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ON DIFFERENTIABILITY WITH RESPECT TO A PARAMETER OF A WEAK SOLUTION OF A SECOND ORDER DIFFERENTIAL EQUATION IN BANACH SPACE

O RÓŻNICZKOWALNOŚCI WZGLĘDEM PARAMETRU SŁABEGO ROZWIĄZANIA DRUGIEGO RZĘDU W PRZESTRZENI BANACHA

A b s t r a c t

The purpose of this paper is to present some theorems on continuity and differentiability with respect to a parameter h of a weak solution of the evolution equation $\ddot{u}(t) = A_h u(t) + f_h(t)$.

Keywords: *operator, cosine family, weak solution*

S t r e s z c z e n i e

Celem artykułu jest przedstawienie twierdzeń o ciągłej zależności od parametru oraz różniczkowalności względem parametru h słabego rozwiązania ewolucyjnego równania różniczkowego $\ddot{u}(t) = A_h u(t) + f_h(t)$.

Słowa kluczowe: *operator, rodzina cosinusów, słabe rozwiązanie*

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1. Introduction

Let X be a Banach space and X^* will denote its dual space. Let A be a densely defined closed linear operator on a space X, let $T > 0$ and let $f \in L^{-1}(0,T;X)$. Our purpose in this paper is to investigate the regularity with respect to a parameter of suitable defined weak solution of the second order Cauchy problem

$$\begin{cases} \dfrac{d^2}{dt^2}u(t) = Au(t) + f(t), \quad t \in (0,T], \\ u(0) = x, \quad u'(0) = y, \end{cases} \tag{1}$$

where $x, y \in X$.

Definition 1. *A function* $u \in C([0,T], X)$ *is a weak solution of problem* (1) *if for each* $v \in D(A^*)$

(i) *the function* $[0, T] \ni t \to \langle u(t), v\rangle \in \mathbb{R}$ *is differentiable,*

(ii) $t \to \dfrac{d}{dt}\langle u(t), v\rangle$ *is absolutely continuous,*

(iii) $\dfrac{d^2}{dt^2}\langle u(t), v\rangle = \langle u(t), A^* v\rangle + \langle f(t), v\rangle$ *a.e in* $[0,T]$,

(iv) $u(0) = x, \quad \dfrac{d}{dt}\langle u(t), v\rangle\Big|_{t=0} = \langle y, v\rangle.$

S. Kanda in [7] proved the theorem:

Theorem 2. *Let A be a densely defined linear operator and* $f \in L^1(0,T;X)$. *Then the following assertions are equivalent*

(i) *for each* $x, y \in X$ *there exists exactly one weak solution of the problem* (1),

(ii) *A generates a strongly continuous cosine family* $\{C(t)\}_{t\in\mathbb{R}}$ *and in this case a weak solution u is given by*

$$u(t) = C(t)x + S(t)y + \int_0^t S(t-s)f(s)ds \quad t \in [0,T]. \tag{2}$$

2. Preliminaries

Let $B(X)$ be the totality of bounded linear operators. Subset $\{C(t)\}_{t\in\mathbb{R}}$ of $B(X)$ is called a strongly continuous cosine family in X if

(i) $\forall t, s \in \mathbb{R}$: $C(t+s) + C(t-s) = 2C(s)C(t)$,

(ii) $C(0) = I$,

(iii) for each $x \in X$ $\mathbb{R} \ni t \to C(t)x$ is continuous.

The associated sine family is given by

$$S(t)x = \int_0^t C(r)x dr$$

for $x \in X$ and $t \in \mathbb{R}$. The infinitesimal generator is the operator $A : D(A) \to X$ defined by $Ax = \lim_{h\to 0} 2h^{-2}(C(h) - I)x$ for $x \in D(A)$, where $D(A) = \{x \in X : \lim_{h\to 0} 2h^{-2}(C(h) - -I)x$ exists$\}$.

The cosine family in X with generator A is associated with the Cauchy problem for the abstract evolution equation of the second order in X

$$\frac{d^2u}{dt^2} = Au \quad t \in \mathbb{R}; \quad u(0) = x, \quad u'(0) = y.$$

For a strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$ in X with the infinitesimal generator A, we define the set

$$E(A) = \{x \in X : C(\cdot)x \text{ is once continuously differentiable in } t \in \mathbb{R}\}.$$

For convenience of the reader and to establish notation we briefly recall the theory for such cosine and sine families.

Theorem 3. ([11], Prop. 2.1) *Let* $\{C(t)\}_{t\in\mathbb{R}}$ *be a cosine family and* $\{S(t)\}_{t\in\mathbb{R}}$ *be the associated sine family. Then*

$$C(t) = C(-t) \quad \text{for} \quad t \in \mathbb{R}, \tag{3}$$

$$\textit{operators } C(s), S(s), C(t), S(t) \textit{ commute for each } t, s \in \mathbb{R}, \tag{4}$$

$$\textit{a mapping } t \to S(t)x \textit{ is continuous for each } x \in X, \tag{5}$$

$$S(s+t) + S(s-t) = 2S(s)C(t), \tag{6}$$

$$S(s+t) = S(s)C(t) + S(t)C(s), \tag{7}$$

$$S(t) = -S(-t), \tag{8}$$

there exist $M \geq 1$ *and* $\omega \geq 0$ *such that*

$$\forall t \in \mathbb{R} : \|C(t)\| \leq Me^{\omega|t|}, \tag{9}$$

$$\forall s, t \in \mathbb{R} : \|S(t) - S(s)\| \leq M \left| \int_s^t e^{\omega|r|} dr \right|. \tag{10}$$

Theorem 4. ([11], Prop. 2.2) *If the operator A is an infinitesimal generator of a cosine family* $\{C(t)\}_{t\in\mathbb{R}}$ *then*

$$D(A) \textit{ is a dense subspace of } X \textit{ and } A \textit{ is a closed operator}, \tag{11}$$

$$\textit{if } x \in X \textit{ and } r, s \in \mathbb{R}, \textit{ then } z := \int_r^s S(t)x dt \in D(A) \textit{ and } Az = C(s)x - C(r)x, \tag{12}$$

$$\text{if } x \in X \text{ and } r, s \in \mathbb{R}, \text{ then } z := \int_0^s \int_0^r C(u)C(v)x\,du\,dv \in D(A) \tag{13}$$

$$\text{and } Az = 2^{-1}[C(s+r)x - C(s-r)x],$$

$$\text{if } x \in X, \text{ then } S(t)x \in E(A), \tag{14}$$

$$\text{for } x \in E(A) : S(t)x \in D(A) \text{ and } \frac{d^2}{dt^2} S(t)x = \frac{d}{dt} C(t)x = AS(t)x, \tag{15}$$

$$\text{for } x \in D(A) : C(t)x \in D(A) \text{ and } \frac{d^2}{dt^2} C(t)x = AC(t)x = C(t)Ax, \tag{16}$$

$$\text{if } x \in D(A), \text{ then } S(t)x \in D(A) \text{ and } AS(t)x = S(t)Ax, \tag{17}$$

$$C(t+s) - C(t-s) = 2AS(t)S(s), \text{ for constants } M \text{ and } \omega \text{ defined above}, \tag{18}$$

$$\operatorname{Re} \lambda > \omega \Rightarrow \lambda^2 \in \rho(A), \tag{19}$$

$$\text{for } x \in X : \lambda R(\lambda^2, A)x = \int_0^\infty e^{-\lambda t} C(t)x\,dt, \tag{20}$$

$$\text{for } x \in X : R(\lambda^2, A)x = \int_0^\infty e^{-\lambda t} S(t)x\,dt, \tag{21}$$

$$\text{for } k = 0, 1, 2 \ldots : \left\| \frac{d^k}{d\lambda^k} [\lambda(A - \lambda^2)^{-1}] \right\| \le Mk!(\operatorname{Re} \lambda - \omega)^{-(k+1)}. \tag{22}$$

3. Dependence on a parameter

Let Ω be a compact subset of $\mathbb{R}^m$. We investigate the Cauchy problem

$$\begin{cases} \dfrac{d^2}{dt^2} u(t) = A_h u(t) + f_h(t) \quad t \in (0, T], \quad h \in \Omega \\ u(0) = x_h, \quad u'(0) = y_h. \end{cases} \tag{23}$$

In this section we will need the following assumptions:

Assumption (A). Assume that:

1) for each $h \in \Omega$ an operator A_h generates a strongly continuous cosine family $\{C_h(t)\}$ satisfying (9) with constants M and ω independent of h,

2) for $h \in \Omega$: $0 \in \rho(A_h)$,

3) domains $D(A_h^*) =: D^*$ are independent of h,

4) for each $h_0 \in \Omega$ a mapping

$$\Omega \ni h \to \overline{A_{h_0}^{-1}A_h} \in Aut(X)$$

is continuous in h_0.

Note that, by Assumption (A), $A_{h_0}^{-1}A_h$ (see [4–6, 12]) is a properly defined bounded operator. By density of $D(A_h): \overline{A_{h_0}^{-1}A_h} \in Aut(X)$.

Assumption (B). Assume that for each $h \in \Omega$

$$\left\| \lambda R(\lambda^2, A_h \right\| \leq M(\left|\lambda\right| - \omega)^{-1},$$

where M i ω satisfy (9) and $\operatorname{Re}(\lambda) > \omega$.

Theorem 5. *Assume that the family of operators* $\{A_h\}$ *satisfies Assumptions* (*A*) *and* (*B*). *Then the mapping*

$$\Omega \times \mathbb{R} \ni (h,t) \to S_h(t) \in \mathcal{B}(X)$$

is continuous. Moreover, for each $x \in X$ *the mapping*

$$\Omega \times \mathbb{R} \ni (h,t) \to C_h(t)x \in X$$

is continuous.

Proof. One can easily verify, by Theorem 1 [13], that the first statement is true. To prove the second one, note that, by (18)

$$(C_h(2t) - I)x = 2A_h S_h^2(t)x \tag{24}$$

for each $x \in X$, $h \in \Omega$, $t \in \mathbb{R}$. This imply that $A_h S_h^2(t) \in B(X)$ and

$$\left\| A_h S_h^2(t) \right\| \leq \frac{1}{2}\left(\left\| C_h(2t) \right\| + 1 \right) \leq \frac{1}{2}\left(Me^{\omega|2t|} + 1 \right), \tag{25}$$

where constants M and ω are independent of a parameter $h \in \Omega$. Operators A_h and S_h commute on D_h, so for each $x \in D_h$

$$A_h S_h^2(t)x = S_h^2(t)A_h x \tag{26}$$

and

$$A_h S_h^2(t) = \overline{S_h^2(t)A_h}. \tag{27}$$

Observe that for each $x \in D_{h_0}: A_h^{-1}A_{h_0}x \in D_h$ and

$$A_h S_h^2(t) A_h^{-1} A_{h_0} x = S_h^2(t) A_{h_0} x.$$

So, the operator $S_h^2(t)A_{h_0}$ is bounded and with domain dense in X. From the above it follows that

$$A_h S_h^2(t)\overline{A_h^{-1}A_{h_0}} = \overline{S_h^2(t)A_{h_0}}. \tag{28}$$

Let us note that

$$\overline{S_h^2(t)A_{h_0}} - \overline{S_{h_0}^2(t)A_{h_0}} = \overline{[S_h^2(t) - S_{h_0}^2(t)]A_{h_0}}.$$

So, for each $x \in X$

$$C_h(2t)x - C_{h_0}(2t)x = 2A_hS_h^2(t)x - 2A_{h_0}S_{h_0}^2(t)x =$$
$$= 2\left[A_hS_h^2(t) - A_hS_h^2(t)\overline{A_h^{-1}A_{h_0}} + \overline{(S_h^2(t) - S_{h_0}^2(t))A_{h_0}}\right]x =$$
$$= 2\left[A_hS_h^2(t) - \left(I - \overline{A_h^{-1}A_{h_0}}\right) + \overline{(S_h^2(t) - S_{h_0}^2(t))A_{h_0}}\right]x.$$

By Assumption (A) and (25)

$$\lim_{(h,t)\to(h_0,t_0)} A_hS_h^2(t)\left(I - \overline{A_h^{-1}A_{h_0}}\right) = 0. \tag{29}$$

Assume that $x \in D(A_{h_0})$. By continuity of the mapping $(h,t) \to S_h(t)$,

$$\lim_{(h,t)\to(h_0,t_0)} \overline{S_h^2(t)A_{h_0}}\,x = \overline{S_{h_0}^2(t_0)A_{h_0}}\,x. \tag{30}$$

From (28) and (25), it follows that

$$\left\|\overline{S_h^2(t)A_{h_0}}\right\| \le C,$$

for each $t \in [a,b]$ and each $h \in \Omega$. By the Banach-Steinhaus Theorem, (30) is satisfied for each $x \in X$. The proof of Theorem 5 is complete. □

Corollary. If the family of operators $\{A_h\}$ satisfies Assumptions (A) and (B), then

$$\lim_{h\to h_0} S_h(t)x = S_{h_0}(t)x$$

$$\lim_{h\to h_0} C_h(t)x = C_{h_0}(t)x$$

uniformly in $(t,x) \in [0,T] \times K$, where K is a compact subset of X.

Theorem 6. *If the family of operators* $\{A_h\}$ *satisfies Assumptions* (*A*) *and* (*B*) *and the mappings* $\Omega \ni h \to x_h \in X$, $\Omega \ni h \to y_h \in X$, $\Omega \ni h \to f_h \in L^1(0,T;X)$ *are continuous, then*

$$\lim_{h\to h_0} u_h(t)x = u_{h_0}(t)$$

uniformly in $t \in [0,T]$, *where the function*

$$u_h(t) = C_h(t)x_h + S_h(t)y_h + \int_0^t S_h(t-s)f_h(s)ds$$

is the unique weak solution of the problem (23).

Proof. Observe that

$$\begin{aligned} u_h(t)-u_{h_0}(t) = \left[C_h(t)-C_{h_0}(t)\right]x_h + C_{h_0}(t)(x_h - x_{h_0}) + \\ +[S_h(t)-S_{h_0}(t)]y_h + S_{h_0}(t)(y_h - y_{h_0}) + \\ +\int_0^t [S_h(t-s)-S_{h_0}(t-s)] f_{h_0}(s)ds + \\ +\int_0^t S_h(t-s)[f_h(s)-f_{h_0}(s)]ds. \end{aligned}$$

Let K be a compact set and let $h_0 \in K$.

$K_1 := \{x_h : h \in K\}$, $K_2 := \{y_h : h \in K\}$, $K_3 := \{\varphi(s) : s \in [0,T]\}$, where $\varphi \in C([0,T], X)$ satisfies: $\left\| f_{h_0} - \varphi \right\|_{L^1} \le \varepsilon$. The rest of the proof is similar to the proof of the Theorem 2 in [3]. □

We need the following.

Theorem 7. *Suppose that A is a generator of a strongly continuous cosine family. If* $x, y \in X$, $f \in L^1(0,T;X)$ *and* $u \in C([0,T], X)$ *is a weak solution of the problem* (1) *then a function*

$$w(t) := \int_0^t \int_0^s u(r)drds$$

is a classical solution of a Cauchy problem

$$\begin{cases} \dfrac{d^2}{dt^2} w(t) = Aw(t) + \displaystyle\int_0^t \int_0^s f(r)drds + x + ty \\ w(0) = 0, \quad w'(0) = 0. \end{cases} \tag{31}$$

Proof. Firstly, we will prove that $w(t) \in D(A)$. Observe that

$$w(t) = \int_0^t \int_0^s C(r)xdrds + \int_0^t \int_0^s S(r)ydrds + \int_0^t \int_0^s \int_0^r S(r-\xi) f(\xi)d\xi drds. \tag{32}$$

By Theorem 3, $\int_0^t \int_0^s C(r)xdrds \in D(A)$, $\int_0^t \int_0^s S(r)ydrds \in D(A)$ and

$$\int_0^t \int_0^s \int_0^r S(r-\xi) f(\xi)d\xi drds = \int_0^t \int_0^s \underbrace{\int_\xi^s S(r-\xi) f(\xi)dr}d\xi ds \in D(A).$$

The function u is a weak solution of the problem

$$\begin{cases} \dfrac{d^2}{dt^2} u(t) = Au(t) + f(t), \quad t \in (0,T], \\ u(0) = x, \quad u'(0) = y. \end{cases}$$

so

$$\int_0^t \int_0^s \frac{d^2}{dr^2} \langle u(r), \upsilon \rangle drds = \int_0^t \int_0^s \langle u(r), A^* \upsilon \rangle drds.$$

This implies that

$$\int_0^t \left[\frac{d}{ds} \langle u(s), \upsilon \rangle - \frac{d}{ds} \langle u(s), \upsilon \rangle \big|_{s=0} \right] ds = \left\langle \int_0^t \int_0^s u(r)drds, A^* \upsilon \right\rangle + \left\langle \int_0^t \int_0^s f(r)drds, \upsilon \right\rangle$$

and

$$\frac{d^2}{dt^2} \langle w(t), \upsilon \rangle - \langle x, \upsilon \rangle - t \langle y, \upsilon \rangle = \langle Aw(t), \upsilon \rangle + \left\langle \int_0^t \int_0^s f(r)drds, \upsilon \right\rangle.$$

Therefore

$$\frac{d^2}{dt^2} \langle w(t), \upsilon \rangle = \left\langle Aw(t) + \int_0^t \int_0^s f(r)drds + x + ty, \upsilon \right\rangle. \tag{33}$$

By $w \in C^2([0,T], X)$

$$\frac{d^2}{dt^2} \langle w(t), \upsilon \rangle = \left\langle \frac{d^2}{dt^2} w(t), \upsilon \right\rangle$$

for each $\upsilon \in D(A^*)$. By w^* – density of $D(A^*)$ in X^* and (33), the function w is a classical solution of the Cauchy problem (31). □

Theorem 8. *Assume that the family of operators* $\{A_h\}$ *satisfies Assumptions* (*A*) *and* (*B*) *and assume that domains* $D_h = D$ *are independent of* $h \in \Omega$. *If*:

(i) *a mapping* $h \to A_h$ *is R-differentiable, i.e. there exist a Banach space Z and a bounded, bijective linear operator* $B : Z \to D$ *such that* $A_h B$ *is bounded and differentiable. In this case* $A'_{h_0} := \left(\frac{d}{dh}(A_h B)\big|_{h=h_0} \right) B^{-1}$,

(ii) *mappings* $\Omega \ni h \to x_h \in X$, $\Omega \ni h \to y_h \in X$ *are continuously differentiable,*

(iii) *a mapping*: $h \to f_h$ *is continuously differentiable, then functions* $(h,t) \to w_h(t)$ *and* $(h,t) \to \langle u_h(t), \upsilon \rangle$, *where* $\upsilon \in D^*$, u_h *is a weak solution of the Cauchy problem* (23) *and*

$$w_h(t) := \int_0^t \int_0^s u_h(r)drds,$$

is differentiable with respect to h and

$$\frac{\partial}{\partial h}\left\langle u_h(t),\upsilon\right\rangle\Big|_{h=h_0}=\left\langle \frac{\partial}{\partial h}w_h(t)\Big|_{h=h_0}, A_{h_0}^{*}\upsilon\right\rangle+\left\langle A'_{h_0}w_{h_0}(t),\upsilon\right\rangle+\left\langle x'_{h_0},\upsilon\right\rangle+$$
$$+\left\langle ty'_{h_0},\upsilon\right\rangle+\left\langle \int_0^t\int_0^s f'_{h_0}(r)drds,\upsilon\right\rangle.$$

Proof. By Theorem 7, $\dfrac{w_h-w_{h_0}}{h-h_0}$ is a solution of a problem

$$\begin{cases}\dfrac{\partial^2}{\partial t^2}\left(\dfrac{w_h-w_{h_0}}{h-h_0}\right)(t)=A_h\dfrac{w_h-w_{h_0}}{h-h_0}(t)+F_h(t),\\ \dfrac{w_h-w_{h_0}}{h-h_0}(0)=0,\quad \dfrac{\partial}{\partial t}\left(\dfrac{w_h-w_{h_0}}{h-h_0}\right)(t)\Big|_{t=0}=0,\end{cases}\tag{34}$$

where

$$F_h(t)=\begin{cases}\dfrac{A_h-A_{h_0}}{h-h_0}w_{h_0}(t)+\displaystyle\int_0^t\int_0^s\frac{f_h-f_{h_0}}{h-h_0}(r)drds+\frac{x_h-x_{h_0}}{h-h_0}+t\frac{y_h-y_{h_0}}{h-h_0} & \text{dla}\quad h\neq h_0,\\ A'_{h_0}w_{h_0}(t)+\displaystyle\int_0^t\int_0^s f'_{h_0}(r)drds+x'_{h_0}+ty'_{h_0} & \text{dla}\quad h=h_0.\end{cases}$$

Firstly, note that for each $t\in[0,T]$

$$\frac{A_h-A_{h_0}}{h-h_0}w_{h_0}(t)=\frac{A_h-A_{h_0}}{h-h_0}BB^{-1}w_{h_0}(t)=\frac{A_hB-A_{h_0}B}{h-h_0}B^{-1}w_{h_0}(t)\to A'_{h_0}w_{h_0}(t).\tag{35}$$

It is well known, that if the family A_h is *R*-differentiable, then for every Banach space Z and bounded, bijective linear operator $B:Z\to D(A_h)\subset X$, a mapping $h\to A_hB$ is bounded and differentiable and $A'_{h_0}=\left(\dfrac{d}{dh}(A_hB)\Big|_{h=h_0}\right)B^{-1}$. Let $B:=R(\lambda^2,A_{h_0})$, for some $\operatorname{Re}\lambda>\omega$. By *R*-differentiability of $h\to A_h$, there exists a constant $K>0$ such that

$$\left\|\frac{A_h-A_{h_0}}{h-h_0}B\right\|\leq K.$$

By Theorem 7, w_{h_0} is a classical solution of the problem (31), so

$$B^{-1}w_{h_0}(t)=A_{h_0}w_{h_0}(t)-\lambda^2 w_{h_0}(t)=$$
$$=\left[u_{h_0}(t)-\int_0^t\int_0^s f_{h_0}(r)drds-x_{h_0}-ty_{h_0}-\lambda^2 w_{h_0}(t)\right]\in C([0,T],X).$$

It follows that

$$\left\| \frac{A_h - A_{h_0}}{h - h_0} w_{h_0}(t) \right\| \le \left\| \frac{A_h - A_{h_0}}{h - h_0} B \right\| \left\| B^{-1} w_{h_0}(t) \right\| < \infty. \tag{36}$$

By (35), (36) and the Lebesgue Theorem

$$\lim_{h \to h_0} \frac{A_h - A_{h_0}}{h - h_0} w_{h_0} = A'_{h_0} w_{h_0}$$

in the norm of the space $L^1(0,T;X)$. One can easily verify that

$$\lim_{h \to h_0} F_h = F_{h_0}$$

in the norm of the space $L^1(0,T;X)$. It is easy to prove (see Theorem 8 [6]) that

$$\exists \lim_{h \to h_0} \frac{w_h - w_{h_0}}{h - h_0}(t) =: w'_{h_0}(t)$$

and w'_{h_0} is weak solution of the Cauchy problem (34), for $h = h_0$. Moreover

$$\frac{\partial^2}{\partial t^2} \left\langle w'_{h_0}(t), \upsilon \right\rangle = \left\langle w_{h_0}(t), A^*_{h_0} \upsilon \right\rangle + \left\langle F_{h_0}(t), \upsilon \right\rangle.$$

On the other hand

$$\frac{\partial^2}{\partial t^2} \left\langle w'_{h_0}(t), \upsilon \right\rangle = \frac{\partial^2}{\partial t^2} \left\langle \frac{\partial}{\partial h} w_h(t) \Big|_{h=h_0}, \upsilon \right\rangle$$

$$\frac{\partial}{\partial h} \left(\left\langle \frac{\partial^2}{\partial t^2} w_h(t), \upsilon \right\rangle \right) \Big|_{h=h_0} = \frac{\partial}{\partial h} \left\langle u_h(t), \upsilon \right\rangle \Big|_{h=h_0} \quad \square$$

The proof of Theorem 8 is complete. □

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