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QUANTUM PHYSICS METHODS IN SHARE OPTION VALUATION

METODY FIZYKI KWANTOWEJ W WYCENIE OPCJI NA AKCJE

A b s t r a c t

This paper deals with European share option pricing using quantum physics methods. These contingent claims are usually priced using the Black-Scholes equation. This nonlinear parabolic equation is based on geometric Brownian motion model of the stock price stochastic process. Similar processes also appear among quantum particles and are described by the time-dependent Schrödinger equation. In this paper, the option pricing based on the Schrödinger equation approach is proposed. Using Wick transformation, the Black-Scholes equation is transformed into the equivalent Schrödinger equation. The Fourier separation method is used to find analytical solutions to this equation. The last square method is used to calibrate the Schrödinger model based on real market data. Numerical results are provided and discussed.

Keywords: *option pricing, econophysics, quantum physics methods*

S t r e s z c z e n i e

Artykuł dotyczy wyceny europejskich opcji na akcje z użyciem metod fizyki kwantowej. Tego typu obliczenia zazwyczaj przeprowadza się z wykorzystaniem równania Blacka-Scholesa. To nieliniowe, paraboliczne równanie, oparte jest na geometrycznym modelu ruchu Browna procesu stochastycznego cen akcji. Podobne procesy dotyczą także cząstek kwantowych i są opisane zależnym od czasu równaniem Schrödingera. Zaproponowano wycenę opcji na akcje z wykorzystaniem równania Schrödingera. Używając transformacji Wicka, równanie Blacka-Scholesa przekształcone jest do równoważnej postaci równania Schrödingera. W celu znalezienia analitycznego rozwiązania tego równania, zastosowano metodę separacji zmiennych Fouriera. Metoda najmniejszych kwadratów została użyta w celu kalibracji modelu Schrödingera dla danych giełdowych. Dostarczono i przedyskutowano wyniki numeryczne.

Słowa kluczowe: *wycena opcji, ekonofizyka, metody fizyki kwantowej*

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1. Introduction

Collective phenomena appearing in economics, social sciences or ecology pose intriguing theoretical challenges for researchers. In view of the empirical abundance of non-trivial fluctuation patterns and statistical regularities, they are very attractive not only for economists or sociologists but also for physicists, especially statistical physicists. In recent years, many physical theories like theories of turbulence, scaling, random matrix theory or renormalization group were successfully applied to economy giving a boost to modern computational techniques of data analysis, risk management, artificial markets, and macro-economy.

The term econophysics was introduced by H.E. Stanley in the mid-nineties to describe the host of papers written by physicists to explain economic and financial phenomena. Econophysics is regarded as an interdisciplinary research field, applying theories and methods originally developed by physicists in order to solve problems in economics, usually those including uncertainty or stochastic processes (random and disspersion models) and nonlinear dynamics (chaos, criticality, power laws). Both areas involve the study of complex systems formed by a large number of smaller subsystems.

The term econophysics can also be understood to mean the physics of finance, because it attempts to understand the global behavior of financial markets [13] from a scientific standpoint. It has its roots in ancient history. N. Copernicus and I. Newton were two luminaries who applied statistical physical concepts to economic problems. Physics involves trying to understand how macroscopic effects are bought about by a huge number of microscopic interactions, so some of the tools used by statistical physicists can be used to more accurately understand market dynamics. For instance, studies of entropy have been applied to gain a better understanding of salary distribution in a free market. Many similarities have been found between data on stock markets and earthquakes. This could help economists better understand and perhaps even predict stock market crashes.

The following problem is considered in many papers that try to describe economy using quantum mechanics. One of the most pioneering works is presented by [8] and is based on the Black-Scholes [2] transformation into time-dependent Schrödinger [1] equation. The solution is given by applying semiclassical methods, of common use in theoretical physics, to find an approximate analytical solution of the Black-Scholes equation. The semiclassical approximation is performed for different arbitrage bubbles (step, linear, parabolic). This model can be interpreted as a Schrödinger equation in imaginary time for a particle of mass $\frac{1}{\sigma^2}$ with a wave function in an external field force generated by the arbitrage potential.

This paper introduces similar Black-Scholes into the Schrödinger equation transformation, but the final solution of option price is not given using a semiclasscal limit but is given by an analytical function. The asset price function is approximated by n-degree polynomial. Arbitrage existence is not considered. The option price is given by the wave function that is solved for the Schrödinger equation in real time for free particle of mass equal to $\frac{2}{\sigma^2}$. A particle interacts with constant potential $\widehat{U} = \frac{r}{2}$.

1.1. Schrödinger equation

The time-dependent Schrödinger equation [1] is a linear partial differential equation, first order in time, second order in the spatial variables. This equation is often written in the form:

$$\widehat{H}\psi(x,t) = i\hbar\frac{\partial\psi(x,t)}{\partial t} \tag{1}$$

where:

$$\widehat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \widehat{U}(x,t) \tag{2}$$

$\widehat{H}$ is called Hamiltonian and is interpreted as system energy (kinetic and potential energy of the particles constituting the system). $\widehat{U}(x,t)$ is the potential energy operator. In quantum physics, the Schrödinger equation is the same what Newton's second law of motion to classical mechanics. It describes how a physical system will change over time. In classical mechanics, we have positions and momenta of all particles at every time t: that give a full description of the system. In quantum mechanics, the information about the system is contained in the solution to Schrödinger's equation, a wave function Ψ. The square of the absolute value of the wave function, $|\psi(x, t)|^2$ gives the probability density for finding the particle at position x. But it is also possible to solve Schrödinger's equation for many particle systems and to find wave functions for other observable quantities, for example the momenta of the particles. We want to know if it is possible to describe economic systems in the same way as physical systems. If it is possible to describe economic systems by Schrödinger equation then it is solution – the wave function can explain behavior of economic forms like options.

1.2. Black-Scholes model assumptions

In 1973, Fischer Black and Myron Scholes developed a formula [2] for the valuation of European contingent claims based on a geometric Brownian motion model for the stock price process. Robert Merton [3] developed another method to derive the mentioned formula that turned out to have very wide applicability.

The Black-Scholes model is based on two assets, a (risky) stock with price governed by the stochastic process $S = S_t$, $t \in [0, T]$ and a (riskless) bond with price process $B = B_t$, $t \in [0, T]$.

1.3. Ito rules

Using Ito's [4] formula, we can write the stock price $S(t)$ process and the bond price process equations in the following form:

$$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dW_t,\ t \in [0, T] \tag{3}$$

where dW_t is a differential of a continuous-time stochastic process called the Wiener [5] process. dW_t is also called Brownian motion. It is one of the best known Lévy processes, so stochastic processes with stationary independent increments, and occurs frequently in physics. It means that process S should be treated as a diffusion process. Moreover:

$$\mu(S_t, t) = \mu S_t \quad \text{and} \quad \sigma(S_t, t) = \sigma S_t \tag{4}$$

as well as:

$$dB_t = rB_t dt,\ t \in [0, T] \tag{5}$$

where μ is stock price average value (expected value), σ is standard deviation, and σ^2 is variance.

The key to the correct interpretation of dW_t is to interpret $dW_t = W_{t+dt} - W_t$ as a random variable with a mean of 0 and an infinitesimal small variance dt:

$$dW_t \approx N(0, dt) \tag{6}$$

where $N(\mu, \sigma)$ is a Gaussian distribution [6] of random variable x and is defined as:

$$N(\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) \tag{7}$$

For $\mu = 0$ and $\sigma = dt$ Gaussian distribution becomes a normal distribution of random variable x and is defined:

$$dW_t \approx N(0, dt),\ dt > 0 \tag{8}$$

Now let us write some basic relations called Ito rules, between dW_t and dt:

$$E(dW_t^2) = dt, \quad Var(dW_t^2) = dt^2, \quad E(dW_t) = 0 \tag{9}$$

$$E(dW_t^2) = dt, \quad E(dW_t) = 0 \tag{10}$$

$$E(dW_t^2 dt^2) = 0, \quad dW_t dt = 0, \quad dW_t^2 = dt \tag{11}$$

where E symbol means expected value, and Var is variance.
We assume that dt is infinitesimally small, thus any power of dt can be omitted, in particular:

$$dt^2 = 0 \tag{12}$$

Let us define:

$$dS^2 = dSdS \tag{13}$$

Consider continuous and differentiable function $V(S, t)$. Using Taylor expansion [7], and omitting higher than second order parts, we obtain:

$$dV(S,t) = \frac{\partial V(S,t)}{\partial t}dt + \frac{\partial V(S,t)}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V(S,t)}{\partial^2 t}dt^2 + \tag{14}$$

$$\frac{1}{2}\frac{\partial^2 V(S,t)}{\partial^2 S}dS^2 + \frac{\partial^2 V(\mathrm{s},t)}{\partial t \partial S}dtdS + ...$$

$V(S, t)$ is interpreted as the option price which depends on the asset price $S(t)$ and time t. We know by (12) that $dt^2 = 0$, so equation (14) simplifies to:

$$dV(S,t) = \frac{\partial V(S,t)}{\partial t}dt + \frac{\partial V(S,t)}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V(S,t)}{\partial^2 S}dS^2 + \frac{\partial^2 V(s,t)}{\partial t \partial S}dtdS \tag{15}$$

Now, at the basis of (3) and (13) let us calculate dS^2 and $dtdS$:

$$dS^2 = \mu^2(S, t)dt^2 + 2\mu(S, t)\sigma(S, t)dtdW + \sigma^2(S, t)dW^2 = \sigma^2(S, t)dt \tag{16}$$

$$dtdS = \mu(S, t)dt^2 + \sigma(S, t)dtdW = 0 \tag{17}$$

Inserting (16) and (17) into (15) we get:

$$dV(S,t) = \left(\frac{\partial V(S,t)}{\partial t} + \mu(S,t)\frac{\partial V(S,t)}{\partial S} + \frac{1}{2}\sigma^2(S,t)\frac{\partial^2 V(S,t)}{\partial S^2}\right)dt + \sigma(S,t)\frac{\partial V(S,t)}{\partial S}dW \tag{18}$$

For Brownian motion using (4) we obtain from (18):

$$dV(S,t) = \left(\frac{\partial V(S,t)}{\partial t} + \mu S\frac{\partial V(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V(S,t)}{\partial S^2}\right)dt + \sigma S\frac{\partial V(S,t)}{\partial S}dW \tag{19}$$

1.4. Riskless portfolio

Let us define portfolio Π which consists of one option in short the term, and n shares in the long term. The portfolio can be written as:

$$\Pi = -V(S, t) + nS \tag{20}$$

or in differential form:

$$d\Pi = -dV(S, t) + ndS \tag{21}$$

Substituting (4), (3), (18), into (21) we get:

$$d\Pi = -\left(\frac{\partial V(S,t)}{\partial t} + \mu S\frac{\partial V(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} - n\mu S\right)dt + \left(n - \frac{\partial V(S,t)}{\partial S}\right)\sigma S dW \quad (22)$$

We assume that our portfolio (22) is riskless, so it must be described by a deterministic equation, so cannot contain any stochastic parts. It means that the following statement should be true:

$$n - \frac{\partial V(S,t)}{\partial S} = 0 \quad (23)$$

and it means that the number of shares that are in the portfolio is given by:

$$n = \frac{\partial V(S,t)}{\partial S} \quad (24)$$

It means that the number of shares in the portfolio is equal to the partial derivative of option price at time t. This derivative is also called *delta* and it measures the sensitivity of the value of an option to changes in the price of the underlying stock, assuming all other variables remain unchanged. If a position is delta neutral it means that its instantaneous change in value, for an infinitesimal change in the value of the underlying security, will be zero [14]. Since delta measures the exposure of a derivative to changes in the value of the underlying, a portfolio that is delta neutral is effectively hedged. The overall value will not change for small changes in the price of its underlying instrument.

Equation (22) simplifies to the following form:

$$d\Pi = -\left(\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2}\right)dt \quad (25)$$

As we assumed that our portfolio is riskless then the rate of return on this portfolio must be equal to the rate of return on any other riskless instrument; otherwise, there would be opportunities for arbitrage [8]:

$$d\Pi = \Pi r dt \quad (26)$$

where r is the risk-free rate of return. Comparing (25) and (26) we get:

$$-\left(\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2}\right) = \Pi r \quad (27)$$

Finally substituting (20) and (24) into (27) we get the Black-Scholes equation:

$$\frac{\partial V}{\partial t}+\frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0 \tag{28}$$

with boundary and initial conditions:

$$\text{for } t=0: V(S(0),0)=V_p \tag{29}$$

$$\text{for } t=T: V(S(T),T)=\max\{S(t=T)-K,0\}=V_T$$

$$S \in [S_0, S_T]$$

$$t \in [0,T],\ T>0$$

$$V, S, t \in R$$

where K is the strike price (exercise price) of an option. Parameter $\sigma \in R$ denotes the volatility of the stock's returns. This is the square root of the quadratic variation of the stock's log price process, $r \in R$ is the annualized risk-free interest rate, continuously compounded (the force of interest), $\mu \in R$ is annualized drift rate of S.

2. Black-Scholes to Schrödinger equation transformation

Quantum [12] mechanics is the theory describing the micro world. We want to to apply quantum mechanics in the stock market in which the stock index/option is based on the statistics of the share prices of many representative stocks. Let us consider the index/option as a macro scale object, it is reasonable to view every stock, which constitutes the index, as a micro system. Stock is always traded at certain prices, which presents its corpuscular property. Stock price fluctuates in the market, which presents the wave property. It means that there is particle – wave dualism, we suppose the micro scale stock as a quantum system. Rules are different between quantum and classical mechanics. In order to describe the quantum characters of the stock, we are going to build a price model on the basic hypotheses of quantum mechanics.

We will use several mathematical operations to transform the Black-Scholes equation into Schrödinger's [1] form. If the transformation is possible, then we will be able to use a quantum physics interpretation to explain the economy's options pricing issues. We are going to use quantum methods, because it seems that the real world of economics might not be describable merely in terms of conventional macroeconomic variables (unemployment rate, GNP, aggregate demand etc.) that is why conceptual innovation is needed. An economic system is in some ways like a mechanism as is recognized in all theories. In the following, we present some very tentative, preliminary conjectures about what an economic theory based on the quantum physics analogy. In this paper, the transformed equation will be called: the Black-Scholes-Schrödinger (BSS) equation.

2.1. Introducing new variables

Let us start by introducing the new variable:

$$x = LnS,\ x \in R \tag{30}$$

We obtain:

$$\frac{\partial V}{\partial S} = \frac{1}{S}\frac{\partial V}{\partial x} \tag{31}$$

and:

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial V}{\partial x}\right) = \frac{-1}{S^2}\frac{\partial V}{\partial x} + \frac{1}{S}\frac{\partial^2 V}{\partial x^2}\frac{\partial x}{\partial S} = \frac{1}{S^2}\left(\frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial x}\right) \tag{32}$$

Using (31), Black-Scholes equation (28) takes the form:

$$\frac{\partial V}{\partial t} + \frac{\partial^2}{2}\frac{\partial^2 V}{\partial x^2} + \frac{\partial V}{\partial x}\left(r - \frac{\sigma^2}{2}\right) - rV = 0 \tag{33}$$

with boundary and initial conditions:

$$\text{for } t = 0 : V(x(0), 0) = V_p \tag{34}$$

$$\text{for } t = T : V(x(T), T) = \max\{x(t = T) - K, 0)\} = V_T$$

$$x \in [Ln(S_0), Ln(S_T)]$$

$$t \in [0, T],\ T > 0$$

$$V, x, t \in R$$

We want to exclude from (33) the expression that contains $\left(r - \frac{\sigma^2}{2}\right)$ element, because we want to liken (33) to Schrödinger form.

Let us introduce new variable y:

$$y = x - \left(r - \frac{\sigma^2}{2}\right)t \tag{35}$$

Previously we had assumed that x does not depend on time. Data related to the market showed that stock price S is time dependent, so the logarithm of that price described in this paper as x, should be also time dependent. The variation of x using time dependent part, generates a new variable y which is time dependent. This procedure was necessary to reproduce the behavior of the real market.
A similar approach has been used in [8].
Let us calculate new partial derivatives:

$$V_y \stackrel{\text{def}}{=} \frac{\partial V}{\partial y} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial y} \tag{36}$$

Second partial derivative is given by:

$$\frac{\partial^2 V}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) = \frac{\partial V_y}{\partial y} = \frac{\partial V_y}{\partial x}\frac{\partial x}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\frac{\partial x}{\partial y}\right)\frac{\partial x}{\partial y} \tag{37}$$

Because $\frac{\partial x}{\partial y} = 1$ and using (35), equation (37) simplifies to $\frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial x^2}$.
Designating x from (35) and use it to calculate time dependent partial derivative of option price:

$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial V}{\partial x}\left(r - \frac{\sigma^2}{2}\right) \tag{38}$$

Inserting (37), (38) into (33), we get:

$$2\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial y^2} - rV = 0 \tag{39}$$

and:

$$-\frac{\sigma^2}{2}\frac{\partial^2 V}{\partial y^2} - rV = 2\frac{\partial V}{\partial t} \tag{40}$$

Let us put (40) into Heat-equation form:

$$\left(-\frac{\sigma^2}{4}\frac{\partial^2}{\partial y^2} + \frac{r}{2}\right)V = \frac{\partial V}{\partial t} \tag{41}$$

We want to transform (41) which represents the Heat-equation [15] into the Schrödinger form, to do this, we have to introduce a new variable τ defined as imaginary time. Using the Wick rotation, we get:

$$\tau = -it \tag{42}$$

Finally (41) equation is represented by Schrödinger's equation:

$$\left(-\frac{\sigma^2}{4}\frac{\partial^2}{\partial y^2}+\frac{r}{2}\right)V = i\frac{\partial V}{\partial \tau} \tag{43}$$

2.2. Quantum interpretation

Equation (43) can be interpreted as Schrödinger's equation for free particle of mass equal to:

$$m = \frac{2}{\sigma^2} \tag{44}$$

and our particle interacts with constant potential:

$$\hat{U} = \frac{r}{2} \tag{45}$$

Remark: equation (43) is written in natural units. The use of $\hbar = c = 1$ units can simplify particle physics notation considerably.
We have proved that it is possible to use quantum physics to describe the option pricing. We showed that it is possible to write Schrödinger's equation for the selected option. Now we have to find its solution.

2.3. Fourier separation method

We want to find the solution of (43), after using the Fourier separation method [9], we get:

$$V(y, \tau) = \psi(y)\, T(\tau) \tag{46}$$

where the boundary and the initial conditions are below:

$$\Psi(x) \in [\Psi_0 = \Psi(x_0), \Psi_T = \Psi(x_T)] \text{ – boundary condition} \tag{47}$$

$$T(\tau) \in [T_0^i, T_t^i] \text{ – initial condition}$$

We obtain two independent equations that describe dependency on variable y and imaginary time variable τ:

$$\left[-\frac{1}{2m}\frac{\partial^2}{\partial y^2}+\widehat{U}\right]\psi(y)=k\psi(y) \tag{48}$$

$$i\frac{\partial T(\tau)}{\partial \tau}=k \tag{49}$$

where m and $\widehat{U}$ are given by (44) and (45). Constant k can be interpreted as particle energy, and we assume that:

$$k>0 \quad \text{and} \quad k<\widehat{U} \tag{50}$$

The solution for (49) has the form:

$$T(\tau) = T_0^{\,i} \exp(-ik\tau) \tag{51}$$

As we remember, all computations in physics and economy are based on time t that is real. That is why we cannot use (51) form as it is based on imaginary time τ. This has to be replaced by an equivalent equation that uses real time t. Going back to real variables means that (51) turns into:

$$T(t) = T_0 \exp(-ik) \tag{52}$$

Our general form (46) moves into the following equation:

$$V(y, t) = \psi(y)\ T(t) \tag{53}$$

with boundary conditions:

$$\Psi(y) \in [\Psi_0 = \Psi(y_0), \Psi_T = \Psi(y_T)] \tag{54}$$

and initial condition:

$$T(t) \in [T_0, T_t] \tag{55}$$

Constant T_0 is determinated from the following initial condition:

$$T_0 = T(t = 0) \neq 0 \tag{56}$$

Function $\psi(y)$ is given by:

$$\psi(y)=D\exp\left(\sqrt{2}\sqrt{m\left(\widehat{U}-k\right)y}\right)+E\exp\left(\sqrt{2}\sqrt{m\left(\widehat{U}-k\right)y}\right) \tag{57}$$

where D and E are constants given by the boundary condition.
Remark: In order to represent a physically observable system, the wave function must satisfy certain constraints [10]:

- it must be a solution to the Schrödinger equation,
- it must be a continuous function of y,
- the slope of the function in y must be continuous. Specifically $\frac{\partial \psi}{\partial x}$ must be continuous,
- it must be normalizable. This implies that the wave function approaches zero as x approaches infinity.

To be consistent with the mentioned assumptions, we have to remove the part that grows to infinity. This means that constant D should be equal to 0. From the economy part it means that if t approaches infinity, the option price function should approach 0. Even though the stock price approaches infinity, the option price will not grow to infinity.

2.4. General solution for the Black-Scholes-Schrödinger equation

From (52), (53) and (57) we have general solution for Black-Scholes-Schrödinger equation:

$$V(y,t) = C\exp(-kt)E\exp\left(\sqrt{2}\sqrt{m\left(\hat{U}-k\right)}y\right) \tag{58}$$

which is equal to:

$$V(y,t) = A\exp\left(-kt - \sqrt{2}\sqrt{m\left(\hat{U}-k\right)}y\right) \tag{59}$$

for given mass (44) and potential (45), equation (59) becomes:

$$V(y,t) = A\exp\left(-kt - \frac{2\sqrt{\left(r/2-k\right)}}{\sigma}y\right) \tag{60}$$

where:

$$A = CE \tag{61}$$

The key to compute BSS equation (60) for the given option is to understand $y(t)$ behavior. We know that $y(t)$ depends on $Ln(S)$ and also contains the linear time dependent part.

Unfortunately we do not know the exact form of $Ln(S)$, that is why $y(t)$ will be interpolated by the first degree polynomial function and then will be used to predict option price using equation (60). Additionally, similar computations will be performed, but $y(t)$ will be interpolated by the second degree polynomial function. Fig. 1, shows $Ln(S)$ behavior for the WIG20 index which is listed on Polish market.

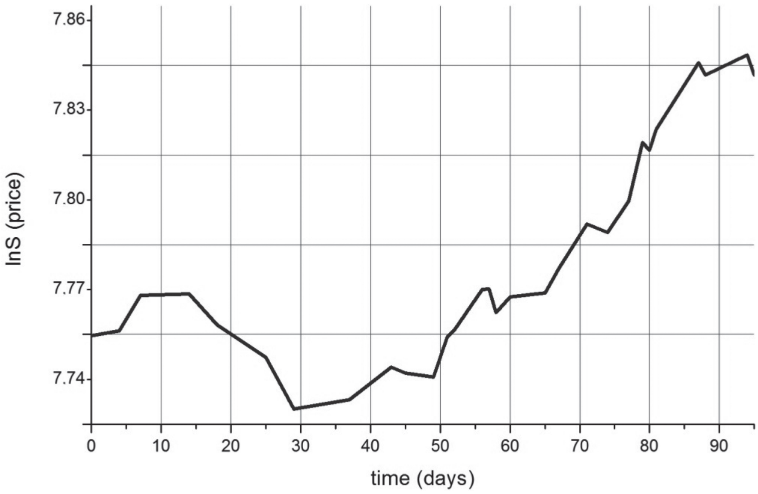

Fig. 1. *Ln*(*S*) time behavior

We will also check how our model fits to market data, we will check this by calculating the residual sum of squares and Pearson's correlation factor.

3. Numerical computations

In this chapter, we want to check how our BSS model (60) fits market data. We will perform computations for cases. The first case assumes that $y(t)$ is modelled by the first order polynomial, the second case assumes that $y(t)$ is modelled by the second order polynomial. Following computations and figures have been generated using scipy library. The mentioned library was implemented into python scripts. The WIG20 index is used as asset price, the OW20F3280 is used as anoption.
Remark: after inserting interpolated $y(t)$ form into BSS equation (60), our $V(y, t)$ becomes $V(t)$ function.

3.1. $y(t)$ is modelled by first order polynomial

Equation (60) can be written in the following form:

$$V(y,t) = \exp\left(-kt - \frac{2\sqrt{(r/2-k)}}{\sigma}y + g\right) \tag{62}$$

where g is constant.
Let us assume that $y(t)$ behavior is interpolated by the first order polynomial:

$$y(t) = f(t) \approx zt + d \tag{63}$$

where z and d are constants that will be designated on the basis of market data using the last squares method.
Let us put (63) into (62):

$$V(t) = \exp\left(-kt - \frac{2\sqrt{(r/2-k)}}{\sigma}(zt+d) + g \right) \tag{64}$$

Additionally we see that (64) can also be interpolated, but using an exponential function with the linear expression:

$$V(t) = \exp\left(-kt - \frac{2\sqrt{(r/2-k)}}{\sigma}(zt+d) + g \right) = \exp(at+b) \tag{65}$$

Constant a and b will be designated on the basis of market data using the last squares method, if z and d are also known, then we can designate constant k by solving following equations:

$$a = -\left(k + \frac{2\sqrt{(r/2-k)}}{\sigma} z \right) \tag{66}$$

and:

$$b = g - \frac{2\sqrt{(r/2-k)}}{\sigma} d \tag{67}$$

We can choose only that values for k that are allowed by (50) equation. Fig. 2 shows the WIG20 index interpolated by the first order polynomial, Fig. 3 shows the OW20F3280 interpolation using exp($at + b$) function. OW20F3280 is the call option for WIG20 asset.

We have calculated residual sum of squares and Pearson's correlation factor for the BSS model for the OW20F3280 option. The residual sum of squares is equal to **210.92**, and the correlation is equal to **0.87**.
Now, we want to check if we should choose the second order polynomial to interpolate the WIG20 index then we will have better results in OW20F3280 option pricing.

3.2. $y(t)$ is modelled by the second order polynomial

Let us assume that $y(t)$ behavior is interpolated by the second order polynomial:

$$y(t) = f(t) \approx zt^2 + td + e \tag{68}$$

where z, d and e are constants that will be designated on the basis of market data using the last squares method.

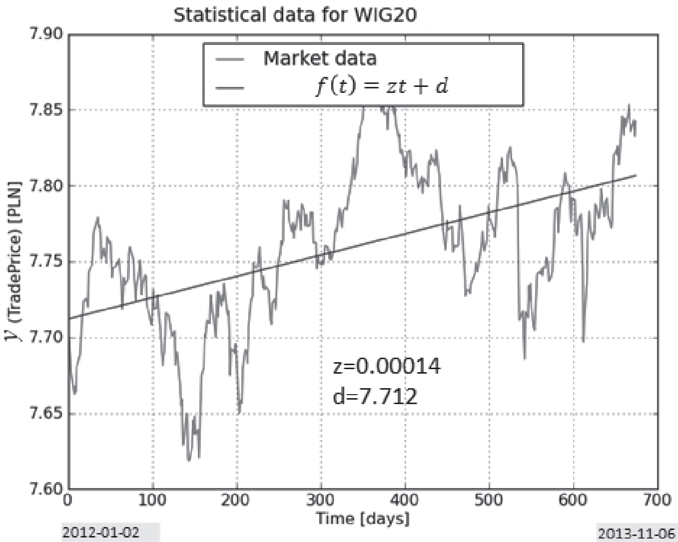

Fig. 2. WIG20 index interpolated by the first order polynomial

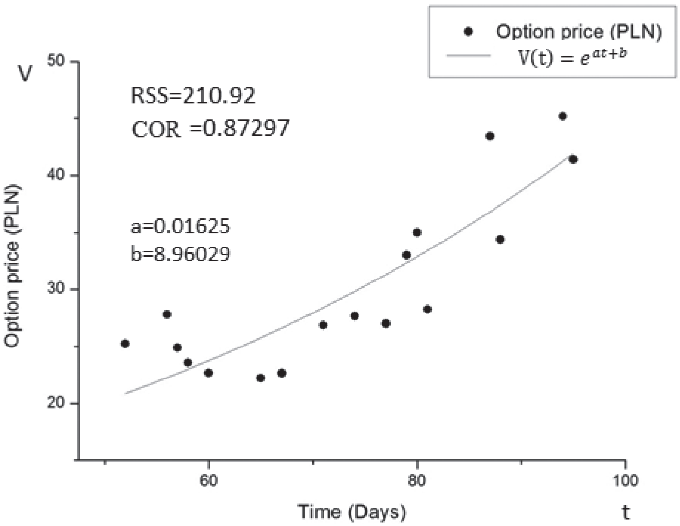

Fig. 3. OW20F3280 interpolation using $\exp(at + b)$ function

Let us insert (68) into (62):

$$V(t) = \exp\left(-kt - \frac{2\sqrt{(r/2-k)}}{\sigma}\left(zt^2 + td + e\right) + g\right) \tag{69}$$

Additionally we see that (69) can be also interpolated, but using an exponential function with quadratic expression:

$$V(t) = \exp\left(-kt - \frac{2\sqrt{(r/2-k)}}{\sigma}\left(zt^2 + td + e\right) + g\right) = \exp\left(at^2 + bt + c\right) \tag{70}$$

Constant a, b and c will be designated on the basis of market data using the last squares method.

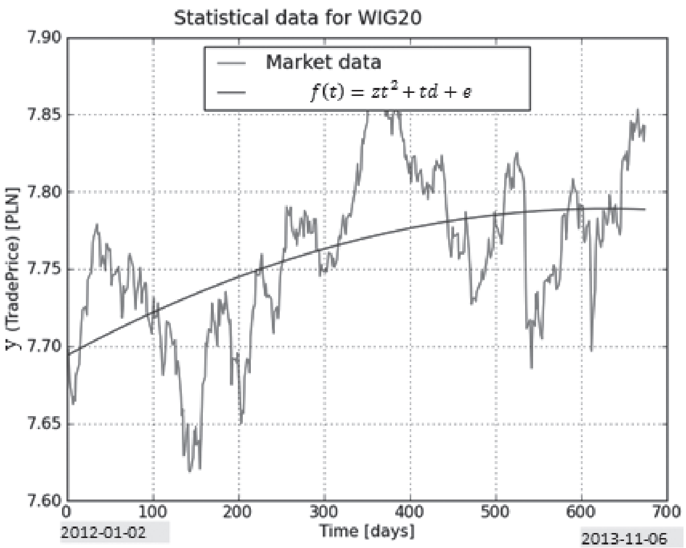

Fig. 4. WIG20 index interpolated by the second order polynomial

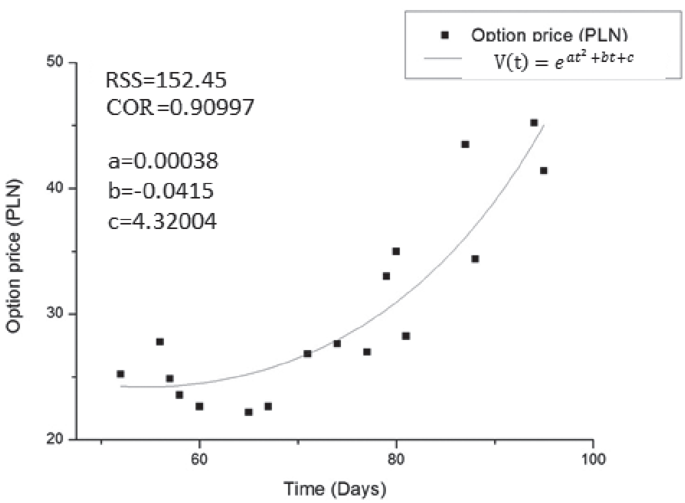

Fig. 5. OW20F3280 interpolation using $\exp(at^2 + bt + c)$ function

Constant k can be designated from:

$$a = -z\frac{2\sqrt{(r/2-k)}}{\sigma} \tag{71}$$

$$b = -\left(k + \frac{2\sqrt{(r/2-k)}}{\sigma}d\right) \tag{72}$$

and:

$$b = g - \frac{2\sqrt{(r/2-k)}}{\sigma}e \tag{73}$$

We can choose only that values for k that are allowed by (50) equation. Fig. 4 shows the WIG20 index interpolated by the second order polynomial, Fig. 5 shows OW20F3280 interpolation using $\exp(at^2 + bt + c)$ function.

We have calculated the residual sum of squares and Pearson's correlation factor for the BSS model for the OW20F3280 option. The residual sum of squares is equal to **152.45**, and the correlation is equal to **0.91**. As we see, choosing the second order polynomial in the WIG20 interpolation, gave better results.

4. Concluding remarks

We have verified that it is possible to transform the Black-Scholes equation into Schrödinger's form. It has been performed by introducing new variables and also by making additional transformations and simplifications. We tried to describe trade price relations in time by using polynomial interpolation. Increasing rhe polynomial order gave us better results. We have developed our own procedures based on Python scripts and this approach will be used in further research.

Our next calculations will be related to searching for the new functions that can describe $y(t)$ behavior. We will test the new list of wave functions with trade price modelled by different functions and we will calculate if that model fits market data. We will use numerical experiments to check if that approach is reasonable. Otherwise we would need to have a fundamental change in our assumptions. One of those assumptions is to take into consideration, that arbitrage [11] exists.

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