A LINEAR ABSTRACT INITIAL VALUE PROBLEM WITH BESOV FUNCTION

ABSTRAKCYJNY LINIOWY PROBLEM POCZĄTKOWY Z FUNKCJĄ BESOVA

Abstract

This article is devoted to the investigation of the abstract linear initial value problem:

\[ \begin{align*}
    \frac{du}{dt}(t) + A(t)u(t) &= f(t) \\
    u(s) &= x
\end{align*} \]

in a Banach space in a parabolic case with Besov function \( f \). We give sufficient condition for existence and uniqueness of the solution of the problem \((*)\) which may have weak singularity at the origin.

Keywords: Besov space, semigroup with singularity

Streszczenie

Niniejszy artykuł dotyczy abstrakcyjnego problemu początkowego: \((*)\)

\[ \begin{align*}
    \frac{du}{dt}(t) + A(t)u(t) &= f(t) \\
    u(s) &= x
\end{align*} \]

w przestrzeni Banacha z funkcją Besova \( f \). Podane są warunki wystarczające na istnienie i jednoznaczność rozwiązania problemu \((*)\) ze słabą osobliwością w punkcie początkowym.

Słowa kluczowe: przestrzeń Besova, półgrupa z osobliwością

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Let $X$ be a Banach space, and let $\{A(t)\}_{t \in [0,T]}$ be a family of closed densely defined linear operators from $X$ to $X$ with domains $D = D(A(t))$ independent on $t$. We consider the linear abstract initial value problem

$$\begin{aligned}
\frac{du}{dt}(t) + A(t)u(t) &= f(t), \quad t \in (s,T], \\
u(s) &= x,
\end{aligned}$$

where $f : [0,T] \to X$ is a Besov function. We prove a theorem on the existence and uniqueness of the solution of problem (1).

**Definition 1.** A function $u : [s,T] \to X$ is said to be a classical solution of the problem (1) if

(i) $u : [s,T] \to X$ is continuous,

(ii) $u : [s,T] \to X$ is of class $C^1$,

(iii) $u(t) \in D$ for each $t \in (s,T]$,

(iv) $\frac{du}{dt}(t) + A(t)u(t) = f(t)$ for each $t \in (s,T]$.

2. Semigroups with singularity

We will use the following assumptions (see [6]):

(Z1). The domain $D$ of $A(t)$ is independent on $t$, $D$ is dense in $X$, and for $x \in D$ the function $[0,T] \ni t \mapsto A(t)x \in X$ is of class $C^1$.

(Z2). For all $t \in [0,T]$, $\lambda \in \sum_{b_0}^\omega \{\lambda \in \mathbb{C} : \arg(\lambda - b_0) < \omega\}$, where $b_0 < 0$, $\omega \in \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ there exist $R_\lambda(t) = -(\lambda + A(t))^{-1} \in B(X)$ and there exist constants $M \geq 1$, $\theta \in \left(\frac{2}{3}, 1\right)$, independent on $t$ and $\lambda$, such that

$$\|R_\lambda(t)\| \leq \frac{M}{(1 + |\lambda|)^\theta} \quad \text{for} \quad \lambda \in \sum_{b_0}^\omega t \in [0,T].$$

For a fixed $s \in [0,T]$ we may use results of [10] to get the following remark:

**Remark 2.** For any fixed $s \in [0,T]$ operators

$$S_y(t) := \frac{-1}{2\pi i} \int_{b^1} e^{\lambda t} R_\lambda(s)d\lambda,$$
where \( \Gamma_b = \{ re^{i\omega} + b; 0 \leq r < \infty \} \cup \{ re^{i\omega} + b; 0 \leq r < \infty \}, \ b \in (b_0, +\infty), \) form an analytic semigroup with singularity generated by \((-A(s))\) (see [9]). Moreover

\[
\frac{d^n}{dt^n} S_s(t) = (-1)^n A^n(s) S_s(t) \quad \text{for} \quad s \in [0, T], \ t > 0,
\]

and for any fixed \( s \in [0, T] \) the estimate

\[
\left\| A^n(s) S_s(t) \right\| \leq M_n t^b \quad \text{for} \quad s \in [0, T], \ t > 0,
\]

(3) holds, where \( M_n \) depends only on \( \theta \) and \( M \) (by assumption (Z2)).

**Definition 3.** Let us define (see [9])

\[
A^{-3}(s) := \frac{-1}{2\pi \i} \int_{\Gamma_b} (-\mu)^{-3} R_{\beta}(s)d\mu,
\]

for \( \beta \in (1-0,1), \ s \in [0, T], \ b \in (b_0,0), \) where for the function \((-\mu)^{-3}\) we mean a branch whose arguments lie between \(-\beta \pi\) and \(\beta \pi\) and which is analytic in the region obtained by omitting the positive real axis (for details see [9]).

**Remark 4.** Under assumptions (Z1), (Z2) the linear operator \( A^{-3}(s) \) is bounded and injective. Thus \( A^3(s) := (A^{-3}(s))^{-1} \) is well defined and for \( \beta \in (1-0,0) \) we have \( D(A^3(s)) \supset D \) (for details see [9]).

We can write the main theorem of [6] in the following way:

**Theorem 5.** Assumptions (Z1), (Z2) guarantee that the problem

\[
\begin{cases}
du/dt(t) + A(t)u(t) = f(t), & t \in (s, T], \ s \in [0, T], \\
u(s) = x, & x \in D = \bigcup_{s \in [0, T]} D(A^3(r)), \ \beta \in (2(1-0),0),
\end{cases}
\]

has the unique two-parameter family of bounded operators \( \{ U(t,s) \}_{(t,s) \in \Delta_T} \) (called fundamental solution) such that:

1° \( U(t,s) : X \to D \) is bounded and

\[
\left\| U(t,s) \right\| \leq C |t-s|^{b-1} \quad \text{for} \quad s < t,
\]

2° For any fixed \( \beta \in (2(1-0),0) \)

\[
\lim_{s \to t} U(t,s)x = x \quad \text{for} \quad x \in \bigcup_{s \in [0, T]} D(A^3(r)),
\]

3° The function \( (s,T) \to U(t,s)x \) is of class \( C^1 \) for \( s \in [0, T], \ x \in X \) and

\[
\frac{\partial U}{\partial t}(t,s)x + A(t)U(t,s)x = 0 \quad \text{for} \quad x \in X, \ 0 \leq s < t \leq T.
\]
Moreover, \[ \left\| \frac{\partial U}{\partial t}(t,s) \right\| \leq C|t-s|^{\alpha-2} \] and
\[ \left| A(t)U(t,s)A^{-1}(s) \right| \leq C|t-s|^{\alpha-1} \quad \text{for } 0 \leq s < t \leq T. \]

4° The function \( [0,t] \ni s \mapsto U(t,s)x \in X \) is differentiable for \( x \in D, \ t \in (0,T] \) and
\[ \frac{\partial U}{\partial s}(t,s)x = U(t,s)A(s)x. \]

**Remark 6.** In [6] it is proved that
\[ U(t,s) = S_s(t-s) + W(t,s) = S_s(t-s) + \int_s^t S_r(t-r)P(r,s)dr, \]
where \( \{S_s(t)\}_{t>0} \) is semigroup generated by \((-A(s))\), and operator \( P(r,s) \) is linear and bounded on \( X \).

**Lemma 7.** (See [6]) If the function \( f \) is continuous then the function
\[ G(t) = \int_t^s W(t,r)f(r)dr \]
is of class \( C^1 \) and
\[ \frac{\partial}{\partial t} G(t) = \frac{\partial}{\partial t} \int_t^s W(t,r)f(r)dr = \int_t^s (A(t)S_s(t-s)P(t,r) \]
\[ - A(p)S_p(t-p)P(p,r)df(r)dr + \int_t^s S_r(t-r)P(r,s)f(r)dr. \]

It is easy to prove the following technical lemma:

**Lemma 8.** For each \( d \in (-1,\infty) \), there is \( C > 0 \) such that for each \( a \in (0,\infty) \)
\[ \int_0^a t^d e^{-at}dt = Ca^{-d-1}. \quad (4) \]

### 3. Besov spaces

For definition of spaces of Besov functions and their properties we refer to [5] and [4]. We recall some of them. We need the following definitions for characterization of Besov functions.

**Definition 9.** (See [4, 5, 10]) Let \( I = (a,b) \) where \(-\infty < a < b < \infty\). We define \( K_0(I) \) as the set of functions \( \varphi: \mathbb{R}^2 \to \mathbb{R} \) of class \( C^\infty \), such that 1° and 2° holds:

1° For all compact \( K \subset \mathbb{R} \) there exists a compact set \( K_1 \subset \mathbb{R} \) such that, \( \text{supp } \varphi(t,.) \subset K_1 \) for \( t \in K \).
2° For all compact \( K \subset \mathbb{R} \) there exists compact set \( K_2 \subset I \) such that supp \( \varphi(t, (t - \tau) / \tau) \subset K_2 \) for \( t \in K \) and \( \tau \in (0,1] \).

**Definition 10.** (See [5]) Let \( I \) be as in the precedent definition

\[
K_m(I) = \left\{ \frac{\partial^m \varphi}{\partial y^m}(t,s) : \varphi \in K_0(I) \right\}.
\]

Let \( I = (a,b) \) be an interval and let the function \( \varphi_0 \) be of class \( C^m \) with support in \( I \) such that \( \int \varphi_0(t)dt = 1 \). Let us define for each integral \( m \) functions

\[
e_m(t,s) = \sum_{j=0}^{m-1} \frac{\partial^j}{\partial s^j} \left( \frac{1}{j!} s^j \varphi_0(t-s) \right),
\]

\[
e'_m(t,s) = 2e_m(t,s) - \int e_m(t,r)e_m(t-cr,s-r)dr.
\]

**Theorem 11.** (See [5] Theorem 1) A distribution \( f \) is in \( B^\varrho_{p,q}(I) \), if and only if exists \( m > \sigma \) such that following conditions holds

\[
\left\langle \varphi, \left( t, \frac{t-s}{c} \right), f(s) \right\rangle_s \in L^p(I,X,dt) \quad \text{for all} \quad \varphi \in K_0(I),
\]

\[
\tau^{-\sigma} \left\langle \tau^{-1}\varphi, \left( t, \frac{t-s}{\tau} \right), f(s) \right\rangle_s \in L^p \left( (0,c), L^q(I,X,dt), \frac{d\tau}{\tau} \right)
\]

for all \( \varphi \in K_m(I) \).

**Theorem 12.** (See [5] Theorem 2) Let \( f \in B^\varrho_{\infty,1}(I,X) \), where \( I = (a,b) \), and let \( h, l \in \{0,1,2,3,\ldots\} \), \(-h < \sigma < l\), \( m = h + l \). Under above assumption the sequence \( \{f_n\}_{n=1}^\infty \) defined by \( f_n(t) = \int_a^b n e_m(t,n(t-r))f(r)dr \) converges to \( f \) in \( B^\varrho_{\infty,1}(I,X) \). If the function \( f : (a,b) \to X \) is continuous then \( \{f_n\}_{n=1}^\infty \) are in \( C^\varrho((a,b)) \cap B^\varrho_{\infty,1}(I,X) \). Moreover, \( \{f_n\} \) converges to \( f \) in \( L^1(I,X) \). By definition of norm in \( B^\varrho_{\infty,1}(I,X) \) for \( \sigma > 0 \), the sequence \( f_n \) converges to \( f \) in \( L^\varrho((a,b);X) \), while \( n \to \infty \).

**4. The linear case**

We consider the following linear Cauchy problem

\[
\begin{align*}
\frac{du}{dt}(t) + A(t)u(t) &= f(t), & t \in (s,T], & s \in [0,T] \\
u(s) &= x, & x \in D^\varrho,
\end{align*}
\]
where \( f : [0, T] \to X \) is Besov function from \( B_{-1,1}^{1-0} \). We investigate the existence and uniqueness of the classical solution. We shall prove that function \( u \) given by

\[
u(t) = U(t, s)x + \int_{s}^{t} U(t, r)f(r)dr
\]

is the unique classical solution of the initial value problem.

We shall show that \( u(\cdot) \) is differentiable. We note that \( U(\cdot, s)x \) is the solution of initial value problem with \( f = 0 \). By the form of \( U(t, s)x \) and by differentiability of \( G(\cdot) \), given in Lemma 7, it is sufficient to show the differentiability of \( F(\cdot) \), given by

\[
F(t) = \int_{s}^{t} S_{\tau}(t - \tau)f(\tau)d\tau.
\]

By the inclusion in Corollary 2.1 from [10] it is sufficient consider function \( F \in B_{-1,1}^{1} \). The following lemmas will show that \( u \) satisfies the equation. We shall prove this by approximating the function \( u \) by solution of equations with the right-hand side function of class \( C^{1} \). We can do that by Theorem 12.

**Lemma 13.** Assume \( Z_{1} - Z_{2} \) and let \( q, z \in [0, T] \). Then there exists \( C \) independent of \( z \) and \( q \), such that

\[
\left\| \frac{\partial^{k}}{\partial q^{k}} \frac{\partial}{\partial z} S_{\tau}(q) \right\| \leq Cq^{2k-2-2}
\]

**Proof.** By (2) and (Z2), we have

\[
\left\| \frac{\partial^{k}}{\partial q^{k}} \frac{\partial}{\partial z} S_{\tau}(q) \right\| \leq C\left( \int_{\tau}^{\tau} \frac{\partial^{k}}{\partial q^{k}} \frac{\partial}{\partial z} e^{\lambda q} R_{\lambda}(z)d\lambda \right) \leq Cq^{2k-2}(1+q^{1-0}) \leq Cq^{2k-2}(1+q^{1-0}) \leq Cq^{2k-2}.
\]

**Lemma 14.** Let \( \varphi \in K_{0}(s+\varepsilon, T) \), \( \varepsilon > 0 \), \( q_{0} \in (0, T-s) \), \( q \in (0, T) \), \( p \in (s+\varepsilon, T) \). Moreover, let function \( f : [0, T] \to X \) be continuous and

\[
M = \sup_{\tau \in (0, 1)} \left\{ \varphi \left( p, \frac{p-q-r}{\tau} \right) \right\} ; \ p, q, r \in (0, T), \ \tau \in (0, 1).
\]

Then

\[
\left\| \int_{s}^{T-q_{0}} \int_{z}^{T-q} \frac{\partial^{k+1} S_{\tau}(q)}{\partial q^{k+1}} \varphi \left( p, \frac{p-q_{0}-r}{\tau} \right) f(r)drdz \right\| \leq \varepsilon_q^{k+1-2} \sum_{\tau}^{T-q_{0}} \varphi \left( p, \frac{p-q_{0}-r}{\tau} \right) f(r)drdz + CMq^{k+1}q^{2k-2+2} \left\| f \right\|_{L^{\infty}}.
\]
Proof. We shall show that \( M < \infty \). For fixed \( p \in (0, T) \) and \( \tau(0,1] \),
\[
\text{supp } \varphi\left( p, \frac{p-q_0-r}{\tau} \right) \subset [0,T].
\]
\[
\sup \left\| \varphi\left( p, \frac{p-q_0-r}{\tau} \right) \right\| ; \ p,q_0,r \in (0,T), \ \tau \in (0,1] \leq \sup \left\| \varphi\left( p,z-q_0-r \right) \right\| ; \ z,q_0,r \in (0,\tau T), \ p \in (0,T), \tau \in (0,1] < \infty
\varphi \in C^\infty \text{ hence } \sup \{ \varphi(t,s) : (t,s) \in [0,T] \times [-2T,T] \} \leq \infty.
\]
Let \( \sup \varphi\left( p, \frac{p-q_0-r}{\tau} \right) \subset [a,b] \subset [q_0 - K \tau, q_0 + K \tau] \) and \( [a,b] \subset (s+\varepsilon,T) \), where \( K \) is independent of \( \tau \). We have
\[
\left\| \int_s^{T-q} \frac{\partial^k}{\partial q^k} \partial z \varphi (q) \right\|^2 \left\| f(r)dr \right\| \leq \left\| \int_s^{T-q} \frac{\partial^k}{\partial q^k} \partial z \varphi (q) \right\|^2 \left\| f(r)dr \right\| + \left\| \int_s^{T-q} \frac{\partial^k}{\partial q^k} \partial z \varphi (q) \right\|^2 \left\| f(r)dr \right\| + \left\| \int_s^{T-q} \frac{\partial^k}{\partial q^k} \partial z \varphi (q) \right\|^2 \left\| f(r)dr \right\|
\]
Estimating each term we prove Lemma 14.
\[
\square
\]
**Lemma 15.** (See [10]) Let \( I = (s,T) \), \( I_{\varepsilon} = (s+\varepsilon,T) \), \( q', q \in (0,T) \), \( \varphi \in K_0(I_{\varepsilon}) \cap K_0(I) \).
If \( f \in L_1(I,X) \) then
\[
\left\| \int_s^{T-q} \frac{\partial^k}{\partial q^k} \partial z \varphi (q) \right\|^2 \left\| f(r)dr \right\| \leq C_{\varepsilon} \left\| f \right\|_{L_1(I,X)}.
\]
**Lemma 16.** (See [10]) Let \( I = (s,T) \), \( I_{\varepsilon} = (s+\varepsilon,T) \), \( 0 \leq q \leq \varepsilon \). If \( f \in L^1(I,X) \) and \( \varphi \in K_0(I_{\varepsilon}) \cap K_0(I) \) then
\[
\left\| \int_s^{T-q} \frac{\partial^k}{\partial q^k} \partial z \varphi (q) \right\|^2 \left\| f(r)dr \right\| \leq \sum_{j=0}^{\varepsilon} \left\| \int_s^{T-q} \frac{\partial^j}{\partial q^j} \partial z \varphi (q) \right\|^2 \left\| f(r)dr \right\| + C_q \left\| f \right\|_{L_1(I,X)},
\]
where \( \Phi_{\varepsilon,j}(t,s) = \frac{\partial^j}{\partial t^{j+1}} \phi(t,s) \).
Proof. By Taylor formula
\[
\varphi(t, q) = \sum_{j=0}^{2} q_j \varphi_{j,0}(t-q, q) + \frac{q^3}{2} \int_0^1 \eta^2 \varphi_{3,0}(t-\eta q, q') d\eta,
\]
for \( t \in I_\varepsilon \) support \( \varphi \left( t, \frac{t-q}{\tau} \right) \subset (s+\varepsilon-q, T-q) \subset (s, T-q) \) so integrals over \((s, T-q)\) and over \((s, T)\) are equal. By Lemma 6 follows (7).

Lemma 17. Let function \( f : [s, T] \to X \) be continuous function in Besov space \( B^{1,0}_{\infty,1}(s; T) \) and \( \varphi \in K_0(I_\varepsilon) \cap K_0(I) \). Then for every \( c \in (0, 1] \) we have
\[
\int_s^T \varphi \left( \frac{p}{c} \frac{r-t}{c} \right) F(t) dt \in L^\infty(I_\varepsilon; X; dp),
\]
where \( F : [s, T] \to X \) is given by
\[
F(t) = \int_s^T S_s(t-r) f(r) dr.
\]

Proof. By (9) we have
\[
\int_s^T \varphi \left( \frac{p}{c} \frac{r-t}{c} \right) F(t) dt \leq \int_s^T \varphi \left( \frac{p}{c} \frac{r-t}{c} \right) S_s(t-r) f(r) dr dt r =
\]
\[
= \int_s^T \int_0^T \varphi \left( \frac{p}{c} \frac{r-q-r}{c} \right) S_s(q) f(r) dq dr r.
\]
By assumption \( S_{1/1}(q) \) is of class \( C^1 \). So
\[
\int_s^T \varphi \left( \frac{p}{c} \frac{r-t}{c} \right) F(t) dt = \int_s^T \int_0^T \varphi \left( \frac{p}{c} \frac{r-t}{c} \right) S_s(q) f(r) dr dq dr +
\]
\[
+ \int_s^T \int_0^T \varphi \left( \frac{p}{c} \frac{r-q-r}{c} \right) S_s(q) f(r) dq dr r.
\]
The second term is in \( L^\infty(I_\varepsilon; X; dp) \) (proof as in [10] because \( A(s) \) is constant).
By Remark 6 and Lemma 14 for the first one we have
\[
\left\| \int_0^T \int_s^T \int_0^T \varphi \left( \frac{p}{c} \frac{r-t}{c} \right) S_s(q) f(r) dr dq dr \right\| \leq C(1+2^2) \| f \|_{L^2} + Cc^2(T-s)^{2^{-1}} \| f \|_{L^p}.
\]

Lemma 18. Let continuous function \( f : [s, T] \to X \) be in \( B^{1,0}_{\infty,1}(s; T) \), and function \( F : [s, T] \to X \) be defined by (9). Then there exists \( c \in (0, 1] \) such that for every \( \varphi \in K_3(I_\varepsilon) \cap K_3(I) \) the following condition holds
\( \tau^{-3} \int_{s}^{T} \varphi \left( p, \frac{p-r}{\tau} \right) F(t) dt \in L^{1}_{\varepsilon}(0,c); L^{\infty}(I; X; dt); d\tau. \)

**Proof.** By definition of \( F \) and using the fact that \( S_{\varepsilon}(q)x \) is of class \( C^{1} \) we have

\[
J = \tau^{-3} \int_{s}^{T} \varphi \left( p, \frac{p-r}{\tau} \right) F(t) dt = \tau^{-3} \int_{s}^{T} \int_{0}^{T-r} \varphi \left( p, \frac{p-r}{\tau} \right) S_{\varepsilon}(q) f(r) dq \, dr =
\]

\[
\tau^{-3} \int_{s}^{T} \int_{0}^{T-r} \frac{\partial \varphi}{\partial z} S_{\varepsilon}(q) \varphi \left( p, \frac{p-r}{\tau} \right) f(r) dz \, dq \, dr +
\]

\[
+ \tau^{-3} \int_{s}^{T} \int_{0}^{T-r} S_{\varepsilon}(q) \varphi \left( p, \frac{p-r}{\tau} \right) f(r) dz \, dq \, dr.
\]

The second term is in \( L^{\infty}(I; X; dp) \) (proof as in [10] \( A(s) \) is constant). We consider the first term. Changing the order of integration we have

\[
\tau^{-3} \int_{s}^{T} \int_{0}^{T-r} \frac{\partial \varphi}{\partial z} S_{\varepsilon}(q) \varphi \left( p, \frac{p-r}{\tau} \right) f(r) dz \, dq \, dr =
\]

\[
= \tau^{-3} \left( \int_{0}^{T-r} + \int_{T-r}^{T} \right) \int_{s}^{T-r} \frac{\partial \varphi}{\partial z} S_{\varepsilon}(q) \varphi \left( p, \frac{p-r}{\tau} \right) f(r) dz \, dq \, dr = J_{1} + J_{2}.
\]

We estimate \( J_{1} \) by Lemma 14 and Lemma 16

\[
\left\| J_{1} \right\| \leq C \sum_{j=0}^{2} \tau \left\| \int_{s}^{T} \tau^{-3+r} \varphi_{j,0} \left( p, \frac{p-r}{\tau} \right) f(r) dr \right\|_{L^{\infty}(I;X)} dq +
\]

\[
+ C \tau^{0} \left\| f \right\|_{L^{1}(I;X)} + C \tau^{2} \left\| f \right\|_{L^{2}}.
\]

By assumption \( f \in B_{L^{1}}^{0} \) we have that \( J_{1} \in L^{1}_{\varepsilon}(0,c); L^{\infty}(I; X; dt); d\tau \). We estimate \( J_{2} \)
in similiar way using Lemma 14 and Lemma 16.

Hence \( J \in L_{\varepsilon}^{1}(0,c); L^{\infty}(I; X; dt); d\tau \) as sum of two elements of this space. \( \square \)

**Remark 19.** By proofs of above lemmas the following inequality holds

\[
\left\| F \right\|_{L^{1}_{d\tau}} \leq C \left\| f \right\|_{L^{1}_{d\tau}} + C \left\| f \right\|_{L^{2}} + C \left\| f \right\|_{L^{2}}.
\]

(10)

**Theorem 20.** Assume \( Z_{1} \sim Z_{k} \) and let continuous function \( f : [0,T] \to X \) be in \( B_{L^{1}}^{0}(0,T) \). Then problem (5) has the unique solution, given by

\[
u(t) = U(t,s)x + \int_{s}^{t} U(t,r) f(r) dr.
\]

(11)

**Proof.** \( U(t,s)x \) is the solution of problem (5) with \( f \equiv 0 \). So, the first term in (11) is of class \( C^{1} \). We consider the second term
The function $f$ is continuous. So Remark 6 gives that $\int_s^t W(r, r) f(r) dr$ is of class $C^1$.

By Lemma 17 and Lemma 18, also, $F(t) = \int_s^t S_r(t-r) f(r) dr$ is of class $C^1$.

Theorem 12 implies the existence of sequence $\{f_n\}_{n=1}^{\infty}$, such that

$$f_n \in B_{w,1}^{1-\varepsilon} \cap C^1([s, T]; X)$$

and

$$f_n \to f \quad \text{in} \quad B_{w,1}^{1-\varepsilon} \cap L^1((s, T); X) \cap L^\infty([s, T]; X).$$

Let us denote

$$F_n(t) = \int_s^t S_r(t-r) f_n(r) dr,$$

$$u_n(t) = U(t, s) x + \int_s^t U(t, r) f_n(r) dr = U(t, s) x + F_n(t) + G_n(t),$$

where $G(t) = \int_s^t W(t, r) f(r) dr$. Then

$$u_n \in C^1((s, T]; X), \quad u_n(t) \in D \quad \text{for} \quad t \in (s, T]$$

$$\frac{du_n}{dt}(t) = -A(t)u_n(t) + f_n(t) \quad \text{for} \quad t \in (s, T]. \quad (12)$$

Substituting $f$ by $f - f_n$ in inequality (10) we have

$$\| F - F_n \|_{B_{w,1}^{1-\varepsilon}} \leq C \| f - f_n \|_{B_{w,1}^{1-\varepsilon}} + C \| f - f_n \|_{L^\infty} + C \| f - f_n \|_{L^1}.$$

So

$$\| F - F_n \|_{B_{w,1}^{1-\varepsilon}} \to 0, \quad \text{while} \quad n \to \infty.$$

On the other hand, by (12), we have

$$\left\| A(t)u_n(t) + \frac{du_n}{dt}(t) + f(t) \right\|_X \leq \left\| -f_n(t) \frac{dF_n}{dt}(t) + \frac{dF}{dt}(t) + f(t) \right\|_X + \left\| -f_n(t) \frac{dG_n}{dt}(t) + \frac{dG}{dt}(t) - \frac{dG_n}{dt}(t) \right\|_X \leq \left\| f(t) - f_n(t) \right\|_X + \left\| \frac{dF}{dt}(t) - \frac{dF_n}{dt}(t) \right\|_X + \left\| \frac{dG}{dt}(t) - \frac{dG_n}{dt}(t) \right\|_X.$$
By Theorem 3 in [5], Remark 19 and definition of Besov function
\[ \left\| f_n(t) - f(t) \right\|_{B^s_{0,1}} \leq C \left\| f_n(t) - f(t) \right\|_{B^s_{0,1}} \to 0, \quad \text{when} \quad n \to \infty. \]
\[ \left\| \frac{dF}{dt}(t) - \frac{dF_n}{dt}(t) \right\|_{B^s_{0,1}} \leq \left\| F(t) - F_n(t) \right\|_{B^s_{0,1}} \to 0, \quad \text{when} \quad n \to \infty. \]

By Lebesgue convergence theorem and Remark 7
\[ \left\| \frac{dG}{dt}(t) - \frac{dG_n}{dt}(t) \right\| \to 0, \quad \text{when} \quad n \to \infty. \]

By closedness of \( A(t) \) for \( t \in (s,T) \) the theorem is proved.

\[
5. \text{ Example}
\]

We give example of continuous functions
\[ f : [0,1] \times [0,1] \to (-\infty, \infty) \]
and \( u : [0,1] \to [0,1] \), such that

I. For fixed \( r \) function \( f \) satisfies the Lipschitz condition with respect to the second variable \( \exists L \geq 0 \forall r \in [0,1] \forall t_1, t_2 \in [0,1] \)
\[ |f(r,t_1) - f(r,t_2)| \leq L |t_1 - t_2|. \]

II. For fixed \( t \) function \( f(\cdot, t) \) is in Besov space \( B^s_{0,1} \), where \( \sigma \in (0,1) \), i.e. \( \exists M \geq 0 \)
\[ \int_0^1 h^{-1-s} \sup_{r \in [0,T-h]} \left| f(t+h, r) - f(t, r) \right| dh \leq M. \]

III. Function \( u \) is of class \( C^\infty \).

IV. Composition function \( f(\cdot, u(\cdot)) : [0,1] \to (-\infty, \infty) \) is not in Besov space, that is
\[ \int_0^1 h^{-1-s} \sup_{r \in [0,T-h]} \left| f(t+h, u(t+h)) - f(t, t) \right| dh = \infty. \]

This shows that the theorem on existence of the solution, for semilinear case requires additional assumption.
6. Construction

Let us define

\[ p_n(t) = \begin{cases} \frac{2^{-n\sigma}}{n \log^2 2} & \text{for } t = 0, \\ \frac{2^{-n\sigma} - n^\sigma}{n \log^2 2 \log^2 t} & \text{for } |t| \in \left(0, \frac{1}{2^n}\right], \\ 0 & \text{for } |t| \geq \frac{1}{2^n}, \end{cases} \]

\[ q_n(t,r) = \begin{cases} \left(1 - \frac{n \log^2 2}{2^{-n\sigma}} |r|\right) p_n(t) & \text{for } |r| \leq \frac{2^{-n\sigma}}{n \log^2 2}, \\ 0 & \text{for } |r| > \frac{2^{-n\sigma}}{n \log^2 2}, \end{cases} \]

\[ f(t,r) = \begin{cases} q_n \left(t - \frac{3}{4} \frac{2^{-m}}{2^n}, r - \frac{3}{4} \frac{2^{-m}}{2^n}\right) & \text{for } t \in \left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right], \\ 0 & \text{for } t = 0, \end{cases} \]

where \( n_m = k + km, k \geq \max \left\{2, \frac{1}{\sigma}\right\}. \)

For fixed \( r \) easy computations shows that

\[ \| f \|_{B^\sigma_{\infty}} \int_0^1 h^{-1-\sigma} \sup_{\mathcal{E}(0, t-h)} |f(t+h,r) - f(t,r)| dh \leq 6. \]

For fixed \( r \) semi-norm of function \( f(\cdot,r) \) in Besov function \( B^\sigma_{\infty,1} \) is no greater than 6. Function \( f \) satisfies the Lipschitz condition with respect to the second variable with constant 1.

Let \( u(t) = t \). Then \( u(\cdot) \) is of class \( C^\infty \) and

\[ \int_0^1 h^{-1-\sigma} \sup_{\mathcal{E}(0, t-h)} |f(t+h,t+h) - f(t,t)| dh = \infty. \]

This means that \( f(\cdot, u(\cdot)) \in B^\sigma_{\infty,1}. \)
References