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Compact conformally Kähler Einstein-Weyl manifolds

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Abstract We give a description of compact conformally Kähler Einstein-Weyl manifolds whose Ricci tensor is Hermitian.

0 Introduction

In this paper we shall investigate compact Einstein-Weyl structures $(M, [g], D)$ on a complex manifold (M, J), $\dim M \geq 4$, which are conformally Kähler and whose Ricci tensor ρ^D is Hermitian, i.e., ρ^D is J-invariant.

We give a complete classification of compact Einstein-Weyl structures $(M, [g], D)$ with $\dim M \geq 4$ such that $(M, [g], J)$ is conformally Kähler, i.e., there exists a metric $g_0 \in [g]$ such that (M, g_0, J) is Kähler and whose Ricci tensor ρ^D is J-invariant, i.e.,

$$\rho^D(JX, JY) = \rho^D(X, Y).$$

Conformally Kähler Einstein manifolds were classified by Derdziński and Maschler in [4]. Compact Gray bi-Hermitian manifolds are partially classified in [10]. It is proved in [9] that compact Einstein-Weyl manifolds are also Gray manifolds (see [8]). The compact Einstein-Weyl 3-D manifolds are studied in [17]. The compact Einstein-Weyl manifolds on complex manifolds compatible with complex structure are studied in [15,16,13,18]. In [6] there are studied Riemannian manifolds (M, g) in four dimensions which are locally conformally Kähler. The Einstein-Weyl conformally Kähler structures studied in [13,18] have the J-invariant Ricci tensor ρ^D of the Weyl structure $(M, [g], D)$. In the first section of the paper we recall some facts from [9] and describe compact Einstein-Weyl manifolds with the Gauduchon metric as Gray manifolds. In Sect. 2 we describe the Riemannian structure of Gray manifold corresponding to Einstein-Weyl manifold with the Gauduchon metric. In Sect. 3 we prove that compact Einstein-Weyl manifold with Hermitian Ricci tensor admits a holomorphic Killing

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vector field with special Kähler-Ricci potential and consequently $M = \mathbb{CP}^n$ or $M = \mathbb{P}(L \oplus \mathcal{O})$ where L is a holomorphic line bundle over Kähler–Einstein manifold. In Sect. 4 we prove that if $M = \mathbb{P}(L \oplus \mathcal{O})$ then L is a holomorphic line bundle over Kähler–Einstein manifold with positive scalar curvature.

1 Einstein-Weyl geometry and Killing tensors

We start with some basic facts concerning Einstein-Weyl geometry. For more details see [17,15,16].

Let M be a n-dimensional manifold with a conformal structure $[g]$ and a torsion-free affine connection D. This defines an Einstein-Weyl (E-W) structure if D preserves the conformal structure, i.e., there exists a 1-form ω on M such that

$$Dg = \omega \otimes g \tag{1.1}$$

and the Ricci tensor ρ^D of D satisfies the condition

$$\rho^D(X, Y) + \rho^D(Y, X) = \bar{\Lambda} g(X, Y) \text{ for every } X, Y \in TM$$

for some function $\bar{\Lambda} \in C^\infty(M)$. Gauduchon proved ([7]) the fundamental theorem that if M is compact then there exists a Riemannian metric $g_0 \in [g]$ for which $\delta\omega_0 = 0$ and g_0 is unique up to homothety. We shall call g_0 a standard metric of E-W structure $(M, [g], D)$. Let ρ be the Ricci tensor of (M, g) and let us denote by S the Ricci endomorphism of (M, g), i.e., $\rho(X, Y) = g(X, SY)$. We recall two important theorems (see [17,15]):

Theorem 1.1 *A metric g and a 1-form ω determine an E-W structure if and only if there exists a function $\Lambda \in C^\infty(M)$ such that*

$$\rho^\nabla + \frac{1}{4}(n-2)\mathcal{D}\omega = \Lambda g \tag{1.2}$$

where $\mathcal{D}\omega(X, Y) = (\nabla_X \omega)Y + (\nabla_Y \omega)X + \omega(X)\omega(Y)$ and $n = \dim M$. If (1.2) holds then

$$\bar{\Lambda} = 2\Lambda + div\omega - \frac{1}{2}(n-2) \parallel \omega^\sharp \parallel^2 \tag{1.3}$$

Tod proved [17] that the Gauduchon metric admits a Killing vector field, more precisely he proved:

Theorem 1.2 *Let M be a compact E-W manifold and let g be the standard metric with the corresponding 1-form ω. Then the vector field $\omega^\sharp$ dual to the form ω is a Killing vector field on M.*

By $\tau = \mathrm{tr}_g\, \rho$ we shall denote the scalar curvature of (M, g). Compact E-W manifolds with the Gauduchon metric are Gray manifolds. To define Gray manifolds we define first a Killing tensor.

Definition A self-adjoint $(1, 1)$ tensor on a Riemannian manifold (M, g) is called a Killing tensor if

$$g(\nabla S(X, X), X) = 0$$

for arbitrary $X \in TM$.

Remark The condition $g(\nabla S(X,X),X)=0$ is equivalent to

$$\mathfrak{C}_{X,Y,Z}\, g(\nabla S(X,Y),Z)=0$$

for arbitrary $X,Y,Z\in\mathfrak{X}(M)$, where $\mathfrak{C}$ denotes the cyclic sum.

Now we can give

Definition A Riemannian manifold (M,g) will be called a Gray $\mathcal{A}\oplus\mathcal{C}^{\perp}$ manifold if the tensor $\rho-\frac{2\tau}{n+2}g$ is a Killing tensor.

In this paper, Gray $\mathcal{A}\oplus\mathcal{C}^{\perp}$ manifolds will be called for short Gray manifolds or $\mathcal{A}\oplus\mathcal{C}^{\perp}$ manifolds. Gray manifolds were first defined by Gray ([8]).

In the next two theorems, which are proved also in [9], we characterize compact E-W manifolds (M,g) with the Gauduchon metric as Gray ([8]) manifolds and show that the eigenvalues λ_0,λ_1 of the Ricci tensor satisfy the equation $(n-4)\lambda_1+2\lambda_0=C_0=\text{const}$ which we shall use later. We sketch the proofs of the theorems. Our motivation is to use the structure of Gray manifolds to use ideas from [10], where we classified a different class of Gray manifolds. From the above theorems it follows (see [9])

Theorem 1.3 *Let $(M,[g])$ be a compact E-W manifold, $n=dimM\geq 3$, and let g be the standard metric on M. Then (M,g) is an $\mathcal{A}\oplus\mathcal{C}^{\perp}$-manifold. The manifold (M,g) is Einstein or the Ricci tensor ρ^{∇} of (M,g) has exactly two eigenfunctions $\lambda_0\in C^{\infty}(M),\lambda_1=\Lambda$ satisfying the following conditions:*

(a) $(n-4)\lambda_1+2\lambda_0=C_0=const$
(b) $\lambda_0\leq\lambda_1$ *on* M
(c) $\dim ker(S-\lambda_0 Id)=1, \dim ker(S-\lambda_1 Id)=n-1$ *on* $U=\{x:\lambda_0(x)\neq\lambda_1(x)\}$,
(d) $\lambda_1-\lambda_0=\frac{n-2}{4}\parallel\xi\parallel^2$ *where* $\xi=\omega^{\sharp}\in\mathfrak{iso}(M)$.

In the addition $\lambda_0=\frac{1}{n}Scal_g^D$ where $Scal_g^D=tr_g\rho^D$ denotes the conformal scalar curvature of (M,g,D).

Proof Note that $\omega(X)=g(\xi,X)$ where $\xi\in\mathfrak{iso}(M)$ and the formula

$$\rho^{\nabla}+\frac{1}{4}(n-2)\omega\otimes\omega=\Lambda g \tag{1.4}$$

holds. Thus, $\nabla_X(\omega\otimes\omega)(X,X)=0$. From (1.4) it follows that

$$(\nabla_X\rho)(X,X)=(X\Lambda)g(X,X). \tag{1.5}$$

It means that $(M,g)\in\mathcal{A}\oplus\mathcal{C}^{\perp}$ and $d(\Lambda-\frac{2}{n+2}\tau)=0$ ([9], Lemma 1.5) , where τ is the scalar curvature of (M,g). From (1.5) it follows that the tensor $T=S-\Lambda Id$ is a Killing tensor. Note that $\rho(\xi,\xi)=(\Lambda-\frac{1}{4}(n-2)\parallel\xi\parallel^2)\parallel\xi\parallel^2$ and if $X\perp\xi$ then $SX=\Lambda X$. Hence, the tensor S has two eigenfunctions $\lambda_0=\Lambda-\frac{1}{4}(n-2)\parallel\xi\parallel^2$ and $\lambda_1=\Lambda$. This proves (b). Note that

$$\tau=\lambda_0+(n-1)\lambda_1=n\Lambda-\frac{1}{4}(n-2)\parallel\xi\parallel^2 .$$

and $2\tau-(n+2)\Lambda=C_0=const$. Thus, $C_0=(n-2)\Lambda-\frac{1}{2}(n-2)\parallel\xi\parallel^2$. However, $(n-4)\lambda_1+2\lambda_0=(n-2)\Lambda-\frac{1}{2}(n-2)\parallel\xi\parallel^2$ which proves (a). Note also that

$$\frac{1}{n}s_g^D=\Lambda-\frac{n-2}{4}\parallel\xi\parallel^2=\lambda_0 \tag{1.6}$$

which finishes the proof. □

In the next theorem we show that conversely every Gray manifold satisfying the above conditions is in fact E-W manifold (see [9]).

Theorem 1.4 *Let (M, g) be a compact $\mathcal{A}\oplus\mathcal{C}^{\perp}$ manifold. Let us assume that the Ricci tensor ρ of (M, g) has exactly two eigenfunctions λ_0, λ_1 satisfying the conditions:*

(a) $(n-4)\lambda_1 + 2\lambda_0 = C_0 = const$
(b) $\lambda_0 \leq \lambda_1$ *on* M
(c) $\dim ker(S - \lambda_0 Id) = 1, \dim ker(S - \lambda_1 Id) = n-1$ *on* $U = \{x : \lambda_0(x) \neq \lambda_1(x)\}$.

Then there exists a twofold Riemannian covering (M', g') of (M, g) and a Killing vector field $\xi \in \mathfrak{iso}(M')$ such that $(M', [g'])$ admits two different E-W structures with the standard metric g' and the corresponding 1-forms $\omega_{\mp} = \mp\xi^{\flat}$ dual to the vector fields $\mp\xi$. In addition, $\lambda_1 - \lambda_0 = \frac{n-2}{4} \parallel \xi \parallel^2$. The condition (b) may be replaced by the condition

(b1) there exists a point $x_0 \in M$ such that $\lambda_0(x_0) < \lambda_1(x_0)$.

Proof Note that $\tau = (n-1)\lambda_1 + \lambda_0$ and $C_0 = (n-4)\lambda_1 + 2\lambda_0$. It follows that

$$\lambda_1 = \frac{2\tau - C_0}{n+2}, \quad \lambda_0 = \frac{(n-1)C_0 - (n-4)\tau}{n+2}. \tag{1.7}$$

In particular $\lambda_0, \lambda_1 \in C^{\infty}(M)$. Let S be the Ricci endomorphism of (M, g) and let us define the tensor $T := S - \lambda_1 Id$. Since from (1.7) we have $d\lambda_1 = \frac{2}{n+2} d\tau$ it follows that T is a Killing tensor with two eigenfunctions: $\mu = 0$ and $\lambda = \lambda_0 - \lambda_1$. Hence there exists a twofold Riemannian covering $p : (M', g') \to (M, g)$ and a Killing vector field $\xi \in \mathfrak{iso}(M')$ (see [9], th. 2.10) such that $S'\xi = (\lambda_0 \circ p)\xi$ where S' is the Ricci endomorphism of (M', g'). Note also that $\parallel \xi \parallel^2 = |\lambda - \mu| = |\lambda_0 - \lambda_1|$. Let us define the 1-form ω on M' by $\omega = c\xi^{\flat}$ where $c = 2\sqrt{\frac{1}{n-2}}$. It is easy to check that with such a choice of ω Eq. (1.4) is satisfied and $\delta\omega = 0$. Thus (M', g', ω) defines an E-W structure and g' is the standard metric for $(M', [g'])$. Note that $(M, g', -\omega)$ gives another E-W structure corresponding to the field $-\xi$. □

Corollary 1.5 *Let (M, g) be a compact simply connected manifold satisfying the assumptions of Th. 1.4. Then $(M, [g])$ admits two E-W structures with the standard metric g.*

2 Killing tensors

In this section, we describe the Riemannian manifold (M, g) where $g \in [g]$ is the standard metric of E-W structure $(M, [g], D)$.

We say, that a distribution (not necessarily integrable) $\mathcal{D}$ is totally geodesic, if $\nabla_X X \in \Gamma(\mathcal{D})$ for every $X \in \Gamma(\mathcal{D})$. Note that if (M, g) is a compact E-W manifold with the Gauduchon metric then the distribution $\mathcal{D}_{\lambda_1}$ is totally geodesic since is orthogonal to the distribution spanned by a Killing vector field.

We start with:

Lemma 2.1 *Let S be a self-adjoint tensor on (M, g) with exactly two eigenvalues λ, μ. If the distributions $\mathcal{D}_\lambda, \mathcal{D}_\mu$ are both umbilical, $\nabla\lambda \in \Gamma(D_\mu)$, $\nabla\mu \in \Gamma(D_\lambda)$ and the mean curvature normals ξ_λ, ξ_μ of the distributions D_λ, D_μ respectively satisfy the equations*

$$\xi_\lambda = \frac{1}{2(\mu - \lambda)}\nabla\lambda, \qquad \xi_\mu = \frac{1}{2(\lambda - \mu)}\nabla\mu,$$

then S is a Killing tensor.

Proof We have to show that $g(\nabla S(Z,Z),Z)=0$ for arbitrary $Z\in TM$. Let $Z=X+Y$ where $X\in\mathcal{D}_\lambda$, $Y\in\mathcal{D}_\mu$. Then

$$g(\nabla S(Z,Z),Z)=g(\nabla S(X,X),X)+2g(\nabla S(X,X),Y)+g(\nabla S(Y,X),X)$$
$$+2g(\nabla S(Y,Y),X)+g(\nabla S(X,Y),Y)+g(\nabla S(Y,Y),Y).$$

Since $\nabla S(X,X)=(\lambda-\mu)g(X,X)\xi_\lambda$, $\nabla S(Y,Y)=(\mu-\lambda)g(Y,Y)\xi_\mu$ and

$$g(\nabla S(X,X),X)=0,\ g(\nabla S(Y,Y),Y)=0$$

one can easily check that $g(\nabla S(Z,Z),Z)=0$. □

Remark We call here a vector field ξ the mean curvature normal of a umibilical distribution $\mathcal{D}$ if for every $X\in\Gamma(\mathcal{D})$ we have $\pi(\nabla_X X)=g(X,X)\xi$ where π is a projection on the orthogonal complement of $\mathcal{D}$. Note that $\mathcal{D}$ may not be integrable.

Proposition 2.2 *Let (M,g) be a 2n-D Riemannian manifold whose Ricci tensor ρ has two eigenvalues $\lambda_0(x),\lambda_1(x)$ of multiplicity 1 and $2n-1$, respectively, at every point x of M. Assume that the eigendistribution $\mathcal{D}_{\lambda_1}$ corresponding to λ_1 is totally geodesic. Then (M,g) is a Gray manifold if and only if $2\lambda_0+(2n-4)\lambda_1$ is constant and $\nabla\tau\in\Gamma(\mathcal{D}_{\lambda_1})$.*

Proof Let S_0 be the Ricci endomorphism of (M,g), i.e., $\rho(X,Y)=g(S_0X,Y)$. Let S be the tensor defined by the formula

$$S_0=S+\frac{\tau}{n+1}\,\mathrm{id}\,. \tag{2.1}$$

Then

$$\operatorname{tr} S=-\frac{(n-1)\tau}{n+1}. \tag{2.2}$$

Let λ_0,λ_1 be the eigenfunctions of S_0 and let us assume that

$$2\lambda_0+(2n-4)\lambda_1=C \tag{2.3}$$

where $C\in\mathbb{R}$. Note that S also has two eigenfunctions which we denote by λ_0',λ_1', respectively. It is easy to see that $\lambda_0'=-\frac{n-1}{n+1}\tau+C\frac{2n-1}{2(n+1)}$, $\lambda_1'=-\frac{C}{2(n+1)}$ and $\lambda_0=-\frac{\tau(n-2)}{n+1}+C\frac{2n-1}{2(n+1)}$, $\lambda_1=\frac{\tau}{(n+1)}-\frac{C}{2(n+1)}$. Since the distribution $\mathcal{D}_{\lambda_0}$ is umbilical we have $\nabla_X X_{|\mathcal{D}_{\lambda_1}}=g(X,X)\xi$ for any $X\in\Gamma(\mathcal{D}_{\lambda_0})$ where ξ is the mean curvature normal of $\mathcal{D}_{\lambda_0}$. Since the distribution $\mathcal{D}_{\lambda_1}$ is totally geodesic we also have $\nabla_X X_{|\mathcal{D}_{\lambda_0}}=0$ for any $X\in\Gamma(\mathcal{D}_{\lambda_1})$. Let $\{E_1,E_2,E_3,E_4,\ldots,E_{2n-1},E_{2n}\}$ be a local orthonormal basis of TM such that $\mathcal{D}_{\lambda_0}=\operatorname{span}\{E_1\}$ and $\mathcal{D}_{\lambda_1}=\operatorname{span}\{E_2,E_3,E_4,\ldots,E_{2n}\}$. Then $\nabla_{E_i}E_{i|\mathcal{D}_{\lambda_0}}=0$ for $i\in\{2,3,4,\ldots,2n\}$ and

$$\nabla_{E_1}E_{1|\mathcal{D}_{\lambda_1}}=\xi.$$

Consequently (note that $\nabla\lambda'_{0|\mathcal{D}_{\lambda_0}}=0$ if and only if $\nabla\tau_{|\mathcal{D}_{\lambda_0}}=0$),

$$\begin{aligned}\operatorname{tr}_g\nabla S=\sum_{i=1}^{n}\nabla S(E_i,E_i)&=-(S-\lambda_0\,\mathrm{id})(\nabla_{E_1}E_1)+\nabla\lambda_{0|\mathcal{D}_{\lambda_0}}\\&=-(\lambda_1-\lambda_0)\xi\end{aligned} \tag{2.4}$$

if we assume that $\nabla\tau_{|\mathcal{D}_{\lambda_0}} = 0$. On the other hand, $\mathrm{tr}_g\,\nabla S_0 = \frac{\nabla\tau}{2}$ and $\mathrm{tr}_g\,\nabla S = \mathrm{tr}_g\,\nabla S_0 - \frac{\nabla\tau}{n+1}$. Consequently,

$$\mathrm{tr}_g\,\nabla S = \frac{(n-1)\nabla\tau}{2(n+1)} = -\frac{1}{2}\nabla\lambda_0'. \tag{2.5}$$

Thus, $\xi = -\frac{1}{2(\lambda_0-\lambda_1)}\nabla\lambda_0'$. From the Lemma it follows that (M, g) is an $\mathcal{A}\oplus\mathcal{C}^\perp$-manifold if $2\lambda_0 + (n-4)\lambda_1$ is constant and $\nabla\tau \in \Gamma(\mathcal{D}_{\lambda_1})$. These conditions are also necessary since $\nabla\lambda_1' = 0$ if (M, g) is an $\mathcal{A}\oplus\mathcal{C}^\perp$-manifold and $\mathcal{D}_{\lambda_1}$ is totally geodesic. Analogously $\xi = -\frac{1}{2(\lambda_0'-\lambda_1')}\nabla\lambda_0'$ and $\nabla\lambda_0' = -\frac{n-1}{(n+1)}\nabla\tau \in \Gamma(\mathcal{D}_{\lambda_1})$, where ξ is the mean curvature normal of the umbilical distribution $\mathcal{D}_{\lambda_0}$, if (M, g) is an $\mathcal{A}\oplus\mathcal{C}^\perp$-manifold. □

3 Conformally Kähler E-W manifolds

Let g be the standard metric of $(M, [g])$. Now let us recall that $\rho^D(X, Y) = \lambda_0 g(X, Y) + \frac{n}{4}d\omega(X, Y)$. Let us assume that (M, J) is complex and $[g]$ is Hermitian i.e. $g(JX, JY) = g(X, Y)$. It follows that ρ^D is J-invariant if and only if $d\omega$ is a (1, 1) form, $d\omega(JX, JY) = d\omega(X, Y)$. Since $\omega(X) = g(\xi, X)$ it follows that $d\omega$ is a (1, 1) form iff $\nabla_{JX}\xi = J\nabla_X\xi$.

Proposition 3.1 *Let (M, J) be a compact complex manifold with conformal Hermitian structure $[g]$. Let us assume that $[g]$ is conformally Kähler and $f^2 g$ is a Kähler metric on (M, J) where g is the standard metric and $f \in C^\infty(M)$. If $(M, [g])$ is E-W with J-invariant Ricci tensor ρ^D then $J\xi$ is colinear with ∇f in $U = \{x : \xi_x \neq 0\}$ and ξ is a holomorphic Killing field on $(M, f^2 g, J)$.*

Proof Let ∇ be a Levi-Civita connection of the standard metric g and ∇^1 be a Levi-Civita connection of the Kähler metric $g_1 = f^2 g$. Note that ξ is a conformal field on (M, g_1), $L_\xi g_1 = L_\xi(f^2 g) = 2\xi \ln f g_1 = \sigma g_1$. Every conformal field on a compact Kähler manifold is Killing (see [12]), hence consequently $\xi f = 0$ and $\xi \in \mathfrak{iso}(M, g_1)$. On a Kähler compact manifold every Killing vector field is holomorphic (see [14]). Thus, $\xi \in \mathfrak{hol}(M, J)$. Note that

$$\nabla_X\xi = \nabla^1_X\xi - \mathrm{d}\ln f(X)\xi - \mathrm{d}\ln f(\xi)X + g_1(X, \xi)\nabla^1 \ln f.$$

Thus,

$$\begin{aligned}\nabla_{JX}\xi - J\nabla_X\xi = &-\mathrm{d}\ln f(JX)\xi - \mathrm{d}\ln f(\xi)JX + g_1(JX, \xi)\nabla^1 \ln f\\ &+\mathrm{d}\ln f(X)J\xi + \mathrm{d}\ln f(\xi)JX - g_1(X, \xi)J\nabla^1 \ln f.\end{aligned}$$

Hence, $\nabla_{JX}\xi = J\nabla_X\xi$ if

$$-\mathrm{d}\ln f(JX)\xi + g_1(JX, \xi)\nabla^1 \ln f + \mathrm{d}\ln f(X)J\xi - g_1(X, \xi)J\nabla^1 \ln f = 0.$$

Put $X = \xi$ then we get $g_1(\xi, \xi)J\nabla^1 \ln f = -\mathrm{d}\ln f(J\xi)\xi$. It follows that in $U = \{x \in M : \xi_x \neq 0\}$ there exists a smooth function ϕ such that $\nabla^1 f = \phi J\xi$. □

Let us recall the definition of a special Kähler-Ricci potential ([5,3]).

Definition A nonconstant function $\tau \in C^\infty(M)$, where (M, g, J) is a Kähler manifold, is called a special Kähler-Ricci potential if the field $X = J(\nabla\tau)$ is a Killing vector field and at every point with $d\tau \neq 0$ all nonzero tangent vectors orthogonal to the fields X, JX are eigenvectors of both $\nabla d\tau$ and the Ricci tensor ρ of (M, g, J).

Now our aim is to prove

Theorem 3.2 *Let us assume that $(M, [g], J)$ is a compact, conformally Kähler E-W manifold with Hermitian Ricci tensor ρ^D which is not conformally Einstein. Then the conformally equivalent Kähler manifold (M, g_1, J) admits a holomorphic Killing field with a Kähler-Ricci potential. Thus, $M = \mathbb{P}(L \oplus \mathcal{O})$ where L is a holomorphic line bundle over a compact Kähler Einstein manifold (N, h) of positive scalar curvature or is a complex projective space $\mathbb{CP}^n$.*

Proof Let ρ, ρ^1 be the Ricci tensors of conformally related riemannian metrics $g, g_1 = f^2 g$. Then

$$\rho = \rho^1 + (n-2)f^{-1}\nabla^1 df + [f^{-1}\Delta^1 f - (n-1)f^{-2}g_1(\nabla^1 f, \nabla^1 f)]g_1.$$

Note that for arbitrary $X, Y \in \mathfrak{X}(M)$ we have $\nabla^1 df(X,Y) = g_1(\nabla^1_X \nabla^1 f, Y) = g_1(X\phi J\xi, Y) + \phi g_1(J\nabla^1_X \xi, Y)$. Thus for any $X, Y \in \mathfrak{X}(M)$:

$$\begin{aligned}\rho(X,Y) - (n-2)fX\phi g(J\xi, Y) = \rho^1(X,Y) &+ (n-2)f^{-1}\phi g_1(J\nabla^1_X\xi, Y)\\ &+[f^{-1}\Delta^1 f - (n-1)f^{-2}g_1(\nabla^1 f, \nabla^1 f)]\\ &g_1(X,Y),\end{aligned} \tag{3.1}$$

where $\Delta^1 f = tr_{g_1}\nabla^1 df$.

We shall show that ξ has zeros on M. If $\xi \neq 0$ on M then the function ϕ would be defined and smooth on the whole of M. Since M is compact it would imply that there exists a point $x_0 \in M$ such that $d\phi = 0$ at x_0. On the other hand, the eigenvalues λ_0, λ_1 of the Ricci tensor ρ satisfy $\lambda_0 - \lambda_1 = Cg(\xi, \xi)$ where $C \neq 0$ is a real number. Since $\xi \neq 0$ it follows that the eigenvalues of ρ do not coincide at any point of M. In particular ρ is not J-invariant at x_0, a contradiction, since the right hand part of (3.1) is J-invariant. It implies that ξ is a holomorphic Killing vector field with zeros and thus has a potential τ (see [11]), i.e., there exists $\tau \in C^\infty(M)$ such that $\xi = J\nabla^1\tau$. Hence, $\mathrm{d}f = -\phi\mathrm{d}\tau$ and $\mathrm{d}\phi \wedge \mathrm{d}\tau = 0$. It implies that $\mathrm{d}\phi = \alpha\mathrm{d}\tau$. Thus, we have for arbitrary $X, Y \in \mathfrak{X}(M)$:

$$\begin{aligned}&\rho(X,Y) + (n-2)f^{-1}\alpha\mathrm{d}\tau(X)\mathrm{d}\tau(Y) = \rho^1(X,Y)\\ &\quad-(n-2)f^{-1}\phi H^\tau(X,Y) - [f^{-1}\alpha Q + f^{-1}\phi\Delta^1\tau + (n-1)f^{-2}\phi^2 Q]g_1(X,Y).\end{aligned} \tag{3.2}$$

where $Q = g_1(\xi, \xi)$.

Note that the tensor $\widetilde{\rho}(X,Y) = \rho(X,Y) + (n-2)f^{-1}\alpha\mathrm{d}\tau(X)\mathrm{d}\tau(Y)$ is J-invariant. In particular $\widetilde{\rho}(\xi,\xi) = \lambda_0 g(\xi,\xi) = \lambda_o\frac{Q}{f^2}$. On the other hand, $\widetilde{\rho}(\nabla^1\tau, \nabla^1\tau) = \lambda_1\frac{Q}{f^2} + (n-2)f^{-1}\alpha Q^2$. Hence, $(\lambda_0 - \lambda_1)\frac{Q}{f^2} = (n-2)f^{-1}\alpha Q^2$. Since $\lambda_0 - \lambda_1 = -\frac{1}{4}(n-2)\frac{Q}{f^2}$ we get $\alpha = -\frac{1}{4f^3}$. Hence

$$\mathrm{d}\phi = -\frac{1}{4f^3}\mathrm{d}\tau = \frac{1}{4f^3}\frac{df}{\phi}, \tag{3.3}$$

and we get $8\phi\mathrm{d}\phi = -\mathrm{d}(\frac{1}{f^2})$. Hence, $\mathrm{d}(4\phi^2 + \frac{1}{f^2}) = 0$ and $4\phi^2 + \frac{1}{f^2} = C = \text{const}$.

Let us denote $\chi = (n-2)f^{-1}\phi$, $\sigma_0 = f^{-1}\alpha Q + f^{-1}\phi\Delta^1\tau + (n-1)f^{-2}\phi^2 Q$. Note also that the vector field $v = \nabla^1\tau$ is holomorphic and consequently $i_v\rho^1 = -\frac{1}{2}\mathrm{d}\Delta^1\tau = -\frac{1}{2}\mathrm{d}Z$ where $Z = \Delta^1\tau$. From the equation

$$\begin{aligned}\widetilde{\rho}(X,Y) = \rho^1(X,Y) &- (n-2)f^{-1}\phi H^\tau(X,Y) -\\ &[f^{-1}\alpha Q + f^{-1}\phi\Delta^1\tau + (n-1)f^{-2}\phi^2 Q]g_1(X,Y).\end{aligned} \tag{3.4}$$

valid for arbitrary $X, Y \in \mathfrak{X}(M)$ we get

$$\frac{\lambda_0}{f^2}\mathrm{d}\tau = -\frac{1}{2}\mathrm{d}Z + \frac{1}{2}\chi\mathrm{d}Q - \sigma_0\mathrm{d}\tau,$$

and

$$\mathrm{d}Z = \chi\mathrm{d}Q - 2\sigma\mathrm{d}\tau \tag{3.5}$$

where $\sigma = \sigma_0 + \frac{\lambda_0}{f^2}$. From (3.5) we obtain

$$\mathrm{d}\chi \wedge \mathrm{d}Q - 2\mathrm{d}\sigma \wedge \mathrm{d}\tau = 0 \tag{3.6}$$

Since $\mathrm{d}\chi = \gamma\mathrm{d}\tau$ we have $\mathrm{d}\tau \wedge (\gamma\mathrm{d}Q + 2\mathrm{d}\sigma) = 0$ which implies $\gamma\mathrm{d}Q + 2\mathrm{d}\sigma = \kappa\mathrm{d}\tau$ for a certain function κ. Note that

$$\mathrm{d}\sigma_0 = f^{-1}\alpha\mathrm{d}Q + f^{-1}\phi\mathrm{d}Z + (n-1)f^{-2}\phi^2\mathrm{d}Q + h\mathrm{d}\tau = \\ \left[-\frac{1}{4f^4} + (n-1)f^{-2}\phi^2\right]\mathrm{d}Q + f^{-1}\phi\mathrm{d}Z + h\mathrm{d}\tau,$$

for a certain function h. On the other hand, $\lambda_0 = -\frac{(n-4)Q}{4f^2} + \frac{C_0}{n-2}$. Hence, $\mathrm{d}(\frac{\lambda_0}{f^2}) = -\frac{(n-4)}{4f^4}\mathrm{d}Q + k\mathrm{d}\tau$ and

$$\mathrm{d}\sigma = \left[-\frac{1}{4f^4} + (n-1)f^{-2}\phi^2 - \frac{(n-4)}{4f^4}\right]\mathrm{d}Q + f^{-1}\phi\mathrm{d}Z + l\mathrm{d}\tau$$

for some functions k, l. Since $\mathrm{d}\chi = (n-2)\mathrm{d}(f^{-1}\phi) = -\frac{(n-2)}{4f^4}(1 - 4\phi^2 f^2)\mathrm{d}\tau$ we have $\gamma = -\frac{(n-2)}{4f^4}(1 - 4\phi^2 f^2)$ and

$$\left[-\frac{3n-8}{4f^4} + \frac{(3n-4)\phi^2}{4f^2}\right]\mathrm{d}Q + 2f^{-1}\phi\mathrm{d}Z = m\mathrm{d}\tau, \tag{3.7}$$

for a certain function m.

From Eqs. (3.5) and (3.7) it follows that

$$\mathrm{d}Q \wedge \mathrm{d}\tau = \mathrm{d}Z \wedge \mathrm{d}\tau = 0 \tag{3.8}$$

on a dense subset of M and hence everywhere.

Define a distribution $\mathcal{D} = span\{\xi, J\xi\}$ and let $\mathcal{D}^{\perp}$ be an orthogonal (with respect to g so also with respect to g_1) complement to $\mathcal{D}$. Both distributions are defined in an open dense subset $U = \{x : \xi_x \neq 0\}$. Let $\pi_{\mathcal{D}}, \pi_{\mathcal{D}^{\perp}}$ be orthogonal projections on $\mathcal{D}, \mathcal{D}^{\perp}$, respectively. Let us define $\omega_{\mathcal{D}}(X, Y) = g_1(J\pi_{\mathcal{D}}X, Y), \omega_{\mathcal{D}^{\perp}}(X, Y) = g_1(J\pi_{\mathcal{D}^{\perp}}X, Y)$. Then $\omega_{\mathcal{D}} + \omega_{\mathcal{D}^{\perp}} = \Omega$ where $\Omega(X, Y) = g_1(JX, Y)$ is the Kähler form of (M, g_1, J). Note that $\omega_{\mathcal{D}} = \frac{1}{Q}d\tau \wedge d^c\tau$. Since ξ is a holomorphic Killing field on (M, g_1, J) it follows that $H^{\tau}(JX, Y) = \frac{1}{2}dd^c\tau(X, Y)$. Since $\nabla^1_v v = -\frac{1}{2}\nabla^1 Q = cv$ it follows that $\mathcal{D}$ is an eigendistribution of both ρ^1 and $dd^c\tau$. We have (we denote the Ricci form also by ρ^1)

$$\rho^1 = \lambda\omega_{\mathcal{D}} + \omega_1 \tag{3.9a}$$

$$\frac{1}{2}\mathrm{d}\mathrm{d}^c\tau = \mu\omega_{\mathcal{D}} + \omega_2, \tag{3.9b}$$

where λ, μ are eigenvalues of ρ^1, H^{τ} corresponding to an eigen distribution $\mathcal{D}$. The eigenvalue μ satisfies an equation $\mu Q = H^{\tau}(\nabla^1\tau, \nabla^1\tau) = -\frac{1}{2}\mathrm{d}Q(\nabla^1\tau) = -\frac{1}{2}\beta Q$ where

$\mathrm{d}Q = \beta d\tau$. Hence, $\mu = -\frac{1}{2}\beta$ and $\mathrm{d}\mu \wedge \mathrm{d}\tau = 0$. From (3.5) it is clear that also $\mathrm{d}\lambda \wedge \mathrm{d}\tau = 0$. Now we have

$$\widetilde{\rho} = \rho^1 - \frac{1}{2}\chi \mathrm{d}\mathrm{d}^c\tau - \sigma_0\Omega, \tag{3.10}$$

and consequently

$$\frac{\lambda_0}{f^2}\omega_{\mathcal{D}} = \lambda\omega_{\mathcal{D}} - \chi\mu\omega_{\mathcal{D}} - \sigma_0\omega_{\mathcal{D}} \tag{3.11}$$

and

$$\frac{\lambda_1}{f^2}\omega_{\mathcal{D}^\perp} = \omega_1 - \chi\omega_2 - \sigma_0\omega_{\mathcal{D}^\perp}. \tag{3.12}$$

From (3.11) we obtain $\lambda - \mu\chi = \sigma_0 + \frac{\lambda_0}{f^2}$. Hence,

$$\omega_1 - \chi\omega_2 = (\sigma_0 + \frac{\lambda_1}{f^2})\omega_{\mathcal{D}^\perp} = (\lambda - \mu\chi + \frac{\lambda_1 - \lambda_0}{f^2})\omega_{\mathcal{D}^\perp} = \sigma_1\omega_{\mathcal{D}^\perp}. \tag{3.13}$$

From (3.9) we get

$$\lambda \mathrm{d}\omega_{\mathcal{D}} = -\mathrm{d}\omega_1, \ \mu \mathrm{d}\omega_{\mathcal{D}} = -\mathrm{d}\omega_2.$$

Equation (3.13) implies that

$$\mathrm{d}\omega_1 - \mathrm{d}\chi \wedge \omega_2 - \chi \mathrm{d}\omega_2 = \mathrm{d}\sigma_1 \wedge \omega_{\mathcal{D}^\perp} + \sigma_1 \mathrm{d}\omega_{\mathcal{D}^\perp},$$

thus

$$(-\lambda + \mu\chi + \sigma_1)\mathrm{d}\omega_{\mathcal{D}} = \mathrm{d}\chi \wedge \omega_2 + \mathrm{d}\sigma_1 \wedge \omega_{\mathcal{D}^\perp}.$$

Note that $\mathrm{d}\omega_{\mathcal{D}} = \mathrm{d}(\frac{1}{Q}\mathrm{d}\tau \wedge \mathrm{d}^c\tau) = -\frac{1}{Q}\mathrm{d}\tau \wedge \mathrm{d}\mathrm{d}^c\tau = -\frac{2}{Q}\mathrm{d}\tau \wedge (\mu\omega_{\mathcal{D}} + \omega_2) = -\frac{2}{Q}\mathrm{d}\tau \wedge \omega_2$ and $\lambda_1 - \lambda_0 = \frac{1}{4}(n-2)\frac{Q}{f^2}$. Let us write $d\sigma_1 = \psi \mathrm{d}\tau$, then we obtain

$$\mathrm{d}\tau \wedge \left(\frac{n-2}{4f^4}(-1 - 4f^2\phi^2)\omega_2 - \psi\omega_{\mathcal{D}^\perp}\right) = 0. \tag{3.14}$$

From (3.14) it is clear that in U we have $\omega_2 = \kappa_2\omega_{\mathcal{D}^\perp}$ for a certain function $\kappa_2 \in C^\infty(U)$. Hence, also $\omega_1 = \kappa_1\omega_{\mathcal{D}^\perp}$ for a certain function $\kappa_1 \in C^\infty(U)$. It follows that the function τ is a Kähler-Ricci potential. The fact that the Einstein–Kähler manifold (N, h) has a positive scalar curvature is proved below. It is easy to check that also for $\dim M = 4$ the manifold (N, h) has constant scalar curvature. The E-W structure on these manifolds is described in [13], [18]. □

4 Eigenvalues of the Ricci tensor

In our construction we shall follow Bérard Bergery (see [1,10]). Let (N, h, J) be a compact Kähler Einstein manifold and $\dim N = 2m$, $s \geq 0$, $L > 0$, $s \in \mathbb{Q}$, $L \in \mathbb{R}$, and $g : [0, L] \to \mathbb{R}$ be a positive, smooth function on $[0, L]$ which is even at 0 and L, i.e., there exists an $\epsilon > 0$ and even, smooth functions $g_1, g_2 : (-\epsilon, \epsilon) \to \mathbb{R}$ such that $g(t) = g_1(t)$ for $t \in [0, \epsilon)$ and $g(t) = g_2(L - t)$ for $t \in (L - \epsilon, L]$. Let $f : (0, L) \to \mathbb{R}$ be positive on $(0, L)$, $f(0) = f(L) = 0$ and let f be odd at the points $0, L$. Let P be a circle bundle over N classified by the integral cohomology class $\frac{s}{2}c_1(N) \in H^2(N, \mathbb{R})$ if $c_1(N) \neq 0$. Let q be the unique positive integer such that $c_1(N) = q\alpha$ where $\alpha \in H^2(N, \mathbb{R})$ is an indivisible integral class. Such a q exists if N is simply connected or dim $N = 2$. Note that every Kähler Einstein manifold with positive scalar curvature is simply connected. Then

$$s = \frac{2k}{q}; k \in \mathbb{Z}.$$

It is known that $q = n$ if $N = \mathbb{CP}^{n-1}$ (see [2], p. 273). Note that $c_1(N) = \{\frac{1}{2\pi}\rho_N\} = \{\frac{\tau_N}{4m\pi}\omega_N\}$ where $\rho_N = \frac{\tau_N}{2m}\omega_N$ is the Ricci form of (N, h, J), τ_N is the scalar curvature of (N, h) and ω_N is the Kähler form of (N, h, J). We can assume that $\tau_N = \pm 4m$. In the case $c_1(N) = 0$ we shall assume that (N, h, J) is a Hodge manifold, i.e., the cohomology class $\{\frac{s}{2\pi}\omega_N\}$ is an integral class. On the bundle $p : P \to N$ there exists a connection form θ such that $\mathrm{d}\theta = sp^*\omega_N$ where $p : P \to N$ is the bundle projection. Let us consider the manifold $U_{s,f,g} = (0, L) \times P$ with the metric

$$k = \mathrm{d}t^2 + f(t)^2\theta^2 + g(t)^2 p^*h. \tag{4.1}$$

It is known that the metric (4.1) extends to a metric on the sphere bundle $M = P \times_{S^1} \mathbb{CP}^1$ if and only if a function g is positive and smooth on $[0, L]$, even at the points $0, L$, the function f is positive on $(0, L)$, smooth and odd at $0, L$ and additionally

$$f'(0) = 1, \qquad f'(L) = -1 \tag{4.2}$$

Then, the metric (4.1) is bi-Hermitian (see [10]). Note that $M = \mathbb{P}(L \oplus \mathcal{O})$ where $L = P \times_{S^1} \mathbb{C}$ with S^1 acting in a standard way on $\mathbb{C}$ and $\mathcal{O}$ is the trivial line bundle over N.

The metric $k = k_{f,g}$ extends to a metric on $\mathbb{CP}^n$ if and only if the function g is positive and smooth on $[0, L)$, even at 0, odd at L, the function f is positive, smooth and odd at $0, L$ and additionally

$$f'(0) = 1, \qquad f'(L) = -1, \qquad g(L) = 0, \qquad g'(L) = -1. \tag{4.3}$$

Let us assume that (N, h) is a $2(n-1)$-D Kähler–Einstein manifold of scalar curvature $4(n-1)\epsilon$ where $\epsilon \in \{-1, 0, 1\}$. Using the results in Sect. 3 and [10] we obtain the following formulae for the eigenvalues of the Ricci tensor ρ of $(U_{s,f,g}, k_{f,g})$:

$$\begin{aligned} \lambda_0 &= -2(n-1)\frac{g''}{g} - \frac{f''}{f}, \\ \lambda_1 &= -\frac{f''}{f} + 2(n-1)\left(\frac{s^2 f^2}{4g^4} - \frac{f'g'}{fg}\right), \\ \lambda_2 &= -\frac{g''}{g} + \left(\frac{s^2 f^2}{4g^4} - \frac{f'g'}{fg}\right) + \frac{2\epsilon}{g^2} - \frac{3s^2 f^2}{4g^4} - (2n-3)\frac{(g')^2}{g^2}. \end{aligned} \tag{4.4}$$

We shall show that in fact $\epsilon = 1$, i.e., the scalar curvature of the Einstein manifold (N, h, J) is positive. From [9], p. 17, th. 3.8. it follows that the conformal scalar curvature of E-W manifold and hence λ_1 is nonnegative. We also have for the Gauduchon metric $\lambda_0 = \lambda_2$ and $\lambda_1 + C^2 f^2 = \lambda_0$ for a positive constant C. Since $f(0) = 0 = f(L)$ it follows that f attains a maximum at a point $t_0 \in (0, L)$. Then $f'(t_0) = 0$ and $f''(t_0) \leq 0$. Hence at t_0 we have

$$\lambda_1 = -\frac{f''}{f} + 2(n-1)\frac{s^2 f^2}{4g^4} > 0$$

and

$$-\frac{g''}{g} + \frac{2\epsilon}{g^2} - \frac{s^2 f^2}{2g^4} - (2n-3)\frac{(g')^2}{g^2} = -2(n-1)\frac{g''}{g} - \frac{f''}{f}$$

and hence

$$\frac{2\epsilon}{g^2} = \frac{s^2 f^2}{2g^4} + (2n-3)\frac{(g')^2}{g^2} - (2n-3)\frac{g''}{g} - \frac{f''}{f}.$$

From (4.2) it follows that at t_0

$$-2(n-1)\frac{g''}{g} = 2(n-1)\frac{s^2 f^2}{4g^4} + C^2 f^2 > 0$$

and consequently $\epsilon > 0$.

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References

1. Bérard Bergery, L.: Sur de nouvelles variétés riemanniennes d'Einstein. Publ. de l'Inst. E. Cartan (Nancy) **4**, 1–60 (1982)
2. Besse, A.: Einstein manifolds. Springer, Berlin, HD (1987)
3. Derdziński, A., Maschler, G.: Local classification of conformally-Einstein Kähler metrics in higher dimension. Proc. London Math. Soc. (3) **87**(3), 779–819 (2003)
4. Derdziński, A., Maschler, G.: A moduli curve for compact conformally-Einstein Kähler manifolds. Compos. Math. **141**(4), 1029–1080 (2005)
5. Derdziński, A., Maschler, G.: Special Kähler-Ricci potentials on compact Kähler manifolds. J. Reine Angew. Math. **593**, 73–116 (2006)
6. Dunajski, M., Tod, K.P.: Four dimensional metrics conformal to Kähler. In: Mathematical Proceedings of the Cambridge Philosophical Society, Cambridge (2010)
7. Gauduchon, P.: La 1-forme de torsion d'une variete hermitienne compacte. Math. Ann. **267**, 495–518 (1984)
8. Gray, A.: Einstein-like manifolds which are not Einstein. Geom. Dedicata **7**, 259–280 (1978)
9. Jelonek, W.: Killing tensors and Einstein-Weyl geometry. Colloq. Math. **81**(1), 5–19 (1999)
10. Jelonek, W.: Higher dimensional Gray Hermitian manifolds. J. London Math. Soc. (2) **80**(3), 729–749 (2009)
11. Kobayashi, S.: Transformation groups in differential geometry. Springer, Berlin (1972)
12. Lichnerowicz, A.: Geometrie des groupes de transformations. Dunod, Paris (1958)
13. Madsen, B., Pedersen, H., Poon, Y., Swann, A.: Compact Einstein-Weyl manifolds with large symmetry group. Duke Math. J. **88** (1997), 407–434 (Corrigendum: Duke Math. J. **100** 167–167 (1999))
14. Moroianu, A.: Lectures on Kähler Geometry. LMSST 69. Cambrige University Press, Cambrige (2007)
15. Pedersen, H., Swann, A.: Riemannian submersions, four-manifolds and Einstein-Weyl geometry. Proc. London Math. Soc **66**(3), 381–399 (1993)
16. Pedersen, H., Swann, A.: Einstein-Weyl geometry, the Bach tensor and conformal scalar curvature. J. Reine Angew. Math. **441**, 99–113 (1993)
17. Tod, K.P.: Compact 3-dimensional Einstein-Weyl structures. J. London Math. Soc. **45**(2), 341–351 (1992)
18. Wang, J., Wang, M.Y.: Einstein metrics on S^2-bundles. Math. Ann. **310**, 497–526 (1998)