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PROCEDURE OF DISCRETE DETERMINATION OF SIGNAL MAXIMISING THE INTEGRAL-SQUARE CRITERION

PROCEDURA DYSKRETNEGO WYZNACZANIA SYGNAŁU MAKSYMALIZUJĄCEGO KRYTERIUM KWADRATOWE

A b s t r a c t

The paper presents procedure of discrete determination of the signals maximizing the integral-square criterion while the magnitude and rate of change constraints are imposed on it. It is possible to prove that maximal value of this criterion can be obtained if signal reaches one of the constraints imposed. The algorithm for determining the signal was considered in detail in this paper. The correctness of the procedure has been proved on the example of fourth order low-pass object by means of computer program implemented in C language and genetic algorithm.

Keywords: *dynamic error*, *discrete convolution*, *genetic algorithm*

S t r e s z c z e n i e

Artykuł przedstawia procedurę dyskretnego wyznaczania sygnału maksymalizującego kryterium całkowo-kwadratowe przy ograniczeniach amplitudy i prędkości narastania. Można wykazać, że maksymalna wartość tego kryterium jest osiągalna, jeśli sygnał spełnia jedno z nałożonych na niego ograniczeń. W artykule przedstawiono algorytm umożliwiający wyznaczenie rozpatrywanego sygnału, a poprawność wyników potwierdzono na przykładzie dolnoprzepustowego obiektu rzędu czwartego za pomocą programu zaimplementowanego w języku C, z wykorzystaniem metody algorytmu genetycznego.

Słowa kluczowe: *błąd dynamiczny*, *splot cyfrowy*, *algorytm genetyczny*

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1. Introduction

The signals maximizing the chosen objective criteria are very useful in many technical domains e.g. in control, automatic, measurements etc. One of the most important is integral-square criterion which in $t \in [0, T]$ is defined as follows

$$I = \int_0^T \left[\int_0^t k(t-\tau) \cdot x(\tau) d\tau \right]^2 dt = \int_0^T [y(t)]^2 dt \tag{1}$$

where $x(t)$ and $y(t)$ is input and output signal respectively, $k(t)$ – impulse response of measurement system.

Let's apply the Simpson method for numerical integration of (1). Then we obtain the discrete form of (1) according to (2).

$$\tilde{I} = \frac{N\Delta}{6k} \left[y(0) + y(N\Delta) + 2\sum_{i=1}^{k-1} y\left(i\frac{N\Delta}{k} \right)^2 + 4\sum_{i=1}^{k} y\left[\left(i - \frac{1}{2} \right) \frac{N\Delta}{k} \right]^2 \right]$$

$$k = \frac{N}{2}, \quad N = T/\Delta, \quad \Delta \text{ – sample interval} \tag{2}$$

The advantage of this formula results from the fact that it makes possible to determine the maximum value of $\int_0^T [y(t)]^2 dt$ in a simple way by means of any discrete method.

2. Discrete procedure for searching of signals maximising the integral-square criterion

In [4, 5] it was proved that if only one constraint is imposed to the magnitude, the maximising signal is always of the „bang-bang" type, of the values $\pm A$ and with switching instants $t_1, t_2, \ldots, t_r$. For the discrete data $t_1, t_2, \ldots, t_r$ are represented by the values of the consecutive samples $n_1, n_2, \ldots, n_r$, where $n_1 < n_2 < \ldots < n_r$.

The exemplary $x(n)$, „bang-bang" input signal, described by the following parameters: $A = \pm 1$, $N = 1000$, $\Delta = 0{,}01$, $r = 5$ and $(n_1, n_2, \ldots, n_5) = (90, 245, 425, 720, 920)$ is presented in Fig. 1.

For $n = N$ output $y(n)$ is described by the following formula

$$\begin{aligned} y(N) = {} & \sum_{i=0}^{90} (+1) \cdot k(N-i)\Delta + \sum_{i=91}^{245} (-1) \cdot k(N-i)\Delta + \sum_{i=246}^{425} (+1) \cdot k(N-i)\Delta + \\ & + 426 \sum_{i=246}^{720} (-1) \cdot k(N-i)\Delta + \sum_{i=721}^{920} (+1) \cdot k(N-i)\Delta + \sum_{i=921}^{N} (-1) \cdot k(N-i)\Delta \end{aligned} \tag{3}$$

where $\Delta = 0.01$.

The maximum number of possible solutions of the $x(n)$ input signals is determined by the expression

$$P = N!/[(N-r)!\,r!] \tag{4}$$

In this way the time necessary for examine all possible solutions depends on r and N, as well as computation power of computer and dedicated numerical procedure. For $r \leq 3$ and $N \leq 100$ in order to determine the signal maximising (2) we can apply Eq. (4) using simple any numerical generator. For $r > 3$ and $N > 100$ the number of possible solutions achieves large dimension and the realisation of numerical procedures becomes very time consuming process. In this case, in order to determine this signal it is convenient to use one of the evolution program e.g. using the genetic algorithm technique.

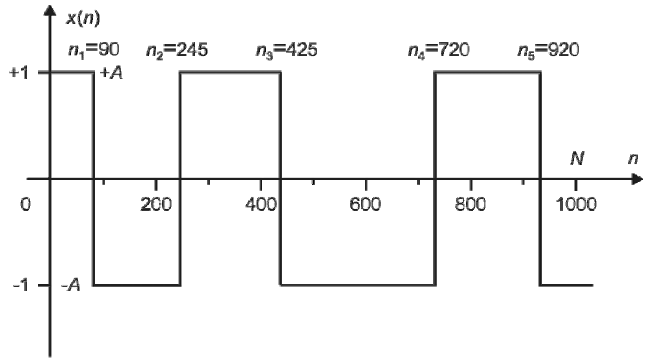

Fig. 1. Exemplary „bang-bang" signal
Rys. 1. Przykładowy sygnał typu "bang-bang"

The procedure is executed in the following steps:

1. The number of searching cycles C valid for every r, where r = 1, 2, 3, … is assumed.
2. For r = 1 and c = 1,where c = 1, 2, …, C, the initial vector $x(n)$ of switchings is generated.
3. Calculations, according to (2)–(3) are completed.
4. The searching procedure „remembers" the value of criterion $\tilde{I}$ from step 3 and corresponding to it vector of signal switchings $x(n)$.
5. The value of c is increased by 1.
6. New vector of switchings $x(n)$ is generated.
7. Calculations (2)–(3) are repeated.
8. The value of criterion $\tilde{I}$ is calculated and is compared with the value stored in the system's memory from the previous step. If the current value of criterion is greater than the previous one, it is now stored in the memory, together with the corresponding switchings vector, and if this value is smaller that the previous one, then this from the previous step are stored in the system's memory.
9. After the value of c reached the value of C, the value of r is increased by 1 and the procedure reverts to step 6.
10. The work of the algorithm stops, when after increasing the value of r for the second time and after value of c reached the value of C, the value of the criterion remains unchanged. The switchings corresponding with this criterion describe the maximising signal.

The values of criterion determined using the signals constrained in magnitude only reach relatively very high. This is caused by dynamics of the „bang-bang" signals, which have infinitely high derivative values in the instants of switching and therefore they are not matched to the dynamics of physically existing systems. In order to match the input signals to the dynamics of typical systems we will impose an additional constrain on these signals.

Let us consider signals maximising the integral-square criterion constrained simultaneously in the magnitude and in the rate of change ϑ.

I can be proved that the maximum value of our criterion can be obtained only with the use of signals, which in the specified time interval reach some of the imposed constrains [2, 3].

In such a case the maximising signal can be a triangular with the slope inclination of $\pm\alpha$ or a trapezoidal with the slope inclination of $\pm\alpha$ and height A, where

$$\vartheta = \mathrm{tg}\,\alpha \tag{5}$$

An exemplary $x(n)$ input signal with two constrains is presented in Fig. 2.

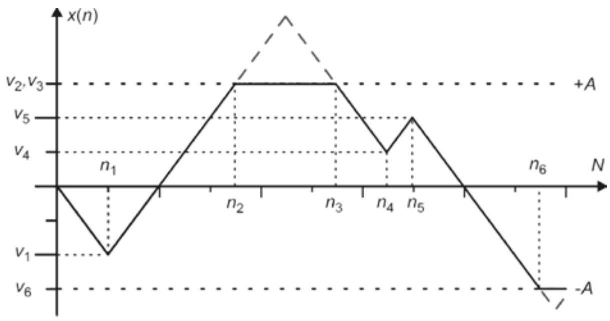

Fig. 2. Exemplary signal with two-constrains
Fig. 2. Przykładowy sygnał z dwoma ograniczeniami

where:

$(n_1, n_2, \ldots, n_6)$ – vector of signal switchings $x(n)$,

$(v_1, v_2, \ldots, v_6)$ – vector of values representing switchings of the $x(n)$.

The slopes of the $x(n)$ presented in Fig. 2, for the particular intervals, are described by the following relations:

for $0 < n \le n_1$

$$x(n) = \frac{v_1}{n_1} n \tag{6}$$

for $n > n_1$

$$x(n) = \frac{v_{i+1} - v_i}{n_{i+1} - n_i}(n - n_i) + v_i, \quad \text{for} \quad i = 1, 2, \ldots, r \tag{7}$$

where r – number of switchings.

For the assumed values A and ϑ, the values of $y(n)$ are described by the following formulae:

for $n \le n_1$

$$y(n) = \frac{v_1}{n}\Delta\sum_{i=0}^{n} k(n-i) \tag{8}$$

for $n_1 < n \le n_2$

$$y(n) = \frac{v_1}{n_1}\Delta\sum_{i=0}^{n_1} k(n-i)i + \frac{v_2 - v_1}{n_2 - n_1}\Delta\sum_{i=n_1}^{n} k(n-i)(i-n_1) + v_1\Delta\sum_{i=n_1}^{n_1} k(n-i) \tag{9}$$

for $n_i < n \le n_{i+1}$

$$\begin{aligned} y(n) = {} & \frac{v_1}{n_1}\Delta\sum_{i=0}^{n_1} k(n-i)i + \sum_{j=2}^{n}\Big[\frac{v_j - v_{j-1}}{n_j - n_{j-1}}\Delta\sum_{i=n_{j-1}}^{n_j} k(n-i)(i-n_{j-1}) + v_{j-1}\Delta\sum_{i=n_{j-1}}^{n_j} k(n-i)\Big] + \\ & + \frac{v_{i+1} - v_i}{n_{i+1} - n_i}\Delta\sum_{i=n_i}^{n} k(n-i)(i-n_i) + v_i\Delta\sum_{i=n_i}^{n} k(n-i) \end{aligned} \tag{10}$$

where:

$n_{r+1} = N$, v_{r+1} – values of $x(n)$ for N.

The algorithm for determining the maximising signal is realised similarly to the procedure applied in case of only one constraint in magnitude. The difference being in this case lies in the fact, that the peaks of the triangular signal are generated – Fig. 2, and the trapezoidal signal is obtained by imposing the magnitude constraint on the triangular signal. Applying such a procedure one can obtain the number of switchings in the interval from 1 to $(2T + 1)$, where T is the number of peaks of the triangular signal [8].

Searching algorithm

According to the specific of the genetic algorithms (AG), searching for the maximising signal is executed in three steps [1, 2, 7]:

- reproduction,
- crossing,
- mutation.

In the first stage of work of AG an initial population, comprised of an even number of chromosomes, is selected (Table 1). Each chromosome comprises of detectors. The number of detectors represents the number of switchings in case of a signal with one constraint, or the number of peaks T of the triangular signal.

The value of index (2) is determined for every chromosome and then, on the basis of the obtained results, and on the basis of the relation (11), the so called adaptation coefficient – the percentage participation of each chromosome in the value of summarised criterion – is calculated

$$\begin{gathered} \tilde{I}_s = \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3 + \ldots + \tilde{I}_m \\ \tilde{I}_1{}' = \frac{\tilde{I}_1}{\tilde{I}_s}\cdot 100\,[\%] \\ \tilde{I}_m{}' = \frac{\tilde{I}_m}{\tilde{I}_s}\cdot 100\,[\%] \end{gathered} \tag{11}$$

where $\tilde{I}_s$ is a value of summarised criterion, whereas $\tilde{I}_1', \tilde{I}_2', ..., \tilde{I}_m'$ represent the percentage participation of the particular adaptation coefficients in the summarised criterion.

Table 1

Population of chromosomes and adaptation coefficient for each chromosome

Chromosome	Detectors					Adaptation coefficient
	1	2	3	…	r	
1	n_{11}	n_{12}	n_{13}	. . .	n_{1r}	$\tilde{I}_1$
2	n_{21}	n_{22}	n_{23}	. . .	n_{2r}	$\tilde{I}_2$
3	n_{31}	n_{32}	n_{33}	. . .	n_{3r}	$\tilde{I}_3$
. . .	. . .	. . .	. . .	. . .	. . .	. . .
m	n_{m1}	n_{m2}	n_{m3}	. . .	n_{mr}	$\tilde{I}_m$

The determination of adaptation coefficients for each of the chromosomes leads to estimating their usefulness in the particular population. In case, when the differences between the obtained values of the adaptation coefficients are small, it is necessary to conduct the scaling operation of the adaptation coefficient, as the following steps of the work of the genetic algorithm might not give the desired results. In this work, the very popular linear scaling was applied.

In the next step, the reproduction operation was executed where, according to probability calculated from (11), the chromosomes are selected from the initial population set. Depending on the value of the adaptation coefficient, the particular chromosome, has smaller or greater chances of being selected to the next generation. There are several methods of calculating those „chances" for the particular chromosomes. The most popular method is the roulette wheel method which means, that the sampling process is conducted the number of times equal to the number of chromosomes in the population and the sampling results are transferred to the new – descendant population. All chromosomes are characterised by different, proportional to the value of the adaptation coefficient, probability of being selected. Thus as a result of the above reproduction procedure a new population of chromosomes is obtained.

The next stage is the crossing procedure. From the fact that the detectors assume the form of real positive numbers, it can be derived that if one pair of chromosomes takes part in the crossing process – then all pairs of detectors are crossed. The chromosomes are paired coincidentally and for the assumed crossing probability P_c the number from the [0, 1] interval is sampled. If the sampled number is from the [0, P_c] interval, then the crossing process takes place. Otherwise, the corresponding detectors of the associated chromosomes are not crossed. The crossing P_c is usually set at a high level, around 0.9.

The crossing process is conducted in accordance to the following formula [6]

$$\begin{aligned} n'_{11} &= (1-\alpha)n_{11} + n_{21} \\ n'_{21} &= \alpha n_{11} + (1-\alpha)n_{21} \end{aligned} \tag{12}$$

where n'_{11} is a detector of the first descendant chromosome, and n'_{21} is a detector of the second descendant chromosome.

The α coefficient has to be selected in such way, that the crossing process does not cause the descendant detectors on i-th position will assume values greater than the detectors on position $(i + 1)$ and lower than detectors in position $(i - 1)$.

The last stage of the AG operation is the process of mutation. In case of every detector, being a part of the descendant chromosomes, we ask, whether the mutation process shall be done or not. This process is usually conducted with a very low probability (usually at the level of 0.01).

Often encountered variation of the mutation process is the linear mutation, conducted with the use of the following formula [6]

$$\begin{aligned} & n''_{1r} = (n_{1r+1} - n_{1r-1})\alpha + n'_{1r-1} \\ & \alpha \in <0,1>, \quad r = 1, 2, ..., m \end{aligned} \tag{13}$$

after completing the mutation process, the process of work of the genetic algorithm is repeated. The number of the populations of the algorithm realising the search in the space of $x(n)$ signals shall be as large as possible. However, it shall be taken into consideration, that for a larger population, the time of work of the algorithm lengthens significantly.

3. Exemplary test results

As an example, let's examine the system described by the equation

$$K_m(s) = \frac{1}{(s^2 + 1.848s + 1) \cdot (s^2 + 0.765s + 1)} \tag{14}$$

and its standard in form of

$$K_s(s) = \frac{1}{(s^2 + 1.76s + 1) \cdot (s^2 + 0.09s + 1)} \tag{15}$$

The model of the standard was determined on the basis of the numerical procedures, involving the optimisation of the parameters of the fourth order model with regard to the execution of the non-distortion transformation. Such transformation assures the flat of the amplitude characteristic and the linear phase characteristic.

Procedures described in points 2.1 2.2 and 2.3 were used in order to determine the maximising signal with two constraints, maximising the functional (2). The magnitude constraint $A = 1$ and the rate of change $\vartheta = 0.46$ were assumed.

The discretised differences of impulse responses of the model and the sample $k(n)$, shape of the maximising signal and the corresponding to this signal output $y(n)$ are presented in Fig. 3.

Signal from Fig. 3 is described by switchings:

$$u_0 \Rightarrow +\vartheta\ [0, 204], +1[205, 295], -\vartheta\ [296, 602], +\vartheta\ [603, 709],$$
$$+\vartheta\ [710, 909], +\vartheta\ [910, 1000]$$

for which the value (2) equals 1.51 $V^2 \cdot s$.

The afore results were obtained with the use of a software application implemented in C language.

The calculations were conducted by the discretisation of time T = 10 s with step Δ = 0.01 s., what corresponds to N = 1000 samples.

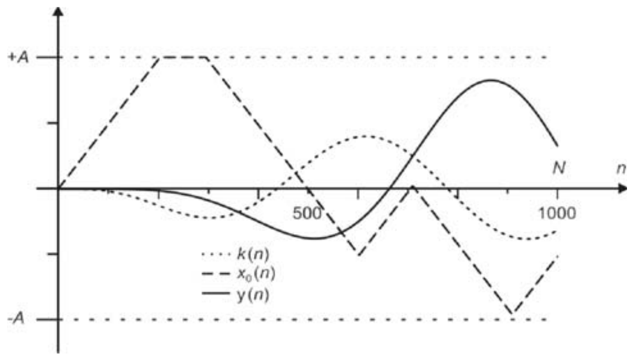

Fig. 3. Impulse response $k(n)$, $x_0(n)$ – maximising signal and output $y(n)$
Rys. 3. Odpowiedź impulsowa $k(n)$, $x_0(n)$ – sygnał maksymalizujący oraz wyjście $y(n)$

The following parameters of the genetic algorithm were assumed: number of populations intended for searching equals 10 000, number of chromosomes in each of the populations equals 32, probability of crossing P_k = 0.9 probability of mutation P_m = 0.07.

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