TECHNICAL TRANSACTIONS | **CZASOPISMO TECHNICZNE**
CIVIL ENGINEERING | BUDOWNICTWO

2-B/2013

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HIGHER-ORDER FEM FOR ANALYSIS OF COUPLED PROBLEMS

MES WYŻSZEGO RZĘDU W ANALIZIE PROBLEMÓW SPRZĘŻONYCH

S t r e s z c z e n i e

W artykule przedstawiono zastosowanie hp-adaptacyjnej MES do analizy numerycznej problemów sprzężonych. W szczególności omówiono program *HP3D* i pewne wątki dotyczące adaptacji siatki oraz numerycznych aspektów modelowania problemów sprzężonych. Niektóre możliwości nowego programu ilustruje przykład numeryczny motywowany rzeczywistym inżynieryjnym problemem. Wskazano również na możliwe i pożądane przyszłe kierunki rozwoju tego oprogramowania.

Słowa kluczowe: hp-adaptacyjna MES, modelowanie problemów sprzężonych, termo-mechanika

A b s t r a c t

The paper presents the application of the hp-adaptive FEM to numerical analysis of coupled problems. *HP3D* code is especially discussed, as well as some topics concerning mesh adaptation and numerical aspects of modeling coupled problems on the example of weakly coupled thermo-mechanical problem. Some possibilities of the new code are illustrated by a numerical example motivated by a real engineering problem. Finally, some future possible and desirable directions of development for this code are noted.

Keywords: hp-adaptive FEM, coupled problems modeling, thermo-mechanical modeling

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Denotations

C – elasticity tensor
σ – stress tensor
ε – strain tensor
ε^T – thermal strain tensor
ε^S – shrinkage strain tensor
s – (concrete) shrinkage strain rate
κ – thermal conductivity
α – thermal expansion coefficient
θ – temperature
Q – heat source
δ_{ij} – Kronecker delta
f_i – body forces
u – displacements

1. Introduction

Many engineering problems are of a multi-physical nature and include different physical phenomena, e.g. mechanics and heat transfer, mechanics and acoustics, mechanics of solids and fluids, etc. The numerical analysis of such problems very often requires special treatment.

The Finite Element Method (FEM) is a widely used tool for the approximation of solutions to mechanical problems that can be defined in terms of differential equations. Its variety evolved over the decades. One of these is *higher-order FEM*, which has the main advantage of being able to perform calculations with use of high order approximation functions. This leads directly to an ability to model various problems in which the satisfactory solution is obtained with use of relatively small number of degrees of freedom.

Specialized software, such as the new *HP3D* code, allows for the modelling of various, especially coupled, problems with use of higher-order FEM. Additionally, it allows the user fairly easy application of *hp* mesh refinement. Furthermore, a very important aspect of using *HP3D* [5] is ability to model coupled problems, e.g. thermo-mechanical fields. Other issues which are significant in ground of numerical techniques are presented in this paper.

Thermo-mechanical coupling is significant in analysis of many structures and materials. For example, the issue of crack initiation and fracture propagation, humidity nursing, effective designing and the separation of the phases of forming massive concrete elements became a vast issue for activities in the individual field of engineering science [1, 10].

The issue bordering these thermo-mechanical problems is shrinkage (also expansion; in general: change in volume) field, which can have its source both in thermal field, which penetrates body and humidity field changes, which in turn strongly influences chemical reactions that lead to changes in volume.

The analyzed examples are reduced to modeling the issue of volume changes taking into account only thermal influence. This problem is in fact a weakly coupled one. Therefore, it can easily be decoupled and solved sequentially, i.e. after heat transfer steady state is obtained, stress/strain fields can be derived from the temperature field data. However, even for such a weakly coupled case this approach seems to be a good option. Moreover, adding an additional coupling term (e.g. heat generation by friction) would change the case diametrically to strongly coupled without a significant complication of the numerical analysis undertaken in *HP3D*.

2. Classification of coupled field problems in engineering

A vast amount of processes which lie directly in the scope of interest for mechanical engineering are of coupled and often multi-physical nature [8]. In general, they have to be analyzed by sophisticated computer codes which need to allow the user to describe various fields that interact with each other and are governed by different laws and involve dependent variables. The coupled problems may be classified and characterized in several ways [2]. One may use:

- weakly coupled (also called one-way or load transfer or loose) – where dependent variables can be eliminated or
- strongly coupled (also called two-way or direct or tight) – where dependent variables usually cannot be eliminated.

These dependent variables result from either physical problems (e.g. displacement and temperature) or the mathematical formulation itself (e.g. displacement and stresses).
In addition, one may easily distinguish the domain and scale of analysis, which may be:

- multi-domain, where coupling effects occurs on an interface of two domains,
- one-domain, with the coupling in the bulk,

and

- one scale analysis, as well as
- multiscale analysis.

Considering the type of physical effects accounted for, one may distinguish the following problems:

1. Exclusively mechanical problems due to independent treatment of:
   - displacement and stress,
   - displacement, strain and stress.
2. Mechanical processes coupled with other physical effects, that induce strain distortions in solids resulting from, for example,
   - temperature change (thermo-mechanical problems),
   - shrinkage or expansion of a composite component, e.g. shrinkage of concrete, development of rust in the steel reinforcement elements within the concrete (chemo-mechanical problems).
3. Fluid-structure interaction e.g. porous media, aeroelasticity, offshore structures (fluid-solid coupling).
4. Acoustic-elastic problems.
5. Bio-heat generation and transfer.
6. Electro-mechanical problems.
7. More than two-field problems, like thermo-hydro-mechanical, welding (CFD, EM, heat, solidification), electro-magnetic-fluid.
8. Other.

The details of approximation significantly influence the computer code structure designated for analysis of coupled field problems. The most important aspects include:

1. Type of coupling:
   - multi-disciplinary, in which one code generates data for another, e.g. temperature field obtained from one code is transferred to another, where in turn the displacement field can be computed and post-processing stages executed,
   - multi-physics, in which all data is stored in one code and can fully operate on weakly or strongly coupled problems.
2. Data base (boundary conditions, subdomains) should account for the type of coupling.
3. Parallel computing may be particularly profitable in this type of modeling.

3. General description of new *HP3D* code

HP3D software was created by: prof. Leszek Demkowicz et al. [3, 5], mainly from *Institute for Computational Engineering and Science, The University of Texas, Austin*, and has a rich library [7] of finite elements: prisms, tetrahedrons and pyramids of arbitrary order up to 8th. To generate the finite element mesh, first a geometry need to be introduced. This step is done with use of GMP (Geometrical Modeling Package). Definition of the domain geometry is obtained by introducing information about "geometrical element mesh" which comes, similarly as the finite elements, in the form of four available solids: tetrahedron, prism, hexahedron and pyramid (see Fig. 1). A Built-in pre and post-processor is used for creating the numerical model and visualizing results in graphical form.

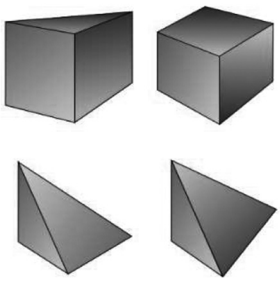

Fig. 1. GMP library members which forms geometry of the modeled object: prism, hexahedron, tetrahedron, pyramid [4]

Then, with use of transfinite interpolation (originally proposed by W. Gordon and C. Hall in 1973 in works regarding four node element and curvilinear coordinate systems applied to mesh generation), which has the important advantage of imitating the exact values of any surface function on its domain boundaries, the strict geometry of the analyzed element is reproduced. This also allows analysis of engineering problems that require very precise modeling of geometry for numerical calculations.

Geometrical blocks consist of the following parts: vertexes, edges, faces. These need to be connected by giving appropriate parameters, as follows: node coordinates (and its sequence), edges (in case of arcs, coordinates of two nodes that belong to the arc and its center), numeration of surfaces in appropriate order which constitutes order for introducing the boundary conditions, and finally, whole solids represented by the node numbers written in appropriate order.

Initial geometry blocks can be partitioned into finite elements of various order of approximation and sizes. Solution improvements obtained by mesh refinement, e.g. building four new, smaller elements from one (division of the base element can be controlled by the user, and executed in any spatial direction separately) is available in two different ways: remeshing the whole domain in three spatial directions separately/simultaneously (uniform *h*-adaptivity) or in selected elements (interactive *h*-adaptivity) by specifying their numbers.

The method of *p*-adaptation is based on the idea of rising up the approximating polynomial order. This gives the ability to model strongly elongated elements without locking effect. An example of anisotropic *hp*-refinement is shown in Fig. 2.

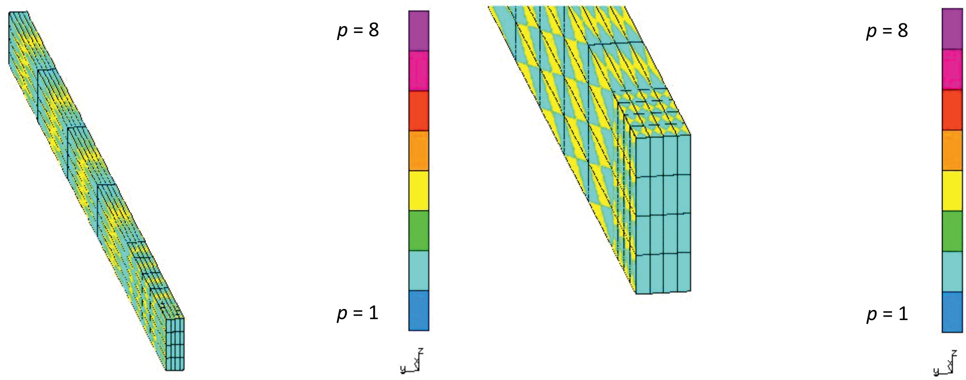

Fig. 2. Elongated elements. *hp*-adaptation in different directions

One of the main objectives of developing the *HP3D* software is introducing mesh refinement criteria (e.g. based on error estimation procedures), from which an appropriate (depending on expectations regarding the quality of solution) order of approximation and density of mesh could be concluded. Experiences and mathematical considerations show [3, 5, 6] that the best convergence (exponential) is obtained when an *hp*-adaptation is applied (see Fig. 3). These criteria should provide answers to a significant question of how to obtain an optimal mesh refinement.

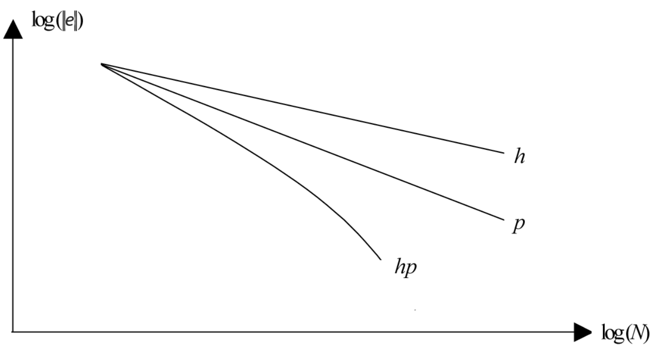

Fig. 3. Idea of convergence rate depending on type of adaptation

Since there is a distinction between geometrical elements that build geometry of the model, and finite elements that provide the solution, pre-processing and refinement procedures cooperate with the GMP module, which is responsible for continuous mapping of exact model geometry, especially after several *h*-adaptations (e.g. rounded objects stay rounded after dividing elements into smaller ones).

HP3D code allows the user relatively easy specification of own problems and procedures, especially when the analyzed issue is of a coupled nature.

4. Formulation of weakly coupled thermo-mechanical problem

The thermo-mechanical problem considered in this paper is weakly coupled. Therefore, the solution may be obtained in the following steps: determination of temperature distribution followed by mechanical analysis. The new *HP3D* code allows solution of all coupled problems in one step, this such an approach is described in this chapter.

Formulation of weakly coupled thermo-mechanics (weakly, because only the temperature field affects changes in the strain field, while deformations do not result in changes in the temperature field) in strong form (considering only elastic model of the material) comes down to the following statements:

$$-\sigma_{i,j,j} = f_i \tag{1}$$

$$\sigma_{i,j} = C_{ijkl}\left(\varepsilon_{kl} - \varepsilon_{kl}^{T} - \varepsilon_{kl}^{s}\right) \tag{2}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{3}$$

$$-\kappa\theta_{,ii} = Q \tag{4}$$

Stated above are the equations of equilibrium (1), stress-strain constitutive law with distortions of different type (2), Cauchy strain-displacement relation (3), and finally differential form of Fourier's law of thermal conduction (4) where C_{ijkl}, κ are material parameters and f_i, Q are given functions. The strain distortions (following will be treated as imposed strains in the procedure) are as follows:

$$\varepsilon_{ij}^{T} = \alpha\theta\delta_{ij} \tag{5}$$

$$\varepsilon_{ij}^{s} = s\delta_{ij} \tag{6}$$

where *s* is the example of concrete shrinkage strain rate [9], i.e.

$$s = \varepsilon_{c,as} = \varepsilon_{c,aso}(f_{cm})\beta_{as}(t) \tag{7}$$

$$\varepsilon_{c,aso}(f_{cm}) = -\alpha_{as}\left(\frac{\frac{f_{cm}}{f_{cmo}}}{6+\frac{f_{cm}}{f_{cmo}}}\right)^{2.5} 10^{-6} \tag{8}$$

$$\beta_{as}(t) = 1-\exp\left[-0.2\left(\frac{t}{t_1}\right)^{0.5}\right] \tag{9}$$

and α_{as} is a coefficient depending on the type of cement used, f_{cm} [MPa] is the average strength of concrete after 28 days ($f_{cm} = f_{ck}$ + 8 MPa)), f_{cmo} = 10 MPa, t_1 = 1 day.

The boundary conditions for mechanical fields:

$$\sigma_{ij} n_j = q_i \Big|_{\delta\Omega_t} \tag{10}$$

$$u_i = h_i \Big|_{\delta\Omega_u} \tag{11}$$

and for thermal quantities:

$$\kappa \frac{\partial\theta}{\partial n} = S \Big|_{\delta\Omega_F} \tag{12}$$

$$\theta = T \Big|_{\delta\Omega_T} \tag{13}$$

The corresponding weak form can be stated as follows:
find $u \in H_0^1(\Omega) + \widehat{h}$ and $\theta \in H_0^1(\Omega) + \widehat{T}$, such that:

$$\begin{cases} \int_{\Omega}^{-} \nu_{i,j} C_{ijkl} u_{k,l} d\Omega - \int_{\delta\Omega_t}^{-} \nu_i q_i ds + \int_{\delta\Omega_t}^{-} \nu_{i,i} s ds, \quad \forall \nu_i \in H_0^1(0) \\ \int_{\Omega}^{-} \psi_{,i} \kappa \theta_{,i} d\Omega = \int_{\sigma\Omega_F}^{-} \psi Q ds, \quad \forall \psi \in H_0^1(\Omega) \end{cases} \tag{14}$$

After discretization one obtains the following system of linear algebraic equations (at this level, weak coupling can be seen as a zero element, responsible for coupling mechanical field with temperature, present in the second row of the stiffness matrix):

$$\begin{bmatrix} B_{uu} & B_{u\theta} \\ 0 & B_{\theta\theta} \end{bmatrix} \begin{bmatrix} d_u \\ d_\theta \end{bmatrix} = \begin{bmatrix} L_u \\ L_\theta \end{bmatrix} \tag{15}$$

where d_u, denotes degrees of freedom for displacements and temperature fields, and L_u, L_θ are components of the load vector.

5. Numerical example

The example considers a facade precast member made of polystyrene with thin layer of plaster on it. The real element was built-in the structure and undergone cracking (see Fig. 4). The motivation to model this issue was to implement a weakly coupled problem into *HP3D* code. Weakly coupling was taking into account thermal influence of solar heat, which probably lead to damage of the element.

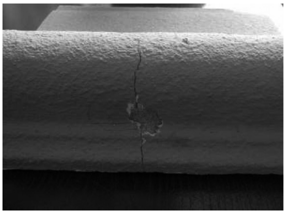

Fig. 4. Facade precast member which undergone cracking

To allow analysis of weakly coupled problems, some changes had to be done to the *HP3D* code. Correctness of these was confirmed by solution of selected benchmark problems and then the software was used to analyze the described real-life problem. The whole part of the element was modeled, although the element was cut from the bigger precast member of facade – this fact is represented in the kinematic boundary conditions.

The following thermal conditions were used: steady state due to the uniform temperature increment inside the domain by ΔT = 20°C (assumed on the basis of daily temperature differences); linear formula (5) for resulting thermal strain ε_{ij}^{T}; adiabatic thermal boundary conditions are assumed only on the exterior face of the element (see Fig. 5 – green color), since only this face stays in contact with the environment.

Material parameters are the same for both materials: thermal expansion coefficient

$\alpha_T = 70x\dfrac{10^{-6}m}{m}°C$, Young modulus E = 3 GPa and Poisson ratio v = 0.35.

The following kinematic boundary conditions were assumed: fixing on every side surface and internal one (sticking to the wall) – blue color in Fig. 5. Free displacements were allowed only at the external face of the member – green color in Fig. 5.

Results did not exclude that the cause of cracking was the high tensile stress in the element due to the heat exposure. Nevertheless, to state more comprehensive conclusions, further analysis must be done.

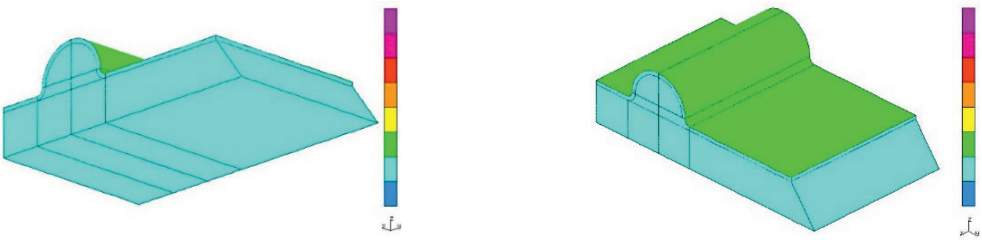

Fig. 5. Kinematic and thermal boundary conditions – see description in the text

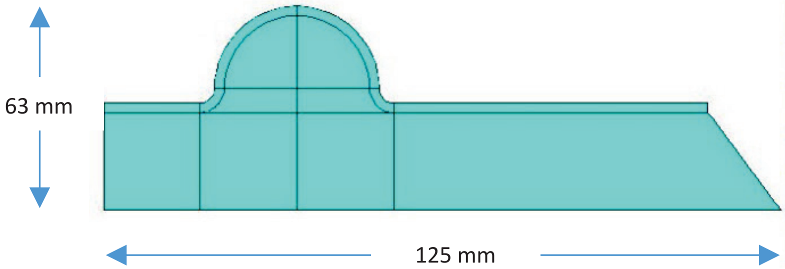

Fig. 6. Dimensions of the model: depth – 80 mm. Thickness of plaster – 3 mm

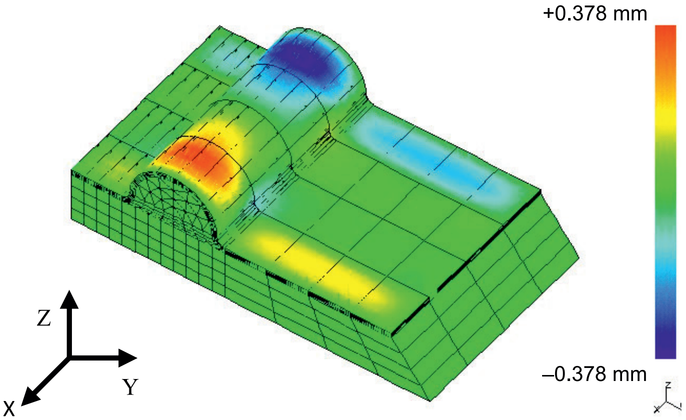

Fig. 7. Displacements at *x* – direction

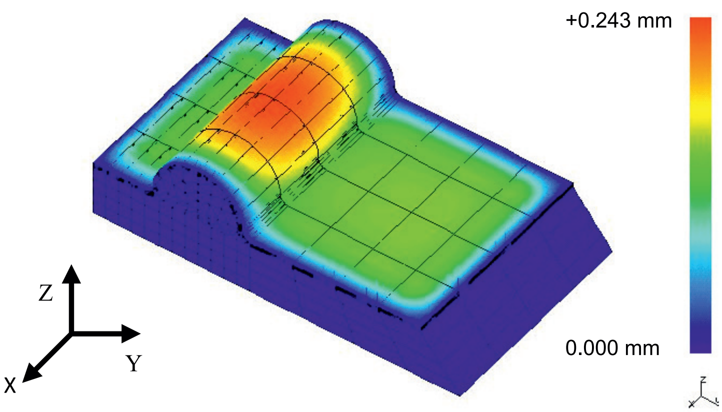

Fig. 8. Displacements at *z* – direction

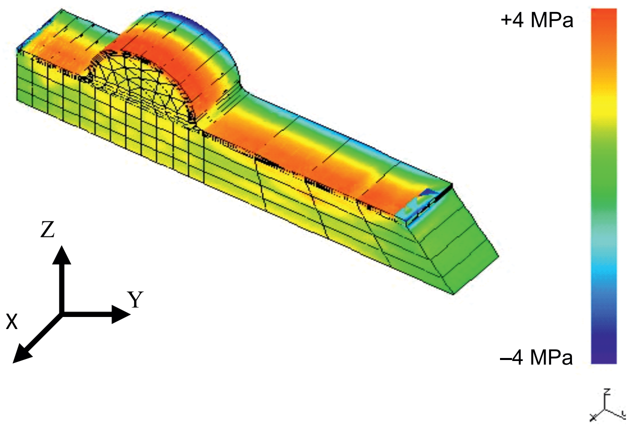

Fig. 9. σ_x strains in longitudinal cross-section

6. Concluding remarks

Development of *hp-adaptive FEM* can contribute to a significant efficiency increase of software used to solve complex engineering issues. To make computations more reliable, algorithms of automatic *hp* adaptation need to be constantly improved and introduced to practice. Effective error estimators should efficiently benefit from *hp* adaptation indicating places where the mesh should be refined and what type of refinement should be performed (*h* – division of elements; *p* – increase in order of approximation; *hp* – both at once).

Solving weakly and strongly coupled problems is also a current challenge. One such problem was presented in the paper [12] and in our future research, the problem of stress/strain fields determination in braking systems induced by heating up of friction pair will be considered. In this case, strongly coupled, nonlinear phenomena are to be considered. The influence of temperature field on mechanics is obvious, whereas the essentially very complex process of energy dissipation, resulting in heat generation, is a direct consequence of friction between two or more elements of the braking system. Therefore, such thermal phenomena have their origin in analysis of nonlinear contact problems.

This research was supported by the European Union through the European Social Fund within; Cracow University of Technology program – top quality teaching for the prospective Polish engineers, University of the 21st century; project(contract no.UDAPOKL.04.01.01-00-029/10-00).

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