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AN IMPROVED MATHEMATICAL PROOF FOR „ZERO-STIFFNESS" POSTBUCKLING

UDOSKONALONY DOWÓD MATEMATYCZNY NA ZACHOWANIE POKRYTYCZNE PRZY „ZEROWEJ SZTYWNOŚCI"

A b s t r a c t

The aim of this paper is to explain zero-stiffness postbuckling, a special form of transition from imperfection sensitivity to imperfection insensitivity, in the framework of Koiter's postbuckling analysis and to provide means to predict the occurrence of this phenomenon as well as a criterion that rules out zero-stiffness postbuckling for certain structures. The application of the presented theory will be shown for a numerical example.

Keywords: bifurcation buckling, zero-stiffness postbuckling, imperfection sensitivity, Koiter's postbuckling analysis

S t r e s z c z e n i e

Artykuł jest poświęcony wyjaśnieniu zjawiska zachowania pokrytycznego przy „zerowej sztywności", czyli specjalnej formie przejścia od wrażliwości na imperfekcje do braku wrażliwości w ramach Koiterowskiej analizy pokrytycznej. Autorzy pokazują metody przewidywania tego zjawiska, jak również kryterium, które wyklucza zachowanie pokrytyczne pewnych konstrukcji przy „zerowej sztywności". Zastosowanie przedstawionej teorii zilustrowano za pomocą przykładu numerycznego

Słowa kluczowe: wyboczenie bifurkacyjne, zachowanie pokrytyczne przy „zerowej sztywności", wrażliwość na imperfekcje, Koiterowska analiza pokrytyczna

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1. Introduction

In the analysis of geometrically nonlinear elastic structures, zero-stiffness postbuckling may occur as a special form of postbuckling behavior. It features a strictly horizontal secondary path, allowing the structure to take on every displacement along the postbuckling path without a change of the external load. Thus, each point on a zero-stiffness postbuckling path can be viewed as a neutral state of equilibrium. It plays a role in sensitivity analysis as a special mode of transition from imperfection sensitivity to imperfection insensitivity.

Research on this topic was triggered by Tarnai [8], who first investigated the phenomenon thoroughly. Further work was done by Jia et al. [2] and Mang et al. [4].

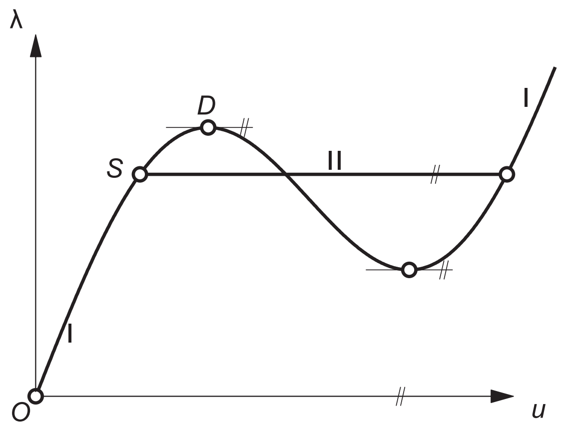

Fig. 1. Nonlinear primary path (I) and zero-stiffness secondary path (II)

Rys. 1. Nieliniowa ścieżka pierwotna (I) i ścieżka wtórna (II) przy „zerowej sztywności"

The main topic of this work is to provide criteria to decide whether or not zero-stiffness postbuckling occurs in a structure, on the basis of quantities that can be calculated in the course of Koiter's postbuckling analysis, i.e. without using branch switching and calculating the secondary path. This is non-trivial since Koiter's postbuckling analysis provides series expansions around the bifurcation point, whereas zero-stiffness postbuckling is a phenomenon that involves the whole secondary path.

In section 2, we will concisely review how zero-stiffness postbuckling is expressed in the frame of Koiter's postbuckling analysis. In section 3, we will present sufficient and necessary conditions for zero-stiffness postbuckling, section 4 is dedicated to an example problem and in Section 5 conclusions from the preceding considerations will be drawn.

2. Zero-stiffness postbuckling in the frame of Koiter's postbuckling analysis

Employing Koiter's postbuckling analysis, a series expansion for the load and the displacement along the secondary path emanating from a simple bifurcation point can be found. The difference in the load with respect to the load level at the stability limit is given by

$$\Delta\lambda(\kappa,\eta) = \lambda_1(\kappa)\eta + \lambda_2(\kappa)\eta^2 + \lambda_3(\kappa)\eta^3 + \lambda_4(\kappa)\eta^4 + O(\eta^5), \quad (1)$$

where η denotes an independent path parameter and κ represents a design parameter of the structure. A criterion for imperfection insensitivity using the coefficients of (1) is given by [7]

$$\exists k \in \{2,4,6,\ldots\}: \quad \lambda_k > 0, \quad \lambda_i = 0 \forall i \in \mathbb{N} \text{ with } i < k. \quad (2)$$

In most cases, this simply reduces to $\lambda_1 = 0$, $\lambda_2 > 0$. Following the lines of Bochenek [1], a condition for stability at the bifurcation point and thus imperfection insensitivity is

$$\lambda_{,\varphi}(\varphi)\,\mathrm{sign}\,(\varphi) \geq 0, \qquad \forall \varphi \in [\varphi_1, \varphi_2], \quad (3)$$

where $\lambda(\varphi)$ denotes the load level along the secondary path as a function of a degree of freedom φ of the system under consideration and $[\varphi_1, \varphi_2]$ stands for an interval containing the buckling configuration. In the case of zero-stiffness postbuckling occurring for a specific value $\kappa = \overline{\kappa}$ of the design parameter

$$\Delta\lambda(\overline{\kappa},\eta) = 0 \quad \forall_\eta \Rightarrow \lambda_i(\overline{\kappa}) = 0 \quad \forall i \in \mathbb{N}. \quad (4)$$

Thus, (2) is not satisfied. On the other hand, zero-stiffness postbuckling is classified imperfection insensitive according to (3). In [4] it was shown that zero-stiffness postbuckling is indeed imperfection insensitive, i.e. for a system that differs from a perfect one with zero-stiffness postbuckling only by a sufficiently small imperfection, the load along the equilibrium path increases strictly monotonously. Steinboeck et al. [6] have shown that

$$\lambda_1(\kappa) = 0 \quad \forall \kappa \quad (5)$$

is a necessary condition for imperfection insensitivity and, consequently, for zero-stiffness postbuckling. Making use of (5), the coefficients $\lambda_2(\kappa)$, $\lambda_3(\kappa)$, and $\lambda_4(\kappa)$ are obtained as follows [5]

$$\lambda_2(\kappa) = d_1(\kappa), \quad (6a)$$

$$\lambda_3(\kappa) = b_1(\kappa)\lambda_2(\kappa) + d_2(\kappa), \quad (6b)$$

$$\lambda_4(\kappa) = a_1(\kappa)\lambda_2(\kappa)^2 + b_2(\kappa)\lambda_2(\kappa) + b_1(\kappa)\lambda_3(\kappa) + d_3(\kappa). \quad (6c)$$

The full expressions for $a_1(\kappa)$, $b_1(\kappa)$, $b_2(\kappa)$, $d_1(\kappa)$, $d_2(\kappa)$, and $d_3(\kappa)$ can be found in [5]. They are calculated by means of Koiter's postbuckling analysis.

A characteristic property of systems subject to (5) is that frequently the first non-vanishing term in the series expansion (1) has an even subscript. This is expressed by

$$d_1(\kappa = \overline{\kappa}) = 0 \;\overset{(6a)}{\Rightarrow}\; \lambda_2(\kappa = \overline{\kappa}) = 0 \;\Rightarrow\; d_2(\kappa = \overline{\kappa}) \;\overset{(6b)}{\Rightarrow}\; \lambda_3(\kappa = \overline{\kappa}) = 0, \quad (7)$$

$$d_1(\kappa = \overline{\kappa}) = 0, \quad d_3(\kappa = \overline{\kappa}) = 0 \;\overset{(6c)}{\Rightarrow}\; \lambda_4(\kappa = \overline{\kappa}) = 0 \;\Rightarrow\; \lambda_5(\kappa = \overline{\kappa}) = 0. \quad (8)$$

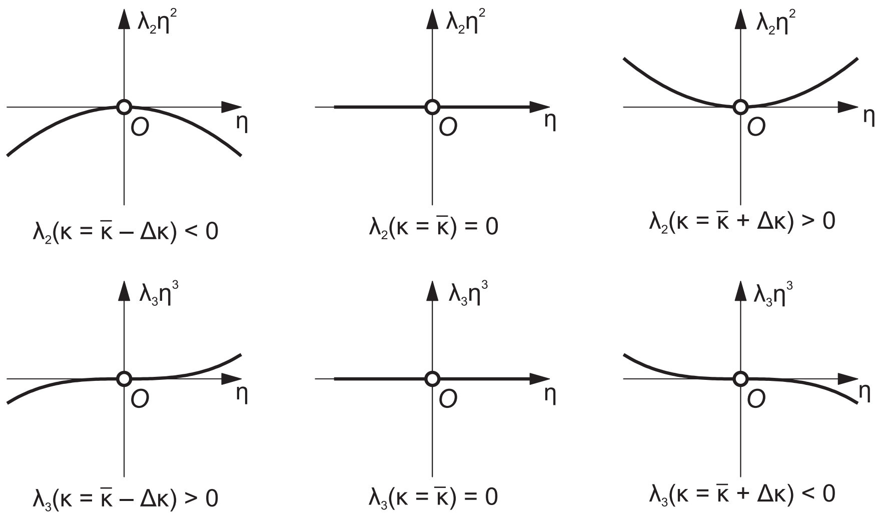

Fig. 2. Transition from imperfection sensitivity to imperfection insensitivity in the course of sensitivity analysis of the initial postbuckling path, λ_2 and λ_3 vanish for the same value of κ. (Sequence of sign of λ_3 may be in reverse order)

Rys. 2. Przejście od wrażliwości do braku wrażliwości na imperfekcje w trakcie analizy wrażliwości ze względu na początkową ścieżkę pokrytyczną, λ_2 i λ_3 znikają dla tej samej wartości κ (kolejność znaku λ_3 może być odwrotna)

3. Sufficient and necessary conditions for zero-stiffness postbuck- ling

Bending in the prebuckling domain precludes zero-stiffness postbuckling [4]. A mathematical condition which is necessary for a prebuckling membrane state of stress is therefore necessary for zero-stiffness postbuckling. The equation [3]

$$\mathbf{v}_1^T \cdot \tilde{\mathbf{K}}_{T,\lambda\lambda}(\lambda) \cdot \mathbf{q}_{,\lambda\lambda}(\lambda)\Big|_{\lambda=\lambda_S} = 0 \tag{9}$$

constitutes such a condition. $\tilde{\boldsymbol{K}}_{T,\lambda\lambda}$ denotes the second derivative of the tangent stiffness matrix $\tilde{\boldsymbol{K}}_T$ in the direction of a tangent to the primary path, $\mathbf{v}_1$ denotes the eigenvector of $\tilde{\boldsymbol{K}}_T$ corresponding to the bifurcation buckling mode, $\mathbf{q}_{,\lambda\lambda}$ is the second derivative of the displacement of the primary path, and λ_S is the critical load at which bifurcation buckling occurs. If the system under consideration exhibits only membrane stresses (axial stresses in beams) [3], (9) is satisfied. This includes structures with linear prebuckling paths and linear stability problems.

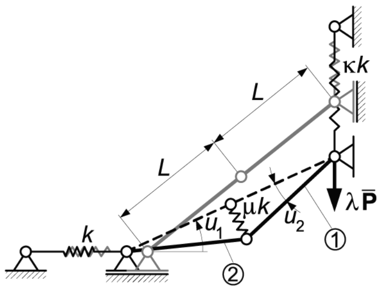

Fig. 3. Pin-jointed two-bar system [4]

Rys. 3. Układ dwóch prętów połączonych przegubowo [4]

Thus

$$\mathbf{v}_1^T \cdot \tilde{\mathbf{K}}_{T,\lambda\lambda}(\lambda) \cdot \mathbf{q}_{,\lambda\lambda}(\lambda)\Big|_{\lambda=\lambda_S} \neq 0 \tag{10}$$

is a sufficient condition to exclude zero-stiffness postbuckling. Within the class of problems for which (9) holds, there are such ones with as well as without zero-stiffness postbuckling. Hence, an additional criterion is needed to predict zero-stiffness postbuckling. This is given by the joint vanishing of λ_2 and λ_4 for a specific value $\kappa = \overline{\kappa}$ of the design parameter [4]. Thus,

$$\mathbf{v}_1^T \cdot \tilde{\mathbf{K}}_{T,\lambda\lambda} \cdot \mathbf{q}_{,\lambda\lambda}\Big|_{\lambda=\lambda_S} = 0, \quad \lambda_1(\kappa) = 0 \quad \forall\kappa, \quad \lambda_2(\overline{\kappa}) = 0, \; \lambda_3(\overline{\kappa}) = 0, \quad \lambda_4(\overline{\kappa}) = 0 \tag{11}$$

defines a set of necessary and sufficient conditions for zero-stiffness postbuckling.

4. Example

As only few simple structures exhibiting zero-stiffness postbuckling are known, we cite here an example that was already presented in [4]. We study the planar, static, conservative system that is outlined in Fig. 3. Both of its rigid bars, 1 and 2, have the length L, and in the non-buckled state they are in-line. The bars are linked at one end and supported by turning-and-sliding joints at their other ends. A horizontal linear elastic spring of stiffness k and a vertical linear elastic spring of stiffness $k\kappa$ are attached to turning-and-sliding joints. A spring of stiffness μk 'pulls' the two bars back into their in-line position. The system is loaded by a vertical load P at the vertical turning-and-sliding joint. The two displacement coordinates are the angles u_1 and u_2, summarized in the vector $\mathbf{u} = (u_1, u_2)^T$. κ is the design parameter that was varied in a sensitivity analysis. The remaining parameters were taken as $\mu = 3/5$ and $u_{10} = 0.67026$, in which case hilltop buckling occurs for $\kappa = 0$. The load-displacement path for hilltop buckling is shown in Fig. 4a). O labels the unloaded state. For parametrization of the secondary path, where $\eta \equiv u_2$, the relevant coefficients of the series expansion (1) follow as

$$\lambda_1 = 0, \quad \lambda_2 = \frac{kL}{p} \frac{(\kappa - \mu/4)}{\sqrt{1 - \dfrac{\cos^2(u_{10})}{(1-\mu/4)^2}}}, \quad \lambda_3 = 0, \quad \lambda_4 = \frac{\lambda_2}{12} \frac{1 - 4\dfrac{\cos^2(u_{10})}{(1-\mu/4)^2}}{1 - \dfrac{\cos^2(u_{10})}{(1-\mu/4)^2}}. \tag{12}$$

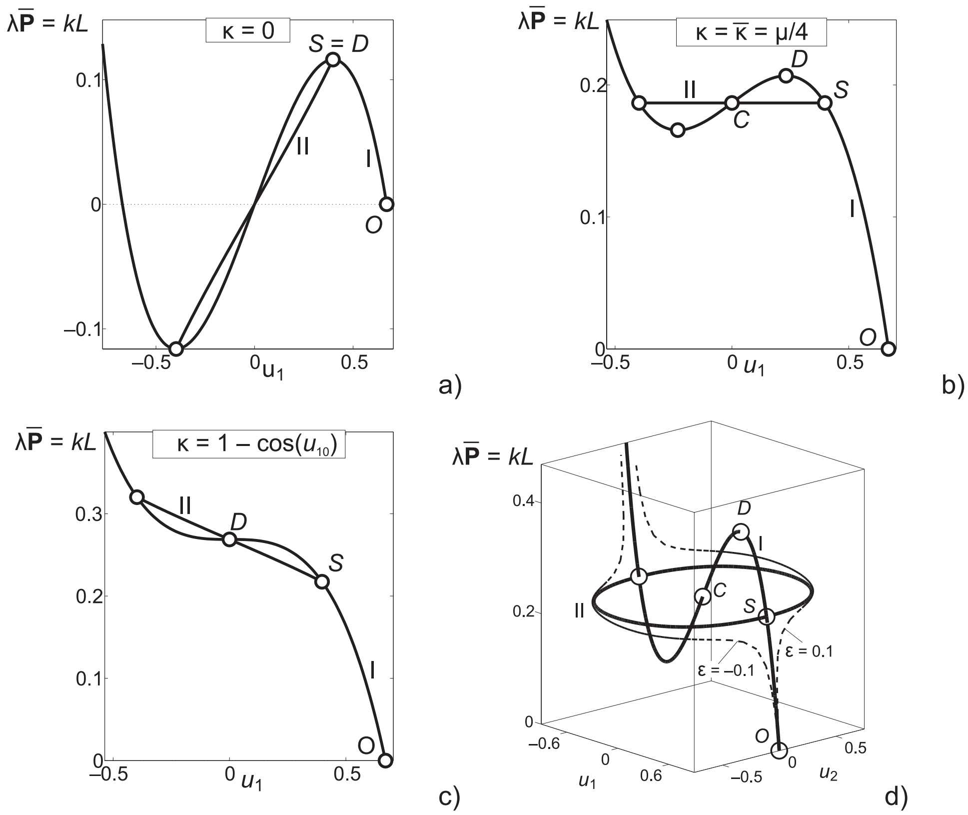

Fig. 4. Selected results from sensitivity analysis of the initial postbuckling behavior of the pin-jointed two-bar system shown in Fig. 3: a)–c) Projections of load-displacement paths onto the plane $u_2 = 0$ for hilltop buckling, zero-stiffness postbuckling, and the beginning of monotonically increasing prebuckling paths, d) Load-displacement paths of the perfect system with zero-stiffness postbuckling in comparison with a disturbed system with load eccentricities, showing imperfection insensitivity of zero-stiffness postbuckling [4]

Rys. 4. Wybrane wyniki analizy wrażliwości ze względu na początkową ścieżkę pokrytyczną dla układu dwóch prętów połączonych przegubowo i pokazanych na rys. 3: a)–c) rzuty ścieżek obciążenie – przemieszczenie na płaszczyznę $u_2 = 0$ dla przypadku zbieżności punktu bifurkacji z punktem przeskoku, zachowania pokrytycznego przy „zerowej sztywności" i początku monotonicznie wzrastających ścieżek przedkrytycznych, d) ścieżki obciążenie – przemieszczenie dla układu idealnego z zachowaniem pokrytycznym przy „zerowej sztywności" w porównaniu z układem nieidealnym zawierającym imperfekcje obciążenia, wykazującym brak wrażliwości na imperfekcje w zakresie zachowania pokrytycznego przy „zerowej sztywności" [4]

For $\kappa = 0$ (hilltop buckling), the system is imperfection sensitive ($\lambda_2 < 0$), and λ_S exceeds the ultimate load of any imperfect system. Increasing the parameter κ, i.e. the stiffness of the vertical spring, improves the postbuckling behavior insofar as λ_2 eventually begins increasing monotonically. The system is imperfection insensitive for $\kappa \geq \mu/4$.

Figure 4b) refers to the case $\kappa = \bar{\kappa} = \mu / 4$, for which $\lambda = \lambda_S$ holds along the whole postbuckling path. Following from (12.4)

$$\frac{\lambda_4(\kappa)}{\lambda_2(\kappa)} = -\frac{1}{12} \frac{1 - 4\dfrac{\cos^2(u_{10})}{(1-\mu/4)^2}}{1 - \dfrac{\cos^2(u_{10})}{(1-\mu/4)^2}} = \text{const} . \tag{13}$$

Consequently

$$\lambda_2(\kappa = \bar{\kappa}) = 0 \quad \Rightarrow \quad \lambda_4(\kappa = \bar{\kappa}) = 0 \tag{14}$$

The specific choice of generalized coordinates in this case implies that the buckling mode $\mathbf{v}_1$ and the whole primary path are contained in distinct orthogonal subspaces and the tangent stiffness $\tilde{\boldsymbol{K}}_T$ (and thus its derivatives with respect to λ becomes a diagonal matrix along the primary path. Under these conditions, (9) holds. Hence, the sufficient and necessary conditions for zero-stiffness postbuckling, as given in (11), are satisfied. Figure 4(d) shows the effect of imperfections, in this case a load eccentricity, on the load-displacement behavior. The bifurcation disappears, but for sufficiently small imperfections, the load-displacement path is close to the one of the perfect system. In the case of imperfection insensitivity, the load along the secondary path increases monotonically.

As κ is further increased, the system becomes markedly imperfection insensitive. Eventually, at $\kappa = 1 - \cos(u_{10})$, the snap through point D of the primary path becomes a saddle point at $\mathbf{q} = \mathbf{0}$. This situation is shown in Figure 4c).

5. Conclusions

1) The predictability of zero-stiffness postbuckling on the basis of a finite number of terms calculated at the stability limit is nontrivial.
2) $\lambda_1(\kappa) = 0$ $\forall\kappa$ must hold for the whole range of values of the design parameter κ. If it holds and if $\lambda_2(\kappa = \bar{\kappa}) = \lambda_3(\kappa = \bar{\kappa}) = 0$, the first non-vanishing term in the series expansion of $\Delta\lambda$ (κ, η) that may not be zero, is $\lambda_4(\kappa = \bar{\kappa})$.
3) Zero-stiffness postbuckling is imperfection insensitive.
4) Zero-stiffness postbuckling is only possible if the prebuckling domain is characterized by a membrane state of stress.
5) With

$$\mathbf{v}_1^T \cdot \tilde{\mathbf{K}}_{T,\lambda\lambda} \cdot \mathbf{q}_{,\lambda\lambda} \Big|_{\lambda=\lambda_S} = 0, \quad \lambda_1(\kappa) = 0 \quad \forall\kappa, \quad \lambda_2(\bar{\kappa}) = 0, \quad \lambda_3(\bar{\kappa}) = 0, \quad \lambda_4(\bar{\kappa}) = 0$$

necessary and sufficient conditions for zero-stiffness postbuckling are at hand.

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