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THE NON-EXISTENCE OF THE FEJER–RIESZ TYPE RESULT FOR SOME WEIGHTED BERGMAN SPACES IN THE UNIT DISC

O NIEISTNIENIU PEWNYCH OSZACOWAŃ TYPU FEJERA–RIESZA W PRZESTRZENIACH BERGMANA Z WAGĄ W KOLE JEDNOSTKOWYM

A b s t r a c t

In this note, we consider the analogues of the classical Fejer–Riesz inequality for some weighted Hilbert spaces of analytic functions in the unit disc. We prove that for some class of such spaces, the Fejer-Riesz inequality type results do not hold.

Keywords: *Fejer-Riesz inequality, Bergman spaces of analytic functions*

S t r e s z c z e n i e

W artykule rozważa się nierówności podobne do klasycznej nierówności Fejera–Riesza w przestrzeniach Hilberta funkcji analitycznych z wagą. Dowodzi się, że w pewnych klasach takich przestrzeni nie zachodzi odpowiednik nierówności Fejera–Riesza.

Słowa kluczowe: *nierówność Fejera–Riesza, przestrzenie Bergmana funkcji analitycznych*

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1. Introduction

Let U be the unit disc in $\mathbb{C}$. For functions in the space $H^2(U)$ the following well-known Fejer-Riesz inequality holds (see e.g. [3], p. 46):

If $f \in H^2(U)$, and f^* denotes the radial boundary values of f on ∂U, f^* being defined a.e. on ∂U and L^2-integrable with respect to the linear Lebesgue measure on ∂U, then

$$\int_{-1}^{1} |f(x)|^2 dx \le \frac{1}{2}\int_0^{2\pi} \left|f^*(e^{i\theta})\right|^2 d\theta. \tag{1}$$

It follows from this inequality that in particular for every $f \in H^2(U)$ and every $z \in \partial U$,

$$\int_0^1 |f(tz)|^2 \, dt < +\infty. \tag{2}$$

(One should mention that the inequality (1) with 2 replaced by p also holds for all H^p-spaces with $1 \le p < +\infty$).

The space $H^2(U)$ can be viewed as one of some family of weighted Hilbert spaces of analytic functions in the unit disc in $\mathbb{C}$; this family can be described as follows:

Given $s > -1$, set

$$A^{2,s}(U) = \left\{ f \text{ holomorphic in } U : \int_U |f(z)|^2 (1-|z|^2)^s \, dm(z) < +\infty \right\}, \tag{3}$$

where m is planar Lebesgue measure in U. Such spaces, also called weighted Bergman spaces, were considered by many authors; see e.g. [1, 2, 7, 8, 9].

If f is holomorphic in U, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in U$, and $s > -1$, then one can prove by integrating in polar coordinates that

$$f \in A^{2,s}(U) \text{ iff } \sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{1+s}} < +\infty. \tag{4}$$

E.g. for $s = 0$ we obtain $A^{2,0}(U) = L^2H(U)$, the so called Bergman space of all holomorphic functions in U with

$$\int_U |f(z)|^2 \, dm(z) < +\infty. \tag{5}$$

The condition (4) for $s = 0$ is

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} < +\infty.$$

For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ holomorphic in U we have

$$\int_U |f(z)|^2 (1-|z|^2)^s \, dm(z) = \sum_{n=0}^{\infty} |a_n|^2 \int_U |z|^{2n} (1-|z|^2)^s \, dm(z) = 2\pi \sum_{n=0}^{\infty} |a_n|^2 \int_0^1 t^{2n+1}(1-t^2)^s \, dt.$$

If $s \leq -1$, we have for $n = 0, 1, 2, \ldots$

$$\int_0^1 t^{2n+1}(1-t^2)^s\,dt = +\infty.$$

Hence for $s \leq -1$ the integral condition (3) gives the space consisting only of the zero function. But the series condition (4) yields non-zero Hilbert spaces of holomorphic functions in U. Therefore we set for $s \leq -1$

$$A^{2,s}(U) = \left\{ f \text{ holomorphic in } U : \sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{1+s}} < +\infty \right\}. \tag{6}$$

Such definition of the space $A^{2,s}(U)$ for $s \leq -1$ seems to be correct also by the fact that for $s = -1$ we obtain from (4) or (6) the condition $\sum_{n=0}^{\infty} |a_n|^2 < +\infty$, which is the well--known condition for f to belong to the space $H^2(U)$; therefore the definition (6) gives $A^{2,-1}(U) = H^2(U)$.

Having placed the space $H^2(U)$ as $A^{2,-1}(U)$ in the above described family of spaces $A^{2,s}(U)$, $s \in \mathbb{R}$, we could ask whether the Fejer-Riesz inequality (1) valid for $H^2(U) = A^{2,-1}(U)$ also has analogues for the other spaces $A^{2,s}(U)$, $s \in \mathbb{R}$.

In [5] we have proved the result similar to (2) for the spaces $A^{2,s}(U)$ with $s > 0$:

Proposition 1. ([5], *Theorem 1*). *Let s be a positive number. Suppose that $f \in A^{2,s-1}(U)$. Then for every $z \in \partial U$.*

$$\int_0^1 |f(tz)|^2 (1-t^2)^s\,dt < +\infty. \tag{7}$$

As was already mentioned above, for $s \leq -1$ the spaces $A^{2,s}(U)$ are defined by the series condition (6). Note that for $s > -1$ the condition (7) makes sense. Hence one can try to prove for s with $-2 < s \leq -1$ the result similar to that in Proposition 1:

If s is a number with $-2 < s \leq -1$, and $f \in A^{2,s}(U)$, i.e. if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic in U, it satisfies also according to (6)

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{1+s}} < +\infty, \tag{8}$$

then for every $z \in \partial U$,

$$\int_0^1 |f(tz)|^2 (1-t^2)^{s+1}\,dt < +\infty. \tag{9}$$

In [6] we have proved only a weakened version of the aforementioned result; it is described in [6], conditions (10) and (11). For the convenience of the reader we recall it here.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfy (8). Let $\{b_k\}_{k=0}^{\infty}$ be a new sequence which is obtained from $\{a_n\}_{n=0}^{\infty}$ in such a way that we delete all numbers a_n with $a_n = 0$ and then reorder the remaining numbers a_n to obtain a new sequence $\{a'_k\}_{k=0}^{\infty}$ with $|a'_0| \geq |a'_1| \geq \ldots;$ we define then $b_k = |a'_k|,\ k = 0,1,\ldots$. We can then also prove that

$$\sum_{n=0}^{\infty} \frac{b_n^2}{(n+1)^{1+s}} < \infty.$$

The additional condition which we assume is as follows:

$$\text{The sequence } \left(\frac{b_n^2}{(n+1)^{1+s}}\right)_{n=0}^{\infty} \text{ is decreasing.} \tag{10}$$

We have proved in [6]:

Proposition 2. ([6], *Proposition 2*). *Suppose that* $-2 < s \leq -1$. *Let the function f, holomorphic in U,* $f(z) = \sum_{n=0}^{\infty} a_n z^n$, *satisfy (8) (i.e.* $f \in A^{2,s}(U)$*). Suppose also that the condition (10) holds. Then for every* $z \in \partial U$

$$\int_0^1 |f(tz)|^2 (1-t^2)^{s+1}\, dt < \infty.$$

As mentioned above, we are still not able to prove Proposition 2 without assuming (10), although this conditions seems to be superfluous.

Consider now the spaces $A^{2,s}(U)$ with $s \leq -2$. As mentioned above, in this case $A^{2,s}(U)$ is defined by the series condition (6). Moreover, if $f \in A^{2,s}(U)$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and the coefficients $\{a_n\}_{n=0}^{\infty}$ are non-negative, then $f(t)$ is bound away from zero say for $\frac{1}{2} < t < 1$, and so the integral condition (9) holds only for f equal zero. On the other hand, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in U, $f \in A^{2,s}(U)$ with $s > -2$, and the coefficients a_n are non-negative, then as explained in [6], the expression

$$\int_0^1 |f(tz)|^2 (1-t^2)^{s+1}\, dt$$

is estimated from below and from above by a constant time the sum of the double series

$$\sum_{k,l=0}^{\infty} a_k a_l \frac{1}{(1+k+l)^{s+2}}.$$

Hence we have assumed in [6] that for $s \leq -2$, the right analogue of the Fejer-Riesz type results, described in Propositions 1 and 2, would be the following:

If f is holomorphic in U, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and $f \in A^{2,s}(U)$, i.e.

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{1+s}} < +\infty, \tag{11}$$

then one should have that

$$\sum_{k,l=0}^{\infty} |a_k||a_l| \frac{1}{(1+k+l)^{s+2}} < +\infty. \tag{12}$$

As observed in [6], such a result is not true. Namely, we have proved the following:

Proposition 3. ([6], *Proposition 4*). *Let s be a real number with s ≤ –2. Then there exists a holomorphic function f such that* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *with* a_n *satisfying (11), but yielding divergent series (12).*

The example of such a function, which therefore does not satisfy the Fejer-Riesz type result for $s \le -2$, is the function $f(z) = \sum_{n=2}^{\infty} a_n z^n$ with

$$a_n = \frac{1}{(n+1)^{-s/2} \log n}, \quad n = 2, 3, \ldots. \tag{13}$$

In the present note we show that the same function does not satisfy the Fejer-Riesz type results mentioned above, in some sharper sense; we prove namely.

Proposition 4. *Let s < –2 be given. Let the function* $f(z) = \sum_{n=2}^{\infty} a_n z^n$ *be defined by (13). Then*

$$\sum_{n=2}^{\infty} \frac{|a_n|^2}{(n+1)^{1+s}} < +\infty, \tag{14}$$

but for every σ with $s \le \sigma \le \frac{s}{2} - 1$ *the series*

$$\sum_{k,l=2}^{\infty} |a_k||a_l| \frac{1}{(1+k+l)^{\sigma+2}} \tag{15}$$

is divergent.

If $s \le -2$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic in U, and the series $\sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{1+s}} < +\infty$, but the series $\sum_{k,l=0}^{\infty} |a_k||a_l| \frac{1}{(1+k+l)^{\sigma+2}}$ is divergent for some $\sigma \ge s$, then the value $\sigma = \frac{s}{2} - 1$ is the largest possible; we have the following result:

Proposition 5. *If s is a real number with* $s \le -2$, *and* $f \in A^{2,s}(U)$, $f(z)=\sum_{n=0}^{\infty} a_n z^n$, *then for every* $\sigma > \frac{s}{2}-1$ *the series*

$$\sum_{k,l=0}^{\infty} |a_k||a_l| \frac{1}{(1+k+l)^{\sigma+2}} < +\infty. \tag{16}$$

Proof of Proposition 4. Fix $s < -2$. Note that we then have $s < \frac{s}{2}-1 < -2$; therefore the condition $s \le \sigma \le \frac{s}{2}-1$ makes sense. With s as above let f be defined by (13), i.e.

$$f(z)=\sum_{n=2}^{\infty} \frac{1}{(n+1)^{-s/2}\log n} z^n. \tag{17}$$

We have, for $n \ge 2$,

$$\sqrt[n]{\frac{1}{(n+1)^{-s/2}\log n}} = \frac{1}{(\sqrt[n]{n+1})^{-s/2}\sqrt[n]{\log n}}$$

and this last sequence converges to 1; so the series in the right-hand side of (17) is convergent for every $z \in U$, and hence f is holomorphic in U.

The facts that for $s < -2$ and $a_n = \frac{1}{(n+1)^{-s/2}\log n}$ the series (14) converges, but the series (15) with $\sigma = s$ is divergent were already proved in [6], Proposition 4. Therefore, let $s < \sigma \le \frac{s}{2}-1$. Then $\sigma = s + \varepsilon$ with some $0 < \varepsilon \le -1-\frac{s}{2}$. We have

$$\sum_{k,l=2}^{\infty} |a_k||a_l| \frac{1}{(1+k+l)^{\sigma+2}} = \sum_{k,l=2}^{\infty} \frac{1}{(1+k+l)^{s+\varepsilon+2}(k+1)^{-s/2}\log k(l+1)^{-s/2}\log l}. \tag{18}$$

Consider the subseries of the series in the right-hand side of (18), consisting of terms for which $k = 2, 3, \ldots$ is arbitrary and $l = 2$. In this way we obtain the series

$$\sum_{k=2}^{\infty} \frac{1}{(k+3)^{s+\varepsilon+2}(k+1)^{-s/2}\log k 3^{-s/2}\log 2}.$$

We easily see that the terms in this last series are estimated from below by a constant time of

$$\sum_{k=2}^{\infty} \frac{1}{k^{s+\varepsilon+2}k^{-s/2}\log k} = \sum_{k=2}^{\infty} \frac{1}{k^{s/2+\varepsilon+2}\log k}. \tag{19}$$

Since $0 < \varepsilon \le -1 - \frac{s}{2}$, then $\frac{s}{2} + \varepsilon + 2 \le 1$. Therefore, the series in the right-hand side of (19), as well as the series in (18), i.e. the series in (15), diverge.

Proof of Proposition 5. We have, by Hölder's inequality,

$$\sum_{k,l=0}^{\infty} |a_k||a_l| \frac{1}{(1+k+l)^{\sigma+2}} = \sum_{k,l=0}^{\infty} \frac{|a_k||a_l|}{(k+1)^{(s+1)/2}(l+1)^{(s+1)/2}} \frac{(k+1)^{(s+1)/2}(l+1)^{(s+1)/2}}{(1+k+l)^{\sigma+2}} \le$$
$$\le \left(\sum_{k,l=0}^{\infty} \frac{|a_k|^2 |a_l|^2}{(k+1)^{s+1}(l+1)^{s+1}} \right)^{1/2} \left(\sum_{k,l=0}^{\infty} \frac{(k+1)^{s+1}(l+1)^{s+1}}{(1+k+l)^{2(\sigma+2)}} \right)^{1/2}. \tag{20}$$

Note that since $f \in A^{2,s}(U)$, then by (6)

$$\sum_{k,l=0}^{\infty} \frac{|a_k|^2 |a_l|^2}{(k+1)^{s+1}(l+1)^{s+1}} = \left(\sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{s+1}} \right)^2 < +\infty.$$

Consider the second series in the right-hand side of (20). This series has the same behavior as the series

$$\sum_{k,l=0,k+l>0}^{\infty} \frac{k^{s+1} l^{s+1}}{(k+l)^{2(\sigma+2)}}. \tag{21}$$

The terms of this last series are decreasing with respect to product order, so we can apply to the series (21) the so-called Cauchy's concentration principle (for double series). It follows from this that in order to verify that the series (21) is convergent, it is sufficient to prove that the series

$$\sum_{r,t=1}^{\infty} \frac{2^r 2^t}{(2^r + 2^t)^{2(\sigma+2)}} 2^{r(s+1)} 2^{t(s+1)} \tag{22}$$

is convergent. We have

$$\sum_{r,t=1}^{\infty} \frac{2^r 2^t}{(2^r + 2^t)^{2(\sigma+2)}} 2^{r(s+1)} 2^{t(s+1)} = \sum_{r,t=1}^{\infty} \frac{2^r 2^{r+p}}{(2^r + 2^{r+p})^{2(\sigma+2)}} 2^{r(s+1)} 2^{(r+p)(s+1)} =$$
$$= \sum_{r=1}^{\infty} \frac{2^{2r+2r(s+1)}}{(2^r)^{2(\sigma+2)}} \sum_{p=1}^{\infty} \frac{2^{p+p(s+1)}}{(1+2^p)^{2(\sigma+2)}}. \tag{23}$$

The terms of the second series in the right-hand side of (23) are estimated from above by

$$\frac{2^{p+p(s+1)}}{(2^p)^{2(\sigma+2)}}.$$

Therefore, the right-hand side of (23) is estimated from above by

$$\sum_{r=1}^{\infty} 2^{2r+2r(s+1)-2r(\sigma+2)} \sum_{p=1}^{\infty} 2^{p+p(s+1)-2p\sigma-4p} = \sum_{r=1}^{\infty} \left(\frac{1}{2^{2(\sigma-s)}} \right)^r \sum_{p=1}^{\infty} \left(\frac{1}{2^{2+2\sigma-s}} \right)^p. \tag{24}$$

Since $\sigma > \frac{s}{2} - 1 > s$, then $2 + 2\sigma - s > 0$, and $\sigma - s > 0$. Therefore both series in (24) are convergent. Hence the series in (22), as well as the second series in the right-hand side of (20) are convergent. This proves Proposition 5.

Note that if $\sigma \le \frac{s}{2} - 1$, then $2 + 2\sigma - s \le 0$, and the second series in the right-hand side of (24) diverges; therefore, we do not obtain by the above reasoning that the series in (16) is convergent for $\sigma \le \frac{s}{2} - 1$.

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The Referee also asked about the zero sets of the functions from the spaces $A^{2,s}(U)$, or more generally, $A^{p,s}(U)$, $p > 0$, for different values of s; in particular whether those zero sets depend on s. Up to now, we have not obtained the results in this direction, but it can be an interesting subject of further investigations.

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