NAUKI PODSTAWOWE
CZASOPISMO TECHNICZNE
TECHNICAL TRANSACTIONS
FUNDAMENTAL SCIENCES
WYDAWNICTWO
POLITECHNIKI KRAKOWSKIEJ
1-NP/2011
ZESZYT 22
ROK 108
ISSUE 22
YEAR 108

LUDWIK BYSZEWSKI, TERESA WINIARSKA*

INTEGRODIFFERENTIAL EVOLUTION NONLOCAL PROBLEM FOR THE FIRST ORDER EQUATION

RÓŻNICZKOWO-CAŁKOWE NIELOKALNE ZAGADNIENIE EWOLUCYJNE DLA RÓWNANIA PIERWSZEGO RZĘDU

A b s t r a c t

The aim of the paper is to prove two theorems on the existence and uniqueness of mild and classical solutions of a semilinear integrodifferential evolution nonlocal Cauchy problem for the first order equation.
The method of semigroups and the Banach fixed point theorem are applied to prove the existence and uniqueness of the solutions of the problem considered.

Keyword: *integrodifferential*, *evolution*, *nonlocal problem*, *Banach spaces*

S t r e s z c z e n i e

W pracy dowodzimy dwóch twierdzeń o istnieniu i jednoznaczności całkowych i klasycznych rozwiązań semiliniowego różniczkowo-całkowego ewolucyjnego nielokalnego zagadnienia Cauchy'ego dla równania rzędu pierwszego. Aby udowodnić istnienie i jednoznaczność rozwiązań rozważanego zagadnienia stosowana jest metoda półgrup i twierdzenie Banacha o punkcie stałym.

Słowa kluczowe: *różniczkowo-całkowe, ewolucyjne, nielokalne zagadnienie, przestrzenie Banacha*

* Dr hab. Ludwik Byszewski, prof. PK, dr hab. Teresa Winiarska, prof. PK, Instytut Matematyki, Wydział Fizyki, Matematyki i Informatyki, Politechnika Krakowska.

BIBLIOTEKA CYFROWA POLITECHNIKI KRAKOWSKIEJ

1. Introduction

In this paper we prove two theorems on the existence and uniqueness of mild and classical solutions of semilinear integrodifferential evolution nonlocal Cauchy problem for a first order equation. To do it the method of semigroups and the Banach fixed point theorem will be used. The Cauchy problem considered here is of the form:

$$u'(t) + Au(t) = f(t, u(t), u(b(t))) + \int_{t_0}^{t} f_1(t, s, u(s))ds +$$

$$+\int_{t_0}^{t_0+a} f_2(t, s, u(s))ds, \quad t \in (t_0, t_0 + a], \tag{1}$$

$$u(t_0) + g(u) = u_0 \tag{2}$$

where $t_0 \geq 0$, $a > 0$, $-A$ is the infinitesimal generator of a $\mathcal{C}_0$ semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space E, f, f_i (i = 1, 2), g and b are given functions satisfying some assumptions and $u_0 \in E$.

The results obtained in the paper, are generalizations of some results from [1]–[5].

2. Preliminaries

In this paper we shall assume that E is a Banach space with norm $\|\cdot\|$, $-A$ is the infinitesimal generator of a $\mathcal{C}_0$ semigroup $\{T(t)\}_{t\geq 0}$ on E and $\mathcal{D}(A)$ is the domain od A.

Throughout the paper we shall use the notation:

$$I = [t_0, t_0 + a], \quad \text{where} \quad t_0 \geq 0 \quad \text{and} \quad a > 0,$$

$$\Delta = \{(t, s) : t_0 \leq s \leq t \leq t_0 + a\},$$

$$M = \sup\{\|T(t)\|, t \in [0, a]\}$$

and

$$X = \mathcal{C}(I, E).$$

Consider the Cauchy problem

$$u'(t) + Au(t) = k(t), \quad t \in I \setminus \{t_0\}, \tag{3}$$

$$u(t_0) = x. \tag{4}$$

A function $u : I \to E$ is said to be a classical solution of the problem (3)–(4) if:

(i) u is continuous on I and continuously differentiable on $I \setminus \{t_0\}$,

(ii) $u'(t) + Au(t) = k(t)$ for $t \in I \setminus \{t_0\}$,

(iii) $u(t_0) = x$.

Theorem 2.1. *Assume that E is a reflexive Banach space, $k : I \to E$ is Lipschitz continuous on I and $x \in \mathcal{D}(A)$.*

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Then the Cauchy problem (3)–(4) has the only one classical solution u and it is given by the formula

$$u(t) = T(t-t_0)x + \int_{t_0}^{t} T(t-s)k(s)ds, \quad t \in I.$$

Proof. One can find a proof of this theorem in [5, Section 4.2].

3. Theorem about a mild solution

A function $u \in \mathcal{C}(I, E)$ and satisfying the integral equation

$$u(t) = T(t-t_0)u_0 - T(t-t_0)g(u) + \int_{t_0}^{t} T(t-s)\big(f(s, u(s, u(b(s)))) +$$

$$+\int_{t_0}^{s} f_1(s,\tau,u(\tau))d\tau + \int_{t_0}^{t_0+a} f_2(s,\tau,u(\tau))d\tau)ds\big), \quad t \in I,$$

is said to be a mild solution of the integrodifferential evolution nonlocal Cauchy problem (1)–(2).

Theorem 3.1. *Assume that*:

(i) $f : I \times E^2 \to E$ *is continuous with respect to the first variable in I,* $f_i : \Delta \times E \to E$ (i = 1, 2) *are continuous with respect to the first and second variables on* Δ, $g : X \to E$, $b : I \to I$ *are continuous and there exist positive constants L,* L_i (i = 1, 2) *and K such that*

$$\| f(s, z_1, z_2) - f(s, \tilde{z}_1, \tilde{z}_2) \| \le L \sum_{i=1}^{2} \| z_i - \tilde{z}_i \| \quad \text{for} \quad s \in I, z_i, \tilde{z}_i \in E \quad (i = 1, 2), \tag{5}$$

$$\| f_i(s,\tau,z) - f_i(s,\tau,\tilde{z}) \| \le L_i \| z - \tilde{z} \| \quad (i = 1, 2) \quad \text{for} \quad (s,\tau) \in \Delta, z, \tilde{z} \in E \tag{6}$$

and

$$\| g(w) - g(\tilde{w}) \| \le K \| w - \tilde{w} \|_x \quad \text{for} \quad w, \tilde{w} \in X. \tag{7}$$

(ii) $M[a(2L + aL_1 + aL_2) + K] < 1.$

(iii) $u_0 \in E.$

Then the integrodifferential evolution nonlocal Cauchy problem (1)–(2) *has a unique mild solution.*

Proof. Introduce the operator $F : X \to X$ given by the formula

$$(Fw)(t) := T(t-t_0)u_0 - T(t-t_0)g(w) + \int_{t_0}^{t} T(t-s)\big(f(s.w(s)), w(b(s)) +$$

$$+\int_{t_0}^{s} f_1(s,\tau,w(\tau))d\tau + \int_{t_0}^{t_0+a} f_2(s,\tau,w(\tau))d\tau \big)ds, \quad t \in I \tag{8}$$

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From (8) and (5)–(7),

$$\begin{aligned}
\| (Fw)(t) - (F\tilde{w})(t) \| \le \| T(t-t_0) \| \, \| g(w) - g(\tilde{w}) \| + \\
+ \int_{t_0}^{t} \| T(t-s) \| \, \| f(s, w(s), w(b(s))) - f(s, \tilde{w}(s), \tilde{w}(b(s))) \| ds + \\
+ \int_{t_0}^{t} \| T(t-s) \| \left(\int_{t_0}^{s} \| f_1(s,\tau, w(\tau)) - f_1(s,\tau,\tilde{w}(\tau)) \| d\tau \right) ds + \\
+ \int_{t_0}^{t} \| T(t-s) \| \left(\int_{t_0}^{s} \| f_2(s,\tau, w(\tau)) - f_2(s,\tau,\tilde{w}(\tau)) \| d\tau \right) ds \le MK \| w - \tilde{w} \|_X + \\
+ ML \int_{t_0}^{t} \left(\| w(s) - \tilde{w}(s) \| + \| w(b(s)) - \tilde{w}(b(s)) \| \right) ds + \\
+ ML_1 \int_{t_0}^{t} \left(\int_{t_0}^{s} \| w(\tau) - \tilde{w}(\tau) \| d\tau \right) ds + ML_2 \int_{t_0}^{t} \left(\int_{t_0}^{s} \| w(\tau) - \tilde{w}(\tau) \| d\tau \right) ds \le \\
\le M(a(2L + aL_1 + aL_2) + K) \| w - \tilde{w} \|_X \quad \text{for} \quad w, \tilde{w} \in X \quad \text{and} \quad t \in I. \qquad (9)
\end{aligned}$$

If we define

$$q = M(a(2L + aL_1 + aL_2) + K)$$

then, by (9) and assumption (ii),

$$\| Fw - F\tilde{w} \| \le q \| w - \tilde{w} \|_X \quad \text{for} \quad w, \tilde{w} \in X \qquad (10)$$

with $0 < q < 1$. This shows that F is a contraction on X.

Consequently, by (10), operator F satisfies all the assumptions of the Banach fixed point theorem. Therefore, in space X there is the only one fixed point of F and this point is the mild solution of the integrodifferential evolution nonlocal Cauchy problem (1)–(2). So, the proof of Theorem 3.1 is complete.

4. Theorem about a classical solution

A function $u : I \to E$ is said to be a classical solution of the integrodifferential evolution nonlocal Cauchy problem (1)–(2) if:

(i) u is continuous on I and continuously differentiable on $I \setminus \{t_0\}$,

(ii) $u'(t) + Au(t) = f(t, u(t)), u(b(t))) + \int_{t_0}^{t} f_1(t, s, u(s)) ds + \int_{t_0}^{t_0 + a} f_2(t, s, u(s)) ds$ for $t \in I \setminus \{t_0\}$,

(iii) $u(t_0) + g(u) = u_0$.

Theorem 4.1. *Assume that*:

(i) E is a reflexive Banach space and $u_0 \in E$.

(ii) $f : I \times E^2 \to E$, $f_i : \Delta \times E \to E$ $(i = 1, 2)$ *are continuous with respect to the second variables in I,* $g : X \to E$, $b : I \to I$ *is continuous on I and there exist positive constants C,* C_i $(i = 1, 2)$ *and K such that:*

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$$\| f(s,z_1,z_2)-f(\tilde{s},\tilde{z}_1,\tilde{z}_2)\| \le C\left(|s-\tilde{s}|+\sum_{i=1}^{2}\| z_i-\tilde{z}_i\|\right) \tag{11}$$
$$for \quad s,\tilde{s}\in I, z_i,\tilde{z}_i\in E \quad (i=1,2),$$

$$\| f_i(s,\tau,z)-f_i(\tilde{s},\tau,\tilde{z})\| \le C_i\left(|s-\tilde{s}|+\| z-\tilde{z}\|\right) \quad (i=1,2) \tag{12}$$
$$for \quad (s,\tau),(\tilde{s},\tau)\in\Delta, z,\tilde{z}\in E$$

and

$$\| g(w)-g(\tilde{w})\| \le K\| w-\tilde{w}\|_X \quad for \quad w,\tilde{w}\in X. \tag{13}$$

(iii) $M[a(2C+aC_1+aC_2)+K]<1.$

Then the integrodifferential evolution nonlocal Cauchy problem (1)–(2) *has a unique mild solution (which is denoted by) u. Moreover, if $u_0 \in \mathcal{D}(A)$, $g(u) \in \mathcal{D}(A)$ and if there exists a positive constant $\mathcal{H}$ such that:*

$$\|u(b(s))-u(b(\tilde{s}))\| \le H\|u(s)-u(\tilde{s})\| \quad for \quad s,\tilde{s}\in I \tag{14}$$

then u is the unique classical solution of the problem (1)–(2).

Proof. Since all the assumptions of Theorem 3.1 are satisfied, it is easy to see that the problem (1)–(2) possesses a unique mild solution which according to the last assumption is denoted by u.

Now, we shall show that u is the classical solution of the problem (1)–(2). To this end, introduce:

$$N := \max_{s\in I}\| f(s,u(s),u(b(s)))\|, \tag{15}$$

$$N_i := \max_{(s,\tau)\in\Delta}\| f_i(s,\tau,u(\tau))\| \quad (i=1,2) \tag{16}$$

and observe that:

$$\begin{aligned}
&u(t+h)-u(t)=(T(t+h-t_0)u_0-T(t-t_0)u_0)-(T(t+h-t_0)g(u)-T(t-t_0)g(u))+\\
&+\int_{t_0}^{t+h}T(t+h-s)(f(s,u(s),u(b(s)))+\int_{t_0}^{s}f_1(s,\tau,u(\tau))d\tau+\int_{t_0}^{t_0+a}f_2(s,\tau,u(\tau))d\tau)ds+\\
&-\int_{t_0}^{t}T(t-s)(f(s,u(s),u(b(s)))+\int_{t_0}^{s}f_1(s,\tau,u(\tau))d\tau+\int_{t_0}^{t_0+a}f_2(s,\tau,u(\tau))d\tau)ds=\\
&=T(t-t_0)(T(h)-I)u_0-T(t-t_0)(T(h)-I)g(u)+\int_{t_0}^{t_0+h}T(t+h-s)(f(s,u(s),u(b(s)))+\\
&+\int_{t_0}^{s}f_1(s,\tau,u(\tau))d\tau+\int_{t_0}^{t_0+a}f_2(s,\tau,u(\tau)d\tau)ds+\int_{t_0}^{t}T(t-s)\big(f(s+h,u(s+h),u(b(s+h)))+\\
&-f(s,u(s),u(b(s)))\big)ds+\int_{t_0}^{t}T(t-s)\left(\int_{t_0}^{s}(f_1(s+h,\tau,u(\tau))-f_1(s,\tau,u(\tau)))d\tau\right)ds+\\
&+\int_{t_0}^{t}T(t-s)\left(\int_{s}^{s+h}f_1(s+h,\tau,u(\tau))d\tau\right)ds+\int_{t_0}^{t}T(t-s)\left(\int_{t_0}^{t_0+a}f_2(s+h,\tau,u(\tau))-f_2(s,\tau,u(\tau))d\tau\right)ds
\end{aligned} \tag{17}$$

for $t\in[t_0,t_0+a]$, $h>0$ and $t+h\in(t_0, t_0+a]$.

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Consequently, by (17), (15), (16), (11), (12) and (14),

$$\begin{aligned}\|u(t+h)-u(t)\| &\le hM\|Au_0\|+hM\|Ag(u)\|+hM(N+aN_1+aN_2)+\\&+MCah+MC\int_{t_0}^{t}\left(\|u(s+h)-u(s)\|+\|u(b(s+h))-u(b(s))\|\right)ds\le\\&\le Ma^2C_1h+MaN_1h+Ma^2C_2h=\\&=C_*h+MC\int_{t_0}^{t}\left(\|u(s+h)-u(s)\|+\|u(b(s+h))-u(b(s))\|\right)ds\le\\&\le C_*h+MC(1+\mathcal{H})\int_{t_0}^{t}\|u(s+h)-u(s)\|ds \qquad (18)\end{aligned}$$

for $t \in [t_0, t_0 + a)$, $h > 0$ and $t + h \in (t_0, t_0 + a]$.

From (18) and Gronwall's inequality:

$$\|u(t+h)-u(t)\|\le C_*e^{aMC(1+\mathcal{H})}h$$

for $t \in [t_0, t_0 + a)$, $h > 0$ and $t + h \in (t_0, t_0 + a]$. Hence u is Lipschitz continuous on I.

The Lipschitz continuity of u on I combined with the Lipschitz continuity of f on $I \times E^2$ and f_i ($i = 1, 2$) with respect to the first variables in I imply that the function:

$$I \ni t \to f(t,u(t),u(b(t)))+\int_{t_0}^{t} f_1(t,s,u(s))ds+\int_{t_0}^{t_0+a} f_2(t,s,u(s))ds$$

is Lipschitz continuous on I. This property of f together with the assumptions of Theorem 4.1 imply , by Theorems 2.1 and 3.1, that the linear Cauchy problem:

$$\upsilon'(t)+A\upsilon(t)=f(t,u(t),u(b(t)))+\int_{t_0}^{t} f_1(t,s,u(s))ds+\int_{t_0}^{t_0+a} f_2(t,s,u(s))ds, \quad t\in I\setminus\{t_0\},$$

$$\upsilon(t_0)=u_0-g(u)$$

has a unique classical solution υ and it is given by:

$$\begin{aligned}\upsilon(t)=T(t-t_0)u_0-T(t-t_0)g(u)+\int_{t_0}^{t}T(t-s)\Big(f(s,u(s),u(b(s)))+\\+\int_{t_0}^{s} f_1(s,\tau,u(\tau))d\tau+\int_{t_0}^{t_0+a} f_2(s,\tau,u(\tau))d\tau\Big)ds=u(t), \quad t\in I.\end{aligned}$$

Consequently, u is the unique classical solution of the integrodifferential evolution nonlocal Cauchy problem (1)–(2) and, therefore, the proof of Theorem 4.1 is complete.

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