes the subspace of $\mathcal{D}(\Omega)$ which consists of functions with supports in $K$. Each $\mathcal{D}_K(\Omega)$ is a Fréchet space with the family of seminorms $(q_m)_{m=0}^{\infty}$,

$$q_m(\varphi) = \sup_{x \in \Omega} \sup_{|\alpha| \le m} \left| D^{\alpha} \varphi(x) \right|$$

for all $\varphi \in \mathcal{D}_K(\Omega)$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and $D^{\alpha} = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$. The family of maps $\mathcal{D}_K(\Omega) \hookrightarrow \mathcal{D}(\Omega)$, for all compact $K \subset \Omega$, introduces the inductive limit topology in $\mathcal{D}(\Omega)$. The same topology can be obtained using the family $\mathcal{D}_{K_\nu}(\Omega) \hookrightarrow \mathcal{D}(\Omega)$, for an arbitrary sequence $(K_\nu)$ such that $\bigcup_{\nu=1}^{\infty} K_\nu = \Omega,\ K_\nu \subset \operatorname{int} K_{\nu+1}$. Moreover, for every compact $K \subset \Omega$ the topology that $\mathcal{D}_K(\Omega)$ inherits from $\mathcal{D}(\Omega)$ coincides with the topology of the Fréchet space.

Let $\mathcal{D}'(\Omega, B)$ denote the space of distributions that is the space of all linear continuous mappings of $\mathcal{D}(\Omega)$ into $B$. If $T_\nu$ is a sequence of distributions in $\mathcal{D}(\Omega)$, the statement

$$T_\nu \to T \qquad \text{in} \qquad \mathcal{D}'(\Omega, B)$$

refers to the weak$^*$-topology which means that $T_\nu(\varphi) \to T(\varphi)$ for every $\varphi \in \mathcal{D}(\Omega)$. Observe that for any fixed compact $K \subset \Omega$ the family of injections

$$\mathcal{L}((\mathcal{D}_K(\Omega), q_m), B) \hookrightarrow \mathcal{L}(\mathcal{D}_K(\Omega), B)$$

for $m = 0, 1, 2, \ldots$ introduces the inductive limit topology in $\mathcal{L}(\mathcal{D}_K(\Omega), B)$ (comp. (2)). □

**Example 2.2.** *Schwartz functions and tempered distributions.* Let $\mathcal{S}(\mathbb{R}^n)$ be the space of all $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$ for which $P \cdot D^\alpha\varphi$ is a bounded function, for every polynomial $P$ and for every multi-index $\alpha$. It is known that $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space with the family of seminorm

$$q_m(\varphi) = \sup_{x \in \mathbb{R}^n} \sup_{|\alpha| \le m} (1 + |x|^2)^m \left| D^{\alpha} \varphi(x) \right|$$

for $m = 0, 1, 2, \ldots$ . Let us remember that a distribution $T \in \mathcal{D}'(\mathbb{R}^n, B)$ is tempered when it is continuous in topology of $\mathcal{S}(\mathbb{R}^n)$. This is equivalent to the fact that there is the unique extension $\overline{T}$ of $T$ to $\mathcal{S}(\mathbb{R}^n)$. It is customary to identify $T$ with its extension $\overline{T}$. Contrary to that, we will avoid this identification and denote by the space of tempered distribution $\mathcal{D}'_{\text{temp}}$ and by the space of their extensions into $\mathcal{S}(\mathbb{R}^n)$ $\mathcal{L}(\mathcal{S}(\mathbb{R}^n), B)$ . Consider the family of mappings

$$\mathcal{L}((\mathcal{S}(\mathbb{R}^n), q_m), B) \hookrightarrow \mathcal{L}(\mathcal{S}(\mathbb{R}^n), B)$$

for all $m = 0, 1, 2, \ldots$ . According to (2) it introduces inductive limit topology into the space $\mathcal{L}(\mathcal{S}(\mathbb{R}^n), B)$. □

Since the linear map

$$\varphi^{**} : \mathcal{L}(\mathcal{S}, B) \ni T \mapsto T(\varphi) \in B$$

is continuous for every $\varphi \in \mathcal{S}$, it is clear that the inductive limit topology in $\mathcal{L}(\mathcal{S}, B)$ is stronger than the weak$^*$-topology. The following example had been communicated to the author by K. Holly. It shows that the inductive limit topology in $\mathcal{L}(\mathcal{S}, B)$ is actually strictly stronger.

**Example 2.3.** Let $K = [0, 1]$, $\mathcal{S} = \mathcal{D}_K(\mathbb{R})$, B = $\mathbb{R}$. Suppose that on the contrary, the inductive limit topology, denoted by top $(\mathcal{D}_K(\mathbb{R}))'$, coincides with the weak$^*$-topology in $(\mathcal{D}_K(\mathbb{R}))'$, Let $(\varphi_\nu)_{\nu=0,1,2,\ldots} \subset \mathcal{D}_K(\mathbb{R})$ be linearly dense in the Hilbert space $L^2([0, 1])$. Consider the linear operator

$$\eta : (\mathcal{D}_K(\mathbb{R}))' \ni T \mapsto \left( \frac{1}{q_0(\varphi_0)} T(\varphi_0), \frac{1}{q_1(\varphi_1)} T(\varphi_1), \ldots \right) \in l^\infty .$$

Since for any $m, n \in \{0, 1, \ldots\}$

$$\left| \frac{1}{q_n(\varphi_n)} T(\varphi_n) \right| \le M \cdot |T|_{q_m} ,$$

so the map $\eta : (\mathcal{D}_K(\mathbb{R}))' \to l^\infty$ is correctly defined and continuous. Thus

$$\left\{ T \in (\mathcal{D}_K(\mathbb{R}))' : |\eta(T)|_{l^\infty} < 1 \right\} \in \text{top } (\mathcal{D}_K(\mathbb{R}))'.$$

We assumed that top $(\mathcal{D}_K(\mathbb{R}))'$ and the weak$^*$-topology in $(\mathcal{D}_K(\mathbb{R}))'$ coincide, thus there are $\psi_1, \ldots, \psi_N \in \mathcal{D}_K(\mathbb{R})$ and $\varepsilon > 0$ such that

$$\sigma = \bigcap_{i=1}^{N} \psi_i^{**}(-\varepsilon, \varepsilon) \subset \{ T \in (\mathcal{D}_K(\mathbb{R}))' : |\eta(T)|_{l^\infty} < 1 \}.$$

Let $f \in L^2([0, 1])$ be such that $|f|_{L^2} > 0$ and $f \perp \psi_1, \ldots, \psi_N$. Define the functional

$$T_f : \mathcal{D}_K(\mathbb{R}) \ni \varphi \mapsto \int_0^1 \varphi(x) f(x)\, dx \in \mathbb{R}.$$

Since

$$\psi_i^{**}(T_f) = T_f(\psi_i) = \int_0^1 \psi_i(x) f(x)\, dx = 0,$$

it follows that $T_f \in \sigma$, and consequently $RT_f \in \sigma$ for any $R \in \mathbb{N}$. Thus

$$\eta(T_f) = 0. \tag{3}$$

On the other hand there is $v \in \{0, 1, \ldots\}$ such that $\varphi_v$, is not orthogonal to $f$, for otherwise $(\varphi_v)_v$ would not be linearly dense in $L^2([0, 1])$. Thus $T_f(\varphi_v) \neq 0$ and so $\eta(T_f) \neq 0$ which contradicts (3). □

## 3. $\mathcal{L}(\mathcal{S}, B)$ – valued functions

Let $M$ be an open interval in $\mathbb{R}$. We shall consider mappings of $M$ into $\mathcal{L}(\mathcal{S}, B)$.

**Definition 3.1.** *A mapping $T : M \to \mathcal{L}(\mathcal{S}, B)$ is continuous if there exists a seminorm $q \in \operatorname{sn} \mathcal{S}$ such that $T(x) \in \mathcal{L}((\mathcal{S}, q), B)$ for all $x \in M$ and the mapping $T : M \to \mathcal{L}(\mathcal{S}, q), B)$ is continuous.*

It is obvious that a mapping $T : M \to \mathcal{L}(\mathcal{S}, B)$ is continuous iff there exists $m \in \mathbb{N}$ such that $T(x) \in \mathcal{L}((\mathcal{S}, q_m), B)$ for all $x \in M$ and the mapping $T : M \to \mathcal{L}((\mathcal{S}, q_m), B)$ is continuous.

**Remark 3.2.** *If $T : M \to \mathcal{L}(\mathcal{S}, B)$ is continuous, then the map $M \ni x \mapsto T(x)\psi \in B$ is continuous for every $\psi \in \mathcal{S}$.*

Indeed, there is a seminorm $q \in \operatorname{sn} \mathcal{S}$ such that $T(x) \in \mathcal{L}((\mathcal{S}, q), B)$ for all $x \in M$ and the map $T : M \to \mathcal{L}((\mathcal{S}, q), B)$ is continuous. Let $\psi \in \mathcal{S}$. Then the operator

$$\psi^{**}_{|\mathcal{L}((\mathcal{S},q),B)} : \mathcal{L}((\mathcal{S},q),B) \to B$$

is linear and continuous. Thus $\psi^{**} \circ T : M \to B$ is continuous and

$$(\psi^{**} \circ T)(x) = \psi^{**}(T(x)) = T(x)\psi$$

for $x \in M$.

**Definition 3.3.** *A mapping $T : M \to \mathcal{L}(\mathcal{S}, B)$ is differentiable if there exists a seminorm $q \in \operatorname{sn} \mathcal{S}$ such that $T(x) \in \mathcal{L}((\mathcal{S}, q), B)$ for all $x \in M$ and the mapping $T : M \to \mathcal{L}((\mathcal{S}, q), B)$ is differentiable.*

Let $q \in \operatorname{sn} \mathcal{S}$ be a seminorm from Definition 3.3, then $\left(\frac{d}{dx}T\right)_q$ denotes the derivative of $T : M \to \mathcal{L}((\mathcal{S}, q), B)$. Suppose that there is another seminorm $p \in \operatorname{sn} \mathcal{S}$ such that $T(x) \in \mathcal{L}((\mathcal{S}, p), B)$ for all $x \in M$ and the mapping $T : M \to \mathcal{L}((\mathcal{S}, p), B)$ is differentiable. Let $r = \max\{p, q\}$. Then $r \in \operatorname{sn} \mathcal{S}$ and $\iota_{qr} : \mathcal{L}((\mathcal{S}, q), B) \hookrightarrow \mathcal{L}((\mathcal{S}, r), B)$, $\iota_{pr} : \mathcal{L}((\mathcal{S}, p), B) \hookrightarrow$ $\hookrightarrow \mathcal{L}((\mathcal{S}, r), B)$ are linear and continuous. Consequently $\iota_{qr} \circ T : M \to \mathcal{L}((\mathcal{S}, r), B)$, $\iota_{pr} \circ T : M \to \mathcal{L}((\mathcal{S}, r), B)$ are differentiable and $\iota_{qr} \circ T = \iota_{pr} \circ T$. Thus

$$\frac{d}{dx}(\iota_{qr} \circ T) = \iota_{qr} \circ \left(\frac{d}{dx}T\right)_q, \quad \frac{d}{dx}(\iota_{pr} \circ T) = \iota_{pr} \circ \left(\frac{d}{dx}T\right)_p,$$

and finally

$$\left(\frac{d}{dx}T\right)_q = \left(\frac{d}{dx}T\right)_p.$$

Therefore the derivative of $T : M \to \mathcal{L}(\mathcal{S}, B)$ does not depend on the choice of a seminorm in Definition 3.3.

**Definition 3.4.** *If a map $T : M \to \mathcal{L}(\mathcal{S}, B)$ is differentiable, then its derivative is defined by*

$$\frac{d}{dx}T = \left(\frac{d}{dx}T\right)_q .$$

Similarly to Remark 3.2 we have

**Remark 3.5.** *If $T : M \to \mathcal{L}(\mathcal{S}, B)$ is differentiable then the map $M \ni x \mapsto T(x)\psi \in B$ is differentiable for every $\psi \in \mathcal{S}$ and*

$$\frac{d}{dx}(T(\cdot)\psi)(x) = \left(\frac{d}{dx}T\right)(\cdot)\psi$$

*for all $x \in M$.*

**Theorem 3.6.** *Let $T : M \to \mathcal{L}(\mathcal{S}, B)$ be differentiable. Additionally consider a Banach space $B_1$ and a locally convex space $\mathcal{S}_1$. Assume that a mapping $L : \mathcal{L}(\mathcal{S}, B) \to \mathcal{L}(\mathcal{S}_1, B_1)$ is linear and satisfies the condition*

$$\forall p \in \operatorname{sn}\mathcal{S}\ \exists q \in \operatorname{sn}\mathcal{S}_1 \quad L(\mathcal{L}((\mathcal{S}, p), B)) \subset \mathcal{L}((\mathcal{S}_1, q), B_1) \quad \textit{and} \tag{4}$$

$$L_{|\mathcal{L}((\mathcal{S},p),B)} : \mathcal{L}((\mathcal{S}, p), B) \to \mathcal{L}((\mathcal{S}_1, q), B_1) \quad \textit{is continuous.}$$

*Then the map $L : \mathcal{L}(\mathcal{S}, B) \to \mathcal{L}(\mathcal{S}_1, B_1)$ is continuous, $L \circ T : M \to \mathcal{L}(\mathcal{S}_1, B_1)$ is differentiable and*

$$\frac{d}{dx}(L \circ T) = L \circ \frac{d}{dx}T.$$

*Proof.* To prove the continuity of $L : \mathcal{L}(\mathcal{S}, B) \to \mathcal{L}(\mathcal{S}_1, B_1)$, with respect to the inductive limit topology on $\mathcal{L}(\mathcal{S}, B)$, it is sufficient to show that for every $p \in \operatorname{sn}\mathcal{S}$ the map $L_{|\mathcal{L}((\mathcal{S},p),B)} : \mathcal{L}((\mathcal{S}, p), B) \to \mathcal{L}(\mathcal{S}_1, B_1)$ is continuous. Let $p \in \operatorname{sn}\mathcal{S}$. According to (4) there is a seminorm $q \in \operatorname{sn}\mathcal{S}_1$ such that $L_{|\mathcal{L}((\mathcal{S},p),B)} : \mathcal{L}((\mathcal{S}, p), B) \to \mathcal{L}((\mathcal{S}_1, q), B_1)$ is continuous. Since $\mathcal{L}(\mathcal{S}_1, B_1)$ is equipped with the inductive topology, the canonical injection $\mathcal{L}(\mathcal{S}_1, q), B_1) \hookrightarrow \mathcal{L}(\mathcal{S}_1, B_1)$ is also continuous. Thus $L_{|\mathcal{L}((\mathcal{S},p),B)} : \mathcal{L}((\mathcal{S}, p), B) \to \mathcal{L}(\mathcal{S}_1, B_1)$ is continuous.

Assume now that $T : M \to \mathcal{L}(\mathcal{S}, B)$ is differentiable. Let $p \in \operatorname{sn}\mathcal{S}$ be such that $T(x) \in \mathcal{L}((\mathcal{S}, p), B)$ for every $x \in M$ and $T : M \to \mathcal{L}((\mathcal{S}, p), B)$ is differentiable. On account of (4) there is $q \in \operatorname{sn}\mathcal{S}_1$ such that $L_{|\mathcal{L}((\mathcal{S},p),B)} : \mathcal{L}((\mathcal{S}, p), B) \to \mathcal{L}((\mathcal{S}_1, q), B_1)$ is linear and continuous. Hence $L_{|\mathcal{L}((\mathcal{S},p),B)} \circ T : M \to \mathcal{L}((\mathcal{S}_1, q), B_1)$ is differentiable and

$$\frac{d}{dx}(L \circ T)_q = \frac{d}{dx}(L_{|\mathcal{L}((\mathcal{S},p),B)} \circ T) = L_{|\mathcal{L}((\mathcal{S},p),B)} \circ \frac{d}{dx}T.$$

□

Note that if $F : S_1 \to S$ is linear and continuous then its transpose

$$F^* : \mathcal{L}(\mathcal{S}, B) \ni T \to T \circ F \in \mathcal{L}(\mathcal{S}_1, B) \tag{5}$$

satisfies (4). Indeed, for any $p \in \operatorname{sn} \mathcal{S}$ there is $q = p \circ F \in \operatorname{sn} \mathcal{S}_1$ such that $F^*(\mathcal{L}((\mathcal{S}, p), B)) \subset \subset \mathcal{L}((\mathcal{S}_1, q), B)$ and the restriction $F^*_{|\mathcal{L}((\mathcal{S},p),B)} : \mathcal{L}((\mathcal{S}, p), B) \to \mathcal{L}(\mathcal{S}_1, q), B)$ is a continuous map between Banach spaces.

On the other hand, if $L : B \to B_1$ is linear and continuous then the map

$$^*L : \mathcal{L}(\mathcal{S}, B) \ni T \mapsto L \circ T \in \mathcal{L}(\mathcal{S}, B_1) \tag{6}$$

satisfies (4). Indeed for every $p \in \operatorname{sn} \mathcal{S}$ the map $^*L_{|\mathcal{L}((\mathcal{S},p),B)} : \mathcal{L}((\mathcal{S}, p), B) \to \mathcal{L}(\mathcal{S}, p), B_1)$ is continuous. In particular, for $L(T) = \lambda T$, $\lambda \in \mathbb{K}$ we have

**Corollary 3.7.** *If* $T : M \to \mathcal{L}(\mathcal{S}, B)$ *is differentiable then* $\lambda T$ *is also differentiable and*

$$\frac{d}{dx}(\lambda T) = \lambda \frac{d}{dx} T.$$

It is also clear that

**Proposition 3.8.** *If* $T_1, T_2 : M \to \mathcal{L}(\mathcal{S}, B)$ *are differentiable then* $T_1 + T_2$ *is also differentiable and*

$$\frac{d}{dx}(T_1 + T_2) = \frac{d}{dx} T_1 + \frac{d}{dx} T_2.$$

## 4. Distribution-valued functions

Let us consider a function $T : M \ni x \mapsto T(x) \in \mathcal{D}'(\Omega, B)$.

**Definition 4.1.** *A map* $T : M \to \mathcal{D}'(\Omega, B)$ *is continuous if for any compact* $K \subset \Omega$ *the map*

$$T_K : M \ni x \mapsto T(x)_{|\mathcal{D}_K(\Omega)} \in \mathcal{L}(\mathcal{D}_K(\Omega), B)$$

is continuous.

We shall write $T(x)_K$ *for* $T(x)_{|\mathcal{D}_K(\Omega)}$.

**Definition 4.2.** *A mapping* $T : M \to \mathcal{D}'(\Omega, B)$ *is differentiable if for any compact* $K \subset \Omega$ *the map*

$$T_K : M \ni x \mapsto T(x)_K \in \mathcal{L}(\mathcal{D}_K(\Omega), B)$$

*is differentiable, and*

$$\left(\frac{d}{dx} T\right)(x)\varphi = \left(\frac{d}{dx} T_K\right)(x)\varphi \tag{7}$$

*for* supp $\varphi \subset K$, $x \in M$.

The following lemma ensures that the definition is meaningful and that the derivative of a map $T : M \to \mathcal{D}'(\Omega, B)$ is a distribution-valued function.

**Lemma 4.3.** *For any compact* $K_1, K_2 \subset \Omega$ *if* $\varphi \in \mathcal{D}_{K_1}(\Omega) \cap \mathcal{D}_{K_2}(\Omega)$ *then*

$$\left(\frac{d}{dx}T_{K_1}\right)(x)\varphi = \left(\frac{d}{dx}T_{K_2}\right)(x)\varphi.$$

*Moreover* $\left(\frac{d}{dx}T\right)(x) \in \mathcal{D}'(\Omega, B)$ *for every* $x \in M$.

*Proof.* Let $x \in M$, $K_1$, $K_2 \subset \Omega$ be compact and $\varphi \in \mathcal{D}_{K_1}(\Omega) \cap \mathcal{D}_{K_2}(\Omega)$. Since maps $\varphi_i^{**} : \mathcal{L}(\mathcal{D}_{K_i}(\Omega), B) \ni T(x)_{K_i} \mapsto T(x)_{K_i}\varphi \in B$ are linear and continuous for $i = 1, 2$ and $(\varphi_1^{**} \circ T_{K_1})(x) = T(x)_{K_1}\varphi$, $(\varphi_2^{**} \circ T_{K_2})(x) = T(x)_{K_2}\varphi$, and both these maps $M \to B$ are differentiable. From Remark 3.5

$$\frac{d}{dx}(\varphi_1^{**} \circ T_{K_1})(x) = \left(\frac{d}{dx}T_{K_1}\right)(x)\varphi, \quad \frac{d}{dx}(\varphi_2^{**} \circ T_{K_2})(x) = \left(\frac{d}{dx}T_{K_2}\right)(x)\varphi,$$

hence

$$\left(\frac{d}{dx}T_{K_1}\right)(x)\varphi = \left(\frac{d}{dx}T_{K_2}\right)(x)\varphi.$$

This proves that the relation given by (7) is a function on the domain $\mathcal{D}(\Omega) = \bigcup\{\mathcal{D}_K(\Omega) : K \subset \Omega, K\text{-compact}\}$. The linearity and continuity of $\left(\frac{d}{dx}T\right)(x) : \mathcal{D}(\Omega) \to B$ is obvious. $\square$

From the above proof it follows that

**Corollary 4.4.** *If* $T : M \to \mathcal{D}'(\Omega, B)$ *is differentiable then for every* $\varphi \in \mathcal{D}(\Omega)$ *the map* $M \ni x \mapsto T(x)\varphi \in \mathbb{K}$ *is differentiable and, for every* $x \in M$

$$\frac{d}{dx}(T(\cdot)\varphi)(x) = \left(\frac{d}{dx}T(x)\right)\varphi.$$

**Remark 4.5.** *If* $T_1, T_2 : M \to \mathcal{D}'(\Omega, B)$ *are differentiable and* $\lambda_1, \lambda_2 \in \mathbb{K}$, *then* $\lambda_1 T_1 + \lambda T_2$ *is also differentiable and*

$$\frac{d}{dx}(\lambda_1 T_1 + \lambda_2 T_2) = \lambda_1 \frac{d}{dx}T_1 + \lambda_2 \frac{d}{dx}T_2.$$

Consider now additionally $\Omega_1 \in \operatorname{top}\mathbb{R}^{n_1}$, $n_1 \in \mathbb{N}$, and a Banach space $B_1$.

**Theorem 4.6.** *Let* $T : M \to \mathcal{D}'(\Omega, B)$ *be differentiable. Suppose that a linear mapping* $L : \mathcal{D}'(\Omega, B) \to \mathcal{D}'(\Omega_1, B_1)$ *satisfies the condition:*

*for any compact* $K \subset \Omega_1$ *there is a compact* $Z \subset \Omega$ *and a linear map* $L_{ZK} : \mathcal{L}(\mathcal{D}_Z(\Omega), B) \to \mathcal{L}(\mathcal{D}_K(\Omega_1), B_1)$ *that satisfies condition (4) and the diagram*

$$\begin{array}{ccc} \mathcal{L}(\mathcal{D}_Z(\Omega), B) & \stackrel{L_{ZK}}{\rightarrow} & \mathcal{L}(\mathcal{D}_K(\Omega_1), B_1) \\ \iota_Z^* \uparrow & & \uparrow \iota_K^* \\ \mathcal{D}'(\Omega, B) & \stackrel{L}{\rightarrow} & \mathcal{D}'(\Omega_1, B_1) \end{array} \tag{8}$$

*commutes.*

*Then the map* $L \circ T : M \to \mathcal{D}'(\Omega_1, B_1)$ *is differentiable and for any* $x \in M$

$$\frac{d}{dx}(L \circ T)(x) = L\left(\frac{d}{dx}T(x)\right).$$

*Proof.* Let $K$ be a compact in $\Omega_1$. Due to (8) there exists a compact $Z \subset \Omega$ and a map $L_{ZK}$. Since $T : M \to \mathcal{D}'(\Omega, B)$ is differentiable, the map $T_Z : M \to \mathcal{L}(\mathcal{D}_Z(\Omega), B)$ is differentiable. We now apply Theorem 3.6 for $\mathcal{S} = \mathcal{D}_Z(\Omega)$, $\mathcal{S}_1 = \mathcal{D}_K(\Omega_1)$, and deduce that $L_{ZK} \circ T_Z : M \to$ $\to \mathcal{L}(\mathcal{D}_K(\Omega_1), B_1)$ is differentiable and for every $x \in M$

$$\frac{d}{dx}(L_{ZK} \circ T_Z)(x) = L_{ZK}\left(\frac{d}{dx}T_Z(x)\right).$$

As the diagram is commutative we get $L_{ZK} \circ \iota_Z^* = \iota_K^* \circ L$, and so

$$L_{ZK} \circ T_Z = L_{ZK} \circ \iota_Z^* \circ T = \iota_K^* \circ L \circ T = (L \circ T)_K.$$

In consequence $L \circ T : M \to \mathcal{D}'(\Omega_1, B_1)$ is differentiable. Moreover, for every $x \in M$

$$\begin{aligned} \frac{d}{dx}(L \circ T)_K(x) &= \frac{d}{dx}(L_{ZK} \circ T_Z)(x) = L_{ZK}\left(\frac{d}{dx}T_Z(x)\right) \\ &= L_{ZK}\left(\frac{d}{dx}(\iota_Z^* \circ T)(x)\right) = (L_{ZK} \circ \iota_Z^*)\left(\frac{d}{dx}T(x)\right) \\ &= (\iota_K^* \circ L)\left(\frac{d}{dx}T(x)\right) = \left(L\left(\frac{d}{dx}T(x)\right)\right)_K. \end{aligned}$$

□

Observe that with any linear and continuous operator $L : B \to B_1$ we may associate the linear operator

$${}^*L : \mathcal{D}'(\Omega, B) \ni T \mapsto L \circ T \in \mathcal{D}'(\Omega, B_1)$$

(compare (6)). It is clear that for every compact $K \subset \Omega$ the diagram

$$\begin{array}{ccc} \mathcal{L}(\mathcal{D}_K(\Omega), B) & \stackrel{({}^*L)_K}{\rightarrow} & \mathcal{L}(\mathcal{D}_K(\Omega), B_1) \\ \iota_K^* \uparrow & & \uparrow \iota_K^* \\ \mathcal{D}'(\Omega, B) & \stackrel{L}{\rightarrow} & \mathcal{D}'(\Omega, B_1) \end{array}$$

commutes, where $({}^*L)_K : \mathcal{L}(\mathcal{D}_K(\Omega), B) \ni T \mapsto L \circ T \in \mathcal{L}(\mathcal{D}_K(\Omega), B_1)$. So we have

**Corollary 4.7.** *If an operator $L : B \to B_1$ is linear and continuous then ${}^*L : \mathcal{D}'(\Omega, B) \ni T \mapsto$ $\mapsto L \circ T \in \mathcal{D}'(\Omega, B_1)$ satisfies condition (8).*

On the other hand, with a linear continuous operator $F : \mathcal{D}(\Omega_1) \to \mathcal{D}(\Omega)$ we may associate its linear transpose

$$F^* : \mathcal{D}'(\Omega, B) \ni T \mapsto T \circ F \in \mathcal{D}'(\Omega_1, B)$$

(compare (5)). The following lemma had been communicated to the author by K. Holly. It indicates a wide class of operators that satisfy (8).

**Lemma 4.8.** *Let $F : \mathcal{D}(\Omega_1) \to \mathcal{D}(\Omega)$ be linear and continuous. For every compact $K \subset \Omega_1$ there is a compact $Z \subset \Omega$ such that $F(\mathcal{D}_K(\Omega_1)) \subset \mathcal{D}_Z(\Omega)$, the restriction $F_K = F_{|\mathcal{D}_K(\Omega_1)}$ is a continuous map of the Frechét spaces $\mathcal{D}_K(\Omega_1)$, $\mathcal{D}_Z(\Omega)$, and the diagram*

$$\begin{array}{ccc} \mathcal{L}(\mathcal{D}_Z(\Omega), B) & \overset{(F_K)^*}{\to} & \mathcal{L}(\mathcal{D}_K(\Omega_1), B) \\ \iota_Z^* \uparrow & & \uparrow \iota_K^* \\ \mathcal{D}'(\Omega, B) & \overset{F^*}{\to} & \mathcal{D}'(\Omega_1, B) \end{array}$$

*commutes.*

*Proof.* Let $F : \mathcal{D}(\Omega_1) \to \mathcal{D}(\Omega)$ be linear and continuous and $K \subset \Omega_1$ be compact. We put $Z = \overline{\bigcup_{\varphi \in \mathcal{D}_K(\Omega_1)} \{F(\varphi) \neq 0\}}$ (the closure in $\mathbb{R}^n$). Suppose that, contrary to our claim, either $Z$ is not a subset of $\Omega$ or $Z$ is not compact. Then there is a sequence $(z_v)_{v \in \mathbb{N}} \subset \bigcup_{\varphi \in \mathcal{D}_K(\Omega_1)} \{F(\varphi) \neq 0\}$ which contains no subsequence convergent in $\Omega$. Let $v \in \mathbb{N}$. There is $\varphi_v \in \mathcal{D}_K(\Omega_1)$ such that $z_v \in \{F(\varphi_v) \neq 0\}$. Then $\varphi_v \neq 0$ and $q_m(\varphi_v) > 0$ for any norm $q_m$ in $\mathcal{D}_K(\Omega_1)$, $m \in \mathbb{N}$. Therefore

$$\psi_v = \frac{1}{v} \cdot \frac{\varphi_v}{q_v(\varphi_v)}$$

is correctly defined for $v \in \mathbb{N}$ and $\psi_v \to 0$ in $(\mathcal{D}_K(\Omega_1), q_m)$ for any $m \in \mathbb{N}$. Thus $\psi_v \to 0$ in the Frechét space $\mathcal{D}_K(\Omega_1)$ and consequently $F(\psi_v) \to 0$ in $\mathcal{D}(\Omega)$. In particular it means that there is a compact set $D \subset \Omega$ such that $\overline{\bigcup_{v \in \mathbb{N}} \{F(\psi_v) \neq 0\}} \subset D$. However $\{F(\psi_v) \neq 0\} = \{F(\varphi_v) \neq 0\}$ for any $v \in \mathbb{N}$. Hence $(z_v)_{v \in \mathbb{N}} \subset D$ and $(z_v)_{v \in \mathbb{N}}$ contains a subsequence which is convergent in $\Omega$. This leads to a contradiction. Therefore $Z$ is compact and $Z \subset \Omega$. Clearly $F(\mathcal{D}_K(\Omega_1)) \subset \mathcal{D}_Z(\Omega)$, and $F_K : \mathcal{D}_K(\Omega_1) \to \mathcal{D}_Z(\Omega)$ is continuous. □

**Corollary 4.9.** *If $F : \mathcal{D}(\Omega_1) \to \mathcal{D}(\Omega)$ is linear and continuous, then $F^* : \mathcal{D}'(\Omega, B) \ni T \mapsto$ $\mapsto T \circ F \in \mathcal{D}'(\Omega_1, B)$ satisfies condition (8).*

Let us recall that if $\Lambda \in \mathcal{D}'(\Omega, B)$, $\alpha$ is a multi-index, then $D^\alpha \Lambda \in \mathcal{D}'(\Omega, B)$ and for any $\varphi \in \mathcal{D}(\Omega)$

$$(D^\alpha \Lambda)\varphi = (-1)^{|\alpha|} \Lambda(D^\alpha \varphi).$$

**Proposition 4.10.** *Let $T : M \to \mathcal{D}'(\Omega, B)$ be differentiable and $\alpha \in \mathbb{N}^n$. Then the map $D^\alpha T : M \ni x \mapsto D^\alpha T(x) \in \mathcal{D}'(\Omega, B)$ is differentiable and for every $x \in M$*

$$\frac{d}{dx} D^\alpha T(x) = D^\alpha \left( \frac{d}{dx} T(x) \right).$$

*Proof.* The operator $F : \mathcal{D}(\Omega) \ni \varphi \mapsto (-1)^{|\alpha|} D^\alpha \varphi \in \mathcal{D}(\Omega)$ is linear, continuous and its transpose is of the form

$$F^* : \mathcal{D}'(\Omega, B) \ni T \mapsto D^\alpha T \in \mathcal{D}'(\Omega, B).$$

Due to Corollary 4.9. $F^*$ satisfies condition (8) and, according to Theorem 4.6, $F^* \circ T : M \to \to \mathcal{D}'(\Omega, B)$ is differentiable and for every $x \in M$

$$\frac{d}{dx}(D^\alpha T)(x) = \frac{d}{dx}(F^* \circ T)(x) = F^* \left( \frac{d}{dx} T(x) \right) = D^\alpha \left( \frac{d}{dx} T(x) \right).$$

Recall also that if $\Lambda \in \mathcal{D}'(\Omega, B)$ and $\eta : \Omega \to \mathbb{K}$ is a smooth function, $g : \Omega_1 \to \Omega$ is a smooth diffeomorphism, then $\eta\Lambda$, $\Lambda \circ g$ are distributions on $\mathcal{D}(\Omega)$, $\mathcal{D}(\Omega_1)$, respectively, and

$$(\eta\Lambda)\varphi = \Lambda(\eta\varphi) \quad \text{for} \quad \varphi \in \mathcal{D}(\Omega),$$

$$(\Lambda \circ g)\varphi = \left|\det g^{-1}\right| \Lambda(\varphi \circ g^{-1}) \quad \text{for} \quad \varphi \in \mathcal{D}(\Omega_1).$$

Thus, similarly to Proposition 4.10, we obtain

**Proposition 4.11.** *Let $T : M \to \mathcal{D}'(\Omega, B)$ be differentiable. Consider a smooth function $\eta : \Omega \to \mathbb{K}$ and a smooth diffeomorphism $g : \Omega_1 \to \Omega$ on an open set $\Omega_1$. Then the mappings: $\eta T : M \ni x \to \eta \cdot T(x) \in \mathcal{D}'(\Omega, B)$, $T \circ g : M \ni x \mapsto T(x) \circ g \in \mathcal{D}'(\Omega_1, B)$ are differentiable and for every $x \in M$*

$$\frac{d}{dx}(\eta \cdot T)(x) = \eta \cdot \frac{d}{dx} T(x),$$

$$\frac{d}{dx}(T \circ g)(x) = \frac{d}{dx} T(x) \circ g.$$

It is known that every locally summable function $u$ defines a distribution, called regular distribution, denoted by $[u]$,

$$[u]\varphi = \int_\Omega u(x)\varphi(x)dx \quad \text{for} \quad \varphi \in \mathcal{D}(\Omega).$$

For $h \in \mathcal{D}(\mathbb{R}^n)$ and $\Lambda \in \mathcal{D}'(\mathbb{R}^n, B)$, the convolution $h * \Lambda$ is well defined by the formula

$$(h * \Lambda)(x) = \Lambda(\tau_x h) \quad \text{for} \quad x \in \mathbb{R}^n,$$

where $\tau_x \varphi(y) = \varphi(x - y)$. Moreover, $h * \Lambda \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $D^\alpha(h * \Lambda) = (D^\alpha h) * \Lambda = h * D^\alpha \Lambda$. For arbitrary fixed $h \in \mathcal{D}(\mathbb{R}^n)$ take

$$F : \mathcal{D}(\mathbb{R}^n) \ni \varphi \mapsto \varphi * \breve{h} \in \mathcal{D}(\mathbb{R}^n),$$

where $\breve{h}(y) = h(-y)$ for $y \in \mathbb{R}^n$. Its transpose is of the form

$$F^* : \mathcal{D}'(\mathbb{R}^n, B) \ni \Lambda \mapsto [h * \Lambda] \in \mathcal{D}'(\mathbb{R}^n, B),$$

and satisfies (8), thus according to Theorem 4.6 we obtain

**Proposition 4.12.** *Let $T : M \to \mathcal{D}'(\mathbb{R}^n, B)$ be differentiable and $h \in \mathcal{D}(\mathbb{R}^n)$. Then the mapping $[h * T] : M \ni x \mapsto [h \cdot T(x)] \in \mathcal{D}'(\Omega, B)$ is differentiable and for every $x \in M$*

$$\frac{d}{dx}[h * T](x) = \left[ h * \frac{d}{dx} T(x) \right].$$

## 5. Tempered distribution-valued functions

Now turn to functions with values in the space of tempered distributions.

**Lemma 5.1.** *Consider a map $T : M \to \mathcal{D}'_{\text{temp}}$ such that $\overline{T} : M \ni x \mapsto \overline{T(x)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), B)$ is differentiable. Then $T : M \to \mathcal{D}'(\mathbb{R}^n, B)$ is differentiable and $\frac{d}{dx} T(x) \subset \frac{d}{dx} \overline{T(x)}$. In particular, the distribution $\frac{d}{dx} T(x)$ is tempered for any $x \in M$.*

*Proof.* Let $K \subset \mathbb{R}^n$ be a compact. Then the injection $F : \mathcal{D}_K(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n)$ is linear and continuous, thus the map $F^* : \mathcal{L}(\mathcal{S}(\mathbb{R}^n), B) \ni \overline{T} \mapsto \overline{T} \circ F \in \mathcal{L}(\mathcal{D}_K(\mathbb{R}^n), B)$ satisfies condition (4) and from Theorem 3.6 the map $F^* \circ \overline{T} : M \to \mathcal{L}(\mathcal{D}_K(\mathbb{R}^n), B)$ is differentiable and for any $x \in M$

$$\frac{d}{dx}(F^* \circ \overline{T})(x) = F^* \left( \frac{d}{dx} \overline{T(x)} \right).$$

Therefore the mapping $T_K = F^* \circ \overline{T} : M \to \mathcal{L}(\mathcal{D}_K(\mathbb{R}^n), B)$ is differentiable for any compact $K \subset \mathbb{R}^n$ and from Definition 4.2, $T : M \to \mathcal{D}'(\mathbb{R}^n, B)$ is differentiable. Moreover for any $x \in M$

$$\left( \frac{d}{dx} T_K \right)(x) = \frac{d}{dx} \overline{T(x)}\Big|_{\mathcal{D}_K(\mathbb{R}^n)}.$$

□

Now we are in the position to consider tempered distribution-valued functions.

**Definition 5.2.** *A mapping $T : M \to \mathcal{D}'_{\text{temp}}$ is continuous iff the mapping $\overline{T} : M \ni x \mapsto \overline{T(x)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), B)$ is continuous.*

**Definition 5.3.** *A map $T : M \to \mathcal{D}'_{\text{temp}}$ is differentiable iff the mapping $\overline{T} : M \ni x \mapsto \overline{T(x)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), B)$ is differentiable. Moreover, for any $x \in M$*

$$\overline{\frac{d}{dx}T(x)} = \frac{d}{dx}\overline{T(x)}.$$

Similarly to the Proposition 4.10, but using now Theorem 3.6 we have

**Proposition 5.4.** *Let* $T : M \to \mathcal{D}'_{\text{temp}}$ *be differentiable and* $\alpha \in \mathbb{N}^n$*. Then* $D^\alpha T : M \ni x \mapsto D^\alpha T(x) \in \mathcal{D}'_{\text{temp}}$ *is differentiable and for any* $x \in M$

$$D^\alpha\left(\frac{d}{dx}T(x)\right) \subset \frac{d}{dx}(\overline{D^\alpha T(x)}).$$

Recall that if $\Lambda \in \mathcal{D}'_{\text{temp}}$, $\eta : \mathbb{R}^n \to \mathbb{K}$ is a smooth function which is polynominally bounded together with all its derivatives, then $\eta\Lambda \in \mathcal{D}'_{\text{temp}}$ and for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\overline{\eta\Lambda}(\varphi) = \overline{\Lambda}(\eta\varphi).$$

Consider $F(\varphi) = \eta\varphi$ for $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then $F^*$ satisfies condition (4), and from Theorem 3.6 we obtain

**Proposition 5.5.** *Let* $T : M \to \mathcal{D}'_{\text{temp}}$ *be differentiable and* $\eta : \mathbb{R}^n \to \mathbb{K}$ *be a smooth function which is polynominally bounded together with all is derivatives. Then* $\eta T : M \ni x \mapsto \eta \cdot T(x) \in \mathcal{D}'_{\text{temp}}$ *is differentiable and for any* $x \in M$

$$\eta \cdot \left(\frac{d}{dx}T(x)\right) \subset \frac{d}{dx}(\overline{\eta \cdot T(x)}).$$

Similarly, taking $F(\varphi) = \varphi * \breve{h}$, where $\breve{h}(y) = h(-y)$ for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we have

**Proposition 5.6.** *Let* $T : M \to \mathcal{D}'_{\text{temp}}$ *be differentiable and* $h \in \mathcal{S}(\mathbb{R}^n)$*. Then* $[h * T] : M \ni x \mapsto [h \cdot T(x)] \in D'_{\text{temp}}$ *is differentiable and for any* $x \in M$

$$\left[h * \left(\frac{d}{dx}T(x)\right)\right] \subset \frac{d}{dx}(\overline{[h * T(x)]}).$$

Let $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ denote Fourier transform, $\mathcal{F}\varphi(\xi) = \hat{\varphi}(\xi) =$

$= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\xi\cdot y}\varphi(y)\,dy$ for $y \in \mathbb{R}^n$. Recall that if $\Lambda \in \mathcal{D}'_{\text{temp}}$ then $\mathcal{F}\Lambda \in \mathcal{D}'_{\text{temp}}$ and

$$\overline{\mathcal{F}\Lambda}(\varphi) = \overline{\Lambda}(\hat{\varphi})$$

for $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

**Proposition 5.7.** *Let* $T : M \ni x \mapsto T(x) \in \mathcal{D}'_{\text{temp}}$ *be differentiable. Then* $\mathcal{F}T : M \ni x \mapsto \mathcal{F}(T(x)) \in \mathcal{D}'_{\text{temp}}$ *is differentiable and for any* $x \in M$

$$\mathcal{F}\left(\frac{d}{dx}T(x)\right) \subset \frac{d}{dx}(\overline{\mathcal{F}(T(x))}).$$

*Proof.* Consider the operator $F: \mathcal{S}(\mathbb{R}^n) \ni \varphi \mapsto \hat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$. It is linear and continuous, so $F^*: \mathcal{L}(\mathcal{S}(\mathbb{R}^n), B) \to \mathcal{L}(\mathcal{S}(\mathbb{R}^n), B)$, where $F^*(\overline{\Lambda}) = \overline{\mathcal{F}\Lambda}$ for $\Lambda \in \mathcal{D}'_{\text{temp}}$. According to Theorem 3.6 the map $F^*\overline{T}: M \to \mathcal{L}(\mathcal{S}(\mathbb{R}^n), B)$ is differentiable, but

$$(F^*\overline{T})(x)\varphi = \overline{T(x)}(\hat{\varphi}) = \overline{\mathcal{F}(T(x))}\varphi$$

for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $x \in M$. Consequently, $M \ni x \mapsto \mathcal{F}(T(x)) \in \mathcal{D}'_{\text{temp}}$ is differentiable and

$$\overline{\mathcal{F}\left(\frac{d}{dx}T(x)\right)} = \frac{d}{dx}(\overline{\mathcal{F}(T(x))}).$$

$\square$

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# BUG FIXING TOOL (BFT) FOR CONTINUOUS INTEGRATION

# NARZĘDZIE BUG FIXING TOOL (BFT) DO CIĄGŁEJ INTEGRACJI

A b s t r a c t

The paper is focused on the bug fixing handling business process rather then just on fixing a bug. The tool presented here is dedicated to supporting the business process of bug fixing and not to bug fixing itself. It is addressed especially to small teams having a common testing team.

*Keywords*: *continuous integration, version control server, bug fixing, bug submission, iterative software development process*

S t r e s z c z e n i e

Artykuł skoncentrowany jest na procesie biznesowym obsługi błędów oprogramowania bardziej niż tylko na kwestiach obsługi błędów. W konsekwencji również narzędzie zaprezentowane w niniejszej pracy służy wspieraniu procesu biznesowego poprawiania błędów, a nie samemu poprawianiu błędów. Jest ono adresowane szczególnie do małych zespołów posiadających wspólny zespół testerów.

*Słowa kluczowe*: *ciągła integracja, serwer wersjonowania,, poprawianie błędów, zgłaszanie błędów, iteracyjny proces rozwoju oprogramowania*

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## 1. Introduction

The authors of the paper have different and complementary experience from the market. One of them is working in a large international software development company and is responsible for configuration management. The other has long lasting experience from the consulting branch of the same business and thus has some observations from many companies different in size – from very small ones to corporations. The authors agree that the process of bug fixing in many companies is handled mainly on the business process level (by business workers – the staff) with very limited tool support. This is why the process is far from automated. As a matter of fact, the possible process automation is blocked by the process inconsistency, by multiplicity of process variations and by the mess of incompatible tools used to support this process. The possible scale of optimization of the process will result in a relatively large number of attractive proposals that are intended to be addressed in succeeding publications.

A simple but useful tool, the Bug Fixing Tool (BFT), supporting existing bug fixing business processes is presented in the paper. This tool communicates with Subversion (SVN) version control server and is intended to be used in a continuous integration environment. At the same time it constitutes the first stage of the implementation of the defect management process improvement concept presented as a whole in paper [8].

## 2. Continuous integration

The concept of continuous integration [1–6] was introduced to support the business need to (almost) always have an up-to-date version of a newly released software product. This need appeared as a result of risk minimization achieved by so-called early risk mitigation. More generally speaking, the early risk mitigation is supported by iterative software development processes very well in many disciplines of the process. For example, Rational Unified Process (RUP) expects to have a new release at least at the end of each iteration (excluding Inception phase). But it does not limit the expectation to have releases more frequently – during iterations – in any way. It seams that XP approaches that are promoted by Agile processes stress the necessity of having such frequent (continuous) releases or at least builds more clearly.

There is a slight umbiguity in understanding the subject of continuous integration. In one case, the product should be understood as the installation program (setup) of the product. In the second case, the subject of release is meant as the build (executable of the product) only. The first approach is more general and maps better to the notion of product. However it is worth noticing here that the product is verified by tests, and the tests are different. In order to perform unit tests, the build is sufficient as tests of this kind are not executed on the installed product but on the build of code in a development environment. In the case of black-box testing, the build is not sufficient and the installation should be the subject of test execution.

Nevertheless, the paper is focused on builds only as the direct results of the integration process. It is also assumed that the bug was already identified and located in the source code correctly at the beginning. So the bugs that are not addressed to code (say wrong version of a database file in the product installation) are not the subject of this paper.

## 3. Bug fixing problem characteristics

This section shows the consequences of finding a bug in a particular project associated to a product for an existing business processes.

A sample configuration tree view for a product's file or directory which is offered by most version control servers is shown in Fig. 1. There are the configuration branches A, B and C. The question of what they represent may appear here. And there are at least two possible answers:

- branches represent different product versions possibly elaborated by different teams,
- branches represent different development or integration branches in a particular project.

So, what should be done when a bug is identified? Let us assume that the bug was identified in branch C during any kind of tests performed by developers (typically white--box testing) or by testers (typically black-box testing). In such a case the special bug-fixing branch should be created from branch C in order to fix the bug just in this newly created branch. The fix branch should not be a development branch as the history of fixing bugs is different to the history of development. Mixing them is a bad practice. That is why Unified Change Management [7] promotes creating special fix branches for bug fixing purposes, nevertheless-manually. The file where the bug was identified is present also in branches A and B. So other teams (or other team members) should be reported to somehow about the bug. They should be able to to pass the report about fixing the bug as well. And this is the place for the subjected tool. Both the characteristic (description) of the bug and fix should be placed somewhere. The best place for it is just the fix branch in the version control server.


Fig. 1. Existing configuration tree view for a file or a directory in version control server

It is worth noting who identifies what in the process of bug identification and fixing. The typical but bad practice is that the team who identifies the bug is responsible for the correct propagation of the bug information to other teams or team members.

This approach is acceptable in a small number of small teams especially when they are supported by a common testing team. In such a situation the team members are able to identify other members of the team or to identify other teams and check if the teams still exist. The question who and how should manage the information of a bug addressed to already not existing team may appear. The common testing team is promoted here because it has enough knowledge to be able to check if the bug maps to other branches and determine which ones. The last assupmption about the common testing team is crucial as the team must be able to know, understand and have access to tests and code. The knowledge and privilidges mentioned above are necessery to make possible the verification, first if the bug exists, and second if the fix of the bug is correct in all branches involved. It must also be underlined here that the product versions in different branches represent different product functionalities. Applying the same approach to large products, large or many teams and many testing teams is not good. So, another approach suitable for this more demanding situation will be proposed in succeeding papers.

The tool presented in the next sections fits best to the simpler software development process as described in the paragraph above.

## 4. Problem solution

The problem of a desirable reaction to the bug finding was described in the previous section. This section is dedicated to the description of the role of the BFT tool. How the tool is related to the existing process is depicted in Fig. 2.

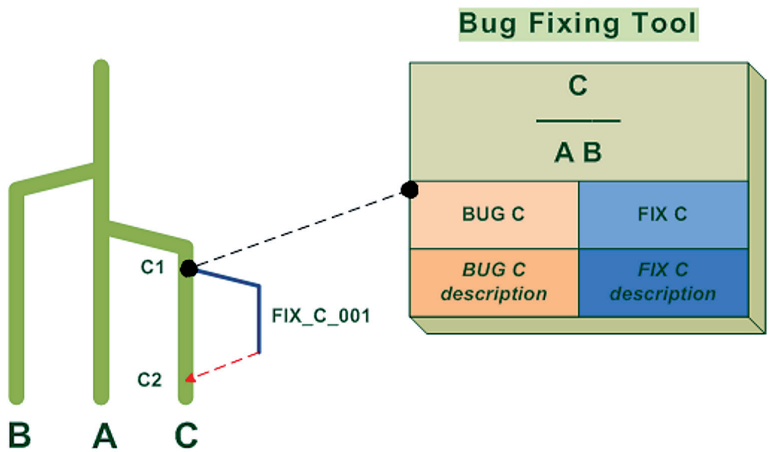


Fig. 2. Location of the Bug Fixing Tool (BFT)

In order to solve the problem with the tool support the following assumptions were made. The tool works for one product but many instances of it may be executed, one for each product. There is one repository per product. The version control server is Subversion (SVN). There are no assumptions regarding SVN clients used by the teams.

The functionality of the product is as follows. When a bug is found by a developer or tester the bug is reported in the tool. The tool is able to store many bug descriptions simultaneously. When the source code file containing the bug is identified the developer responsible for fixing it opens a bug in the tool. At this moment, the automatically tool creates the fix branch based on branch C directly in SVN. It stores the bug description in SVN as associated to the just created branch. The developer fixes the bug and reports this in the tool by associating an appropriate description to this fix. At this moment, the tool automatically merges the fix to the branch C if possible. If it is not possible, the developer does it himself manually. The manual merge may be necessary if there were merges to branch C in the meantime (that is between C1 and C2). So, the good practice is to do fixes fast. More specifically, in order to maximize the chance for this requirement, the fixing process should be started as late as possible (just before starting fixing the bug). This practice could be called lazy bug fixing per analogy to the notion of lazy initialization well known from programming.

This way the bug fixing was performed on branch C only. But how about bug propagation to other branches? This task is performed by the tool semi-automatically. When the developer or tester identifies the bug, it manually checks on the other branches if the bug has (may have) impact on these branches. If he identifies the possible impact he specifies the possibly impacted branches in the tool (A and B below C in the top-most compartment in Fig. 2). The good practice here is not to assume the lack of impact in unclear situations. As a consequence, the default is to assume the impact to all other branches. And this is the weak point of the process (not the tool – it supports the existing process from the assumption). In the case of the common testers team, the existence of the bug may be verified by running appropriate tests (if they exist). Otherwise, the identification of the impact is a matter of guess work. And this is the reason of assumptions of having common testing team as well as the source for the good practice of assuming impact even if this impact does not exist.

The main role of the BFT tool could be characterized as a tool storing all the information about bugs and fixes identified in the product life, performing simple configuration management tasks like branching and merging as well as storing descriptions in the product configuration repository. This way the tool both simplifies the technical process and supports the existing business process of bug fixing by offering a good communication platform.

The tool described here was already implemented and is used in the environment containing:

– Subversion (ver. 1.6.2) for configuration management,
– Ant (ver. 1.7.1) for build process execution,
– Hudson (ver. 1.306) for build automation,
– Tomcat (ver. 6.0.18) for running the tool.

The BFT tool was implemented in the following technologies:

– Java EE 1.5,
– Hibernate (ver. 3.2.1),
– svnkit (ver. 1.3.0).

## 5. Tool advantages

The advantages of the tool are of different kinds as is shown below.

The main advantages are that the tool is very cheap due to the fact that it is based on open source tools and it is easy to implement. As a consequence, the source code of the whole software required by the tool is available which creates the opportunity for more advanced improvements of the BFT tool in the future. The above mentioned advantages make starting the tool usage in a company easy and are not connected to any significant investment at the beginning.

Another group of advantages is strictly connected to the running business of software development. This group of advantages consists of:
– uniform way of bug fixing in the whole company,
– usage of one simple web tool for the bug fixing process which is easily accessible by different teams,
– ease of implementation of the tool,
– small size of the code which limits the likelihood of defects,
– limited but adequate functionality which makes testing the tool easy,
– improvement of company communication regarding bug fixing,
– improvement of statistics that may be performed on version control server via distinguishing between development and bug fixing.

## 6. Conclusions

The bug fixing tool dedicated to the continuous integration approach to software development process described in the paper is very useful due to the advantages presented in the previous section. However, this tool supports existing business processes that have disadvantages mentioned in section 3. Consequently, the approach described here and the tool itself are a good starting point to the further optimization of both the process and the tool concept. This problem of optimization is a subject of investigations that are taking place at the moment. The results of that different problem defined for the purpose of wide and deep optimization and automation are intended to be published soon.

The situation presented in this paper is quite simple, nevertheless, realistic in many cases. A proposal of a solution to much more complex situations not limited to one configuration repository and consisting of many different additional actions that may be performed on the configuration repository is the subject of current investigations and also will be published soon.

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