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ADAM BEDNARZ*, LUDWIK BYSZEWSKI*

ON ABSTRACT NONLOCAL CAUCHY PROBLEM

O NIELOKALNYM ABSTRAKCYJNYM ZAGADNIENIU CAUCHY'EGO

A b s t r a c t

In this paper, we investigate the existence and uniqueness of the classical solution to an abstract nonlocal Cauchy problem. For this purpose, we apply a notion of mild solution and the Banach contraction theorem.

Keywords: *abstract Cauchy problem*, *nonlocal conditions*, *mild and classical solution*

S t r e s z c z e n i e

W artykule zbadano istnienie i jednoznaczność klasycznego rozwiązania abstrakcyjnego nielokalnego zagadnienia Cauchy'ego. W tym celu zastosowano rozwiązanie całkowe i twierdzenie Banacha o kontrakcji.

Słowa kluczowe: *abstrakcyjne zagadnienie Cauchy'ego*, *warunki nielokalne*, *rozwiązania całkowe i klasyczne*

* Ph.D. Adam Bednarz, prof. Ludwik Byszewski, Institute of Mathematics, Faculty of Physics, Mathematics and Computer Science, Cracow University of Technology.

1. Introduction

We study the existence and uniqueness of the classical solution to a functional-differential abstract nonlocal Cauchy problem.

The functional-differential nonlocal problem considered in this paper, is of the form

$$u'(t) = f(t, u(t), u(a_1(t)), \ldots, u(a_r(t))), \quad t \in I, \tag{1.1}$$

$$u(t_0) + \sum_{k=1}^{p} c_k u(t_k) = x_0, \tag{1.2}$$

where $I := [t_0, t_0 + T]$, $t_0 < t_1 < \ldots < t_p \le t_0 + T$, $T > 0$; $f : I \times E^{r+1} \to E$ and $a_j : I \to I$ ($j = 1, \ldots, r$) are given functions satisfying suitable assumptions; E is a Banach space with norm $\|\cdot\|$, $x_0 \in E$, $c_k \neq 0$, ($k = 1, \ldots, p$) and $p, r \in \mathbb{N}$.

If $c_k \neq 0$, ($k = 1, \ldots, p$), then the results of the paper can be applied in kinematics to determine the evolution $t \to u(t)$ of the location of a physical object for which we do not know the positions $u(t_0)$, $u(t_1)$, …, $u(t_p)$, but we know that the nonlocal condition (1.2) holds.

The paper bases on books [3–4] and on papers [1–2].

2. Theorems about the existence and uniqueness of a classical solution

By X we denote the Banach space $C(I, E)$, where $I = [t_0, t_0 + T]$ with the standard norm $\|\cdot\|_X$. So

$$\|w\|_X := \sup_{t \in I} \|w(t)\|, \quad w \in X.$$

Assume that $\sum_{k=1}^{p} c_k \neq -1$. A function $u \in X$, satisfying the integral equation

$$\begin{aligned} u(t) = \breve{c} \left[x_0 - \sum_{k=1}^{p} c_k \int_{t_0}^{t_k} f(\tau, u(\tau), u(a_1(\tau)), \ldots, u(a_r(\tau)))\, d\tau \right] + \\ + \int_{t_0}^{t} f(\tau, u(\tau), u(a_1(\tau)), \ldots, u(a_r(\tau)))\, d\tau, \quad t \in I, \end{aligned} \tag{2.1}$$

where $\breve{c} \overset{df}{=} \left(1 + \sum_{k=1}^{p} c_k \right)^{-1}$, is said to be a **mild solution** of the nonlocal problem (1.1) – (1.2).

A function $u : I \to E$ is said to be a **classical solution** of the nonlocal problem (1.1)–(1.2) if

(i) u is continuous on I and continuously differentiable on I,

(ii) $u'(t) = f(t, u(t), u(a_1(t)), \ldots, u(a_r(t)))$ for $t \in I$,

(iii) $u(t_0) + \sum_{k=1}^{p} c_k u(t_k) = x_0$.

Theorem 2.1. Suppose that $f : I \times E^{r+1} \to E$, $a_j : I \to I$ ($j = 1, \ldots, r$) and $\sum_{t \in I}^{p} c_k \neq -1$. If u is a classical solution of the nonlocal problem (1.1)–(1.2), then u is a mild solution of this problem.

Proof. Let u be a classical solution of the nonlocal problem (1.1)–(1.2). Then u satisfies equation (1.1) and consequently,

$$u(t) = u(t_0) + \int_{t_0}^{t} f(\tau, u(\tau), u(a_1(\tau)), \ldots, u(a_r(\tau)))\, d\tau, \quad t \in I. \tag{2.2}$$

From (2.2),

$$u(t_k) = u(t_0) + \int_{t_0}^{t_k} f(\tau, u(\tau), u(a_1(\tau)), \ldots, u(a_r(\tau)))\, d\tau, \quad (k = 1, \ldots, p). \tag{2.3}$$

By (1.2) and (2.3),

$$u(t_0) + \sum_{k=1}^{p} c_k \left[u(t_0) + \int_{t_0}^{t_k} f(\tau, u(\tau), u(a_1(\tau)), \ldots, u(a_r(\tau)))\, d\tau \right] = x_0. \tag{2.4}$$

Since $\sum_{t \in I}^{p} c_k \neq -1$, then (2.4) implies

$$u(t_0) = \breve{c} \left[x_0 - \sum_{k=1}^{p} c_k \int_{t_0}^{t_k} f(\tau, u(\tau), u(a_1(\tau)), \ldots, u(a_r(\tau)))\, d\tau \right]. \tag{2.5}$$

From (2.2) and (2.5), we obtain that u is a mild solution of the nonlocal problem (1.1)–(1.2). The proof of Theorem 2.1 is complete.

Theorem 2.2. Suppose that $f \in C(I \times E^{r+1})$, $a_j : I \to I$ ($j = 1, \ldots, r$) and $\sum_{k=1}^{p} c_k \neq -1$. If u is a mild solution of the nonlocal problem (1.1)–(1.2), then u is a classical solution of this problem.

Proof. Let u be a mild solution of the nonlocal problem (1.1)–(1.2). Then u satisfies equation (1.1) and, from the continuity of f, $u \in C^1(I, E)$. Now we will show that u satisfies the nonlocal condition (1.2). For this purpose, observe that by (2.1),

$$u(t_0) = \breve{c}\left[x_0 - \sum_{k=1}^{p} c_k \int_{t_0}^{t_k} f(\tau, u(\tau), u(a_1(\tau)), \ldots, u(a_r(\tau)))\,d\tau\right] \tag{2.6}$$

and

$$\begin{aligned} u(t_i) = \breve{c}&\left[x_0 - \sum_{k=1}^{p} c_k \int_{t_0}^{t_k} f(\tau, u(\tau), u(a_1(\tau)), \ldots, u(a_r(\tau)))\,d\tau\right] + \\ &+ \int_{t_0}^{t_i} f(\tau, u(\tau), u(a_1(\tau)), \ldots, u(a_r(\tau)))\,d\tau \quad (i = 1, \ldots, p). \end{aligned} \tag{2.7}$$

From (2.6) and (2.7), and from some computations,

$$\begin{aligned} u(t_0) + \sum_{i=1}^{p} c_i u(t_i) = &\left(x_0 - \sum_{k=1}^{p} c_k \int_{t_0}^{t_k} f(\tau, u(\tau), u(a_1(\tau)), \ldots, u(a_r(\tau)))\,d\tau\right) + \\ &+ \sum_{i=1}^{p} c_i \int_{t_0}^{t_i} f(\tau, u(\tau), u(a_1(\tau)), \ldots, u(a_r(\tau)))\,d\tau = x_0. \end{aligned}$$

Therefore, the proof of Theorem 2.2 is complete.

As a consequence of Theorems 2.1 and 2.2, we obtain:

Theorem 2.3. Suppose that $f \in C(I \times E^{r+1}, E)$, $a_j : I \to I$ $(j = 1, \ldots, r)$ and $\sum_{k=1}^{p} c_k \neq -1$. Then u is the unique classical solution to the nonlocal problem (1.1)–(1.2) if, and only if, u is the unique mild solution to this problem.

Now, we will prove the main theorem of the paper.

Theorem 2.4. Assume that:

(i) $a_j \in C(I, I)$ $(j = 1, \ldots, r)$, $f : I \times E^{r+1} \to E$ is continuous with respect to the first variable on I and there is $L > 0$ such that

$$\|f(s, z_1, \ldots, z_{r+1}) - f(s, \tilde{z}_1, \ldots, \tilde{z}_{r+1})\| \le L \sum_{i=1}^{r+1} \|z_i - \tilde{z}_i\| \quad \text{for} \quad s \in I,\ z_i, \tilde{z}_i \in E\ \ (i = 1, \ldots, r+1), \tag{2.8}$$

(ii) $\sum_{k=1}^{p} c_k \neq -1$

(iii) $(r+1)LT\left(1 + \left|\breve{c}\sum_{k=1}^{p} c_k\right|\right) < 1.$

Then the nonlocal Cauchy problem (1.1)–(1.2) has a unique classical solution u. Moreover, the successive approximations u_n ($n = 0, 1, 2, \ldots$), defined by the formulas

$$u_0(t) = x_0 \qquad \text{for} \qquad t \in I \tag{2.9}$$

and

$$\begin{aligned} u_{n+1}(t) &:= \breve{c}\left[x_0 - \sum_{k=1}^{p} c_k \int_{t_0}^{t_k} f(\tau, u_n(\tau), u_n(a_1(\tau)), \ldots, u_n(a_r(\tau)))\, d\tau \right] + \\ &+ \int_{t_0}^{t_i} f(\tau, u_n(\tau), u_n(a_1(\tau)), \ldots, u_n(a_r(\tau)))\, d\tau \quad \textit{for} \quad t \in I \quad (n = 0,1,2,\ldots), \end{aligned} \tag{2.10}$$

converge uniformly on I to the unique classical solution u.

Proof. Introduce an operator A by the formula

$$\begin{aligned} (Aw)(t) &:= \breve{c}\left[x_0 - \sum_{k=1}^{p} c_k \int_{t_0}^{t_i} f(\tau, w(\tau), w(a_1(\tau)), \ldots, w(a_r(\tau)))\, d\tau \right] + \\ &+ \int_{t_0}^{t} f(\tau, w(\tau), w(a_1(\tau)), \ldots, w(a_r(\tau)))\, d\tau, \quad w \in X, \quad t \in I. \end{aligned} \tag{2.11}$$

It is easy to see that

$$A : X \to X. \tag{2.12}$$

Now, we will show that A is a contraction on X. For this purpose observe that

$$\begin{aligned} &(Aw)(t) - (A\tilde{w})(t) := \\ &= -\breve{c}\sum_{k=1}^{p} c_k \int_{t_0}^{t_k} [f(\tau, w(\tau), w(a_1(\tau)), \ldots, w(a_r(\tau))) - f(\tau, \tilde{w}(\tau), \tilde{w}(a_1(\tau)), \ldots, \tilde{w}(a_r(\tau)))]\, d\tau + \\ &+ \int_{t_0}^{t} [f(\tau, w(\tau), w(a_1(\tau)), \ldots, w(a_r(\tau))) - f(\tau, \tilde{w}(\tau), \tilde{w}(a_1(\tau)), \ldots, \tilde{w}(a_r(\tau)))]\, d\tau, \; w, \tilde{w} \in X, \; t \in I. \end{aligned} \tag{2.13}$$

From (2.13) and (2.8)

$$\|(Aw)(t) - (A\tilde{w})(t)\| \le (r+1)LT\left(1 + \left|\breve{c}\sum_{k=1}^{p} c_k\right|\right)\|w - \tilde{w}\|_X, \quad w, \tilde{w} \in X, \; t \in I. \tag{2.14}$$

Let

$$q := (r+1)LT\left(1 + \left|\breve{c}\sum_{k=1}^{p} c_k\right|\right). \tag{2.15}$$

Then, by (2.14), (2.15) and assumption (iii),

$$\|Aw - A\tilde{w}\|_X \le q\|w - \tilde{w}\|_X \quad \text{for} \quad w, \tilde{w} \in X \tag{2.16}$$

with $0 < q < 1$.

Consequently, by (2.12) and (2.16), operator A satisfies all the assumptions of the Banach contraction theorem. Therefore, in space X there is only one fixed point u of A and this point is the mild solution to the nonlocal problem (1.1)–(1.2). Consequently, from Theorem 2.3, u is the unique classical solution to the nonlocal problem (1.1)–(1.2).

Now, we will prove the second part of the thesis of Theorem 2.4. To this end, observe that by (2.10) and (2.9),

$$\begin{aligned}\|u_1 - u_0\|_X &= \sup_{t\in I}\|u_1(t) - u_0(t)\| \le \left\| -\breve{c}\sum_{k=1}^{p} c_k \int_{t_0}^{t_k} f(\tau, u_0(\tau), u_0(a_1(\tau)), \ldots, u_0(a_r(\tau)))\, d\tau \right\| + \\ &+ \sup_{t\in I}\left\| \int_{t_0}^{t} f(\tau, u_0(\tau), u_0(a_1(\tau)), \ldots, u_0(a_r(\tau)))\, d\tau \right\| \le MT\left(1 + \left|\breve{c}\sum_{k=1}^{p} c_k\right|\right),\end{aligned} \tag{2.17}$$

where

$$M := \sup\left\{\|f(\tau, w(\tau), w(a_1(\tau)), \ldots, w(a_r(\tau)))\| : w \in X, \tau \in I\right\}.$$

Next, assume that

$$\|u_n - u_{n-1}\|_X \le MT\left(1 + \left|\breve{c}\sum_{k=1}^{p} c_k\right|\right) \cdot \left[(r+1)LT\left(1 + \left|\breve{c}\sum_{k=1}^{p} c_k\right|\right)\right]^{n-1} \tag{2.18}$$

for some natural $n \ge 2$.

Then, by (2.10), (2.9), (2.8) and (2.18),

$$\begin{aligned}&\|u_{n+1} - u_n\|_X = \sup_{t\in I}\|u_{n+1}(t) - u_n(t)\| = \\ &= \left\| -\breve{c}\sum_{k=1}^{p} c_k \int_{t_0}^{t_k} [f(\tau, u_n(\tau), u_n(a_1(\tau)), \ldots, u_n(a_r(\tau))) - f(\tau, u_{n-1}(\tau), u_{n-1}(a_1(\tau)), \ldots, u_{n-1}(a_r(\tau)))]\, d\tau \right\| + \\ &+ \sup_{t\in I}\left\| \int_{t_0}^{t} [f(\tau, u_n(\tau), u_n(a_1(\tau)), \ldots, u_n(a_r(\tau))) - f(\tau, u_{n-1}(\tau), u_{n-1}(a_1(\tau)), \ldots, u_{n-1}(a_r(\tau)))]\, d\tau \right\| \le \\ &\le (r+1)LT\left(1 + \left|\breve{c}\sum_{k=1}^{p} c_k\right|\right)\|u_n - u_{n+1}\|_X \le MT\left(1 + \left|\breve{c}\sum_{k=1}^{p} c_k\right|\right) \cdot \left[(r+1)LT\left(1 + \left|\breve{c}\sum_{k=1}^{p} c_k\right|\right)\right]^n.\end{aligned} \tag{2.19}$$

Therefore, from (2.17), (2.18), (2.19), and from induction argument,

$$\|u_n - u_{n-1}\|_X \leq MT\left(1+\left|\breve{c}\sum_{k=1}^{p} c_k\right|\right)\cdot\left[(r+1)LT\left(1+\left|\breve{c}\sum_{k=1}^{p} c_k\right|\right)\right]^{n-1} \tag{2.20}$$

for all $n = 1, 2, \ldots$

Inequalities (2.20) and assumption (iii) imply, by the Weierstrass theorem, the uniform convergence of the series

$$u_1 + \sum_{n=1}^{\infty}(u_{n+1} - u_n)$$

on the integral I and consequently, the uniform convergence of the sequence u_n on I.

Let

$$u_*(\mathrm{t}) := \lim_{n\to\infty} u_n(t) \quad \text{for} \quad t \in I.$$

Since u_n tends uniformly to u_* on I then, by (2.9), (2.10) and (2.8), u_* is a classical solution to the nonlocal problem (1.1)–(1.2) on I. But, from the first part of the thesis of Theorem 2.4, we know that there exists only one classical solution u to the nonlocal problem (1.1)–(1.2) on I. So, $u_* = u$ on I.

The proof of Theorem 2.4 is complete.

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