TECHNICAL TRANSACTIONS | **CZASOPISMO TECHNICZNE**
CIVIL ENGINEERING | BUDOWNICTWO

2-B/2013

ŁUKASZ ZDANOWICZ*, PIOTR NOAKOWSKI**

STIFFNESS NONLINEAR ANALYSIS OF SECTIONS FOR REINFORCED CONCRETE MEMBERS

SZTYWNOŚCIOWA NIELINIOWA ANALIZA PRZEKROJÓW ŻELBETOWYCH

A b s t r a c t

An example of nonlinear mechanics of reinforced concrete based on stiffness of the analysed elements (*stiffness oriented design*) is presented in the paper. To define internal forces in reinforced concrete members, usually a linear relation is used. There is lack of considering an effect of stiffness variation after the first cracking. It often leads to underestimation of cross-sectional forces, which may give incorrect calculation results. The stiffness oriented nonlinear analysis allows for the description of work and behaviour of the structure much more precisely, which leads to an increase of safety and economy of the designed object.

Keywords: stiffness, curvature, nonlinear analysis, cracking, reinforced concrete elements

S t r e s z c z e n i e

W artykule zaprezentowano przykład nieliniowej mechaniki żelbetu opartej na sztywności analizowanych elementów (*stiffness oriented design*). Określenie sił wewnętrznych w elementach żelbetowych zwykle oparte jest na związkach liniowych. Nie uwzględnia się zjawiska zmiany sztywności po pojawieniu się zarysowania. Prowadzi to często do niedoszacowanych wielkości sił przekrojowych, co może skutkować niemiarodajnym wynikiem obliczeń. Nieliniowa analiza zorientowana sztywnościowo pozwala poznać dokładniej pracę i zachowanie się konstrukcji, co prowadzi do podwyższenia bezpieczeństwa i ekonomii projektowanego obiektu.

Słowa kluczowe: sztywność, krzywizna, analiza nieliniowa, zarysowanie, elementy żelbetowe

* M.Sc. Eng. Łukasz Zdanowicz, Institute of Building Materials and Structures, Faculty of Civil Engineering, Cracow University of Technology.
** Prof. Piotr Noakowski, Exponent Düsseldorf, TU Dortmund University.

1. Introduction

The criterion of adequate similarity between the real object and its design model is a condition for the proper designing of building structures. The methods and tools for analysis should be so selected as to fulfil the criterion for the required level of reliability.

The most popular assumption used in the designing of reinforced concrete elements is linear stress-strain (σ-ε) distribution. The classic linear theory [6, 12] takes into consideration the work of reinforced a concrete element in three phases: (1) phase 1 for uncracked element; (2) phase 2 for cracked section; (3) phase 3 treated as exceeding the limit state of this section. However, the possibility of redistribution of static values caused by cracking is omitted. It causes that a change of the element stiffness (local decreases of stiffness in cracked sections) is not taken into account, which is usually a significant value [4]. As a result, it can lead to underestimation of internal forces values that most often occurs in the response of a structure to temperature changes, forced displacements or abrupt changes of stresses. Usually in these cases, using a more precise and proper approach is required, such as, for instance, a nonlinear analysis [1, 11, 13].

Nonlinear analysis allows for the defining of distributions of internal forces and displacements of the structure closer to reality [15], what influences directly its level of reliability. *Meaning unclear, consider rephrasing* The Nonlinear method embraces two aspects here: physical and geometrical. Physical nonlinearity of reinforced concrete is connected with the law of behaviour of this material under current action, whereas geometrical nonlinearity is connected with the geometry and strains of the structure.

This paper focuses on the nonlinear analysis of reinforced concrete in the stiffness aspect (*stiffness oriented design*) [9, 10]. The article is restricted to physical nonlinear problem and omits the effects of geometric nonlinearity of the structure (LNR method, where *L* – geometric linearity, *N* – physical nonlinearity, *R* – real stiffness of the element) [6].

2. Nonlinear analysis

2.1. Assumptions

Current national standard [N2] concerning the design of the concrete structures allows for the use of nonlinear idealisation of the structure response. The regulation in [N2] states: "5.1.1(4)P Analysis shall be carried out using idealisations of […] the behaviour of the structure. The idealisations selected shall be appropriate to the problem being considered". Therefore a proper method should be applied depending on the problem.

In the statically determinate systems, cross-sectional forces are not dependent on their material and geometric attributes (except for the loads implicated from the self-weight). In the statically indeterminate systems, these attributes already become significant. Differences in stiffness of members caused by cracking and nonlinear behaviour of concrete influence the distribution of internal forces in the element [3, 4]. Therefore, values of cross-sectional forces are, among other things, a function of the physical and geometrical attributes.

The advantage of nonlinear approach, as distinct from other advanced methods, e.g. plastic analysis [8], is a possibility of its application for defining both the ultimate limit states (ULS) and serviceability limit states (SLS) [16].

2.2. Relation of bending moment to curvature (M-κ)

The basic assumption of the stiffness nonlinear analysis of reinforced concrete members is the relationship of bending moment to curvature of the member. This relationship, which is based on experimental results, is strongly nonlinear (Fig. 1). The initial, linear character of the M-κ function (phase 1, concrete in cooperation with reinforcement works on its whole height) is saved until achieving the value of the cracking moment M_{cr} in section. First cracks are created after obtaining this value, and as a result, an abrupt variation of the curvature of the element occurs (from this moment, concrete in the cracked section is bearing only compression stresses – the reduction of local stiffness can be observed). The stiffness of the section is represented here by the function of an angle α (tg(α_I)) for phase 1, tg(α_{II}) and for phase 2, tg(α_i) – the stiffness of element after i-th crack. In this area, cracking will occur without any increment of the bending moment. The stiffness of the bent element is considerably reduced until it achieves a stabilized cracking [6]. After the pattern of cracks in the element stabilises, an increment of bending moment will cause an increase in the width of these cracks (the number of cracked sections will remain approximately constant) until it achieves the value of the maximum moment M_y. Then, a breakage of tensile steel reinforcement occurs (the stress in reinforcement is equal to f_y).

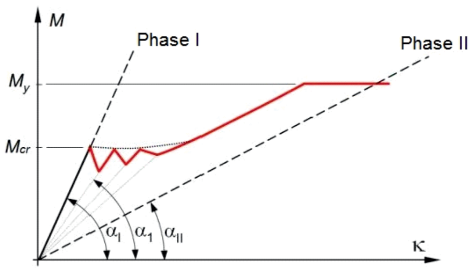

Fig. 1. Relation bending moment – curvature of the cross-section (M-κ) [6, 7, 17]

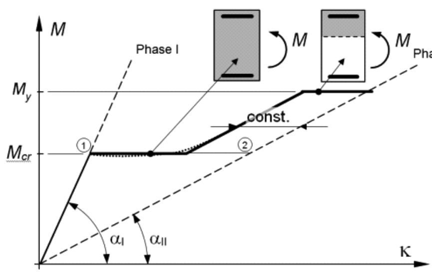

Fig. 2. Simplified relation M-κ [10]

A simplification of the M(κ) function is presented [9, 10]. It is based on replacing the abrupt changes of the curvature in the sections of unstabilised cracking by a horizontal line at the level of value of the cracking moment M_{cr} (Fig. 2). It is assumed here, in agreement with experimental data [5], that occurrence and development of cracks happen without an increase of the bending moment ($M = M_{cr}$). Concrete between the cracked sections cooperates with reinforcement bars by carrying tension stresses and causes strengthening of the areas between cracks (*tension stiffening*). The value of this strengthening (*dk*) in section is constant (it is assumed as 40% of the length 1–2 [10] – confer Fig. 2) and independent from the increment of bending moment (cracks spacing in elements is approximately constant) – confer [2, 5, 14] too.

2.3. Iterative evaluation of the cross-section stiffness

To define the proper values for the element (crack width, deflection), its stiffness shall be defined. It requires using iteration in the nonlinear stiffness method (confer Fig. 3).

The iteration here is done by assuming such value of the element curvature (κ_1) that the value of bending moment for this curvature (M_i) lays on the nonlinear function $M(\kappa)$. The number of iterations (n, on Fig. 3. $n = 4$) depends on the assumed length of the iteration step and accuracy (%) of a ratio of iterative moment M_i to the value of moment readout from the M-κ function, i.e. $\Delta M = M_i / [M_i - M(\kappa)]$).

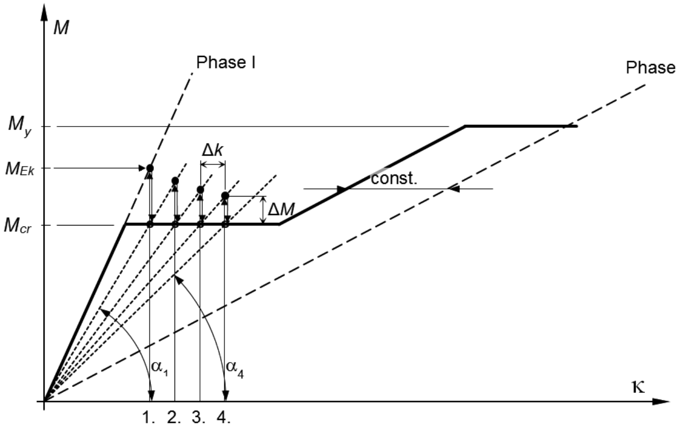

Fig. 3. Relation M-κ – iterative evaluation of stiffness of the cross-section

A block diagram illustrating an iterative procedure is presented below.

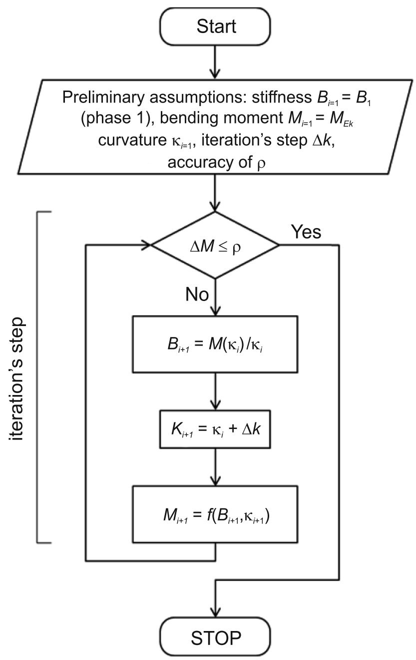

Fig. 4. Block diagram for iteration

3. Design example

As an example, a reinforced concrete two-span beam with a continuous load (cf. Fig. 5) was analyzed, where:
- geometry: $L_{eff} = 10.0$ m, $b = 0.5$ m, $h = 1.0$ m,
- material: concrete C30/37, reinforcement B500St: $A_{s1} = 30$ cm², $A_{s2} = 20$ cm²,
- bending moment in the "B" support $M_{Ek.B} = 500$ kNm.

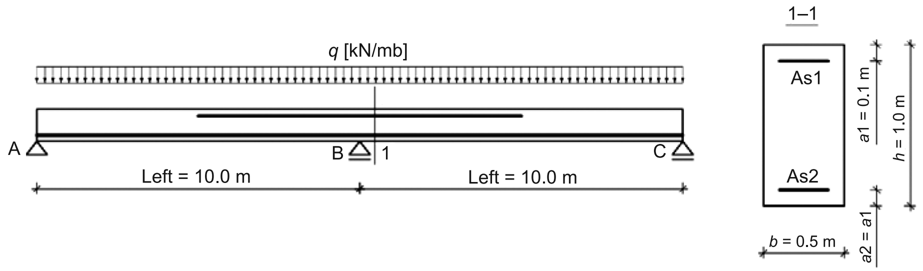

Fig. 5. Two-span beam – geometry and design scheme

It is assumed that:
- step of iteration $\Delta k = 20 \cdot 10^{-4}$%/m,
- relative error of iteration $\rho = 5\%$ (12.1 kNm),
- the cracking moment $M_{cr} = 242$ kNm,
- stiffness of the cross-section $B_{I} = 1.522 \cdot 10^{6}$ kNm² (phase I) and $B_{II} = 0.331 \cdot 10^{6}$ kNm² (phase II).

After execution of $n = 7$ steps of iteration (Fig. 6), the relative error of iteration is equal to $\rho = 4.8\%$, which gives values of bending moment $M_{i=7} = 253.6$ kNm, curvature $\kappa_{i=7} = 436.12 \cdot 10^{-4}$%/m and revised stiffness of the analysed section $B_{i=7} = 0.485$ kNm² (30.18°). The results of the subsequent iteration steps are presented in Table 1. For comparison – stiffness of the same cross-section defined according to EC2 procedure [N2] is equal to $B_{EC2} = 0.405 \cdot 10^{6}$ kNm², where relative error of stiffness $B_{i=7}$ and B_{EC2} is equal to 19.8%.

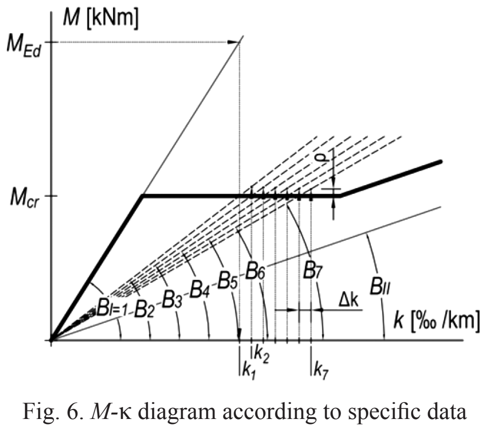

Fig. 6. M-κ diagram according to specific data

Table 1

Results of subsequent iteration steps

Iteration's step (i)	B_i [10^6 kNm^2]	κ_i [10^{-4}%/m]	M_i [kNm]	ρ_i
1.	1.484	316.12	500.0	106.9%
2.	0.579	336.12	257.4	6.5%
3.	0.558	356.12	256.4	6.1%
4.	0.539	376.12	255.6	5.8%
5.	0.519	396.12	254.9	5.5%
6.	0.502	416.12	254.2	5.2%
7.	**0.485**	**436.12**	**253.6**	**4.8%**

The results in Table 1 present speed of iterative convergence in the subsequent steps. This example shows that just after the first step of iteration, the value of an iteration error is approximately $\rho_{i=1}$ = 7%, which should be acknowledged as a satisfactory result. The next iterative steps ($i \geq 2$) reveal moderate convergence. This convergence however is not linear, but decreasing.

4. Conclusions

Nonlinear analysis in the stiffness aspect allows for the defining of the values of internal forces much more precisely in relevance to classical linear theory (here almost 20%). Their scope can be used to analyse problems which significantly exceed the scope of a linear approach (for example: the redistribution of internal forces, design for exciting forces). Additionally, such an aspect of nonlinear analysis might be used – opposite to plastic analysis – to define serviceability limit states (SLS) for the structure.

The presented nonlinear design model based on stiffness oriented design method gives an opportunity to assume any accuracy of values (controllable parameters: numbers of iterations and relative difference of cross-sectional forces). Just after the first step of iteration, the founded values allow for the forecasting of the final result. It is a legible and algorithmic method, suitable for common application in practice and for implementation in design software. Designing according to the nonlinear stiffness method increases reliability of the structures and rationality in forming their geometry.

References

[N1] PN-EN 1990:2002/A1:2008/NA:2010 Eurocode – Basis of structural design (in Polish).
[N2] PN-EN 1992-1-1:2008/Ap1:2010/NA:2010/AC:2011 Eurocode 2: Design of concrete structures – Part 1-1: General rules and rules for buildings (in Polish).
[1] Bathe K.-J. et al., *Nonlinear analysis of concrete structures*, Computers & Structures, Vol. 32, No. 3/4, 1989, 563-590.
[2] Bischoff P.H., *Effects of shrinkage on tension stiffening and cracking in reinforced concrete*, Canadian Journal of Civil Engineering, 28, 2001, 363-374.
[3] Czkwianianc A., Kamińska M., *Analysis of structure*[in:] Plain, reinforced and prestressed concrete structures. Scientific commentary for PN-B-03264:2002, t. 1, Building Research Institute, Warsaw 2005.
[4] Czkwianianc A., Kamińska M., *Method of nonlinear analysis of one-dimensional reinforced concrete memebers*, PAN – IPPT, No. 36, Warsaw 1993.
[5] Lee G.-Y., Kim W., *Cracking and Tension Stiffening Behavior of High-Strength Concrete Tension Members Subjected to Axial* Load, Advanced in Structural Engineering, Vol. 12, No. 2, 2009, 127-137
[6] Knauff M., *Calculation of reinforced concrete structures according to Eurocode 2*, PWN, Warsaw 2013.
[7] Koeberl B., Willam K., *Question of Tension Softening versus Tension Stiffeinig in Plain and Reinforced Concrete*, Journal of Engineering Mechanics, 2008, 804-808,
[8] Nielsen M.P., Hoang L.C., *Limit Analysis and Concrete Plasticity*, CRC Press – Taylor & Francis Group, 2011.
[9] Noakowski P., *Nachweisverfahren für Verankerung, Verformung, Zwangbeanspruchung und Rißbreite*, DAfStb, Heft 394, Beuth Verlag, Berlin 1988.
[10] Noakowski P., Schäfer H.G., *Steifigkeitsorientierte Statik im Stahlbetonbau*, Ernst&Sohn, Berlin 2003.
[11] Polak M.A., Vecchio F.J., *Nonlinear Analysis of Reinforced Concrete Shells*, Publication 93-03, University of Toronto, Department of Civil Engineering, Toronto 1993.
[12] Rombach G.A., *Finite element design of concrete structures*, Thomas Telford, London 2004.
[13] Vecchio F.J., *Non-linear finite element analysis of reinforced concrete at the crossroads?*, Structural Concrete, 2, No. 4, 2001, 201-212,
[14] Whittle R., Jones T., *Technical Report No. 59: Influence of tension stiffeinig on deflection of reinforced concrete structures*, The Concrete Society, Camberley, 2004, http://www.docin.com/p-282876679.html – dostęp: 21.06.2013.
[15] Winnicki A., Cichoń Cz., Waszczyszyn Z., *Nonlinear analysis of reinforced concrete bar members using FEM*, Archiwum Inżynierii Lądowej, 36, z. 1-2, 1990, 29-61.
[16] Wołowicki W., Garstecki A., *Static analysis of structure* [in:] *Basics of structural design of reinforced and prestressed concrete structures according to Eurocode 2*, DWE, Wroclaw 2006 (in Polish),
[17] Wu H. Q., Gilbert R. I., *Modeling short-term tension stiffening in reinforced concrete prisms using a continuum-based finite element model*, Engineering Structures 31, 2009, 2380-2391.