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JAN KUCWAJ*

# VOLUME MESHING USING A COUPLED DELAUNAY TRIANGULATION WITH ADVANCING FRONT TECHNIQUE – MATHEMATICAL FOUNDATION

# TRIANGULACJA OBSZARÓW PRZESTRZENNYCH METODĄ POSTĘPUJĄCEGO FRONTU POŁĄCZONĄ Z TRIANGULACJĄ DELAUNAYA – PODSTAWY MATEMATYCZNE

A b s t r a c t

The paper presents an algorithm of volume meshing by using the Advancing Front Technique (AFT) combined with the Delaunay Triangulation. The tetrahedronization starts with the surface mesh with elements oriented towards the interior 3-D domain. The main idea is based upon AFT, with simultaneous points insertion and tetrahedra creation. The characteristic feature of the approach is the part of AFT in case, when a new calculated point on the current face of the front is not accepted then the existing point in the front is found to create a new tetrahedron by using Delaunay triangulation on the given set of points. Additionally the algorithm takes into account a mesh size function.

*Keyword*: *grid generation*, *grid adaptation*, *Delaunay triangulation*, *advancing front technique*, *geometric modelling*

S t r e s z c z e n i e

Artykuł zawiera podstawowe definicje i własności podziału Dirichleta, wielościanów Voronoi oraz triangularyzacji Delaunaya. W dalszej części przedstawione są twierdzenia Delaunaya będące podstawą algorytmu triangularyzacji łączącego metody frontowe z triangularyzacją Delaunaya. Następnie przedstawiony jest algorytm łączący triangularyzację Delaunaya z metodą postępującego frontu. Artykuł kończy punkt w wynikami numerycznymi w postaci graficznej.

*Słowa kluczowe*: *generowanie siatek*, *adaptacja*, *triangularyzacja Delaunaya*, *metoda postępującego frontu*, *modelowanie geometryczne*

* PhD. Jan Kucwaj, Institute of Computer Modelling, Faculty of Physics, Mathematics and Computer Science, Cracow University of Technology.

## 1. Introduction

The purpose of the paper is to describe a method of grid generation for 3-D domains. The problem has an important role in finite element applications. According to generation methods, we have the following classification:

– structured grid generation techniques,
– unstructured grid generation techniques.

Generation of structured grids takes less computer time, but the range of geometries is restricted. The only possibility to control the size of the meshes is to place points over the surfaces, and then to fix the height of every tetrahedron.

In the case of unstructured volume grids, if the meshes generated over the surfaces have the required sizes, then by using the Advancing Front Technique the placement of new points can be controlled. Additionally, the Advancing Front Technique gives the possibility of tetrahedronization of complex geometries [14]. Unfortunately, the algorithms of generation of unstructured grids are much more complicated. The program's performance is much more time consuming. However, when the Advancing Front Technique is combined with Delaunay triangulation, then very neatly looking meshes can be obtained. These methods have been investigated by Lo, George, Frey, Borouchaki, Wang [2, 19].

The presented here approach is based mainly upon AFT with simultaneous points insertion and triangulation [7, 8], but when different parts of the front "meet" themselves and the point to be inserted at this time is not accepted, then on the considered triangle of the front the tetrahedron is built by using as a lacking vertex the point from the front. It is done by using the idea of the Delaunay triangulation.

Unstructured grid generation is connected with the creation of a mesh on the boundary of a domain [13, 19], i.e. in this case, a closed surface or a disjoint union of surfaces being the boundary of the domain. If the domain is topologically and geometrically complicated the problem of surface grid generation becomes more complex.

The paper consists of 10 sections. Sections 1-4 contain definitions and theorems about Voronoi tesselation and Delaunay triangulation. The author presents only those proofs, which are done originally by him. The presented here theory gives the necessary knowledge on the subject not known in the literature of this form with a strict mathematical approach. Sections 5-9 contain the data structure and the algorithm of volume meshing. Section 10 presents numerical results. The proposed approach and performed computer code give (in comparisons with others, for example [17, 19, 15]) the possibility of adaptation [9] by appropriate mesh size function modification. In the computer code a mesh size function is defined as a routine.

The new elements of the paper are:

– the method of point creation over a chosen face of the front for a new tetrahedron insertion,
– the algorithm of choosing points of the front for the sake of a new element creation,
– taking into account a mesh size function,
– an auxiliary test for checking front-face intersection,
– a new data structure, own computer code and own numerical examples,
– the form of the theoretical approach on the Delaunay triangulation.

## 2. An introduction to simplexes

At this point the principal information about simplex and its properties are introduced. More advanced material can be found in the [16] paper.

It is assumed, that we have a system of $(n + 1)$ points $\mathbf{P}_0, \mathbf{P}_1, ..., \mathbf{P}_n$ in the considered $n$-dimensional space, which do not lie in any hyperplane.

***Definition* 1**. *A simplex S defined on the set of points* $\mathbf{P}_0, \mathbf{P}_1, ..., \mathbf{P}_n$ *is the set of points*:

$$S = \mathbf{P}_0\mathbf{P}_1...\mathbf{P}_n = \left\{ \mathbf{P} = \sum_{i=0}^{n} \lambda_i \mathbf{P}_i : \sum_{i=0}^{n} \lambda_i = 1, \quad \lambda_i > 0 \right\} \tag{1}$$

*The number n is called its dimension.*

In the case, when $n = 2$ we have 3 points $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2$ not lying on one straight line. The simplex de.ned on this set is a triangle.

If $n = 3$ the simplex with its vertices placed at $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ not lying in one hyperplane determines a tetrahedron with vertices located at these points.

***Definition* 2**. *Coeffiecients* $\lambda_0, \lambda_1, \dots, \lambda_n$ *are called barycentric coordinates of the* $\mathbf{P} = \sum_{i=0}^{n} \lambda_i \mathbf{P}_i$.

It is easy to verify that barycentric coordinates are uniquely determined, thus the following lemma can be proved:

***Lemma* 1**. *If the system of points* $\mathbf{P}_0, \mathbf{P}_1, ..., \mathbf{P}_n$ does not lie in one hyperplane in $\mathbb{R}^n$, *then the system of the vectors* $\mathbf{P}_0\mathbf{P}_1, \mathbf{P}_0\mathbf{P}_2, ..., \mathbf{P}_0\mathbf{P}_n$ *is linearly independent*.

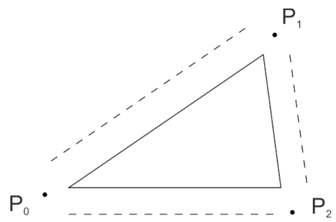


Fig. 1. Partition of a simplex into subsimplexes

Rys. 1. Podział sympleksu na podsystemy

Let us assume that in the considered $n$-dimensional space there is a given set of points $\mathbf{P}_0, \mathbf{P}_1, ..., \mathbf{P}_n$, which do not lie in one hyperplane. The following definition defines subsimplexes which construct the boundary of a simplex in $n$-dimensional space.

***Definition* 3**. *The following set*:

$$\mathbf{P}_{i_0}\mathbf{P}_{i_1}...\mathbf{P}_{i_k} = \left\{ \mathbf{P} = \sum_{j=0}^{k} \lambda_j \mathbf{P}_{i_j} : \sum_{j=0}^{k} \lambda_j = 1, \quad \lambda_j > 0 \right\} \tag{2}$$

*is called the face (edge) of the size* $0 \le k < n$ of the simplex $S = \mathbf{P}_0\mathbf{P}_1\mathbf{P}_2 ... \mathbf{P}_n$.

Additionally, the set $\mathbf{P}_0, \mathbf{P}_1, ..., \mathbf{P}_n$ of points is included in faces (edges). The following formula is true [5, 16] (figure 1):

$$\overline{S} = \bigcup_{k=0}^{n} \bigcup_{\alpha=\{i_0, i_1, ..., i_k\}} \mathbf{P}_{i_0}\mathbf{P}_{i_1}...\mathbf{P}_{i_k} \tag{3}$$

where $\{i_0, i_1, ..., i_k\}$ is a $(k + 1)$ – element combination from the set $\{0, 1, 2, ..., n\}$, and $\overline{S}$ means the closure of the $S$.

***Remark* 1**. *Because all the components of the formula (3) are open sets in the topology induced in an appropriate k-dimensional hyperplane (including vertices of the simplex), so none of the components can be omitted, and all of them are disjoint.*

For example, for $n = 2$ we have a simplex being a triangle $\mathbf{P}_0\mathbf{P}_1\mathbf{P}_2$ (figure 1) and its edges are simplexes $\mathbf{P}_0\mathbf{P}_1$, $\mathbf{P}_0\mathbf{P}_2$, $\mathbf{P}_1\mathbf{P}_2$, and certain points $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$ are 0-dimensional simplexes. Figure 1 illustrates the disjointness among: the triangle, its sides and its vertices. In a general case, we have: $\binom{n+1}{n} = n+1$ $(n – 1)$-dimensional faces.

For example, for $n = 3$ we have four faces and $\binom{n+1}{2} = \binom{4}{2} = 6$ $l$-dimensional simplexes (edges) and $\binom{4}{1}$ 0-dimensional simplexes (vertices).

### 2.1. Triangulation of arbitrary system of points in $I\!R^n$

Let us assume that there is a given set $\mathcal{P} = \{\mathbf{P}_1, ..., \mathbf{P}_m\}$ of points in $I\!R^n$ satisfying the following condition: $\forall i \neq j$ for $i, j = 1, 2, ..., m$ and not lying in one hyperplane. The following de.nition is introduced:

***Definition* 4**. *A set of simplexes (triangles)* $\{T_i\}_{i=1}^{N_T}$ *in n-dimensional space is called a triangulation of a set* $\mathcal{P}$*, if*:

(i) $\bigcup_{i=1}^{N_T} \overline{T_i}$ *– is a convex set,*

(ii) *All the vertices of the triangles belong to the set* $\mathcal{P}$*,*

(iii) *Every point* $\mathbf{P}i$, $i = 1, ..., m$ *is a vertex of at least one* $T_k$*, for* $1 \le k \le N_T$*,*

(iv) $\forall i \neq j$ $i, j = 1, 2, ..., N_T$ : $T_i \bigcap T_j = \emptyset$.

It can be proved that so de.ned triangulation .lls the convex hull of the set $\mathcal{P}$.

### 3. Dirichlet Tesselation

We assume that there is given a set of points $\mathbf{P}_i \in \mathbb{R}^n$, where $i$ = 1, ..., $N$; $\mathbf{P}_i \neq \mathbf{P}_j$, for $i, j$ = 1, 2, ..., $N$, $i \neq j$; and that these points do not lie in one hyperplane. Let us introduce the following definition [4]:

***Definition* 5**. *The Voronoi set associated with the point* $\mathbf{P}_i$, for $i$ = 1, ..., $N$, *is called the set*:

$$V_i = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{P}_i\| < \|\mathbf{x} - \mathbf{P}_j\| \ \forall j = 1,2,\ldots,N,\ j \neq i\} \tag{4}$$

where

$$\|\mathbf{x}\| = \sqrt{\sum_{i=1}^{n} x_i^2} \quad \forall \mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n \tag{5}$$

The following lemma is connected with Voronoi polygons properties and can be proved by standard algebra tools.

***Lemma* 2.** *Let* $\mathbf{A},\mathbf{B} \in \mathbb{R}^n \quad \mathbf{A} \neq \mathbf{B}$,

$$U = \left\{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{A}\| = \|\mathbf{x} - \mathbf{B}\|\right\} \tag{6}$$

*then U is a hyperplane with normal n* = **AB** *and point* $\mathbf{C} = \frac{\mathbf{A}+\mathbf{B}}{2} \notin U$.

The following theorem describes the properties of Voronoi files:

***Theorem* 1.** *Let* $\mathcal{P} = \{\mathbf{P}_i : i = 1, ..., N\}$ *(N may be equal to infinity) be a set of points in* $\mathbb{R}^n$, *then Voronoi files have the following properties*:

(i) $V_i = \bigcap_{j=1,\, j\neq i}^{N} \{x \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{P}_i\| < \|\mathbf{x} - \mathbf{P}_j\|\}$,

(ii) $V_i$ are open sets,

(iii) $\bigcup_{j=1}^{N} \overline{V}_j = \mathbb{R}^n$, *i.e. Voronoi sets fill the whole n-dimensional space*,

(iv) *The boundary of Voronoi files consists of a finite number of sets obtained as the result of an intersection of hyperplanes with halfspaces, what means that every Voronoi file is a polygon.*

(v) *Voronoi polygons are disjoint*,

(vi) *Voronoi polygons are convex sets.*

Proof:

(i) The proof is a direct consequence of the de.nition number 5,

(ii) The set $\{x \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{P}_i\| < \|\mathbf{x} - \mathbf{P}_j\|\}$ for $i, j$ = 1, ..., $N$, $j \neq i$ is an open set as an inverse image of the interval $(-\infty, 0)$ through a continuous mapping: $f(\mathbf{x}) = \|\mathbf{x} - \mathbf{P}_i\| - \|\mathbf{x} - \mathbf{P}_j\|$. A Voronoi polygon $V_i$ is a finite intersection of files of this type. Because the intersection of finite number of open files is an open set it means that the point (ii) is proven.

(iii) It is obvious that $\bigcup_{j=1}^{N} \overline{V}_j \subset I\!R^n$. We will prove that $I\!R^n \subset \bigcup_{j=1}^{N} \overline{V}_j$. Let $\mathbf{x} \in I\!R^n$ and:

$$\|\mathbf{x}-\mathbf{P}_k\| = \min\{\|\mathbf{x}-\mathbf{P}_1\|, \|\mathbf{x}-\mathbf{P}_2\|, \ldots, \|\mathbf{x}-\mathbf{P}_N\|\}, \quad \text{for some} \quad k \in \{1, \ldots, N\} \tag{7}$$

as the finite set has a minimum, so $\|\mathbf{x}-\mathbf{P}_k\| \le \|\mathbf{x}-\mathbf{P}_N\|$ for $\forall j = 1, \ldots, N, j \ne k$. Because the closure of an intersection of finite sets equals to the intesection of their closures and with respect to the point (i) we have: $\mathbf{x} \in \overline{V}_k$, what proves (iii).

(iv) This point follows from the de.nition 5, lemma 2, formula of the point (i) and from the following formula describing the boundary of the intersection set:

$$\partial(A \cap B) = (\overline{A}) \cap \partial(B) \bigcup (\overline{B}) \cap \partial(A) \tag{8}$$

where $\partial(A)$ is the boundary of $A$, and $\overline{A}$ denotes the closure of $A$.

(v) On the contrary, it is assumed that $\exists \mathbf{x} \in V_i \cap V_j$, for some $i, j \in \{1, 2, ..., N\}$, $i \ne j$, also $\mathbf{x} \in V_i$ and $\mathbf{x} \in V_j$. From the first relation and the definition 5 it follows that:

$$\forall k = 1, \ldots, N, \; k \ne i : \|\mathbf{x}-\mathbf{P}_i\| < \|\mathbf{x}-\mathbf{P}_k\| \tag{9}$$

It means that $\|\mathbf{x}-\mathbf{P}_i\| < \|\mathbf{x}-\mathbf{P}_k\|$.

Because of symmetry, by interchanging i with j we analogously obtain:

$\|\mathbf{x}-\mathbf{P}_j\| < \|\mathbf{x}-\mathbf{P}_i\|$. The first inequality denies the second one.

(vi) From the point (i) we have:

$$V_i = \bigcap_{j=1}^{N} H_i, \quad \text{where} \quad H_i = (\mathbf{x} \in I\!R^n : (\mathbf{Ax}, \mathbf{n}) > 0 \tag{10}$$

As the intersection of convex sets is a convex set, it is enough to prove, that the set $H_i$ is convex. Let $\mathbf{x}, \mathbf{y} \in H_i$ for $i = 1, ..., N$. We will prove that $\lambda\mathbf{x} + \mu\mathbf{y} \in H_i$ as $\lambda + \mu = 1$ and $\lambda, \mu \ge 0$. From the above assumptions we have:

$$\lambda(\mathbf{Ax}, \mathbf{n}) \ge 0 \quad \text{and} \, \mu(\mathbf{Ay}, \mathbf{n}) \ge 0 \tag{11}$$

$$\text{what leads to:} \quad (\mathbf{A}(\lambda\mathbf{x}),) \ge 0 \quad \text{and} \quad (\mathbf{A}(\mu\mathbf{y}), n) \ge 0 \tag{12}$$

Adding the inequalities (11), (12), and from the fact that at least one of them is sharp ($\lambda + \mu = 1$), eventually we obtain: $(\mathbf{A}(\lambda\mathbf{x} + \mu\mathbf{y}), \mathbf{n}) > 0$, what means that $\lambda\mathbf{x} + \mu\mathbf{y} \in V_i$.

## 4. A system of points in a space and the "walking sphere"

The following lemmas and theorems are connected with the infinite set of points in $I\!R^n$, which are the vertices of simplexes [5] covering the whole space. In the 2-D case it will be a family of triangles, but in the 3-D case, it is a family of tetrahedra. The assumption that the points are the vertices of a tetrahedra is to ensure the "uniform" overspreading of points in the whole space. It is easy to check that such a family is countable. The following lemma can be proved:

***Lemma* 3**. *Exactly one sphere passes through n* + 1 *points* (*in* $\mathbb{R}^n$ ), *which do not lie in one hyperplane.*

It is easy to prove the following lemma:

***Lemma* 4**. *The common part of two intersecting spheres lies in a uniquely determined hyperplane*.

The purpose of the paper is to present the Delaunay idea [4, 17], which is called the idea of an empty sphere, i.e. the sphere that does not have in its interior any points from the considered set. Such a sphere walking in a space with varying radius and center touches ($n$ + 1) points, which determine a simplex. This idea is the foundation of triangulation methods [6, 19].

***Definition* 6**. *The difference between the square of the distance from a sphere center and a given point and the square of this sphere radius is called the power of the point with respect to a sphere in* $\mathbb{R}^n$.

**Remark 2**. *In the case of a plane the power of a point with respect to a circle is the square of the line segment of tangential line to the circle passing throughout the point and determined by the point and the point of tangency.*

***Theorem* 2**. *(I Delaunay theorem) By the following assumptions*:

(i) *Let* $\mathcal{T} = \{T_i\}_{i=1}^{\infty}$ *be a family of simplexes filling* $\mathbb{R}^n$, i.e. $\mathbb{R}^n = \bigcup_{i=1}^{\infty} \overline{T_i}$ $\forall i \neq j$ $T_i \cap \mathrm{T}_j = \emptyset$,

(ii) *An arbitrary bounded nonempty set has common points with finite number of simplexes from* $\mathcal{T}$,

(iii) Let $\mathcal{P} = \{P_i\}_{i=1}^{\infty}$ *be the set of all vertices of all symplexes from the family* $\mathcal{T}$,

(iv) *Let* $K_T$ *indicates the ball passing throughout n* + 1 *vertices of the simplex* $T \subset \mathbb{R}^n$, *we have the following conclusion:*
$\forall T_i, T_j \in \mathcal{T}$ $K_{T_i}$ *does not contain vertices of* $T_j$ *in its interior (and vice versa), if and only if* $\forall T_i, T_j \in \mathcal{T}$ *that satisfy the condition:* $\overline{T_i} \cap \overline{T_j} =$ *{(n –* 1*)-dimensional simplex}* $K_{T_i}$ *does not contain vertices* $T_j$ *in its interior (and vice versa). (We say then about mutual Delaunay position)*

Proof can be found in [4].

***Definition* 7.** *The system of points* $S = \{S_1, ..., S_k\} \subset \mathcal{P} \subset \mathbb{R}^n$ *has a singular placement, if* $k > n + 1$, *all of its points belong to a sphere in* $\mathbb{R}^n$ *and the sphere does not contain any points from the set* $\mathcal{P}$.

***Definition* 8.** *A set of points* $\mathcal{P} \subset \mathbb{R}^n$ *is called singular, if there exists a subsystem with singular placement* $\mathcal{S} \subset \mathcal{P}$, *otherwise it is called nonsingular.*

***Remark* 3.** *If a system of points in the* $\mathbb{R}^n$ *space is singular, then it can be transformed into a nonsingular system by choosing a linear mapping "infinitesimaly" differing from* *the identity. It means that by a "small" movement of points in a singular system it is possible to turn this system of points into a nonsingular one.*

Indeeed, if we add to a unit matrix representing an identity matrix an apriopriate matrix (such a matrix exists) with its norm close to zero, then the system of points after this transformation will not be singular. It follows from the fact that linear mapping does not keep points on the sphere.

Further, we will assume that we have to deal with nonsingular systems of points. At present, the definition of the infinite system of points "uniformly" distributed over the space will be provided.

Let the system of points $\mathcal{P}$ be a nonsingular system of points $\mathcal{P} \subset I\!R^n$, being the vertices of simplexes, closures of which cover $I\!R^n$. Additionaly, it is assumed that in every bounded set there is a finite number of points from $\mathcal{P}$. Taking into account these considerations the triangulation procedure will be as follows:

a) Starting from an arbitrarily chosen point $\mathbf{P} \in \mathcal{P}$ we can find point $\mathbf{P}' \in \mathcal{P}$, such that the lowest ball K with a sphere $L$ containing segment $\mathbf{PP}'$ has no other points from $\mathcal{P}$ in its interior. By increasing the radius of the sphere $L$ so as to make points $\mathbf{P}$, $\mathbf{P}' \in L$ we get the next point $\mathbf{P}'' \in L$. We continue the procedure till the moment, when we obtain $(n + 1)$ points from $\mathcal{P}$ on the $L$ and none in the $S$.
b) Let us denote by $W_1$ a simplex defined by those $(n + 1)$ points. Such a simplex exists, because considered points do not lie in one hyperplane. Otherwise, it would be possible to augment the radius (keeping these points on the sphere) and to find $(n + 2)$ points lying on it, but it would deny the nonsingularity of the system $\mathcal{P}$.
c) By choosing any from the $(n + 1)$ faces of $W_1$ and moving the center of the sphere, the ball passes throughout the vertices of $W_1$ and "extending it" into the exterior of the chosen face, but keeping the vertices of the face on the sphere ($n$ points from $\mathcal{P}$) till the moment of "catching" a point from $\mathcal{P}$ – a ball is obtained. The obtained ball does not contain in its interior any points belonging to $\mathcal{P}$. Its sphere passes through $(N + 1)$ points from $\mathcal{P}$. The points define the simplex $W_2$. The procedure is kept going till all points of $\mathcal{P}$ become the vertices of at least one of all obtained simplexes.

***Definition* 9.** *The obtained system of simplexes* $\{W_j\}_{j=1}^{\infty}$ *is called the Delaunay triangulation of the set* $\mathcal{P}$.

***Theorem* 3.** *Let* $\mathcal{T} = \{T_i : i = 1,..., \infty\}$ *be the family of simplexes satisfying the conditions:*

*An arbitrary bounded set in* $I\!R^n$ *contains a finite number of simplexes from* $\mathcal{T}$,

$\forall T_i, T_j \in \mathcal{T} \;\; i \neq j \quad T_i \bigcap T_j = \emptyset, \;\; \bigcup_{i=1}^{\infty} \overline{T}_i = I\!R^n.$

$\mathcal{P} = \{P_i : i = 1, \dots, \infty\}$ *is the set of vertices of the family* $\mathcal{T}$.

$\mathcal{T}_D = \{T_i^D : i = 1, ..., \infty\}$ *is the set of Delaunay simplexes constructed on the set* $\mathcal{P}$.

*The following theses are valid:*

a) $\bigcup_{i=1}^{\infty} \overline{T}_i^D = \mathbb{R}^n$,

b) $\overline{T}_i^D \bigcap \overline{T}_j^D$ *is an empty set or a simplex of the size smaller than n.*

Proof:

a) It is obvious that $\bigcup_{i=1}^{\infty} \overline{T}_i^D \subset \mathbb{R}^n$.

Let us take an arbitrary point $\mathbf{x} \in \mathbb{R}^n$. If $\mathbf{x}$ is a vertex of one $T_i^D$ then $\mathbf{x} \in \bigcup_{i=1}^{\infty} \overline{T}_i^D$, otherwise there is an empty ball with a radius $r > 0$ that contains $\mathbf{x}$. In the next step we will augment its radius, while not changing its center, till we meet a vertex. Keeping that vertex on the sphere we move the center of the ball, so as the point $\mathbf{x}$ leaves on the sphere till catching the next vertex on the sphere. Continuing the procedure we obtain an empty ball with $(n + 1)$ points from $\mathcal{P}$ lying on its sphere. If the simplex $T_1$ denoted by $(n + 1)$ points contains $\mathbf{x}$, then the proof is finished. Otherwise, we do as follows:

We find the $(n - 1)$-dimensional face closest to $\mathbf{x}$. Let $T_2$ denotes the simplex having the face common with $T_1$. By the similar reasoning – if $\mathbf{x} \in T_2$, the proof is finished, otherwise by using the same rule we find the simplex $T_3$.

The process must be stopped after a finite number of steps, because according to the beginning assumption, in every bounded set there is a finite number of vertices from $\mathcal{P}$ and $\mathbf{x}$ must belong to some $T_k$.

b) Suppose, we have two arbitrary different simplexes $T_1^D$, $T_2^D$ from the set $\mathcal{T}$. Let $K_1$ and $K_2$ be the spheres passing through the vertices of the $T_1^D$, $T_2^D$. If $K_1$ and $K_2$ are disjoint or tangential, then $T_1^D$ and $T_2^D$ are disjoint too.

In the case, when $\overline{K}_1 \bigcap \overline{K}_2 \neq \emptyset$ and $K_1$ or $K_2$ is not tangent, by applying lemma 4 we conclude that the common part uniquely denotes the hyperplane containing $\overline{K}_1 \bigcap \overline{K}_2$. Every one of the simplexes $T_1^D$ and $T_2^D$ has its vertices on one side of the hyperplane, otherwise the condition of the "empty ball" would not be satisfied. Because the system is nonsingular, there are no more than $n$ of those points. It is like this, because on the common part contained in the hyperplane there are at least $n$ points, which form a simplex, whose size does not exceed $n$.

***Remark* 4.** *The introduction of the notion of the family of simplexes $\mathcal{T}$ is used only in the dedifinition of points uniformly distributed over the whole n dimensional space, and it will be applied in this sense in the next theorem.*

***Theorem* 4**. (II Delaunay theorem)

*Let $\mathcal{T} = \{T_i : i = 1, ..., \infty\}$ be the family of simplexes satisfying the condition*:

1. $\forall T_i, T_j \in \mathcal{T} \quad i \neq j \quad T_i \bigcap T_j = \emptyset$,

2. $\bigcup_{i=1}^{\infty} \overline{T_i} = I\!R^n$,
3. $\mathcal{T}_D = \{T_i^D : i = 1, ..., \infty\}$ *is a set of Delaunay simplexes,*
4. $\mathcal{P} = \{\mathbf{P}_i : i = 1, \dots, \infty\}$ *is a set of the vertices of those simplexes (we assume that the set is nonsingular),*
5. $\{V_i : i = 1, \dots, \infty\}$ *is a Voronoi set.*
6. *Let* $\mathcal{Q}$ $\{\mathbf{Q}_i : i = 1, \dots, \infty\}$ *be the set of the vertices of Voronoi polygons and let for each* $i = 1, \dots, \infty : W_i = \{V_k : \mathbf{Q}_i \in \overline{V_k}\}$.

*From these assumptions we have:*

a. *A set of points* $\{P_k\}_{k=1}^{L} \subset \mathcal{P}$, *which describes Voronoi polygons from the set* $W_i$ *contains an "empty" ball (not containing points from* $\mathcal{P}$*) with the center in* $\mathbf{Q}i$,
b. $L = n + 1$ *and the points* $\{\mathbf{P}_k\}_{k=1}^{L}$ d*escribe Delaunay simplex, where* $L$ *is the number of elements of the set* $W_i$.

Proof can be found in [4].

## 5. Domain representation and data structure description

In this section the following notations will be used:
**F** – a face,
**T** – a tetrahedron.

It is assumed that there exists a generated mesh on the boundary surface of the three dimensional domain Ω. The boundary surface is represented by the following sets of data:

$\mathcal{P}_b = \{\mathbf{P}_i : i = 1, ..., N_{\mathcal{P}}\}$ – the set of boundary points,

$\mathcal{F}_b = \{Fi : i = 1, ..., N_{\mathcal{F}}\}$ – the set of triangles on the boundary surface Ω.

For the sake of data structure description every boundary triangle is represented by three boundary points from $\mathcal{P}_b$. In other words, an arbitrary surface element is represented as a triple of integer numbers, being the number of its vertices.

Furthermore, for the representation volume mesh the following sets are introduced:

$\mathcal{F} = \{F_i : i = 1, ..., N_{\mathcal{F}}\}$ – the set of triangles being the faces of the tetrahedra Ω,

$\mathcal{T} = \{T_i : i = 1, ..., N_{\mathcal{T}}\}$ – the set of the tetrahedra filling Ω.

In the data structure every tetrahedron is represented by its numbers of the four faces. Additionally, the following sets necessary for the algorithm representation are introduced:

$\mathcal{B}$ – a generation front, at the start consisting of boundary faces,

$\mathcal{E}$ – the set of edges of B.

## 6. A boundary surface orientation

A single triangle on the boundary surface can be oriented as every orientable surface by its normal. According to the general algorithm of tetrahedronization it is necessary to make an orientation with normal towards interior of the domain $\Omega$. The applied algorithm is based on the papers of [11, 12].

It is based on establishing the orientation of the arbitrarily chosen boundary face. Then orientations of the rest of the surface triangles are adapted progressively to that one. For the algorithm realization the Advancing Front Technique is used (AFT).

## 7. The orientation of an arbitrary surface triangle

If we have an arbitrary triangle it is checked whether the normal to the face determined by its orientation is inward or outward the domain $\Omega$. The half-line predicted by this normal is led from the barycenter of the triangle. If the half-line intersects the boundary surface the odd number of times, then the normal is inward the domain; otherwise it is outward.

As it was previously mentioned, the triangle $T$ is represented by the sequence of its three points numbers, i.e. $T = klm$. The change of the $T$ orientation relies on the transposition of any of the pairs of numbers from the set $\{k, l, m\}$. For example, in this case the following change can be done: $klm \to kml$.

## 8. A unification of all the boundary triangles

To adapt the boundary surface to AFM for three-dimensional domain triangulation it is necessary that every triangle of the surface would be directed inward the domain $\Omega$ (what is the direction of the normal vector). The closed external surface should have all the triangles directed inward the domain surrounded by the surface, thus internal closed surfaces representing holes, outward the volume surrounded by them.

The algorithm of unification of triangles orientations is due to one closed connected boundary component. It is assumed that there exists a triangle $T_0$ with a requested orientation. The presented method is the modification of the method from the paper of [12], and is based mainly on AFM. The initial front consists of three edges of the triangle $T_0$, then all the triangles, which have as an edge one of the frontal edges are checked and if necessary then reoriented. The frontal edges that are adjacent to the verified triangles are removed from the front and the adjacent edges of those triangles are added to the front. This process will be repeated until all the triangles are verified.

The modification of the algorithm from the work of [12] is based on this that the process continues till all the faces are reviewed, contrary to the condition when the front becomes an empty set. This modification allows to uniformize as well the open surfaces. Both these algorithms have the same computational complexity.

The summary of the algorithm for the uniform front orientation is as follows:

***Algorithm* 1.**

1. $U \leftarrow \mathcal{T}_b$.
2. $U \leftarrow U \setminus \{F_0\}$.

3. Find the set $M = L \bigcap \bigcup_{F \in \mathcal{T}_b} \overline{F}$.

4. IF (#$M$ is an even number) THEN change the orientation of
   $T_0 = (n_1^0, n_2^0, n_3^0)$ onto $T_0 \leftarrow (n_1^0, n_2^0, n_3^0)$
   ENDIF

5. Take as the initial front $\Gamma$ the set of the edges $T_0$
   $\Gamma = (\mathbf{P}_{n_1^0}\mathbf{P}_{n_1^3}, \mathbf{P}_{n_3^0}\mathbf{P}_{n_1^2}, \mathbf{P}_{n_2^0}\mathbf{P}_{n_1^1})$.

6. IF $(U \neq \emptyset)$ THEN
   choose the edge $\mathbf{l} \in \Gamma$, let $\mathbf{l} = \mathbf{PQ}$
   ELSE
   Finish the unification process.
   ENDIF

7. Find such a triangle $F \in U$ that the vector $\mathbf{l}$,
   taken as an edge would be the edge of $F$,
   let $F = \mathbf{P}_{n_1^0}\mathbf{P}_{n_1^2}\mathbf{P}_{n_1^3}$.

8. IF (vector $l = \mathbf{P}_{n_1^0}\mathbf{P}_{n_1^2} \vee l = \mathbf{P}_{n_1^0}\mathbf{P}_{n_1^3} \vee l = \mathbf{P}_{n_3^0}\mathbf{P}_{n_1^1}$) THEN
   change the orientation of $F$, i.e. $F \leftarrow (n_1, n_3, n_2)$,
   ENDIF

9. $\Gamma \leftarrow \Gamma \backslash \{\mathbf{l}\}$.

10. $U \leftarrow U \backslash \{F\}$.

11. **Modify the front**
    (a) IF $(\mathbf{l} = \mathbf{P}_{n_1}\mathbf{P}_{n_2} \vee \mathbf{l} = \mathbf{P}_{n_2}\mathbf{P}_{n_1})$ *THEN*
    $\Gamma \leftarrow \Gamma \bigcup \{\mathbf{P}_{n_1}\mathbf{P}_{n_3}, \mathbf{P}_{n_3}\mathbf{P}_{n_1}\}$
    *ENDIF*.
    (b) IF $(\mathbf{l} = \mathbf{P}_{n_1}\mathbf{P}_{n_3} \vee \mathbf{l} = \mathbf{P}_{n_3}\mathbf{P}_{n_1})$ *THEN*
    $\Gamma \leftarrow \Gamma \bigcup \{\mathbf{P}_{n_1}\mathbf{P}_{n_2}, \mathbf{P}_{n_2}\mathbf{P}_{n_1}\}$
    *ENDIF*.
    (c) IF $(\mathbf{l} = \mathbf{P}_{n_2}\mathbf{P}_{n_3} \vee \mathbf{l} = \mathbf{P}_{n_3}\mathbf{P}_{n_2})$ *THEN*
    $\Gamma \leftarrow \Gamma \bigcup \{\mathbf{P}_{n_2}\mathbf{P}_{n_1}, \mathbf{P}_{n_1}\mathbf{P}_{n_2}\}$
    *ENDIF*.

12. go to6.

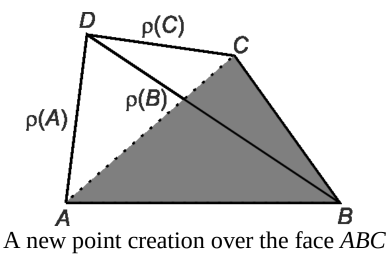


Fig. 2. A new point creation over the face *ABC*

Rys. 2. Tworzenie nowego punktu nad trójkątem *ABC*

## 9. The algorithm of volume tetrahedronization

Using the notations from the previous point the algorithm 3-D domain triangulation can be presented as follows:

***Algorithm* 2.**

1. Adapt the orientation of the boundary faces inward the domain $\Omega$, according to the algorithm from the previous point.
2. $\mathcal{F} \leftarrow \mathcal{F}_b$.
3. Find all the edges of the boundary faces.
4. $\mathcal{B} \leftarrow \mathcal{T}_b$.
5. IF $(\mathcal{B} \neq \emptyset)$ THEN

(a) Choose the last triangle from the front $B$,
let $F = \mathbf{ABC}$,

(b) Find point $\mathbf{D}$ that is on the left side of $F$ (which has the inward direction) (see Fig. 2) and which satisfies the following condition:

$$\|\mathbf{AD}\| = \rho(\mathbf{A}), \quad \|\mathbf{BD}\| = \rho(\mathbf{B}), \quad \|\mathbf{CD}\| = \rho(\mathbf{C}). \tag{13}$$

(c) IF edges $\mathbf{AD}$, $\mathbf{BD}$, $\mathbf{CD}$ do not intersect the front $\mathbf{B}$, and the triangles $\mathbf{ABD}$, $\mathbf{BCD}$, $\mathbf{CAD}$ do not intersect all the frontal egdes $B$, i.e. the set $\mathcal{E}_b$ and

$$dist(\mathbf{D}, \mathcal{B}) > \frac{\rho(\mathbf{A}) + \rho(\mathbf{B}) + \rho(\mathbf{C})}{3 \cdot 2} \tag{14}$$

THEN

i. Add $\mathbf{D}$ to the list of internal points $\mathbf{P}$ and triangles $F_1 = \mathbf{ABD}$, $F_2 = \mathbf{BCD}$, $F_3 = \mathbf{CAD}$, to the list of triangles $T$.

ii. Add tetrahedron $T = FF_1\, F_2\, F_3$ to the list $\mathcal{T}$ of the tetrahedra.

iii. Update the front $B$, i.e. remove $F$ from $B$ and add the faces of the triangles $F_1\, F_2\, F_3$ to $\mathbf{B}$.

iv. Update the set of the frontal edges $\mathcal{E}_b$ by adding the following edges $\mathbf{AD}$, $\mathbf{BD}$, $\mathbf{CD}$.

OTHERWISE

i. By using the Delaunay condition find the point $\mathbf{D} \in \mathcal{B}$. Create a tetrahedron which would have the face $F = \mathbf{ABC}$.

ii. Remove $F$ from $B$.

iii. For $i = 1, 2, 3$.
IF $(F_i \in \mathcal{B})$ *THEN* remove $F_i$ from $\mathbf{B}$

```
        OTHERWISE
        Add F to the list of triangles and to the front B
        ENDIF
        End the loop i.
    iv. Modify the set Eb of the frontal edges.
        Let E1 = AD, E2 = BD, E3 = CD.
        For i = 1, 2, 3
        IF (Ei ∉ Eb) THEN add Ei to the set Eb
        OTHERWISE
        IF (Ei is not an edge
          of the newly created faces) THEN
          remove Ei from the set Eb.
          ENDIF
        ENDIF
        End the loop i.
   ENDIF
OTHERWISE
Finish the tetrahedronization.
ENDIF
```

9.1. The auxiliary test

With respect to the newly generated points it is necessary to check, if they could be accepted. If the point $D$ was created over the face $ABC$ (Fig. 2) to form the tetrahedron with the vertices $A$, $B$, $C$, $D$ it is necessary to check whether the faces $ABD$, $BCD$, $CAD$ intersect the front.

In the papers of [1, 7] the intersection of the tetrahedron faces with every face of the front is also verified.

In this paper the intersection of edges $AD$, $BD$, $CD$ with the frontal faces is performed and as well the intersection of the front edges with the faces $ABD$, $BCD$, $CAD$. It easy to show that the condition ensures no intersection of the newly obtained tetrahedron with the front and such an approach is less time consuming.

Additionally, a test is performed, whether the point $D$ is not too close to the front and the following inequality is taken into consideration:

$$dist(D,\Gamma) = dist(D,P) > \frac{\rho(D)+\rho(P)}{3\times 2} \tag{15}$$

where:

$\Gamma$ – the generation front,

P – a point belonging to $\Gamma$ and realizing the distance $D$ to $\Gamma$,

dist($D$, $\Gamma$) – a distance from $D$ to $\Gamma$.

9.2. Forming a tetrahedron by the Delaunay method

When the point $D$ on the face $ABC$ is not accepted to form a tetrahedron with vertices $A$, $B$, $C$, $D$, then the point from the front is taken to form a tetrahedron which now would have vertices $A$, $B$, $C$, and the point from the front. The algorithm has the following steps:

1. Find the ball $B_{ABCD}$ passing through the points $A$, $B$, $C$ and with a radius $r$ equal to:

$$r = \frac{\rho(A) + \rho(B) + \rho(C)}{3} \tag{16}$$

2. Find the set $S$ of candidate points from the front belonging to the ball $B_{ABCD}$ and lying on the right side of the plane $\Pi_{ABC}$ passing through the face $ABC$.
3. In the set $S$ the following relation $\mathcal{R}$ is introduced:

$$X\mathcal{R} \quad \text{iff} \quad Y \in B_{ABCY}\ \forall X, Y \in S \tag{17}$$

4. The relation $\mathcal{R}$ is the equivalence relation, not an ordering relation, as every two elements in $S$ are in relation (are comparable), so the set $Z$ of the points from $S$ is found that satisfies the condition:

$$\forall X \in Z\ X\mathcal{R}\ Y\ \forall Y \in S \tag{18}$$

5. In the set $Z$ find the point $D$ which satis.es the auxiliary tests.
6. Form a tetrahedron with the vertices $A$, $B$, $C$, $D$ and add it to the list.

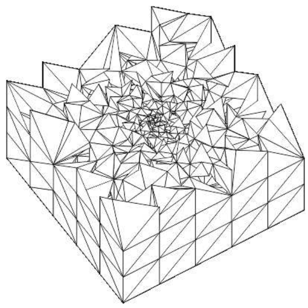

Fig. 3. A cube filled with 4188 tetrahedra, 10629 faces and 1276 points

Rys. 3. Prostopadłościan z 4188 czworościanami, 10629 trójkątami, 1276 punktami

## 10. Numerical examples

In this section some examples of the tetrahedronization of 3-D domains with the mesh size function are presented. The examples illustrated here are obtained in the way that those tetrahedra of triangulated 3-D domain are shown, whose barycenters lie on one side of a given plane.

In figures 3, 4 the tetrahedronized cube with the mesh density function given by the formula (19) is presented.

$$\rho(x, y, z) = 0.33\sqrt{\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 + \left(z - \frac{1}{2}\right)^2} \tag{19}$$

In figures 5, 6 the domain obtained from the union of a cylinder and a part of sphere filled with a tetrahedra is presented. The size function is given by the formula (20):

$$\rho(x,y,z)=\begin{cases}0.3\sqrt{x^2+y^2+z^2}+0.02, & \text{if} \quad r<1.0\\ 0.2 \quad \text{otherwise}\end{cases} \tag{20}$$

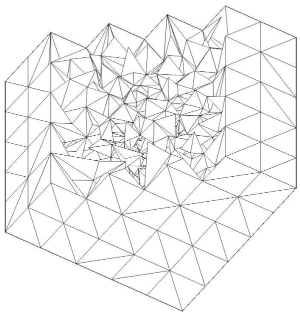

Fig. 4. The meshed domain from figure 3 with an another sharing plane

Rys. 4. Striangulowany obszar z rysunku 3 z inną płaszczyzną dzielącą

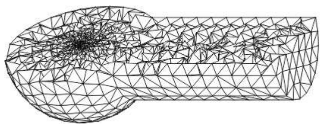

Fig. 5. A cylinder united with a part of the sphere with a symmetrical sharing plane – 80401 faces, 6712 points and 39467 tetrahedra

Rys. 5. Walec z częścią kuli z symetryczną płaszczyzną dzielącą – 80401 trójkątów, 6712 punktów i 39467 czworościanów

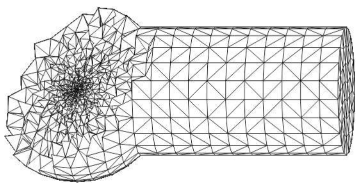

Fig. 6. The tetrahedronized domain from figure 5 with an another sharing plane

Rys. 6. Striangulowany obszar z rysunku 5 z inną płaszczyzną dzielącą

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