MECHANIKA
CZASOPISMO TECHNICZNE
TECHNICAL TRANSACTIONS
MECHANICS
WYDAWNICTWO POLITECHNIKI KRAKOWSKIEJ
2-M/2010
ZESZYT 8
ROK 107
ISSUE 8
YEAR 107

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GENERATION OF THE DIFFERENCE FORMULAS FOR SCATTERED NODES USING RADIAL BASIS FUNCTIONS

GENERACJA WZORÓW RÓŻNICOWYCH DLA NIEREGULARNIE POŁOŻONYCH WĘZŁÓW Z UŻYCIEM RADIALNYCH FUNKCJI BAZOWYCH

Abstract

In the paper the method of the generation of the global difference formulas based on the arbitrary irregular node distribution is presented. The presented method uses approach that comes from DQ technique. As the base functions, radial functions are assumed, which are especially efficient in the approximation of a function given in a large number of scattered nodes. In order to show the accuracy of the formulas some derivatives of an exampled function are computed. The influence of the number of nodes as well as the shape parameter of the radial function on the accuracy and condition number of the system is investigated.

Keywords: meshless methods, radial basis functions, differential quadrature method

Streszczenie

W artykule przedstawiono metodę generacji globalnych wzorów różnicowych opartych na dowolnie nieregularnie rozmieszczonych węzłach. Przedstawiony sposób wykorzystuje podejścia stosowane w metodzie kwadratur różniczkowych. Jako funkcje bazowe przyjęto funkcje radialne, które są szczególnie skuteczne w aproksymacji funkcji zadanych w dużej liczbie nieregularnie rozmieszczonych punktów. Dla zilustrowania dokładności schematów różnicowych obliczono pochodne przykładowej funkcji oraz zbadano wpływ liczby węzłów i współczynnika kształtu funkcji radialnych na dokładność przybliżenia pochodnej i wskaźnik uwarunkowania układu równań.

Słowa kluczowe: metody bezsiatkowe, radialne funkcje bazowe, metoda kwadratur różniczkowych

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1. Introduction

The difference formulas are the basic tools that allow to calculate the approximation of derivatives as well as to solve numerically differential equations. The most often used difference formulas are obtained with the use of the uniform point distribution for the domain discretization. The appropriate weighs are determined using local Taylor series expansion. The method that applies such formulas to solve differential equations is called classical finite difference method and has been used in computational mathematics and mechanics for a long time.

Such difference formulas are not suitable when the domain of the solution has irregular shape or when the derivatives of a function are approximated at scattered nodes. In these cases another approach is required.

The attempt to overcome this inconvenience has been made in several works (for example [1]), that give the beginning of the so-called generalized finite difference method. This method uses arbitrary irregular grids to generate local difference formulas and appropriate weights are calculated with the aid of Taylor series expansion.

The generalized finite difference method falls into a larger group of numerical techniques called meshless or meshfree methods [2]. The common feature of these methods is the use of the arbitrary irregular node distribution. It means that there are no geometrical dependences between the nodes unlike the discretization in finite element method or classical finite difference one. The mentioned feature allows to apply these methods to challenging computational problems.

A lot of meshless methods use the radial basis functions (RBF) to create the shape functions or calculate the weight coefficients. It was found [3] that RBF (especially multiquadrics) are very effective in the interpolation of a large set of scattered data points.

In the present work a method for generation of the global difference formulas with the aid of differential quadrature (DQ) technique [4] that uses multiquadrics RBF is shown. The method leads to obtain difference formulas that accurately approximate derivatives of the function given at irregularly distributed nodes by solving simple algebraic equations.

2. DQ technique based on RBF

In the conventional DQ method a function derivative at a point is approximated by a linear weighted sum of the function values at all nodes along the line parallel to coordinate axis. Therefore the method requires regularly distributed set of nodes. In the present paper the method is generalized in the way that a function derivative at a point $\mathbf{x}_i = (x_1^i, ..., x_n^i)$, where n is the dimension of the problem, is approximated by a linear weighted sum of the function values $f_{x_a}^{(r)}(x_k)$ at all nodes from the domain $k = 1, \ldots, N$. It can be put as follows:

$$f_{,x}^{(r)}(\mathbf{x}_i) = \sum_{k=1}^{N} c_k^{(r)}(\mathbf{x}_i)\, f(\mathbf{x}_k) = \sum_{k=1}^{N} c_{ik}^{(r)}\, f_k\,, \quad i = 1,\ ...,\ N \tag{1}$$

where:

$c_{ik}^{(r)}$ – are unknown weighting coefficients for the rth order derivative of the f function with respect to x variables, where can take values from the set $\alpha = \{1, \ldots, n\}$.

In order to determine the weighting coefficients, the function $f(\mathbf{x})$ is written as a linear combination of some base functions $\varphi_j(\mathbf{x})$. The appropriate interpolation formula takes the form

$$f(\mathbf{x}) = \sum_{j=1}^{N} a_j \varphi_j(\mathbf{x}) \tag{2}$$

Introducing Equation (2) to (1) and changing the order of the summation on the right hand side of the formula one can obtain

$$\sum_{j=1}^{N} a_j \varphi_j^{(r)}(\mathbf{x}_i) = \sum_{j=1}^{N} a_j \sum_{k=1}^{N} c_{ik}^{(r)} \varphi_j(\mathbf{x}_k), \quad i = 1, \ ..., \ N \tag{3}$$

Comparing the values standing next to appropriate coefficients a_j one gets

$$\varphi_j^{(r)}(\mathbf{x}_i) = \sum_{k=1}^{N} c_{ik}^{(r)} \varphi_j(\mathbf{x}_k), \quad i, j = 1, \ ..., \ N \tag{4}$$

Equation (4) can be expressed in a close form using matrix notation

$$\mathbf{\Phi} \mathbf{c}_i = \boldsymbol{\varphi}_i^{(r)}, \quad i = 1, \ ..., \ N \tag{5}$$

where:

$$\mathbf{\Phi} = \begin{bmatrix} \varphi_1(\mathbf{x}_1) & \cdots & \varphi_1(\mathbf{x}_N) \\ \vdots & \ddots & \vdots \\ \varphi_N(\mathbf{x}_1) & \cdots & \varphi_N(\mathbf{x}_N) \end{bmatrix}, \ \mathbf{c}_i = \begin{bmatrix} c_{i1} \\ \vdots \\ c_{iN} \end{bmatrix}, \ \boldsymbol{\varphi}_i^{(r)} = \begin{bmatrix} \varphi_1^{(r)}(\mathbf{x}_i) \\ \vdots \\ \varphi_N^{(r)}(\mathbf{x}_i) \end{bmatrix}.$$

The set of $N \times N$ linear equations given by (5) allows to determine weighting coefficients for the global difference formulas that approximate rth order derivative at each node $\mathbf{x}_i$. The existence of the unique solution of Equation (5) is conditioned by the nonsingularity of the matrix.

Introducing multiquadrics RBF (6) as $\varphi_j(\mathbf{x})$

$$\varphi_j(\mathbf{x}) = \sqrt{(\mathbf{x} - \mathbf{x}_j)^2 + C^2} \tag{6}$$

this nonsingularity depends on the shape parameter C.

Different values of the shape parameter influence the accuracy of the difference formulas as well as the condition number of the matrix $\widetilde{\mathbf{\Phi}}$.

3. Numerical example

In order to show the accuracy of the difference formulas described in chapter 2, some derivatives of the two-variable function $f(x_1, x_2) = \sin(x_1)\cos(x_2)$ are computed.

The values of the function are imposed at the nodes from the range of $x_1, x_2 \in [0, \pi]$. The calculations are carried out applying uniform and random node distribution. The accuracy is determined by the percentage, relative error given by the expression:

$$\delta = \sqrt{\sum_{i=1}^{N}\left(f_i^{(r)} - \overline{f}_i^{(r)}\right)^2} \Bigg/ \sqrt{\sum_{i=1}^{N}\left(\overline{f}_i^{(r)}\right)^2} \cdot 100\% \tag{7}$$

where:

$f_i^{(r)}$ – denotes the approximate value of the rth order derivative at the ith node with respect to appropriate variable,

$\overline{f}_i^{(r)}$ – is the exact value of this derivative.

In any case the condition number of the coefficient matrix Φ of the system (5) is computed as follows

$$\mathrm{cond}(\boldsymbol{\Phi}) = \left\|\boldsymbol{\Phi}\right\|_{\infty} \cdot \left\|\boldsymbol{\Phi}^{-1}\right\|_{\infty} \tag{8}$$

The goal of the computation is to show the influence of the number of nodes N and the shape parameter C on the accuracy of the difference formulas and the conditioning of the problem.

The results obtained using uniform node distribution are presented in Figs. 1–2.

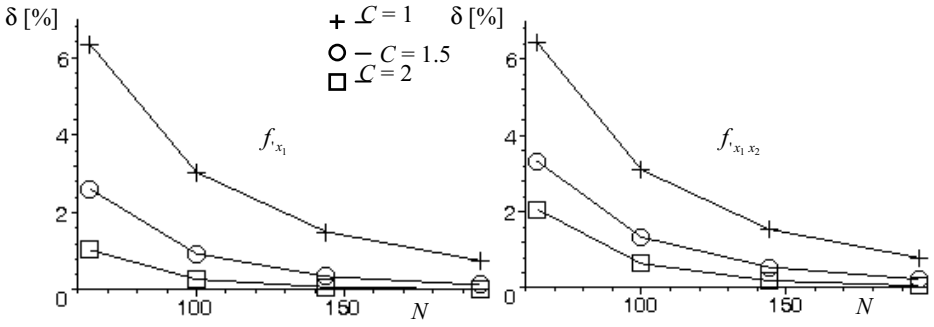

Fig. 1. The error of the approximation of the first $f_{,x_1}$ and second $f_{,x_1x_2}$ order derivatives – uniform node distribution

Rys. 1. Błąd przybliżenia pierwszej $f_{,x_1}$ oraz drugiej $f_{,x_1x_2}$ pochodnej – węzły rozmieszczone równomiernie

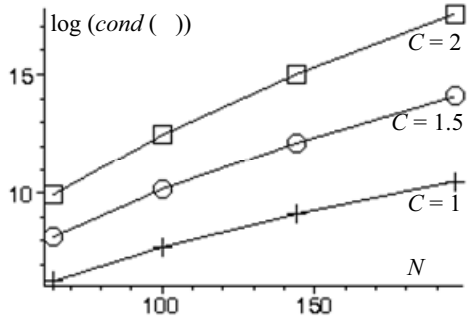

Fig. 2. The condition number of the Φ matrix – uniform node distribution

Rys. 2. Wskaźnik uwarunkowania macierzy Φ – węzły rozmieszczone równomiernie

The results presented in Fig. 1 show that increasing the coefficient C improves the accuracy. The difference formulas that use a larger number of nodes yield the better results without concern for losing the convergence as it takes place in conventional DQ method. In the latter the interpolation polynomial is used to calculate the weight coefficients and it is the main reason that the method may not converge. Unfortunately in the present approach the better results can be achieved at the expense of worse condition number of the matrix Φ in the system (5). It is clearly shown in Fig. 2. In this case one should assume a large enough computational precision (a number of significant digits) or use the special type algorithms.

Figs. 4–5 present the results obtained with the use of random node distribution. The example of this distribution contained $N = 100$ points is shown in Fig. 3. The conclusions follow from Figs. 4–5 are similar to those presented above, where the uniform point distribution is applied. It is worth to notice that in both cases the accuracy of the calculation of the first order derivative is better then the second one.

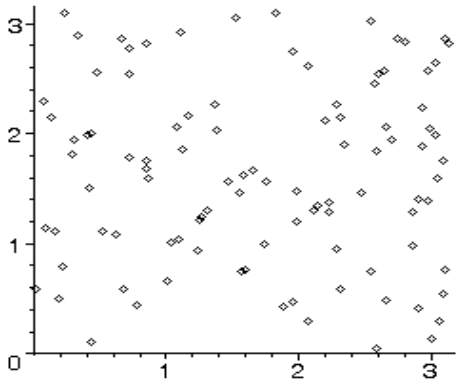

Fig. 3. The example of the random node distribution used in computations ($N = 100$)

Rys. 3. Przykład losowego rozmieszczenia węzłów używany w obliczeniach ($N = 100$)

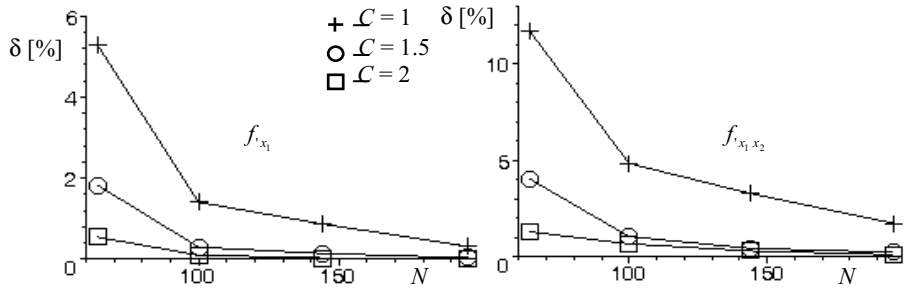

Fig. 4. The error of the approximation of the first $f_{,x_1}$ and second $f_{,x_1x_2}$ order derivatives – random node distribution

Rys. 4. Błąd przybliżenia pierwszej $f_{,x_1}$ oraz drugiej $f_{,x_1x_2}$ pochodnej – węzły rozmieszczone losowo

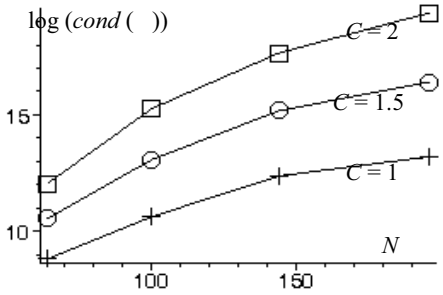

Fig. 5. The condition number of the Φ matrix – random node distribution

Rys. 5. Wskaźnik uwarunkowania macierzy Φ – węzły rozmieszczone losowo

4. Conclusions

In the paper the way of the generation of the global difference formulas that use all nodes to approximate a derivative is presented. This idea comes from the DQ method and gives generally more accurate results than the local approach but has some drawbacks. For a large number of nodes the condition number is poor what makes numerical complications.

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