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APPROXIMATION OF FUNCTIONS OF TWO VARIABLES FROM EXPONENTIAL WEIGHT SPACES

APROKSYMACJA FUNKCJI DWÓCH ZMIENNYCH Z WYKŁADNICZYCH PRZESTRZENI WAGOWYCH

A b s t r a c t

In this paper we study approximative properties of modified Szasz-Mirakyan operators for functions of two variables from exponential weight spaces. We present theorems giving a degree of approximation by these operators for exponential bounded functions.

Keywords: linear positive operators, Bessel function, modulus of continuity, degree of approximation

S t r e s z c z e n i e

W artykule przedstawiono aproksymacyjne własności zmodyfikowanych operatorów typu Szasza-Mirakyana dla funkcji dwóch zmiennych z wykładniczych przestrzeni wagowych. Wyprowadzono twierdzenia podające rząd aproksymacji funkcji ograniczonych wykładniczo przez operatory tego typu.

Słowa kluczowe: dodatni operator liniowy, funkcja Bessela, moduł ciągłości, rząd aproksymacji

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1. Introduction

Let us denote by $C(R_0)$ a set of all real-valued functions continuous on $R_0=[0;+\infty)$. In paper [1] we investigated operators of Szasz-Mirakyan type defined as follows

$$A_n^{\nu}(f;x)=\begin{cases}\dfrac{1}{I_{\nu}(nx)}\displaystyle\sum_{k=0}^{\infty}\dfrac{\left(\frac{nx}{2}\right)^{2k+\nu}}{\Gamma(k+1)\Gamma(k+\nu+1)}f\left(\dfrac{2k}{n}\right), & x>0;\\ f(0), & x=0,\end{cases}$$

where Γ is the Euler-gamma function and I_v the modified Bessel function defined by the formula ([6], p. 77)

$$I_{\nu}(z)=\sum_{k=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{2k+\nu}}{\Gamma(k+1)\Gamma(k+\nu+1)} \tag{1}$$

We studied approximative properties of these operators in exponential weight spaces

$$E_p=\{f\in C(R_0): w_p f \text{ is uniformly continuous and bounded on } R_0\},$$

where w_p was the exponential weight function defined as follows:

$$w_p(x)=e^{-px},\ \ p\in R_+ \tag{2}$$

for $x\in R_0$.

In the spaces we introduced the norm:

$$\|f\|_p=\sup\{w_p(x)|f(x)|: x\in R_0\} \tag{3}$$

and we established ([1], Corollary 1) that operators A_n^{ν} are linear, positive, bounded and transform the space E_p into E_r for some $r>p$.

Notice that a certain modification of A_n^{ν} give a linear, positive and bounded operator L_n^{ν} transforming the space E_p into E_r (Theorem 2.1), namely:

$$L_n^{\nu}(f;x)=\begin{cases}\dfrac{1}{I_{\nu}(nx)}\displaystyle\sum_{k=0}^{\infty}\dfrac{\left(\frac{nx}{2}\right)^{2k+\nu}}{\Gamma(k+1)\Gamma(k+\nu+1)}f\left(\dfrac{2k}{n+p}\right), & x>0;\\ f(0), & x=0.\end{cases} \tag{4}$$

In the present paper we introduce an analogy of the presented theorems for $f\in E_p$, but now we consider the bivariate version of the operator L_n^{ν}

$$L_{n,m}^{\nu,\mu}(f;x,y)= \tag{5}$$

$$\begin{cases} \dfrac{1}{I_\nu(nx)}\dfrac{1}{I_\mu(my)}\displaystyle\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\dfrac{\left(\frac{nx}{2}\right)^{2k+\nu}}{\Gamma(k+1)\Gamma(k+\nu+1)}\dfrac{\left(\frac{my}{2}\right)^{2j+\mu}}{\Gamma(j+1)\Gamma(j+\mu+1)}f\left(\dfrac{2k}{n+p},\dfrac{2j}{m+q}\right), x>0, y>0; \\ \dfrac{1}{I_\nu(nx)}\displaystyle\sum_{k=0}^{\infty}\dfrac{\left(\frac{nx}{2}\right)^{2k+\nu}}{\Gamma(k+1)\Gamma(k+\nu+1)}f\left(\dfrac{2k}{n+p},0\right),\ x>0, y=0; \\ \dfrac{1}{I_\mu(my)}\displaystyle\sum_{j=0}^{\infty}\dfrac{\left(\frac{my}{2}\right)^{2j+\mu}}{\Gamma(j+1)\Gamma(j+\mu+1)}f\left(0,\dfrac{2j}{m+q}\right),\ y>0, x=0; \\ f(0,0),\ x=y=0, \end{cases}$$

for $n, m \in N$, $\nu, \mu \in R_0$ and $f \in E_{p,q}$, where:

$$E_{p,q}=\left\{f\in C\left(R_0^2\right): w_{p,q}f \text{ is uniformly continuous and bounded on } R_0^2\right\},$$

and $w_{p,q}$ is the exponential weight function

$$w_{p,q}(x,y)=e^{-(px+qy)},\ \ p,q\in R_+ \tag{6}$$

for $(x,y)\in R_0^2$.

It is easy to verify that $E_{p,q}$ is a normed space with the norm:

$$\|f\|_{p,q}=\sup\left\{w_{p,q}(x,y)\left|f(x,y)\right|;(x,y)\in R_0^2\right\} \tag{7}$$

Moreover, we use the weighted modulus of continuity defined as follows:

$$\omega\left(f,E_{p,q};t,s\right)=\sup\left\{\left\|\Delta_{h,d}f\right\|_{p,q}: h\in[0,t], d\in[0,s]\right\} \tag{8}$$

where:

$$\Delta_{h,d}f(x,y)=f(x+h,y+d)-f(x,y)$$

for $(x,y),(h,d)\in R_0^2$.

The note was inspired by the results of [3–5] which investigate approximation problems for bivariate operators.

We shall present theorems giving a degree of approximation of function $f \in E_{p,q}$ by operators $L_{n,m}^{\nu,\mu}$.

2. Auxiliary results

At the beginning we recall preliminary results which we immediately obtain from paper [1] and definition (4).

Lemma 2.1 ([1], Lemma 8) *For each* $\nu \in R_0$ *there exists a positive constant* $M(\nu)$ *such that for all* $n \in N$ *and* $x \in R_0$ *we have:*

$$\left|\frac{I_{\nu+1}(nx)}{I_\nu(nx)}\right| \le M(\nu),$$

$$nx\left|\frac{I_{\nu+1}(nx)}{I_\nu(nx)}-1\right| \le M(\nu).$$

By elementary calculations we get

Lemma 2.2 *For each* $n \in N$, $\nu \in R_0$, $p \in R_+$ *and* $x \in R_0$.

$$L_n^\nu(1;x)=1,\ \ L_n^\nu(t;x)=x\frac{n}{n+p}\frac{I_{\nu+1}(nx)}{I_\nu(nx)}=\frac{n}{n+p}A_n^\nu(t;x),$$

$$L_n^\nu(t^2;x)=x^2\left(\frac{n}{n+p}\right)^2\frac{I_{\nu+2}(nx)}{I_\nu(nx)}+x\frac{2n}{(n+p)^2}\frac{I_{\nu+1}(nx)}{I_\nu(nx)}=\left(\frac{n}{n+p}\right)^2A_n^\nu(t;x),$$

$$L_n^\nu(t-x;x)=\frac{n}{n+p}\left(A_n^\nu(t;x)-\frac{p}{n}x\right),$$

$$L_n^\nu\left((t-x)^2;x\right)=\left(\frac{n}{n+p}\right)^2\left(A_n^\nu\left((t-x)^2;x\right)-2x\frac{p}{n}A_n^\nu(t-x;x)+\left(\frac{p}{n}x\right)^2\right)$$

Lemma 2.3 *For each* $n \in N$, $\nu \in R_0$, $p \in R_+$ *and* $x \in R_0$.

$$L_n^\nu\left(e^{pt};x\right)=\frac{I_\nu\left(nx\exp\left(\frac{p}{n+p}\right)\right)}{I_\nu(nx)}\exp\left(-\frac{\nu p}{n+p}\right),$$

$$L_n^\nu\left(te^{pt};x\right)=\frac{nx}{n+p}\frac{I_{\nu+1}\left(nx\exp\left(\frac{p}{n+p}\right)\right)}{I_\nu(nx)}\exp\left(\frac{(1-\nu)p}{n+p}\right),$$

$$L_n^\nu\left(t^2e^{pt};x\right)=\left(\frac{nx}{n+p}\right)^2\frac{I_{\nu+2}\left(nx\exp\left(\frac{p}{n+p}\right)\right)}{I_\nu(nx)}\exp\left(\frac{(2-\nu)p}{n+p}\right)+\frac{2}{n+p}\frac{nx}{n+p}\frac{I_{\nu+1}\left(nx\exp\left(\frac{p}{n+p}\right)\right)}{I_\nu(nx)}\exp\left(\frac{(1-\nu)p}{n+p}\right).$$

Similarly to Lemma 6 ([1]), using basic properties of modified Bessel function (1), we can prove

Lemma 2.4 *For each* $\nu \in R_0$ *and* $p \in R_+$ *there exists a positive constant* $M(\nu,p)$ *such that for all* $n \in N$ *and* $z \in R_0$ *we have:*

$$\frac{I_\nu\left(z\exp\left(\frac{p}{n+p}\right)\right)}{I_\nu(z)} \le M(\nu,p)\exp\left(z\left(\exp\left(\frac{p}{n+p}\right)-1\right)-\frac{p}{2(n+p)}\right).$$

Lemma 2.5 *For all* $v \in R_0$ *and* $p \in R_+$ *there exists a positive constant* $M(v,p)$ *such that for each* $n \in N$ *we have*

$$\left\| L_n^{\nu}\left(1/w_p;\cdot\right)\right\|_p \le M(\nu,p) \tag{9}$$

Proof. Pick $v \in R_0$ and $p \in R_+$. By definition (2) and Lemma 2.3 we get

$$w_p(x)L_n^{\nu}\left(1/w_p(t);x\right)=e^{-px}\exp\left(-\tfrac{\nu p}{n+p}\right)\frac{I_\nu\left(nx\exp\left(\frac{p}{n+p}\right)\right)}{I_\nu(nx)},$$

for $x \in R_0$ and $n \in N$.

Substituting $nx = z$ and applying Lemma 2.4 we get

$$w_p(x)L_n^{\nu}\left(1/w_p(t);x\right)\le M(\nu,p)\exp\left(-\tfrac{zp}{n}\right)\exp\left(-\tfrac{\nu p}{n+p}\right)\exp\left(z\left(\exp\left(\tfrac{p}{n+p}\right)-1\right)-\frac{p}{2(n+p)}\right)=$$

$$M(\nu,p)\exp\left(-\left(\nu+\tfrac{1}{2}\right)\tfrac{p}{n+p}\right)\exp\left(z\left(-\tfrac{p}{n}+\exp\left(\tfrac{p}{n+p}\right)-1\right)\right)\le M(\nu,p)$$

because

$$\exp\left(\tfrac{p}{n+p}\right)-1=\sum_{k=1}^{\infty}\left(\frac{p}{n+p}\right)^k\frac{1}{k!}<\sum_{k=1}^{\infty}\left(\frac{p}{n+p}\right)^k=\frac{p}{n},$$

so we have the following estimation:

$$\exp\left(\tfrac{p}{n+p}\right)-1<\frac{p}{n} \tag{10}$$

for $n \in N$. From these inequalities and definition (3) we obtain (9).

An obvious consequence of the above lemma and definition (3) is

Theorem 2.1 *For all* $v \in R_0$ *and* $p \in R_+$ *there exists a positive constant* $M(v,p)$ *such that for each* $n \in N$ *and* $f \in E_p$ *we have:*

$$\left\| L_n^{\nu}(f;.)\right\|_p \le M(\nu,p)\|f\|_p.$$

Now we present the crucial lemma for the approximating theorems in the next section.

Lemma 2.6 *For all* $v \in R_0$ *and* $p \in R_+$ *there exists a positive constant* $M(v,p)$ *such that for each* $n \in N$ *and* $x \in R_0$ *we have:*

$$w_p(x)\left|L_n^{\nu}\left(\frac{(t-x)^2}{w_p(t)};x\right)\right|\le M(\nu,p)\frac{x(x+1)}{n} \tag{11}$$

Proof. Let us fix $\nu \in R_0$ and $p \in R_+$.
By (3), linearity of the operator L_n^ν and Lemma 2.3 it follows that:

$$w_p(x)\left|L_n^\nu\left(\frac{(t-x)^2}{w_p(t)};x\right)\right| = e^{-px}\left|L_n^\nu\left(t^2e^{pt};x\right)-2xL_n^\nu\left(te^{pt};x\right)+x^2L_n^\nu\left(e^{pt};x\right)\right| =$$

$$e^{-px}\left|\left(\frac{nx}{n+p}\right)^2\frac{I_{\nu+2}\left(nx\exp\left(\frac{p}{n+p}\right)\right)}{I_\nu(nx)}\exp\left(\frac{p(2-\nu)}{n+p}\right)+\frac{2nx}{(n+p)^2}\frac{I_{\nu+1}\left(nx\exp\left(\frac{p}{n+p}\right)\right)}{I_\nu(nx)}\exp\left(\frac{p(1-\nu)}{n+p}\right)\right.$$

$$\left.-\frac{2nx^2}{n+p}\frac{I_{\nu+1}\left(nx\exp\left(\frac{p}{n+p}\right)\right)}{I_\nu(nx)}\exp\left(\frac{p(1-\nu)}{n+p}\right)+x^2\frac{I_\nu\left(nx\exp\left(\frac{p}{n+p}\right)\right)}{I_\nu(nx)}\exp\left(\frac{-p\nu}{n+p}\right)\right| =$$

$$e^{-px}\left(\frac{n}{n+p}\right)^2\exp\left(\frac{-p\nu}{n+p}\right)\left|x^2\frac{I_{\nu+2}\left(nx\exp\left(\frac{p}{n+p}\right)\right)}{I_\nu(nx)}\exp\left(\frac{2p}{n+p}\right)+\frac{2x}{n}\frac{I_{\nu+1}\left(nx\exp\left(\frac{p}{n+p}\right)\right)}{I_\nu(nx)}\exp\left(\frac{p}{n+p}\right)\right.$$

$$-2x^2\frac{I_{\nu+1}\left(nx\exp\left(\frac{p}{n+p}\right)\right)}{I_\nu(nx)}\exp\left(\frac{p}{n}\right)+x^2\frac{I_\nu\left(nx\exp\left(\frac{p}{n+p}\right)\right)}{I_\nu(nx)}+x^2\left(\frac{p}{n}\right)^2\frac{I_\nu\left(nx\exp\left(\frac{p}{n+p}\right)\right)}{I_\nu(nx)}$$

$$\left.x^2\frac{2p}{n}\frac{I_\nu\left(nx\exp\left(\frac{p}{n+p}\right)\right)}{I_\nu(nx)}-2x^2\frac{p}{n}\frac{I_{\nu+1}\left(nx\exp\left(\frac{p}{n+p}\right)\right)}{I_\nu(nx)}\exp\left(\frac{p}{n}\right)\right| \le$$

$$e^{-px}\exp\left(\frac{-p\nu}{n+p}\right)\left(x^2\left|\frac{\exp\left(\frac{2p}{n+p}\right)I_{\nu+2}\left(nx\exp\left(\frac{p}{n+p}\right)\right)}{I_{\nu+1}\left(nx\exp\left(\frac{p}{n+p}\right)\right)}-\exp\left(\frac{p}{n+p}\right)\right|\left|\frac{I_{\nu+1}\left(nx\exp\left(\frac{p}{n+p}\right)\right)}{I_{\nu+1}(nx)}\right|\left|\frac{I_{\nu+1}(nx)}{I_\nu(nx)}\right|+\right.$$

$$x^2\left|1-\frac{\exp\left(\frac{p}{n+p}\right)I_{\nu+1}\left(nx\exp\left(\frac{p}{n+p}\right)\right)}{I_\nu\left(nx\exp\left(\frac{p}{n+p}\right)\right)}\right|\left|\frac{I_\nu\left(nx\exp\left(\frac{p}{n+p}\right)\right)}{I_\nu(nx)}\right|+\frac{2x}{n}\exp\left(\frac{p}{n+p}\right)\left|\frac{I_{\nu+1}\left(nx\exp\left(\frac{p}{n+p}\right)\right)}{I_{\nu+1}(nx)}\right|\left|\frac{I_{\nu+1}(nx)}{I_\nu(nx)}\right|+$$

$$\left.x^2\left(\frac{p}{n}\right)^2\left|\frac{I_\nu\left(nx\exp\left(\frac{p}{n+p}\right)\right)}{I_\nu(nx)}\right|+x^2\frac{2p}{n}\left|1-\frac{\exp\left(\frac{p}{n+p}\right)I_{\nu+1}\left(nx\exp\left(\frac{p}{n+p}\right)\right)}{I_\nu\left(nx\exp\left(\frac{p}{n+p}\right)\right)}\right|\left|\frac{I_\nu\left(nx\exp\left(\frac{p}{n+p}\right)\right)}{I_\nu(nx)}\right|\right).$$

Now applying Lemmas 2.1, 2.4, estimation (10) we get (11).
The definition of the operator $L_{n,m}^{\nu,\mu}$ implies

$$L_{n,m}^{\nu,\mu}(f;x,y)=L_n^\nu(f_1;x)L_m^\mu(f_2;y) \tag{12}$$

for all functions of the form $f(x,y) = f_1(x)f_2(y)$ where $f_1 \in E_p$ and $f_2 \in E_q$, $p, q \in R_+$.
In particular we get

$$L_{n,m}^{\nu,\mu}(1;x,y)=1,$$
$$L_{n,m}^{\nu,\mu}\left(1/w_{p,q};x,y\right)=L_n^\nu\left(1/w_p;x\right)L_m^\mu\left(1/w_q;y\right).$$

From the above facts and Lemma 2.5 we derive

Lemma 2.8 *For all v, μ ∈ R_0 and p, q ∈ R_+ there exists a positive constant M(v, μ, p, q) such that for each n, m ∈ N we have:*

$$\left\|L_{n,m}^{\nu,\mu}\left(1/w_{p,q};\cdot\right)\right\|_{p,q} \le M\left(\nu,\mu,p,q\right).$$

Lemma 2.9 *For all v, μ ∈ R_0 and p, q ∈ R_+ there exists a positive constant M(v, μ, p, q) such that for each n, m ∈ N we have:*

$$\left\|L_{n,m}^{\nu,\mu}\left(f;\cdot\right)\right\|_{p,q} \le M\left(\nu,\mu,p,q\right)\left\|f\right\|_{p,q}.$$

Proof. Applying definition (7), linearity of the operator and connection (12) we get

$$w_{p,q}\left(x,y\right)\left|L_{n,m}^{\nu,\mu}\left(f\left(t,s\right);x,y\right)\right| \le w_{p,q}\left(x,y\right)L_{n,m}^{\nu,\mu}\left(\left|f\left(t,s\right)\right|;x,y\right)=$$
$$w_{p,q}\left(x,y\right)L_{n,m}^{\nu,\mu}\left(w_{p,q}\left(t,s\right)f\left(t,s\right)\frac{1}{w_{p,q}\left(t,s\right)};x,y\right)\le$$
$$\left\|f\right\|_{p,q} w_p\left(x\right)w_q\left(y\right)L_n^{\nu}\left(1/w_p\left(t\right);x\right)L_m^{\mu}\left(1/w_q\left(s\right);y\right)\le M\left(\nu,p\right)M\left(\mu,q\right)\left\|f\right\|_{p,q}$$

Hence the operator $L_{n,m}^{\nu,\mu}$ transforms the space $E_{p,q}$ into $E_{p,q}$.

3. Approximation theorems

The following theorem estimates a weighted error of approximation for functions belonging to the space $E_{p,q}^1=\left\{f\in E_{p,q}:f'\in E_{p,q}\right\}$.

Theorem 3.1 *For all v, μ ∈ R_0, p, q ∈ R_+ and for each function $g\in E_{p,q}^1$ there exists a positive constant M(v, μ, p, q) such that for all n, m ∈ N and $\left(x,y\right)\in R_0^2$ we have:*

$$w_{p,q}\left(x,y\right)\left|L_{n,m}^{\nu,\mu}\left(g;x,y\right)-g\left(x,y\right)\right|\le$$
$$M\left(\nu,\mu,p,q\right)\left(\left\|g_x'\right\|_{p,q}\frac{x+1}{\sqrt{n}}+\left\|g_y'\right\|_{p,q}\frac{y+1}{\sqrt{m}}\right).$$

The proofs of the above and the next theorems are analogous to the proofs of Theorems 3.1, 3.2 ([2]) so we omit it.

The following theorem gives a degree of approximation of functions by operators $L_{n,m}^{\nu,\mu}$.

Theorem 3.2 *For all v, μ ∈ R_0, p, q ∈ R_+ and for each function f ∈ $E_{p,q}$ there exists a positive constant M(v, μ, p, q) such that for all n, m ∈ N and $\left(x,y\right)\in R_0^2$ we have*

$$w_{p,q}\left(x,y\right)\left|L_{n,m}^{\nu,\mu}\left(f;x,y\right)-f\left(x,y\right)\right|\le M\left(\nu,\mu,p,q\right)\omega\left(f,E_{p,q};\frac{x+1}{\sqrt{n}},\frac{y+1}{\sqrt{m}}\right).$$

Theorem 3.2 implies the following corollaries.

Corollary 3.3 *If* ν, $\mu \in R_0$, *p, q* $\in R_+$ *and f* $\in E_{p,q}$ *then for all* $(x,y) \in R_0^2$

$$\lim_{n,m\to\infty} L_{n,m}^{\nu,\mu}(f;x,y) = f(x,y).$$

Moreover, the above convergence is uniform on every set $[x_1,x_2]\times[y_1,y_2]$ *with* $0 \le x_1 < x_2$, $0 \le y_1 < y_2$.

Corollary 3.4 *For all* α, $\beta \in (0,1]$, ν, $\mu \in R_0$, *p, q* $\in R_+$ *and for each f* $\in \mathrm{Lip}(E_{p,q}, \alpha, \beta)$ *there exists a positive constant* $M(\nu, \mu, p, q)$ *such that for all n, m* $\in N$ *and* $(x,y) \in R_0^2$ *we have:*

$$w_{p,q}(x,y)\left|L_{n,m}^{\nu,\mu}(f;x,y) - f(x,y)\right| \le M(\nu,\mu,p,q)\left(\left(\frac{x+1}{\sqrt{n}}\right)^{\alpha} + \left(\frac{y+1}{\sqrt{m}}\right)^{\beta}\right),$$

$$\textit{where } \mathrm{Lip}\left(E_{p,q},\alpha,\beta\right) = \left\{f \in E_{p,q} : \omega\left(f, E_{p,q};t,s\right) = O\left(t^{\alpha} + s^{\beta}\right)\right\}.$$

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