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DISTRIBUTION-VALUED FUNCTIONS

FUNKCJE O WARTOŚCIACH DYSTRYBUCYJNYCH

Mathematics Subject Classification: 46F05, 46M40.

A b s t r a c t

We study continuity and differentiability of a distribution-valued function. It is understood in a strong sense due to inductive limit topology.

Keywords: *distribution-valued function, inductive limit topology*

S t r e s z c z e n i e

W artykule przedstawiono zagadnienie ciągłości i różniczkowalności funkcji o wartościach dystrybucyjnych. Pojęcia te są rozumiane w sensie silnym dzięki zastosowaniu topologii induktywnej.

Słowa kluczowe: *funkcja o wartościach dystrybucyjnych*, *topologia induktywna*

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1. Introduction

Distribution-valued functions are a natural and convenient tool in constructing linear mathematical models for many physics phenomena and solving differential equations. The spaces of distributions and tempered distributions can be treated as duals to the nuclear spaces: the space of test functions $\mathcal{D}(\Omega)$ and rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n)$ respectively, which are equipped with inductive limit topology [3, 6]. Usually, continuity and differentiability of a distribution-valued function is understood in a weak sense, that is a function $T : M \ni x \mapsto T(x) \in \mathcal{D}'(\Omega, B)$ is of class $\mathcal{C}^1$ if the map $M \ni x \mapsto T(x)\varphi \in B$ is of class $\mathcal{C}^1$ for any test function φ [1, 2, 7, 8].

In this paper, we consider distribution-valued functions which are continuous and differentiable in a strong sense due to inductive limit topology. We prove that if a distribution--valued function is differentiable in the sense of Definition 4.2, then it is differentiable in the weak sense (Corollary 4.4), the same refers to tempered distribution-valued functions (Remark 3.2, 3.5, Lemma 5.1). A similar approach was presented in [9] where summable in a strong sense distribution-valued functions were considered, and with the use of absolutely continuous distribution-valued functions, solutions to several Cauchy problems (Dirac equation, Navier-Lamé equation, biparabolic equation) were constructed. On the other hand, in [2] the parameter product of a distribution and a smooth (in a weak sense) distribution-valued function is introduced. This kind of product can be use in quantum electrodynamics, but also in modelling the vibration of a plate with piezoelectric actuators of an arbitrary shape [10].

2. Preliminaries

Let $\mathcal{S}$ be a locally convex space and let sn $\mathcal{S}$ denote the family of all continuouos seminorms on $\mathcal{S}$. Assume there is a decreasing sequence of convex balanced subsets of $\mathcal{S}$ that forms a local base in $\mathcal{S}$ and let q_m be the Minkowski functional of the m-th set of some fixed base of that kind. Then $(q_m)_{m\in\mathbb{N}}$ is a seperating family of continuous seminorms on $\mathcal{S}$ and introduces the same topology on $\mathcal{S}$ as sn $\mathcal{S}$ does.

Let B be a Banach space over a scalar field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) and $\mathcal{L}(\mathcal{S}, B)$ be the space of all $\mathbb{K}$-linear continuous mappings $\mathcal{S} \to B$. For every $p \in$ sn $\mathcal{S}$, let $\mathcal{L}((\mathcal{S}, p), B)$ be the space of all linear p-continuous mappings $\mathcal{S} \to B$. Let $T \in \mathcal{L}((\mathcal{S}, p), B)$ and denote

$$\left|T\right|_p = \sup_{p(\varphi)\le 1} \left|T(\varphi)\right|. \tag{1}$$

It is well known that $\mathcal{L}((\mathcal{S}, p), B)$ with the norm (1) is a Banach space. Moreover

(i) if $p, q \in$ sn $\mathcal{S}$ and $p \le q$ then $\left|\cdot\right|_p \ge \left|\cdot\right|_q$ and the canonical injection $\mathcal{L}((\mathcal{S}, p), B) \hookrightarrow$ $\hookrightarrow \mathcal{L}((\mathcal{S}, q), B)$ is continuous,

(ii) $\mathcal{L}(\mathcal{S}, B) = \bigcup_{p\in \mathrm{sn}\,\mathcal{S}} \mathcal{L}(\mathcal{S}, p), B) = \bigcup_{m=1}^{\infty} \mathcal{L}(\mathcal{S}, q_m), B).$

We consider the space $\mathcal{L}(\mathcal{S}, B)$ endowed with the inductive limit topology with respect to the family of canonical injections

$$\mathcal{L}((\mathcal{S}, q), B) \hookrightarrow \mathcal{L}(\mathcal{S}, B) \tag{2}$$

for all $q \in \operatorname{sn} \mathcal{S}$, that is the finest locally convex topology on $\mathcal{L}(\mathcal{S}, B)$ such that all the mappings (2) are continuous. This topology is also determined by the smaller family of inclusions,

$$\mathcal{L}((\mathcal{S}, q_m), B) \hookrightarrow \mathcal{L}(\mathcal{S}, B)$$

for $m \in \mathbb{N}$.

Example 2.1. *Test functions and distributions. Let* $\mathcal{D}(\Omega)$ denote the linear space of infinitely differentiable functions with compact supports which map $\Omega \in \operatorname{top} \mathbb{R}^n$ into $\mathbb{K}$ and let $\mathbb{K} \subset \Omega$ be compact. Then $\mathcal{D}_K(\Omega)$ denotes the subspace of $\mathcal{D}(\Omega)$ which consists of functions with supports in K. Each $\mathcal{D}_K(\Omega)$ is a Fréchet space with the family of seminorms $(q_m)_{m=0}^{\infty}$,

$$q_m(\varphi) = \sup_{x\in\Omega} \sup_{|\alpha|\le m} \left|D^\alpha \varphi(x)\right|$$

for all $\varphi \in \mathcal{D}_K(\Omega)$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$. The family of maps $\mathcal{D}_K(\Omega) \hookrightarrow \mathcal{D}(\Omega)$, for all compact $K \subset \Omega$, introduces the inductive limit topology in $\mathcal{D}(\Omega)$. The same topology can be obtained using the family $\mathcal{D}_{K_\nu}(\Omega) \hookrightarrow \mathcal{D}(\Omega)$, for an arbitrary sequence (K_ν) such that $\bigcup_{\nu=1}^{\infty} K_\nu = \Omega,\ K_\nu \subset \operatorname{int} K_{\nu+1}$. Moreover, for every compact $K \subset \Omega$ the topology that $\mathcal{D}_K(\Omega)$ inherits from $\mathcal{D}(\Omega)$ coincides with the topology of the Fréchet space.

Let $\mathcal{D}'(\Omega, B)$ denote the space of distributions that is the space of all linear continuous mappings of $\mathcal{D}(\Omega)$ into B. If T_ν is a sequence of distributions in $\mathcal{D}(\Omega)$, the statement

$$T_\nu \to T \qquad \text{in} \qquad \mathcal{D}'(\Omega, B)$$

refers to the weak*-topology which means that $T_\nu(\varphi) \to T(\varphi)$ for every $\varphi \in \mathcal{D}(\Omega)$. Observe that for any fixed compact $K \subset \Omega$ the family of injections

$$\mathcal{L}((\mathcal{D}_K(\Omega), q_m), B) \hookrightarrow \mathcal{L}(\mathcal{D}_K(\Omega), B)$$

for $m = 0, 1, 2, \ldots$ introduces the inductive limit topology in $\mathcal{L}(\mathcal{D}_K(\Omega), B)$ (comp. (2)). □

Example 2.2. *Schwartz functions and tempered distributions.* Let $\mathcal{S}(\mathbb{R}^n)$ be the space of all $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$ for which $P \cdot D^\alpha\varphi$ is a bounded function, for every polynomial P and for every multi-index α. It is known that $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space with the family of seminorm

$$q_m(\varphi) = \sup_{x\in\mathbb{R}^n} \sup_{|\alpha|\le m} (1+|x|^2)^m \left|D^\alpha \varphi(x)\right|$$

for $m = 0, 1, 2, \ldots$. Let us remember that a distribution $T \in \mathcal{D}'(\mathbb{R}^n, B)$ is tempered when it is continuous in topology of $\mathcal{S}(\mathbb{R}^n)$. This is equivalent to the fact that there is the unique extension $\overline{T}$ of T to $\mathcal{S}(\mathbb{R}^n)$. It is customary to identify T with its extension $\overline{T}$. Contrary to that, we will avoid this identification and denote by the space of tempered distribution $\mathcal{D}'_{\text{temp}}$ and by the space of their extensions into $\mathcal{S}(\mathbb{R}^n)$ $\mathcal{L}(\mathcal{S}(\mathbb{R}^n), B)$. Consider the family of mappings

$$\mathcal{L}((\mathcal{S}(\mathbb{R}^n), q_m), B) \hookrightarrow \mathcal{L}(\mathcal{S}(\mathbb{R}^n), B)$$

for all $m = 0, 1, 2, \dots$. According to (2) it introduces inductive limit topology into the space $\mathcal{L}(\mathcal{S}(\mathbb{R}^n), B)$. □

Since the linear map

$$\varphi^{**} : \mathcal{L}(\mathcal{S}, B) \ni T \mapsto T(\varphi) \in B$$

is continuous for every $\varphi \in \mathcal{S}$, it is clear that the inductive limit topology in $\mathcal{L}(\mathcal{S}, B)$ is stronger than the weak*-topology. The following example had been communicated to the author by K. Holly. It shows that the inductive limit topology in $\mathcal{L}(\mathcal{S}, B)$ is actually strictly stronger.

Example 2.3. Let $K = [0, 1]$, $\mathcal{S} = \mathcal{D}_K(\mathbb{R})$, B = $\mathbb{R}$. Suppose that on the contrary, the inductive limit topology, denoted by top $(\mathcal{D}_K(\mathbb{R}))'$, coincides with the weak*-topology in $(\mathcal{D}_K(\mathbb{R}))'$, Let $(\varphi_\nu)_{\nu=0,1,2,\dots} \subset \mathcal{D}_K(\mathbb{R})$ be linearly dense in the Hilbert space $L^2([0, 1])$. Consider the linear operator

$$\eta : (\mathcal{D}_K(\mathbb{R}))' \ni T \mapsto \left(\frac{1}{q_0(\varphi_0)} T(\varphi_0), \frac{1}{q_1(\varphi_1)} T(\varphi_1), \dots \right) \in l^\infty .$$

Since for any $m, n \in \{0, 1, \dots\}$

$$\left| \frac{1}{q_n(\varphi_n)} T(\varphi_n) \right| \le M \cdot |T|_{q_m} ,$$

so the map $\eta : (\mathcal{D}_K(\mathbb{R}))' \to l^\infty$ is correctly defined and continuous. Thus

$$\left\{ T \in (\mathcal{D}_K(\mathbb{R}))' : |\eta(T)|_{l^\infty} < 1 \right\} \in \text{top } (\mathcal{D}_K(\mathbb{R}))'.$$

We assumed that top $(\mathcal{D}_K(\mathbb{R}))'$ and the weak*-topology in $(\mathcal{D}_K(\mathbb{R}))'$ coincide, thus there are $\psi_1, \dots, \psi_N \in \mathcal{D}_K(\mathbb{R})$ and $\varepsilon > 0$ such that

$$\sigma = \bigcap_{i=1}^{N} \psi_i^{**}(-\varepsilon, \varepsilon) \subset \{ T \in (\mathcal{D}_K(\mathbb{R}))' : |\eta(T)|_{l^\infty} < 1 \}.$$

Let $f \in L^2([0, 1])$ be such that $|f|_{L^2} > 0$ and $f \perp \psi_1, \dots, \psi_N$. Define the functional

$$T_f : \mathcal{D}_K(\mathbb{R}) \ni \varphi \mapsto \int_0^1 \varphi(x) f(x)\, dx \in \mathbb{R}.$$

Since

$$\psi_i^{**}(T_f) = T_f(\psi_i) = \int_0^1 \psi_i(x) f(x)\, dx = 0,$$

it follows that $T_f \in \sigma$, and consequently $RT_f \in \sigma$ for any $R \in \mathbb{N}$. Thus

$$\eta(T_f) = 0. \tag{3}$$

On the other hand there is $v \in \{0, 1, \dots\}$ such that φ_v, is not orthogonal to f, for otherwise $(\varphi_v)_v$ would not be linearly dense in $L^2([0, 1])$. Thus $T_f(\varphi_v) \neq 0$ and so $\eta(T_f) \neq 0$ which contradicts (3). ☐

3. $\mathcal{L}(\mathcal{S}, B)$ – valued functions

Let M be an open interval in $\mathbb{R}$. We shall consider mappings of M into $\mathcal{L}(\mathcal{S}, B)$.

Definition 3.1. *A mapping $T : M \to \mathcal{L}(\mathcal{S}, B)$ is continuous if there exists a seminorm $q \in \operatorname{sn} \mathcal{S}$ such that $T(x) \in \mathcal{L}((\mathcal{S}, q), B)$ for all $x \in M$ and the mapping $T : M \to \mathcal{L}(\mathcal{S}, q), B)$ is continuous.*

It is obvious that a mapping $T : M \to \mathcal{L}(\mathcal{S}, B)$ is continuous iff there exists $m \in \mathbb{N}$ such that $T(x) \in \mathcal{L}((\mathcal{S}, q_m), B)$ for all $x \in M$ and the mapping $T : M \to \mathcal{L}((\mathcal{S}, q_m), B)$ is continuous.

Remark 3.2. *If $T : M \to \mathcal{L}(\mathcal{S}, B)$ is continuous, then the map $M \ni x \mapsto T(x)\psi \in B$ is continuous for every $\psi \in \mathcal{S}$.*

Indeed, there is a seminorm $q \in \operatorname{sn} \mathcal{S}$ such that $T(x) \in \mathcal{L}((\mathcal{S}, q), B)$ for all $x \in M$ and the map $T : M \to \mathcal{L}((\mathcal{S}, q), B)$ is continuous. Let $\psi \in \mathcal{S}$. Then the operator

$$\psi^{**}_{|\mathcal{L}((\mathcal{S},q),B)} : \mathcal{L}((\mathcal{S},q),B) \to B$$

is linear and continuous. Thus $\psi^{**} \circ T : M \to B$ is continuous and

$$(\psi^{**} \circ T)(x) = \psi^{**}(T(x)) = T(x)\psi$$

for $x \in M$.

Definition 3.3. *A mapping $T : M \to \mathcal{L}(\mathcal{S}, B)$ is differentiable if there exists a seminorm $q \in \operatorname{sn} \mathcal{S}$ such that $T(x) \in \mathcal{L}((\mathcal{S}, q), B)$ for all $x \in M$ and the mapping $T : M \to \mathcal{L}((\mathcal{S}, q), B)$ is differentiable.*

Let $q \in \operatorname{sn} \mathcal{S}$ be a seminorm from Definition 3.3, then $\left(\frac{d}{dx}T\right)_q$ denotes the derivative of $T : M \to \mathcal{L}((\mathcal{S}, q), B)$. Suppose that there is another seminorm $p \in \operatorname{sn} \mathcal{S}$ such that $T(x) \in \mathcal{L}((\mathcal{S}, p), B)$ for all $x \in M$ and the mapping $T : M \to \mathcal{L}((\mathcal{S}, p), B)$ is differentiable. Let $r = \max\{p, q\}$. Then $r \in \operatorname{sn} \mathcal{S}$ and $\iota_{qr} : \mathcal{L}((\mathcal{S}, q), B) \hookrightarrow \mathcal{L}((\mathcal{S}, r), B)$, $\iota_{pr} : \mathcal{L}((\mathcal{S}, p), B) \hookrightarrow$ $\hookrightarrow \mathcal{L}((\mathcal{S}, r), B)$ are linear and continuous. Consequently $\iota_{qr} \circ T : M \to \mathcal{L}((\mathcal{S}, r), B)$, $\iota_{pr} \circ T : M \to \mathcal{L}((\mathcal{S}, r), B)$ are differentiable and $\iota_{qr} \circ T = \iota_{pr} \circ T$. Thus

$$\frac{d}{dx}(\iota_{qr} \circ T) = \iota_{qr} \circ \left(\frac{d}{dx}T\right)_q, \quad \frac{d}{dx}(\iota_{pr} \circ T) = \iota_{pr} \circ \left(\frac{d}{dx}T\right)_p,$$

and finally

$$\left(\frac{d}{dx}T\right)_q = \left(\frac{d}{dx}T\right)_p.$$

Therefore the derivative of $T : M \to \mathcal{L}(\mathcal{S}, B)$ does not depend on the choice of a seminorm in Definition 3.3.

Definition 3.4. *If a map* $T : M \to \mathcal{L}(\mathcal{S}, B)$ *is differentiable, then its derivative is defined by*

$$\frac{d}{dx}T = \left(\frac{d}{dx}T\right)_q .$$

Similarly to Remark 3.2 we have

Remark 3.5. *If* $T : M \to \mathcal{L}(\mathcal{S}, B)$ *is differentiable then the map* $M \ni x \mapsto T(x)\psi \in B$ *is differentiable for every* $\psi \in \mathcal{S}$ *and*

$$\frac{d}{dx}(T(\cdot)\psi)(x) = \left(\frac{d}{dx}T\right)(\cdot)\psi$$

for all $x \in M$.

Theorem 3.6. *Let* $T : M \to \mathcal{L}(\mathcal{S}, B)$ *be differentiable. Additionally consider a Banach space* B_1 *and a locally convex space* $\mathcal{S}_1$. *Assume that a mapping* $L : \mathcal{L}(\mathcal{S}, B) \to \mathcal{L}(\mathcal{S}_1, B_1)$ *is linear and satisfies the condition*

$$\forall p \in \operatorname{sn}\mathcal{S}\ \exists q \in \operatorname{sn}\mathcal{S}_1 \quad L(\mathcal{L}((\mathcal{S}, p), B)) \subset \mathcal{L}((\mathcal{S}_1, q), B_1) \quad \textit{and} \tag{4}$$

$$L_{|\mathcal{L}((\mathcal{S},p),B)} : \mathcal{L}((\mathcal{S}, p), B) \to \mathcal{L}((\mathcal{S}_1, q), B_1) \quad \textit{is continuous.}$$

Then the map $L : \mathcal{L}(\mathcal{S}, B) \to \mathcal{L}(\mathcal{S}_1, B_1)$ *is continuous,* $L \circ T : M \to \mathcal{L}(\mathcal{S}_1, B_1)$ *is differentiable and*

$$\frac{d}{dx}(L \circ T) = L \circ \frac{d}{dx}T.$$

Proof. To prove the continuity of $L : \mathcal{L}(\mathcal{S}, B) \to \mathcal{L}(\mathcal{S}_1, B_1)$, with respect to the inductive limit topology on $\mathcal{L}(\mathcal{S}, B)$, it is sufficient to show that for every $p \in \operatorname{sn}\mathcal{S}$ the map $L_{|\mathcal{L}((\mathcal{S},p),B)} : \mathcal{L}((\mathcal{S}, p), B) \to \mathcal{L}(\mathcal{S}_1, B_1)$ is continuous. Let $p \in \operatorname{sn}\mathcal{S}$. According to (4) there is a seminorm $q \in \operatorname{sn}\mathcal{S}_1$ such that $L_{|\mathcal{L}((\mathcal{S},p),B)} : \mathcal{L}((\mathcal{S}, p), B) \to \mathcal{L}((\mathcal{S}_1, q), B_1)$ is continuous. Since $\mathcal{L}(\mathcal{S}_1, B_1)$ is equipped with the inductive topology, the canonical injection $\mathcal{L}(\mathcal{S}_1, q), B_1) \hookrightarrow \mathcal{L}(\mathcal{S}_1, B_1)$ is also continuous. Thus $L_{|\mathcal{L}((\mathcal{S},p),B)} : \mathcal{L}((\mathcal{S}, p), B) \to \mathcal{L}(\mathcal{S}_1, B_1)$ is continuous.

Assume now that $T : M \to \mathcal{L}(\mathcal{S}, B)$ is differentiable. Let $p \in \operatorname{sn}\mathcal{S}$ be such that $T(x) \in \mathcal{L}((\mathcal{S}, p), B)$ for every $x \in M$ and $T : M \to \mathcal{L}((\mathcal{S}, p), B)$ is differentiable. On account of (4) there is $q \in \operatorname{sn}\mathcal{S}_1$ such that $L_{|\mathcal{L}((\mathcal{S},p),B)} : \mathcal{L}((\mathcal{S}, p), B) \to \mathcal{L}((\mathcal{S}_1, q), B_1)$ is linear and continuous. Hence $L_{|\mathcal{L}((\mathcal{S},p),B)} \circ T : M \to \mathcal{L}((\mathcal{S}_1, q), B_1)$ is differentiable and

$$\frac{d}{dx}(L \circ T)_q = \frac{d}{dx}(L_{|\mathcal{L}((\mathcal{S},p),B)} \circ T) = L_{|\mathcal{L}((\mathcal{S},p),B)} \circ \frac{d}{dx}T.$$

□

Note that if $F : S_1 \to S$ is linear and continuous then its transpose

$$F^* : \mathcal{L}(\mathcal{S}, B) \ni T \to T \circ F \in \mathcal{L}(\mathcal{S}_1, B) \tag{5}$$

satisfies (4). Indeed, for any $p \in \operatorname{sn} \mathcal{S}$ there is $q = p \circ F \in \operatorname{sn} \mathcal{S}_1$ such that $F^*(\mathcal{L}((\mathcal{S}, p), B)) \subset$ $\subset \mathcal{L}((\mathcal{S}_1, q), B)$ and the restriction $F^*_{|\mathcal{L}((\mathcal{S},p),B)} : \mathcal{L}((\mathcal{S},p),B) \to \mathcal{L}(\mathcal{S}_1,q),B)$ is a continuous map between Banach spaces.

On the other hand, if $L : B \to B_1$ is linear and continuous then the map

$$^*L : \mathcal{L}(\mathcal{S}, B) \ni T \mapsto L \circ T \in \mathcal{L}(\mathcal{S}, B_1) \tag{6}$$

satisfies (4). Indeed for every $p \in \operatorname{sn} \mathcal{S}$ the map $^*L_{|\mathcal{L}((\mathcal{S},p),B)} : \mathcal{L}((\mathcal{S},p),B) \to \mathcal{L}(\mathcal{S},p),B_1)$ is continuous. In particular, for $L(T) = \lambda T$, $\lambda \in \mathbb{K}$ we have

Corollary 3.7. *If* $T : M \to \mathcal{L}(\mathcal{S}, B)$ *is differentiable then* λT *is also differentiable and*

$$\frac{d}{dx}(\lambda T) = \lambda \frac{d}{dx} T.$$

It is also clear that

Proposition 3.8. *If* $T_1, T_2 : M \to \mathcal{L}(\mathcal{S}, B)$ *are differentiable then* $T_1 + T_2$ *is also differentiable and*

$$\frac{d}{dx}(T_1 + T_2) = \frac{d}{dx} T_1 + \frac{d}{dx} T_2.$$

4. Distribution-valued functions

Let us consider a function $T : M \ni x \mapsto T(x) \in \mathcal{D}'(\Omega, B)$.

Definition 4.1. *A map* $T : M \to \mathcal{D}'(\Omega, B)$ *is continuous if for any compact* $K \subset \Omega$ *the map*

$$T_K : M \ni x \mapsto T(x)_{|\mathcal{D}_K(\Omega)} \in \mathcal{L}(\mathcal{D}_K(\Omega), B)$$

is continuous.

We shall write $T(x)_K$ *for* $T(x)_{|\mathcal{D}_K(\Omega)}$.

Definition 4.2. *A mapping* $T : M \to \mathcal{D}'(\Omega, B)$ *is differentiable if for any compact* $K \subset \Omega$ *the map*

$$T_K : M \ni x \mapsto T(x)_K \in \mathcal{L}(\mathcal{D}_K(\Omega), B)$$

is differentiable, and

$$\left(\frac{d}{dx} T\right)(x)\varphi = \left(\frac{d}{dx} T_K\right)(x)\varphi \tag{7}$$

for $\operatorname{supp} \varphi \subset K$, $x \in M$.

The following lemma ensures that the definition is meaningful and that the derivative of a map $T : M \to \mathcal{D}'(\Omega, B)$ is a distribution-valued function.

Lemma 4.3. *For any compact* $K_1, K_2 \subset \Omega$ *if* $\varphi \in \mathcal{D}_{K_1}(\Omega) \cap \mathcal{D}_{K_2}(\Omega)$ *then*

$$\left(\frac{d}{dx}T_{K_1}\right)(x)\varphi = \left(\frac{d}{dx}T_{K_2}\right)(x)\varphi.$$

Moreover $\left(\frac{d}{dx}T\right)(x) \in \mathcal{D}'(\Omega, B)$ *for every* $x \in M$.

Proof. Let $x \in M$, K_1, $K_2 \subset \Omega$ be compact and $\varphi \in \mathcal{D}_{K_1}(\Omega) \cap \mathcal{D}_{K_2}(\Omega)$. Since maps $\varphi_i^{**} : \mathcal{L}(\mathcal{D}_{K_i}(\Omega), B) \ni T(x)_{K_i} \mapsto T(x)_{K_i}\varphi \in B$ are linear and continuous for $i = 1, 2$ and $(\varphi_1^{**} \circ T_{K_1})(x) = T(x)_{K_1}\varphi$, $(\varphi_2^{**} \circ T_{K_2})(x) = T(x)_{K_2}\varphi$, and both these maps $M \to B$ are differentiable. From Remark 3.5

$$\frac{d}{dx}(\varphi_1^{**} \circ T_{K_1})(x) = \left(\frac{d}{dx}T_{K_1}\right)(x)\varphi, \quad \frac{d}{dx}(\varphi_2^{**} \circ T_{K_2})(x) = \left(\frac{d}{dx}T_{K_2}\right)(x)\varphi,$$

hence

$$\left(\frac{d}{dx}T_{K_1}\right)(x)\varphi = \left(\frac{d}{dx}T_{K_2}\right)(x)\varphi.$$

This proves that the relation given by (7) is a function on the domain $\mathcal{D}(\Omega) = \bigcup\{\mathcal{D}_K(\Omega) : K \subset \Omega, K\text{-compact}\}$. The linearity and continuity of $\left(\frac{d}{dx}T\right)(x) : \mathcal{D}(\Omega) \to B$ is obvious. □

From the above proof it follows that

Corollary 4.4. *If* $T : M \to \mathcal{D}'(\Omega, B)$ *is differentiable then for every* $\varphi \in \mathcal{D}(\Omega)$ *the map* $M \ni x \mapsto T(x)\varphi \in \mathbb{K}$ *is differentiable and, for every* $x \in M$

$$\frac{d}{dx}(T(\cdot)\varphi)(x) = \left(\frac{d}{dx}T(x)\right)\varphi.$$

Remark 4.5. *If* $T_1, T_2 : M \to \mathcal{D}'(\Omega, B)$ *are differentiable and* $\lambda_1, \lambda_2 \in \mathbb{K}$, *then* $\lambda_1 T_1 + \lambda T_2$ *is also differentiable and*

$$\frac{d}{dx}(\lambda_1 T_1 + \lambda_2 T_2) = \lambda_1 \frac{d}{dx}T_1 + \lambda_2 \frac{d}{dx}T_2.$$

Consider now additionally $\Omega_1 \in \operatorname{top}\mathbb{R}^{n_1}$, $n_1 \in \mathbb{N}$, and a Banach space B_1.

Theorem 4.6. *Let* $T : M \to \mathcal{D}'(\Omega, B)$ *be differentiable. Suppose that a linear mapping* $L : \mathcal{D}'(\Omega, B) \to \mathcal{D}'(\Omega_1, B_1)$ *satisfies the condition:*

for any compact $K \subset \Omega_1$ there is a compact $Z \subset \Omega$ and a linear map $L_{ZK} : \mathcal{L}(\mathcal{D}_Z(\Omega), B) \to \mathcal{L}(\mathcal{D}_K(\Omega_1), B_1)$ that satisfies condition (4) and the diagram

$$\begin{array}{ccc} \mathcal{L}(\mathcal{D}_Z(\Omega), B) & \xrightarrow{L_{ZK}} & \mathcal{L}(\mathcal{D}_K(\Omega_1), B_1) \\ \iota_Z^* \uparrow & & \uparrow \iota_K^* \\ \mathcal{D}'(\Omega, B) & \xrightarrow{L} & \mathcal{D}'(\Omega_1, B_1) \end{array} \tag{8}$$

commutes.
Then the map $L \circ T : M \to \mathcal{D}'(\Omega_1, B_1)$ is differentiable and for any $x \in M$

$$\frac{d}{dx}(L \circ T)(x) = L\left(\frac{d}{dx}T(x)\right).$$

Proof. Let K be a compact in Ω_1. Due to (8) there exists a compact $Z \subset \Omega$ and a map L_{ZK}. Since $T : M \to \mathcal{D}'(\Omega, B)$ is differentiable, the map $T_Z : M \to \mathcal{L}(\mathcal{D}_Z(\Omega), B)$ is differentiable. We now apply Theorem 3.6 for $\mathcal{S} = \mathcal{D}_Z(\Omega)$, $\mathcal{S}_1 = \mathcal{D}_K(\Omega_1)$, and deduce that $L_{ZK} \circ T_Z : M \to$ $\to \mathcal{L}(\mathcal{D}_K(\Omega_1), B_1)$ is differentiable and for every $x \in M$

$$\frac{d}{dx}(L_{ZK} \circ T_Z)(x) = L_{ZK}\left(\frac{d}{dx}T_Z(x)\right).$$

As the diagram is commutative we get $L_{ZK} \circ \iota_Z^* = \iota_K^* \circ L$, and so

$$L_{ZK} \circ T_Z = L_{ZK} \circ \iota_Z^* \circ T = \iota_K^* \circ L \circ T = (L \circ T)_K.$$

In consequence $L \circ T : M \to \mathcal{D}'(\Omega_1, B_1)$ is differentiable. Moreover, for every $x \in M$

$$\begin{aligned} \frac{d}{dx}(L \circ T)_K(x) &= \frac{d}{dx}(L_{ZK} \circ T_Z)(x) = L_{ZK}\left(\frac{d}{dx}T_Z(x)\right) \\ &= L_{ZK}\left(\frac{d}{dx}(\iota_Z^* \circ T)(x)\right) = (L_{ZK} \circ \iota_Z^*)\left(\frac{d}{dx}T(x)\right) \\ &= (\iota_K^* \circ L)\left(\frac{d}{dx}T(x)\right) = \left(L\left(\frac{d}{dx}T(x)\right)\right)_K. \end{aligned}$$

□

Observe that with any linear and continuous operator $L : B \to B_1$ we may associate the linear operator

$${}^*L : \mathcal{D}'(\Omega, B) \ni T \mapsto L \circ T \in \mathcal{D}'(\Omega, B_1)$$

(compare (6)). It is clear that for every compact $K \subset \Omega$ the diagram

$$\begin{array}{ccc} \mathcal{L}(\mathcal{D}_K(\Omega), B) & \xrightarrow{({}^*L)_K} & \mathcal{L}(\mathcal{D}_K(\Omega), B_1) \\ \iota_K^* \uparrow & & \uparrow \iota_K^* \\ \mathcal{D}'(\Omega, B) & \xrightarrow{L} & \mathcal{D}'(\Omega, B_1) \end{array}$$

commutes, where $({}^*L)_K : \mathcal{L}(\mathcal{D}_K(\Omega), B) \ni T \mapsto L \circ T \in \mathcal{L}(\mathcal{D}_K(\Omega), B_1)$. So we have

Corollary 4.7. *If an operator* $L : B \to B_1$ *is linear and continuous then* ${}^*L : \mathcal{D}'(\Omega, B) \ni T \mapsto$ $\mapsto L \circ T \in \mathcal{D}'(\Omega, B_1)$ *satisfies condition (8).*

On the other hand, with a linear continuous operator $F : \mathcal{D}(\Omega_1) \to \mathcal{D}(\Omega)$ we may associate its linear transpose

$$F^* : \mathcal{D}'(\Omega, B) \ni T \mapsto T \circ F \in \mathcal{D}'(\Omega_1, B)$$

(compare (5)). The following lemma had been communicated to the author by K. Holly. It indicates a wide class of operators that satisfy (8).

Lemma 4.8. *Let* $F : \mathcal{D}(\Omega_1) \to \mathcal{D}(\Omega)$ *be linear and continuous. For every compact* $K \subset \Omega_1$ *there is a compact* $Z \subset \Omega$ *such that* $F(\mathcal{D}_K(\Omega_1)) \subset \mathcal{D}_Z(\Omega)$, *the restriction* $F_K = F_{|\mathcal{D}_K(\Omega_1)}$ *is a continuous map of the Frechét spaces* $\mathcal{D}_K(\Omega_1)$, $\mathcal{D}_Z(\Omega)$, *and the diagram*

$$\begin{array}{ccc} \mathcal{L}(\mathcal{D}_Z(\Omega), B) & \overset{(F_K)^*}{\to} & \mathcal{L}(\mathcal{D}_K(\Omega_1), B) \\ \iota_Z^* \uparrow & & \uparrow \iota_K^* \\ \mathcal{D}'(\Omega, B) & \overset{F^*}{\to} & \mathcal{D}'(\Omega_1, B) \end{array}$$

commutes.

Proof. Let $F : \mathcal{D}(\Omega_1) \to \mathcal{D}(\Omega)$ be linear and continuous and $K \subset \Omega_1$ be compact. We put $Z = \overline{\bigcup_{\varphi \in \mathcal{D}_K(\Omega_1)} \{F(\varphi) \neq 0\}}$ (the closure in $\mathbb{R}^n$). Suppose that, contrary to our claim, either Z is not a subset of Ω or Z is not compact. Then there is a sequence $(z_\nu)_{\nu \in \mathbb{N}} \subset \bigcup_{\varphi \in \mathcal{D}_K(\Omega_1)} \{F(\varphi) \neq 0\}$ which contains no subsequence convergent in Ω. Let $\nu \in \mathbb{N}$. There is $\varphi_\nu \in \mathcal{D}_K(\Omega_1)$ such that $z_\nu \in \{F(\varphi_\nu) \neq 0\}$. Then $\varphi_\nu \neq 0$ and $q_m(\varphi_\nu) > 0$ for any norm q_m in $\mathcal{D}_K(\Omega_1)$, $m \in \mathbb{N}$. Therefore

$$\psi_\nu = \frac{1}{\nu} \cdot \frac{\varphi_\nu}{q_\nu(\varphi_\nu)}$$

is correctly defined for $\nu \in \mathbb{N}$ and $\psi_\nu \to 0$ in $(\mathcal{D}_K(\Omega_1), q_m)$ for any $m \in \mathbb{N}$. Thus $\psi_\nu \to 0$ in the Frechét space $\mathcal{D}_K(\Omega_1)$ and consequently $F(\psi_\nu) \to 0$ in $\mathcal{D}(\Omega)$. In particular it means that there is a compact set $D \subset \Omega$ such that $\overline{\bigcup_{\nu \in \mathbb{N}} \{F(\psi_\nu) \neq 0\}} \subset D$. However $\{F(\psi_\nu) \neq 0\} = \{F(\varphi_\nu) \neq 0\}$ for any $\nu \in \mathbb{N}$. Hence $(z_\nu)_{\nu \in \mathbb{N}} \subset D$ and $(z_\nu)_{\nu \in \mathbb{N}}$ contains a subsequence which is convergent in Ω. This leads to a contradiction. Therefore Z is compact and $Z \subset \Omega$. Clearly $F(\mathcal{D}_K(\Omega_1)) \subset \mathcal{D}_Z(\Omega)$, and $F_K : \mathcal{D}_K(\Omega_1) \to \mathcal{D}_Z(\Omega)$ is continuous. $\square$

Corollary 4.9. *If* $F : \mathcal{D}(\Omega_1) \to \mathcal{D}(\Omega)$ *is linear and continuous, then* $F^* : \mathcal{D}'(\Omega, B) \ni T \mapsto$ $\mapsto T \circ F \in \mathcal{D}'(\Omega_1, B)$ *satisfies condition (8).*

Let us recall that if $\Lambda \in \mathcal{D}'(\Omega, B)$, α is a multi-index, then $D^\alpha \Lambda \in \mathcal{D}'(\Omega, B)$ and for any $\varphi \in \mathcal{D}(\Omega)$

$$(D^\alpha \Lambda)\varphi = (-1)^{|\alpha|} \Lambda(D^\alpha \varphi).$$

Proposition 4.10. *Let* $T : M \to \mathcal{D}'(\Omega, B)$ *be differentiable and* $\alpha \in \mathbb{N}^n$. *Then the map* $D^\alpha T : M \ni x \mapsto D^\alpha T(x) \in \mathcal{D}'(\Omega, B)$ *is differentiable and for every* $x \in M$

$$\frac{d}{dx} D^\alpha T(x) = D^\alpha \left(\frac{d}{dx} T(x) \right).$$

Proof. The operator $F : \mathcal{D}(\Omega) \ni \varphi \mapsto (-1)^{|\alpha|} D^\alpha \varphi \in \mathcal{D}(\Omega)$ is linear, continuous and its transpose is of the form

$$F^* : \mathcal{D}'(\Omega, B) \ni T \mapsto D^\alpha T \in \mathcal{D}'(\Omega, B).$$

Due to Corollary 4.9. F^* satisfies condition (8) and, according to Theorem 4.6, $F^* \circ T : M \to \to \mathcal{D}'(\Omega, B)$ is differentiable and for every $x \in M$

$$\frac{d}{dx}(D^\alpha T)(x) = \frac{d}{dx}(F^* \circ T)(x) = F^* \left(\frac{d}{dx} T(x) \right) = D^\alpha \left(\frac{d}{dx} T(x) \right).$$

Recall also that if $\Lambda \in \mathcal{D}'(\Omega, B)$ and $\eta : \Omega \to \mathbb{K}$ is a smooth function, $g : \Omega_1 \to \Omega$ is a smooth diffeomorphism, then $\eta\Lambda$, $\Lambda \circ g$ are distributions on $\mathcal{D}(\Omega)$, $\mathcal{D}(\Omega_1)$, respectively, and

$$(\eta\Lambda)\varphi = \Lambda(\eta\varphi) \quad \text{for} \quad \varphi \in \mathcal{D}(\Omega),$$
$$(\Lambda \circ g)\varphi = \left|\det g^{-1}\right| \Lambda(\varphi \circ g^{-1}) \quad \text{for} \quad \varphi \in \mathcal{D}(\Omega_1).$$

Thus, similarly to Proposition 4.10, we obtain

Proposition 4.11. *Let* $T : M \to \mathcal{D}'(\Omega, B)$ *be differentiable. Consider a smooth function* $\eta : \Omega \to \mathbb{K}$ *and a smooth diffeomorphism* $g : \Omega_1 \to \Omega$ *on an open set* Ω_1. *Then the mappings:* $\eta T : M \ni x \to \eta \cdot T(x) \in \mathcal{D}'(\Omega, B)$, $T \circ g : M \ni x \mapsto T(x) \circ g \in \mathcal{D}'(\Omega_1, B)$ *are differentiable and for every* $x \in M$

$$\frac{d}{dx}(\eta \cdot T)(x) = \eta \cdot \frac{d}{dx} T(x),$$
$$\frac{d}{dx}(T \circ g)(x) = \frac{d}{dx} T(x) \circ g.$$

It is known that every locally summable function u defines a distribution, called regular distribution, denoted by $[u]$,

$$[u]\varphi = \int_\Omega u(x)\varphi(x)dx \quad \text{for} \quad \varphi \in \mathcal{D}(\Omega).$$

For $h \in \mathcal{D}(\mathbb{R}^n)$ and $\Lambda \in \mathcal{D}'(\mathbb{R}^n, B)$, the convolution $h * \Lambda$ is well defined by the formula

$$(h * \Lambda)(x) = \Lambda(\tau_x h) \quad \text{for} \quad x \in \mathbb{R}^n,$$

where $\tau_x \varphi(y) = \varphi(x - y)$. Moreover, $h * \Lambda \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $D^\alpha(h * \Lambda) = (D^\alpha h) * \Lambda = h * D^\alpha \Lambda$. For arbitrary fixed $h \in \mathcal{D}(\mathbb{R}^n)$ take

$$F : \mathcal{D}(\mathbb{R}^n) \ni \varphi \mapsto \varphi * \check{h} \in \mathcal{D}(\mathbb{R}^n),$$

where $\breve{h}(y) = h(-y)$ for $y \in \mathbb{R}^n$. Its transpose is of the form

$$F^* : \mathcal{D}'(\mathbb{R}^n, B) \ni \Lambda \mapsto [h * \Lambda] \in \mathcal{D}'(\mathbb{R}^n, B),$$

and satisfies (8), thus according to Theorem 4.6 we obtain

Proposition 4.12. *Let $T : M \to \mathcal{D}'(\mathbb{R}^n, B)$ be differentiable and $h \in \mathcal{D}(\mathbb{R}^n)$. Then the mapping $[h * T] : M \ni x \mapsto [h \cdot T(x)] \in \mathcal{D}'(\Omega, B)$ is differentiable and for every $x \in M$*

$$\frac{d}{dx}[h * T](x) = \left[h * \frac{d}{dx} T(x) \right].$$

5. Tempered distribution-valued functions

Now turn to functions with values in the space of tempered distributions.

Lemma 5.1. *Consider a map $T : M \to \mathcal{D}'_{\text{temp}}$ such that $\overline{T} : M \ni x \mapsto \overline{T(x)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), B)$ is differentiable. Then $T : M \to \mathcal{D}'(\mathbb{R}^n, B)$ is differentiable and $\frac{d}{dx} T(x) \subset \frac{d}{dx} \overline{T(x)}$. In particular, the distribution $\frac{d}{dx} T(x)$ is tempered for any $x \in M$.*

Proof. Let $K \subset \mathbb{R}^n$ be a compact. Then the injection $F : \mathcal{D}_K(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n)$ is linear and continuous, thus the map $F^* : \mathcal{L}(\mathcal{S}(\mathbb{R}^n), B) \ni \overline{T} \mapsto \overline{T} \circ F \in \mathcal{L}(\mathcal{D}_K(\mathbb{R}^n), B)$ satisfies condition (4) and from Theorem 3.6 the map $F^* \circ \overline{T} : M \to \mathcal{L}(\mathcal{D}_K(\mathbb{R}^n), B)$ is differentiable and for any $x \in M$

$$\frac{d}{dx}(F^* \circ \overline{T})(x) = F^* \left(\frac{d}{dx} \overline{T(x)} \right).$$

Therefore the mapping $T_K = F^* \circ \overline{T} : M \to \mathcal{L}(\mathcal{D}_K(\mathbb{R}^n), B)$ is differentiable for any compact $K \subset \mathbb{R}^n$ and from Definition 4.2, $T : M \to \mathcal{D}'(\mathbb{R}^n, B)$ is differentiable. Moreover for any $x \in M$

$$\left(\frac{d}{dx} T_K \right)(x) = \frac{d}{dx} \overline{T(x)}\Big|_{\mathcal{D}_K(\mathbb{R}^n)}.$$

□

Now we are in the position to consider tempered distribution-valued functions.

Definition 5.2. *A mapping $T : M \to \mathcal{D}'_{\text{temp}}$ is continuous iff the mapping $\overline{T} : M \ni x \mapsto \overline{T(x)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), B)$ is continuous.*

Definition 5.3. *A map $T : M \to \mathcal{D}'_{\text{temp}}$ is differentiable iff the mapping $\overline{T} : M \ni x \mapsto \overline{T(x)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), B)$ is differentiable. Moreover, for any $x \in M$*

$$\overline{\frac{d}{dx}T(x)} = \frac{d}{dx}\overline{T(x)}.$$

Similarly to the Proposition 4.10, but using now Theorem 3.6 we have

Proposition 5.4. *Let* $T : M \to \mathcal{D}'_{\text{temp}}$ *be differentiable and* $\alpha \in \mathbb{N}^n$*. Then* $D^\alpha T : M \ni x \mapsto D^\alpha T(x) \in \mathcal{D}'_{\text{temp}}$ *is differentiable and for any* $x \in M$

$$D^\alpha\left(\frac{d}{dx}T(x)\right) \subset \frac{d}{dx}(\overline{D^\alpha T(x)}).$$

Recall that if $\Lambda \in \mathcal{D}'_{\text{temp}}$, $\eta : \mathbb{R}^n \to \mathbb{K}$ is a smooth function which is polynominally bounded together with all its derivatives, then $\eta\Lambda \in \mathcal{D}'_{\text{temp}}$ and for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\overline{\eta\Lambda}(\varphi) = \overline{\Lambda}(\eta\varphi).$$

Consider $F(\varphi) = \eta\varphi$ for $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then F^* satisfies condition (4), and from Theorem 3.6 we obtain

Proposition 5.5. *Let* $T : M \to \mathcal{D}'_{\text{temp}}$ *be differentiable and* $\eta : \mathbb{R}^n \to \mathbb{K}$ *be a smooth function which is polynominally bounded together with all is derivatives. Then* $\eta T : M \ni x \mapsto \eta \cdot T(x) \in \mathcal{D}'_{\text{temp}}$ *is differentiable and for any* $x \in M$

$$\eta \cdot \left(\frac{d}{dx}T(x)\right) \subset \frac{d}{dx}(\overline{\eta \cdot T(x)}).$$

Similarly, taking $F(\varphi) = \varphi * \breve{h}$, where $\breve{h}(y) = h(-y)$ for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we have

Proposition 5.6. *Let* $T : M \to \mathcal{D}'_{\text{temp}}$ *be differentiable and* $h \in \mathcal{S}(\mathbb{R}^n)$*. Then* $[h * T] : M \ni x \mapsto [h \cdot T(x)] \in D'_{\text{temp}}$ *is differentiable and for any* $x \in M$

$$\left[h * \left(\frac{d}{dx}T(x)\right)\right] \subset \frac{d}{dx}(\overline{[h * T(x)]}).$$

Let $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ denote Fourier transform, $\mathcal{F}\varphi(\xi) = \hat{\varphi}(\xi) =$ $= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\xi \cdot y}\varphi(y)\,dy$ for $y \in \mathbb{R}^n$. Recall that if $\Lambda \in \mathcal{D}'_{\text{temp}}$ then $\mathcal{F}\Lambda \in \mathcal{D}'_{\text{temp}}$ and

$$\overline{\mathcal{F}\Lambda}(\varphi) = \overline{\Lambda}(\hat{\varphi})$$

for $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Proposition 5.7. *Let* $T : M \ni x \mapsto T(x) \in \mathcal{D}'_{\text{temp}}$ *be differentiable. Then* $\mathcal{F}T : M \ni x \mapsto \mathcal{F}(T(x)) \in \mathcal{D}'_{\text{temp}}$ *is differentiable and for any* $x \in M$

$$\mathcal{F}\left(\frac{d}{dx}T(x)\right) \subset \frac{d}{dx}(\overline{\mathcal{F}(T(x))}).$$

Proof. Consider the operator $F: \mathcal{S}(\mathbb{R}^n) \ni \varphi \mapsto \overset{\wedge}{\varphi} \in \mathcal{S}(\mathbb{R}^n)$. It is linear and continuous, so $F^*: \mathcal{L}(\mathcal{S}(\mathbb{R}^n), B) \to \mathcal{L}(\mathcal{S}(\mathbb{R}^n), B)$, where $F^*(\overline{\Lambda}) = \overline{\mathcal{F}\Lambda}$ for $\Lambda \in \mathcal{D}'_{\text{temp}}$. According to Theorem 3.6 the map $F^*\overline{T}: M \to \mathcal{L}(\mathcal{S}(\mathbb{R}^n), B)$ is differentiable, but

$$(F^*\overline{T})(x)\varphi = \overline{T(x)}(\overset{\wedge}{\varphi}) = \overline{\mathcal{F}(T(x))}\,\varphi$$

for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $x \in M$. Consequently, $M \ni x \mapsto \mathcal{F}(T(x)) \in \mathcal{D}'_{\text{temp}}$ is differentiable and

$$\overline{\mathcal{F}\left(\frac{d}{dx}T(x)\right)} = \frac{d}{dx}(\overline{\mathcal{F}(T(x))}).$$

$\square$

References

[1] Atanacković T.M., Stanković B., *Generalized functions in solving linear mathematical models in mechanics*, Zb. Rad. (Beogr.), **11**, 19, 2006, 5-59.

[2] Embacher H.G., Grubl G., Oberguggenbergen M., *Products of Distributions in Several Variables and Applications to Zero-Mass* QED_2, Zeitschrift f. Analysis u. ihre Anwendungen **11**, 1992, 437-454.

[3] Pietsch A., *Nuclear locally convex spaces*, Springer, 1972.

[4] Rudin W., *Functional Analysis*, McGraw-Hill, New York 1973.

[5] Schwartz L., *Théorie des distributions*, Hermann, Paris 1951.

[6] Thomas E., *Bochner and Bernstein theorems via the nuclear integral representation theorem*, J. Math. Anal. Appl. 297, 2004, No. 2, 612-624.

[7] Treves F., *Basic Linear Partial Differential Equations*, Academic Press, New York 1975.

[8] Vladimirov V., *Generalized Functions in Mathematical Physics*, Mir, Moscow 1979.

[9] Wiciak M., *A solution of the Cauchy problem in the class of absolutely continuous distribution--valued functions*, Univ. Iagel. Acta Math., **42**, 2004, 31-53.

[10] Wiciak M., *Analytical solution of the problem of vibration of plates with piezoelectric actuators with arbitrary shape in distribution formulation*, Acta Phys. Polon. A, **121**, 2012, 142-147.