MECHANIKA
CZASOPISMO TECHNICZNE
TECHNICAL TRANSACTIONS
MECHANICS
WYDAWNICTWO
POLITECHNIKI KRAKOWSKIEJ
4-M/2010
ZESZYT 20
ROK 107
ISSUE 20
YEAR 107

JACEK KRUŻELECKI, ADAM STAWIARSKI*

OPTIMAL DESIGN OF THIN-WALLED COLUMNS FOR BUCKLING UNDER COMBINED LOADINGS CONTROLLED BY DISPLACEMENTS

OPTYMALNE KSZTAŁTOWANIE CIENKOŚCIENNYCH KOLUMN ZE WZGLĘDU NA WYBOCZENIE POD DZIAŁANIEM OBCIĄŻEŃ ZŁOŻONYCH STEROWANYCH PRZEMIESZCZENIOWO

A b s t r a c t

In this paper the problem of the optimal design of thin-walled tubular columns under axial compression and torsion controlled by displacements is investigated. A radius of cross-sectional circular profile varying along the axis of the shell-column as well as a wall thickness, which lead to the maximal displacements caused by loadings before the structure buckles are sought. Both global (buckling of a column) and local (wall buckling of a shell) stability of a structure are taken into account. The geometry of the structure is approximated by the convex Bézier polynomial. The results are obtained using the simulated annealing method.

Keywords: optimization, column, shell, buckling, wall stability, critical displacement

S t r e s z c z e n i e

W artykule badano problem optymalnego kształtowania cienkościennych kołowych kolumn poddanych działaniu osiowego ściskania i skręcania sterowanych przemieszczeniowo. Poszukiwano zmiennego wzdłuż osi kolumny-powłoki promienia kołowego profilu przekroju oraz grubości ścianki, które prowadzą do maksymalnego przemieszczenia spowodowanego obciążeniami do momentu utraty stateczności. Uwzględniono zarówno globalną (wyboczenie kolumny), jak i lokalną (wyboczenie ścianki powłoki) stateczność konstrukcji. Geometrię konstrukcji aproksymowano wypukłymi wielomianami Bèziera. Optymalizację przeprowadzono, stosując metodę symulowanego wyżarzania.

Słowa kluczowe: optymalizacja, kolumna, powłoka, wyboczenie, stateczność ścianki, krytyczne przemieszczenie

* Prof. dr hab. inż. Jacek Krużelecki, Instytut Mechaniki Stosowanej, Wydział Mechaniczny, Politechnika Krakowska.
** Mgr inż. Adam Stawiarski, Instytut Konstrukcji Maszyn, Wydział Mechaniczny, Politechnika Krakowska.

1. Introduction

Optimal design of structures under stability constraints concerns mainly loadings controlled by a system of forces. Then a classical formulation of optimization problem under stability constraints leads to the maximization of a critical force. However, in some practical engineering applications, the loadings, which are controlled by displacements, can also occur. In this case an optimal shape of a structure, which maximizes a critical displacement, is sought. This type of problems is, for example, connected with structures under thermal loadings in the case of immovable supports or under assembly loadings, which are due to initial imperfections of overall dimensions of a structure. Then, the compressive forces, which occur, for example, due to an elevated temperature or due to assembly displacements, depend on geometry of the structure, whereas in the classical optimization problem the forces are independent of the structure. Hence, the results of shape optimization for loadings controlled by displacements can be qualitatively and quantitatively different from those obtained for the classical problem. Such problems were investigated by Krużelecki and Smaś [16] for annular plates and by Krużelecki and Smaś [14, 15] and Smaś [25] for columns with solid cross-sections. They found quite considerable differences in the optimal shapes of structures as well as in the critical axial displacements obtained for loadings controlled by displacements and for the classical problem.

Thin-walled columns with closed cross-sections belong to particular light structures since optimal design under wall stability constraints results in substantial reduction of they weight. The first investigations in this field are due to Feigen [7] and Kriste [13]. Kriste proved that a closed annular cross-section is more efficient than a thin-walled polygonal one since a local buckling is governed by shell stability equations. Janiczek [10] and Volkersen [26] considered parametric optimization of thin-walled tubes whereas Huang and Sheu [9] and Krzyś [20] introduced stress constraints into such optimization problems. Surveys of those problems are given in the monographs by Gajewski and Życzkowski [8] and by Bochenek and Krużelecki [6]. The papers mention above concerning thin-walled structures utilized the classical formulation of optimization problem leading to maximization of a critical force.

On the other hand the problem of optimization of thin-walled columns against buckling under loading controlled by displacements is almost untouched. Only two papers can be mentioned here. The first one by Życzkowski, Krużelecki and Trzeciak [29] is devoted to rotationally symmetric shell under thermal loading and there, in fact, is no column buckling phenomenon taken into account. The recent papers by Krużelecki and Stawiarski [17, 18] deal with such a problem for a thin-walled column simply supported at both ends under axial compression.

In this paper the problem discussed by Krużelecki and Stawiarski [17, 18] is generalized for combined state of loadings, namely for axial compression and torsion. The problem of optimization of elastic annular thin-walled columns under combined compression and torsion controlled by displacements with respect to their stability is investigated. Both, global buckling of a structure treated as a column (Euler's buckling) and wall buckling connected with buckling of a column treated as a thin-walled shell (local buckling), are taken into account. The paper shows also a comparison of the optimal shapes of columns obtained for loadings controlled by displacements and by forces.

2. Stability constraints

There exists a great variety of possible buckling modes of thin-walled columns. If considerations are restricted to compressed and twisted structures with an annular thin-walled cross-section then two main buckling modes should be taken into account. The first one is Euler's buckling, called "the overall buckling", which is the global buckling mode connected with stability of a column. It may be investigated under the assumption of underformable cross-sectional profiles. However, in fact, the profiles may also be subjected to deformation and the second mode, due to this possible deformation of thin-walled bar, is connected with wall buckling, governed by shell stability equations.

A global buckling of a column under combined axial compression and torsion can be treated as a conservative stability problem only for simply supported or clamped column at both ends (Ziegler [28], Bažant and Cedolin [5]). A mode of buckling of a column under axial compression by a force P and twisting moment M_s (both loadings are conservative) is spatial one and can be described by displacements $\overline{y}$ and $\overline{z}$, Fig. 1.

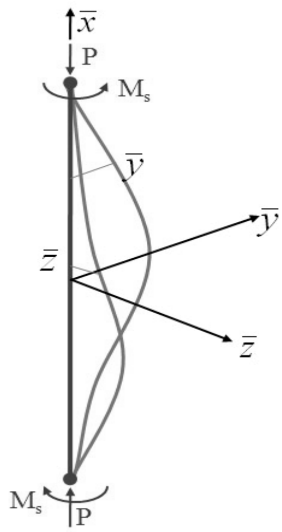

Fig. 1. Spatial deflection of a column characterized by displacements $\overline{y}$ and $\overline{z}$

Rys. 1. Przestrzenna deformacja kolumny scharakteryzowana przemieszczeniami $\overline{y}$ i $\overline{z}$

In this paper only a simply supported column at both ends is considered. For such end conditions a column with a cross-section varying along its axis can be analyzed using the fourth order stability differential equations

$$
\begin{aligned}
(EJ\overline{y}'')'' + P_{cr}\overline{y}'' + M_{scr}\ \overline{z}''' = 0 \\
(EJ\overline{z}'')'' + P_{cr}\overline{z}'' - M_{scr}\ \overline{y}''' = 0
\end{aligned}
\tag{1}
$$

describing a deflection curve in two planes, where $(') = d/d\bar{x}$, E is the Young modulus, J denotes the moment of inertia of the cross-sectional area perpendicular to the axis $\bar{x}$ of the column, $\bar{y}$ and $\bar{z}$ stand for the components of a lateral displacement of a column due to buckling. Such stability equations can be found, for example, in books by Ziegler [28] and Bažant and Cedolin [5]. In the considered case of a simply supported column at both ends, when the co-ordinate system is located in the middle of a column (Fig. 1), the boundary conditions can be written as follows

$$\bar{y} = \bar{z} = 0 \quad \text{for } \bar{x} = -\bar{x}_k \text{ and } \bar{x} = \bar{x}_k \tag{2}$$

$$EJ\bar{y}'' = -M_{\text{scr}}\bar{z}' \text{ and } EJ\bar{z}'' = M_{\text{scr}}\bar{y}' \quad \text{for } \bar{x} = -\bar{x}_k \text{ and } \bar{x} = \bar{x}_k, \tag{3}$$

where $|\bar{x}_k|$ denotes the co-ordinate of the upper/lower end of a column.

On the other hand, wall buckling analysis presents considerable difficulties connected with very complex stability equations of shells, especially if both middle surface and variable thickness of a wall are unknown. To avoid these difficulties, a simplified local formulation of the stability condition may be applied. Instabilities of shells very often has a local form and buckling does not depend essentially on the boundary conditions. It is particularly true in the case of non-uniform geometry of a shell and non-uniform state of stresses. In the other words, instability can be determined by the stress state and the shape of a shell and the buckling is initiated in the weakest zone of a structure. For a shell with a double curvature, Shirshov [24] transformed the problem of overall shell stability to a simpler problem of the local stability of such a structure. Using the linear theory of shell stability, applying the equations given by Wlasow [27] and assuming a sinusoidal deflection mode over a small restricted area, Shirshov obtained a rather simple closed formula for the critical loading parameter q, namely

$$q = 2\sqrt{\text{DEH}}\,\frac{k_\varphi \cos^2\phi + k_\theta \sin^2\phi}{\overline{N}_\theta \cos^2\phi + 2\overline{S}\cos\phi\sin\phi + \overline{N}_\varphi \sin^2\phi}, \tag{4}$$

where $k_\theta = 1/R_\theta$ and $k_\varphi = 1/R_\varphi$ denote circumferential and meridional curvatures, respectively, D stands for the shell stiffness, H is the wall thickness of a shell and ϕ is a certain free parameter assumed by Shirshov to describe a deflection mode with respect to which the loading parameter q should be minimized. In (4), the membrane resultant stresses depend on q, namely $N_\theta = q\overline{N}_\theta$, $N_\varphi = q\overline{N}_\varphi$, $S = q\overline{S}$, where S is a shearing resultant stress due to twisting. Axelrad [1] also obtained the formulae (4) describing the critical membrane resultant stresses using different governing stability equations. Minimization of q with respect to ϕ leads to two solutions

$$q_{\text{cr}1,2} = 2\sqrt{\text{DEH}}\,\frac{k_\varphi z_{1,2}^2 + k_\theta}{\overline{N}_\theta z_{1,2}^2 + 2\overline{S}z_{1,2} + \overline{N}_\varphi}, \tag{5}$$

where

$$z_{1,2} = -\frac{k_\varphi\overline{N}_\varphi - k_\theta\overline{N}_\theta}{k_\varphi\overline{S}} \pm \sqrt{\left(\frac{k_\varphi\overline{N}_\varphi - k_\theta\overline{N}_\theta}{k_\varphi\overline{S}}\right)^2 + 4\frac{k_\theta}{k_\varphi}} \tag{6}$$

The critical value of a loading parameter q_{cr} is determined by one of (5), whichever leads to a smaller value. For the case of a pure axial compression ($S = 0$) of an axially symmetric shell, minimization of q with respect ϕ leads to two solutions: $\phi_1 = 0$, $\phi_2 = \pi/2$ and finally (2) obtained, in general, for combined state of loadings, leads to the simple formula for the critical compressive meridional resultant stress

$$N_{\varphi cr} = \frac{E}{\sqrt{3(1-\nu^2)}} \frac{H^2}{R_\theta} \tag{7}$$

where ν is the Poisson ratio. For the case of a pure torsion ($N_\varphi = N_\theta = 0$) the critical twisting moment can be written as follows

$$M_{scr} = \frac{2\pi E}{\sqrt{3(1-\nu^2)}} \frac{H^2 R^2}{\sqrt{R_\varphi R_\theta}} \tag{8}$$

where R is a radius of the profile of the middle surface. A local stability condition, based on Eq. (5), referring, in general, to a dangerous point of a structure can be written as follows

$$q \le q_{cr} \tag{9}$$

For the limit case of a pure axial compression (9) leads to the following simple inequality

$$N_\varphi \le N_{\varphi cr} \tag{10}$$

whereas for a pure torsion to

$$M_s \le M_{scr} \tag{11}$$

where $N_{\varphi cr}$ and M_{scr} are given by (7) and (8), respectively.

On the other hand, using the hypothesis of locality of buckling, the concept of a shell of uniform stability may be applied. This concept introduced by Życzkowski and Krużelecki [30] can be stated as follows: if a condition of local stability is satisfied in the form of equality not only at the dangerous point but at any point of a shell, such structure is called “the shell of uniform stability”. It was applied to optimization of shells against buckling, for example, by Krużelecki and Trzeciak [19], Barski [2], Barski and Krużelecki [3, 4], Życzkowski, Krużelecki and Trzeciak [29]. This concept is utilized in the present paper.

If both types of the stability conditions, mention above, are expected to be satisfied simultaneously in the form of equality then a buckling mode interaction may occur. In the present paper we confine our investigation to the linear theory of stability. So, we do not going to analyzed the buckling mode interaction, the post-buckling behaviour of structures and also we do not investigate the influence of imperfections on the critical loadings. A detail analysis of such problems is difficult, but the simplest way of avoiding these inconveniences is by introducing into calculations, based on the linear stability equations, certain suitable raised safety factors, which may globally compensate changes connected with possible geometrically nonlinear behaviour of the structure as well as with geometrical imperfections. Such safety factors should be chosen as sufficiently high numerical coefficients. We introduce the safety factor j_g with respect to the global buckling of a column

$$P_{\mathrm{cr}} = j_g P \quad \text{and} \quad M_{\mathrm{scr}} = j_g M_s \tag{12}$$

as well as the safety factor j_l referring to the local buckling of a shell

$$q_{cr} = j_l q \tag{13}$$

Application of two independent safety factors can allow to neutralize different effects, which may occur due to two different stability phenomena considered here.

3. Formulation of optimization problem

Consider an elastic thin-walled column of a length $2L_0$, which is simply supported at both ends. The structure is loaded by an axial displacement U_0 and angle of torsion θ_0, but for comparison, loading controlled by a compressive force P and twisting moment M_s are also considered. It is assumed that both the loadings resulting from the displacements U_0 (P) and θ_0 (M_s) as well as the force P and twisting moment M_s for the classical optimization problem are conservative ones. It is also assumed that axial displacement U_0 and angle of torsion θ_0 as well as the axial force P and twisting moment M_s are proportional loadings

$$m_s = \frac{M_s}{PR_o} \tag{14}$$

where P (U_0) is an independent load and m_s is assumed dimensionless twisting moment parameter.

We look for such geometry of the thin-walled column, which leads to the maximal combined displacement

$$f_0 = w_1 U_0 / U_0^{\mathrm{cyl}} + w_2 \theta_0 / \theta_0^{\mathrm{cyl}} \Rightarrow \max \tag{15}$$

before the column buckles, and which contains an axial displacement and angle of torsion, where w_1 and w_2 denote weighting coefficients satisfying the following condition

$$w_1 + w_2 = 1 \tag{16}$$

In Eq. (15) U_0^{cyl} and θ_0^{cyl} denot e the axial displacement and angle of rotation for a reference cylindrical column, respectively. The reference column is discussed in the Appendix.

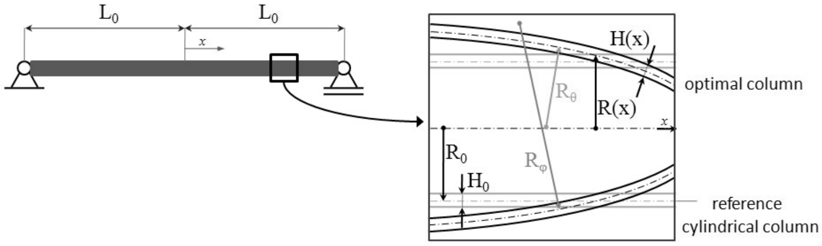

Fig. 2. Geometry of a column

Rys. 2. Geometria kolumny

Geometry of a thin-walled column, which can be also treated as a thin-walled shell, is uniquely defined when the shape R of a middle surface and a wall thickness H are given. Our consideration is restricted to axially symmetrical shapes of the middle surface and to the circular profiles of cross-sections of a structure. The shape of the middle surface is uniquely defined when the radius of meridional (R_φ) and circumferential (R_θ) curvatures are given

$$R_\varphi = \frac{(1+R'^2)^{3/2}}{-R''}, \qquad R_\theta = R(1+R'^2)^{1/2} \tag{17}$$

where R is the distance from the axis of a column to a point at the middle surface – the middle radius of the cross-sectional circular profile, Fig. 2.

Both thickness H and shape of the middle surface, described by the radius R, serve as the design variables. In Fig. 2 the index 0 refers to geometry of a reference cylindrical column, for which geometrical parameters are discussed in the Appendix. Because of symmetry of geometry of the structure only a half of a column $0 \le x \le L_0$ is considered and the condition $R'(0) = 0$ of symmetry of the structure is assumed.

So, we look for geometry of a thin-walled column which leads to the maximal combined displacement f_0, given by (15), before the column buckles, where the axial compression displacement U_0 takes the form

$$U_0(P_{\mathrm{cr}}, m_s) = 2\int_0^{L_0} \varepsilon_\varphi dx \tag{18}$$

and the angle of rotation can be written as follows

$$\theta_0(P_{\mathrm{cr}}, m_s) = 2m_s P_{\mathrm{cr}} R_0 \int_0^{L_0} \frac{dx}{GJ_0} \tag{19}$$

ε_φ is a meridional strain and P_{cr} denotes a critical force for the column, which can be evaluated from (1), whereas G stands for Kirchhoff's modulus and J_0 is a polar moment of inertia of the thin-walled annular cross-section, $J_0 = 2\pi H R^3$.

For an axially symmetric shell loaded by the axial compression force $P_{\mathrm{cr}} = j_g P$ the meridional resultant stress takes the form

$$N_\varphi = j_g P \frac{\sqrt{1+R'^2}}{2\pi R} \tag{20}$$

whereas the circumferential resultant stress, using Laplace's equation, can be written as follows

$$N_\theta = -\frac{R_\theta}{R_\varphi} N_\varphi = j_g P \frac{R''}{2\pi\sqrt{1+R'^2}} \tag{21}$$

In Eqs. (20) and (21) the formulae (17) were utilized and the reverse signs to the classical convention are used. The meridional strain ε_φ, in (18), can be evaluated from Hooke's law

$$\varepsilon_\varphi = \frac{1}{E}(\sigma_\varphi - \nu\sigma_\theta) \tag{22}$$

in which the stresses $\sigma = N / H$ are expressed by the resultants (20) and (21)

$$\varepsilon_\varphi = \frac{j_g P}{2\pi E H}\left(\frac{\sqrt{1+R'^2}}{\mathrm{R}} - \nu \frac{R''}{\sqrt{1+R'^2}}\right) \tag{23}$$

On the other hand the resultant stress caused by torsion can be written as follows

$$S = j_g \frac{M_s}{2\pi R^2} \tag{24}$$

Finally, taking into account Eqs. (16), (18), (19), (23), the functional (15) takes the form

$$f_0 = (1-w_2)\frac{j_g P}{U_0^{\mathrm{cyl}}\pi E}\int_0^{L_0}\left(\frac{\sqrt{1+R'^2}}{HR} - \nu\frac{R''}{H\sqrt{1+R'^2}}\right)dx + w_2\frac{m_s j_g P}{\pi\theta_0^{\mathrm{cyl}} G}\int_0^{\mathrm{L}_0}\frac{dx}{HR^3} \tag{25}$$

It depends on two functional design variables, namely the function determining the wall thickness $H = H(x)$ and relation $R = R(x)$ describing the position of the optimal meridian.

Such an optimization problem is stated under the following constraints. It is assumed that the optimal structure has the same volume of material as the cylindrical reference column with the constant thickness H_0 and the constant radius R_0

$$2\pi L_0 R_0 H_0 = 2\pi\int_0^{L_0} RH\sqrt{1+R'^2}\,dx \tag{26}$$

Additionally, the minimal value of the coordinate R is constrained by the lower bound

$$R(L_0) = R_{\min} \geq R_{\mathrm{adm}} \tag{27}$$

the slope of the meridian is limited by the upper bound

$$\left|R'\right| \le R'_{\text{adm}} \tag{28}$$

and our investigation is restricted to a double convex shell

$$R'' \le 0 \tag{29}$$

where $R_{\text{adm}} > 0$, $R'_{\text{adm}} > 0$ are certain assumed admissible values.

If the condition of local stability (5) is satisfied in the form of equality not only at the dangerous point but at any point of a column, such a structure is called “the shell of uniform stability” and that condition defines, in general, a variable thickness which is a wall thickness of uniform stability. This concept is applied in the present paper. Assuming that the critical loading parameter $q_{\text{cr}} = P_{\text{cr}} = j_g P$, where the safety factor j_g with respect to global buckling is introduced, the local stability condition (5) in the form of equality with the safety factor j_l referring to the local buckling taken into account can be utilized to derive a variable wall thickness of uniform stability. Substituting the membrane resultant stresses given by (20), (21) and (24) into (5) one can obtain the wall thickness of uniform stability for general case of loading considered here, namely

$$H = \sqrt{\frac{j_g P j_1 \sqrt{3(1-\nu^2)}}{2\pi E} \frac{R_\theta^{\ 1/2}}{R} \sqrt[4]{R_\theta^{\ 2} + kR_0^2 m_s^2}} \tag{30}$$

where

$$k = \frac{R_\varphi}{R_\theta} = \frac{1+R'^2}{-RR''} \tag{31}$$

and where R_φ and R_θ are given by (17), j_l is the local safety factor according to (13) whereas m_s is defined by (14). In the case of pure compression ($m_s = 0$) the formula (30) can be simplified to

$$H = \sqrt{\frac{j_g P j_1 \sqrt{3(1-\nu^2)}}{2\pi E} \frac{R_\theta}{R}} \tag{32}$$

To obtain the wall thickness of uniform stability for pure torsion one can take M_{scr} as the critical loading parameter, namely $q_{\text{cr}} = M_{\text{scr}} = j_g M_s$. Then, utilizing (14), the relation (30) can be transformed to the following one

$$H = \sqrt{\frac{j_g M_s j_1 \sqrt{3(1-\nu^2)}}{2\pi \text{E}} \frac{(R_\varphi R_\theta)^{1/4}}{R}} \tag{33}$$

describing the wall thickness of uniform stability for pure torsion. The wall thickness of uniform stability given by: (30) for compression and twisting, (32) for pure compression and (33) for pure twisting, in general, is varying along the axis of a column because it depends on shape of the middle surface via R_φ, R_θ and R. On the other hand, if the concept of a shell of uniform stability is not utilized the buckling constraint (9) should be satisfied locally in the form of equality at the dangerous point only. It leads to evaluation of the minimal necessary thickness $\overline{H} = H_{\max} = H(L_0)$ which is the maximal wall thickness H for a column, evaluated from (30). The optimal structure obtained using such an approach is a constant thickness column with a variable radius of the profile. This variant of optimization is not considered in the present paper.

In Eqs. (25), (30) and (32) $j_g P = P_{cr}$ is defined by (1) and P, via the moment of inertia J, depends on geometry of a structure. For a thin-walled annular cross-section the moment of inertia $J = \pi R^3 \hat{H}$, where the thickness $\hat{H}$ is measured perpendicular to the axis of a column, Fig. 3.

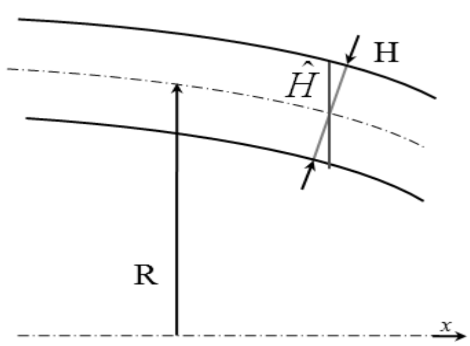

Fig. 3. Thickness $\hat{H}$ measured in a perpendicular direction to the axis x

Rys. 3. Grubość $\hat{H}$ mierzona prostopadle do osi x

The thicknessfor $\hat{H}$ general case of loading can be written as follows

$$\hat{H} = H\sqrt{1+R'^2} = H = \sqrt{\frac{j_g P j_1 \sqrt{3(1-\nu^2)}}{2\pi E}}\,\frac{R_\theta^{\ 1/2}}{R}\sqrt[4]{R_\theta^{\ 2} + kR_0^2 m_s^2}\,\sqrt{1+R'^2} \tag{34}$$

For further calculations, it is convenient to introduce dimensionless quantities
$r = R / R_0$, $r_\theta = R_\theta / R_0$, $r_\varphi = R_\varphi / R_0$, $h = H / R_0$, $y = \overline{y} / L_0$, $z = \overline{z} / L_0$, $\mu = R_0 / L_0$,

$$u_0 = U_0 / L_0\,,\quad p = \frac{2Pj_g}{\pi E R_0^2 \mu^4 j_1 \sqrt{3(1-\nu^2)}}\,.$$

Then the differential equations (1) can be rewrite in the form of eight dimensionless differential equations of the first order

$$\begin{aligned} &\frac{dy}{d\xi} = \varphi_y\,, \quad \frac{d\varphi_y}{d\xi} = -\frac{m_y + \mu m_s p^{1/2}\varphi_z}{r^2 r_\theta^{1/2}\sqrt{1+\mu^2 r'^2}\sqrt[4]{r_\theta^2 + km_s^2}}\,, \quad \frac{dm_y}{d\xi} = p^{1/2}\varphi_y + t_y\,, \quad \frac{dt_y}{d\xi} = 0 \\ &\frac{dz}{d\xi} = \varphi_z\,, \quad \frac{d\varphi_z}{d\xi} = -\frac{m_z - \mu m_s p^{1/2}\varphi_y}{r^2 r_\theta^{1/2}\sqrt{1+\mu^2 r'^2}\sqrt[4]{r_\theta^2 + km_s^2}}\,, \quad \frac{dm_z}{d\xi} = p^{1/2}\varphi_z + t_z\,, \quad \frac{dt_z}{d\xi} = 0 \end{aligned} \tag{35}$$

where

$$r_\varphi = \frac{(1+\mu^2 r'^2)^{3/2}}{-\mu^2 r''}, \qquad r_\theta = r\sqrt{1+\mu^2 r'^2}, \qquad k = \frac{r_\varphi}{r_\theta} = -\frac{1+\mu^2 r'^2}{\mu^2 r r''} \tag{36}$$

Equations (35) with the boundary conditions

$$y(-1) = y(1) = 0, \qquad z(-1) = z(1) = 0 \tag{37}$$

$$m_y(-1) = m_y(1) = 0, \qquad m_z(-1) = m_z(1) = 0 \tag{38}$$

for a simply supported column lead to evaluation of the dimensionless critical force p for a thin-walled column of uniform stability.

If the classical problem of maximization of the critical force is considered then the functional can be written as follows

$$p \Rightarrow \max \tag{39}$$

and the results for the classical optimization problem are also presented in the paper.

Geometry of such a column is defined by a shape of the middle surface described by a dimensionless radius $r(\xi)$, which is subjected to optimization, and by the variable thickness, described by (30) and rewritten here in the dimensionless form

$$h = H / R_0 = \frac{j_l\sqrt{3(1-\nu^2)}\mu^2}{2} p^{1/2} \frac{r_\theta^{1/2}}{r} \sqrt[4]{r_\theta^2 + k m_s^2} \tag{40}$$

Taking into account the displacements U_0^{cyl} and θ_0^{cyl} for a reference column given by (A.15) and (A.17) respectively, the functional (25) takes the dimensionless form

$$f_0 = (1-w_2)\frac{p^{1/2} m_s^2}{2\left(\sqrt{1+m_s^2\mu^2\dfrac{\pi^2}{4}}-1\right)}\left[\int_0^1 \frac{r}{r_\theta^{1/2}\sqrt[4]{r_\theta^2 + k m_s^2}}\left(\frac{\sqrt{1+\mu^2 r'^2}}{r} - \frac{\nu\mu^2 r''}{\sqrt{1+\mu^2 r'^2}}\right)d\xi + \right.$$

$$\left. + w_2\,\mu^2 \int_0^1 \frac{d\xi}{r^2 r_\theta^{1/2}\sqrt[4]{r_\theta^2 + k m_s^2}}\right] \tag{41}$$

whereas the equality constraint (26) of a constant volume of material can be rewritten as follows:

$$\frac{H_0}{R_0} = h_0 = \frac{j_l\sqrt{3(1-\nu^2)}\mu^2}{2} p^{1/2} \int_0^1 \frac{r_\theta^{3/2}}{r} \sqrt[4]{r_\theta^2 + k m_s^2}\, d\xi \tag{42}$$

The ratio $H_0 / R_0 = h_0$, in (42), denotes the dimensionless wall thickness h_0 of a reference column and it should be evaluated from (A.11) for assumed state of loadings (m_s). The inequality constraints (27)–(29) in the dimensionless form can be rewritten as follows

$$r_{\min} \geq r_{\text{adm}}, \quad r'' \leq 0, \quad |r'| \leq r'_{\text{adm}} \tag{43}$$

where $(') = d / d\xi$, $r_{\text{adm}} = R_{\text{adm}} / R_0$, $r'_{\text{adm}} = R'_{\text{adm}} / \mu$. The last constraint from (43) is not used in the examples presented in this paper.

In the general case, two functions describing geometry of a column are looked for, but in view of the unique relation between the wall thickness h defined by (40) and the shape of a middle surface of a shell-column of uniform stability defined by $r(\xi)$, only one shape function – shape of a middle surface remains free and can be subjected to optimization.

4. Generating of a middle surface of a column

The geometry of a column is uniquely defined when the shape of a middle surface and a wall thickness of the thin-walled column are prescribed. A function which describes the middle surface of a shell, besides the geometrical constraints (43), should satisfy several following conditions. It should be smooth, convex and continuous function at least up to the second derivative.

One of the simplest function, which fulfils the above requirements is the parabolic shape written here in the following dimensionless form

$$r(\xi) = r_0 (1 - \gamma \xi^2) \tag{44}$$

where r_0 is a radius of a shell for $\xi = 0$ and γ ($0 \leq \gamma \leq 1$) is a free parameter – both quantities are subjected to optimization. This function was applied by Krużelecki and Stawiarski [17] for parametric optimization of a column under axial compression only, where r_0 and γ are regarded as design variables. Parametric optimization usually leads to a local optimum. To obtain the global optimum more general geometry of a column, satisfying requirements assumed above, should be considered. The appropriate convex Bézier functions, discussed by Kiciak [11], were applied to optimization of thin-walled shells under stability constraints by Barski [2], Barski and Krużelecki [3, 4]. They approximated the radius $r = r(\xi)$, representing the middle surface of a shell, by a simple relation

$$r(\xi) = \alpha p(\xi) \tag{45}$$

where α is a dimensionless parameter introduced to satisfy the constraint (26) and $p(\xi)$ is the Bézier polynomial of the fifth order written as follows

$$p(\xi) = \begin{bmatrix} \xi^5 & \xi^4 & \xi^3 & \xi^2 & \xi & 1 \end{bmatrix} \begin{bmatrix} -1 & 5 & -10 & 10 & -5 & 1 \\ 5 & -20 & 30 & -20 & 5 & 0 \\ -10 & 30 & -30 & 10 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{bmatrix} \tag{46}$$

where r_0, r_1....,r_5 are r co-ordinates of the control points. It is assumed that $r_1 = r_0$ to satisfy the symmetry condition $r'(0) = 0$, where $(') = d/d\zeta$. The values of r_i are generated in optimization procedure.

Krużelecki and Stawiarski [18] showed that high accuracy of solutions can be easier obtained if we approximate the second derivative r'' of radius r instead of a direct approximation of a radius r, since the wall thickness (40) depends on the derivatives of r. It follows a general rule that integration of a function leads to its smoothing whereas differentiation can spoil it. Based on their experience, to approximate shape – the second derivative r'' – we applied the convex Bézier polynomial of the fifth order (46)

$$r'' = -p(\xi) \tag{47}$$

Simple double integration of the Bézier polynomial (47)

$$r(\xi) = \int \{\int [-p(\xi)] d\xi\} d\xi + C\xi + D \tag{48}$$

and application of appropriate conditions

$$r'(0) = 0,\ r(0) = r_{00} \tag{49}$$

leads to a very convenient description of the shape of a middle surface

$$r(\xi) = \overline{p}(\xi) + r_{00} \tag{50}$$

where

$$\overline{p}(\xi) = \int \{\int [-p(\xi)] d\xi\} d\xi \tag{51}$$

r_{00} denotes here the radius for beginning of the coordinate $\xi = 0$ and it can be evaluated from the condition of constant volume (42) whereas coordinates of the key points $r_0 - r_5$ are obtained from the condition of optimality using the SA algorithm.

5. Numerical optimization procedure

For optimization of thin-walled columns under axial compression only Krużelecki and Stawiarski [18] applied the SA algorithm. This method developed by Kirkpatrick *et al.* [12], which becomes a popular one for structural optimization (van Laarhoven [21], Krużelecki and Smaś [15], Barski and Krużelecki [3, 4], Smaś [25]), occurred to be very effective and accurate method for such problems. A description of this algorithm as well as a simple and very effective procedure written in C++ can be found in the book by Masters [22]. This algorithm, with some deal of development, is applied here for optimization of thin-walled columns under stability constraints.

The optimization procedure starts from generating of a middle surface of a shell, which shape is approximated by the convex Bézier polynomials and it is based on , so called, the control points (key points). We divided the dimensionless length (equal to 1) of a half of a column into several segments of equal length which define ξ co-ordinates of the control points. In our case we have six control points r_0, r_1...., r_5. At a given cooling temperature (SA temperature) the SA algorithm draws a set of r_i co-ordinates of the control points and sorts them in the appropriate order, namely from the largest to the smallest value (the maximal value is connected with $\xi = 0$ and the minimal value with $\xi = 1$). In this way the Bézier polynomial is uniquely described. Introducing it into (50) the radius r of a shell is defined and it depends on the known values of r_i (design parameters) and unknown value of the radius r_{00} for the beginning of the coordinate $\xi = 0$. Now substituting (50) into the condition of a constant volume of material (42) one obtains the relation between the radius r_{00} and the critical forces for the reference and optimized columns (the design parameters r_i are known at each stage of drawings). Since p_{cyl} is known (via A10 and A11) the constant volume constraint (42) allows to eliminate the critical force p from the stability equations (35). Hence, the stability equations (35) depend now on the radius r_{00}. Next, for each r, generated by the SA algorithm, r_{00} is evaluated by solution of the boundary value problem (instead of p). If all the geometrical constraints (43) are satisfied a value of the objective function is calculated otherwise a value of the objective function is specially decreased (a method of a penalty function is used). For one cooling temperature a number of random variations of r_i co-ordinates of the control points are considered. If a set of r_i results in the maximal value of the objective function it is accepted as a starting set for a next cooling cycle (temperature). It occurred that 120 random variations of r for each cooling temperature give satisfying results. Such a procedure is repeated until the maximal value of the axial displacement is obtained (for the classical problem the maximum of the critical force p is obtained) or the final cooling temperature is reached.

An initial range of r_i, from which the SA algorithm draws r co-ordinates for the control points, should contain the optimal solution and this range is associated with the so-called initial cooling temperature T_{start}. This temperature and the final temperature T_{stop} should be initially established. The value of cooling temperature is different at each step of iteration. The way of decreasing of a cooling temperature can have a great influence on efficiency of the algorithm. It occurred that formula proposed by Masters [22]

$$T_{K+1} = e^{\left(\frac{\ln T_{stop}/T_{start}}{N-1}\right)} T_K \tag{52}$$

for evaluation of the following cooling temperatures leads to high efficiency of the algorithm, wehere K denotes number of temperature, N is a number of cooling temperatures, whereas T_{start} and T_{stop} stand for the initial and final temperature, respectively. It turned out in our calculations that assuming $T_{start} = 0.2$, $T_{stop} = 0.0001$ (should be very low to obtain a high enough accuracy of solution) and taking $N = 20$ cooling temperatures lead to very accurate optimal solutions.

6. Results of numerical optimization

Optimization of short columns under stability constraints should be based only on the concept of the shell of uniform stability because buckling of a wall is crucial for short shells whereas a global buckling seems to be not dangerous for such structures. For long columns both types of the stability conditions should be satisfied and the optimal structure is considered as a column of equal stability. So, calculations were performed only for rather long columns with the length parameter $\mu = 0.05$ and 0.025, the Poisson ratio $\nu = 0.3$ and assuming $r_{adm} = 0.25$ or

0.5. The objective function for maximization of the combined displacement (41), in general, depends also on weighting coefficients w_1 and w_2, which control contributions of an axial displacement and angle of torsion in f_0. It was decided that calculations were performed only for one value of the weighting coefficients, namely $w_1 = 1$ ($w_2 = 0$). It means that torsion has no direct influence on the objective function because the angle of torsion is not included in (41) for $w_2 = 0$. On the other hand, the twisting moment, via the stability conditions, strongly effects the optimal shapes for both considered types of the objective function.

The results of optimization are presented in Table 1, namely the values of the objective function f_0 for four different state of loadings (different values of m_s, where a larger value of m_s denotes a larger portion of torsion in a global loading) and two considered μ, where $m_s = 0$ means a pure compression case. The full results for this case of loading can be found in papers by Krużelecki and Stawiarski [17, 18]. Table 1 contains the results obtained for the problem of maximization of the axial displacement f_0 at buckling and, for comparison, the results for the classical maximization of the critical force p.

Table 1

Results of optimization

μ	Objective function	r_{adm}	$m_s = 0.0$		$m_s = 0.1$		$m_s = 0.5$		$m_s = 0.85$	
			f_0	p / p_{cyl}	f_0	p / p_{cyl}	f_0	p / p_{cyl}	f_0	p / p_{cyl}
0.05	f_0	0.50	1.2361	1.1479	1.2262	1.1624	1.1329	0.8357	1.0719	0.6613
		0.25	1.3538	1.0645	1.4083	1.2140	1.3649	1.0032	1.3367	0.9091
	p	0.50	1.2305	1.1479	1.2262	1.1624	1.0000	1.0000	1.0000	1.0000
		0.25	1.2811	1.1578	1.4083	1.2140	1.3649	1.0032	1.0000	1.0000
			$m_s = 0.0$		$m_s = 0.1$		$m_s = 0.5$		$m_s = 1.0$	
0.025	f_0	0.50	1.2220	1.0778	1.1964	1.0614	1.0000	1.0000	1.0000	1.0000
		0.25	1.3522	1.0621	1.4009	1.1745	1.3265	0.8811	1.2899	0.7648
	p	0.50	1.2220	1.0778	1.1964	1.0614	1.0000	1.0000	1.0000	1.0000
		0.25	1.2689	1.0898	1.4009	1.1745	1.0000	1.0000	1.0000	1.0000

The shapes of the optimal columns of uniform stability obtained for loading controlled by displacement and the appropriate variable wall thickness h / h_0 related to the wall thickness for a reference cylindrical column for $r_{adm} = 0.25$, μ = 0.05 and all considered m_s are shown in Fig. 4. The optimal profiles of the middle surface for $m_s = 0$, referring to a pure compression, is located below the optimal profile for $m_s = 0.1$ and above the all others. On the other hand the minimal thickness for $\xi = 0$ refers to $m_s = 0.1$ and the wall-thickness increases along the axis of a column for each considered m_s. The maximal value h/h_0 is obtained always at the end of a column for $\xi = 1.0$ and ranges from almost 2.5 for $m_s = 0.1$ to over 3.0 for $m_s = 0.85$ (excluding a pure compression $m_s = 0$). This larger values of the wall-thickness under torsion in the region closed to $\xi = 1.0$ is due to relatively small r in this zone, which must sustain a constant twisting moment.

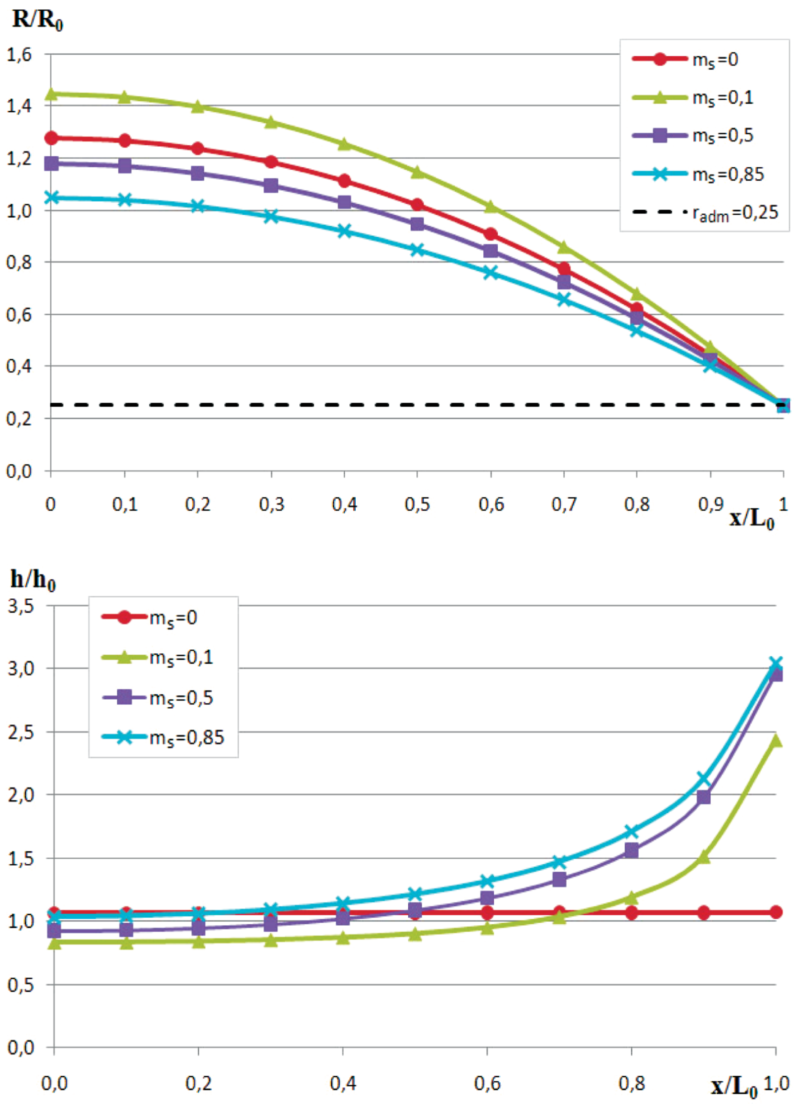

Fig. 4. Columns of uniform stability optimal with respect to f_0 for $\mu = 0.05$ and $r_{adm} = 0.25$

Rys. 4. Kolumny równomiernej stateczności optymalne ze względu na f_0 dla $\mu = 0{,}05$ i $r_{adm} = 0{,}25$

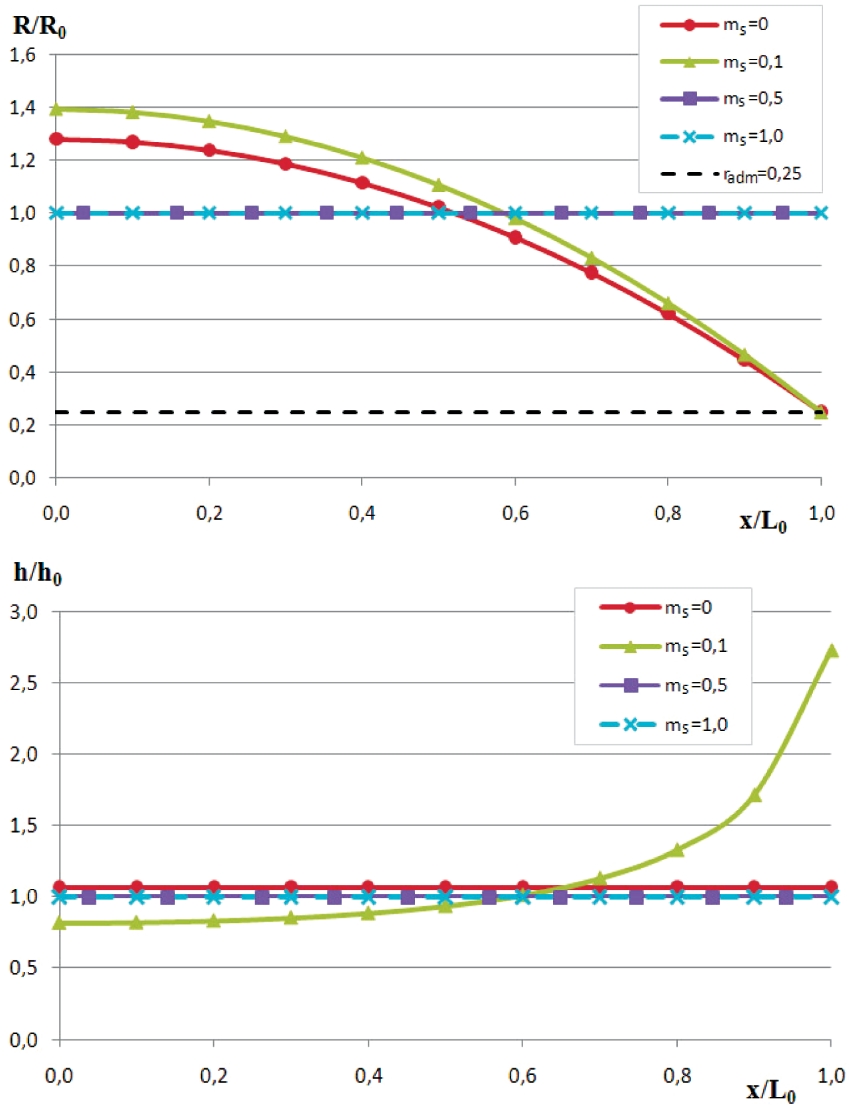

Fig. 5. Columns of uniform stability optimal with respect to f_0 for μ = 0.025 and r_{adm} = 0.25

Rys. 5. Kolumny równomiernej stateczności optymalne ze względu na f_0 dla μ = 0.025 i r_{adm} = 0,25

Figure 5 presents the shapes of the optimal columns of uniform stability obtained for loading controlled by displacement and the appropriate variable wall thickness h / h_0 related to the wall thickness for a reference cylindrical column for r_{adm} = 0.25, μ = 0.025 and considered m_s.

It occurred that for combined compression and torsion, for quite large range of m_s, the numerical optimization process leads practically to

$$r'' = -p(\xi) = \text{const} \tag{53}$$

which shows that optimal $r_i = \text{const}$. It means that optimal middle surface of a column under combined state of loadings can be practically described by the parabolic function of the type (44). That conclusion is valid even for a small torsion whereas for a pure compression the optimal r'' was found (Krużelecki and Stawiarski [18]) to be varying along axis of the optimal column. The minimal value of m_s from which $r'' = \text{const.}$ depends on the length parameter μ. It is not determined precisely here.

It also occurred that for considered values of μ (μ = 0.05 and 0.025) the columns optimized with respect to two different objective functions ($\max(f_0)$ or $\max(p)$) can have the same shape for a certain range of loadings. Such a situation is presented in Fig. 6, where maximal radius r_{max}, representing shape of the optimal column, is plotted versus m_s for μ = 0.05. For $m_s = 0$ (pure compression) the optimal shapes of columns with respect to the both criteria are different since the values of r_{max} are different. For loadings m_s from $0.1 \le m_s \le 0.51$ the both curves merge and the same optimal shape (the same r_{max}) is obtained for the both criteria. From $m_s = 0.51$ the curve, in Fig. 6, splits and the different criteria lead to the different optimal shapes of columns. It can be seen in Table 1, where for the due range of m_s the values of the objective function f_0 for the both optimization problems are equal to each other for the same r_{adm} and the values of the objective function p/p_{cyl} are equal to each other for the same r_{adm}. It means that there are no differences in the optimal shapes of columns obtained for loading controlled by displacements and for the classical problem. On the other hand, for the appropriate larger values of m_s the differences in the optimal shapes obtained for the different objective functions can be very significant. For example, for $m_s = 0.5$, $r_{adm} = 0.25$ and μ = 0.025 optimization with respect to f_0 leads to "barrelled" columns, whereas the classical maximization of the critical force leads to the cylindrical columns with $r(\xi) = 1, \quad h = \text{const.}$, whereas for $m_s = 1$, $r_{adm} = 0.5$ and μ = 0.025 both optimal columns are the cylindrical ones.

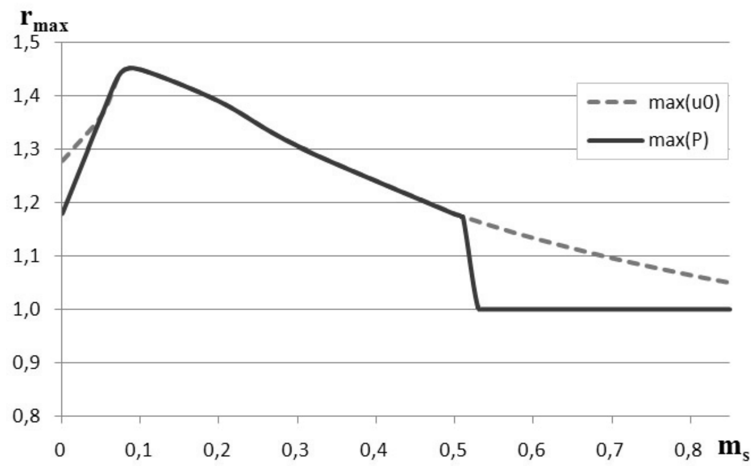

Fig. 6. r_{max} versus m_s for μ = 0.05 and $r_{adm} = 0.25$

Rys. 6. r_{max} w funkcji m_s dla μ = 0,05 i $r_{adm} = 0,25$

In Fig. 7 the shapes and thicknesses of the optimal columns of uniform stability for loading controlled by displacement are compared with the appropriate optimal columns of uniform stability for the classical problem for μ = 0.05 and r_{adm} = 0.25. Differences in shapes and thicknesses between the both types of the optimal structures are noticeable for small and large m_s. For middle values of m_s, which in Fig. 7 represents m_s = 0.2, the optimal shapes and thicknesses show no differences. For large values of m_s (m_s = 0.55 in Fig. 7) the classical problem of maximization of p leads to the cylindrical column of a constant thickness whereas optimization with respect to f_0 leads to "barrelled" columns. To obtain a cylindrical column as an optimal structure with respect to f_0 larger values of m_s should be applied. Such conclusions are valid for any length (any μ) of a column.

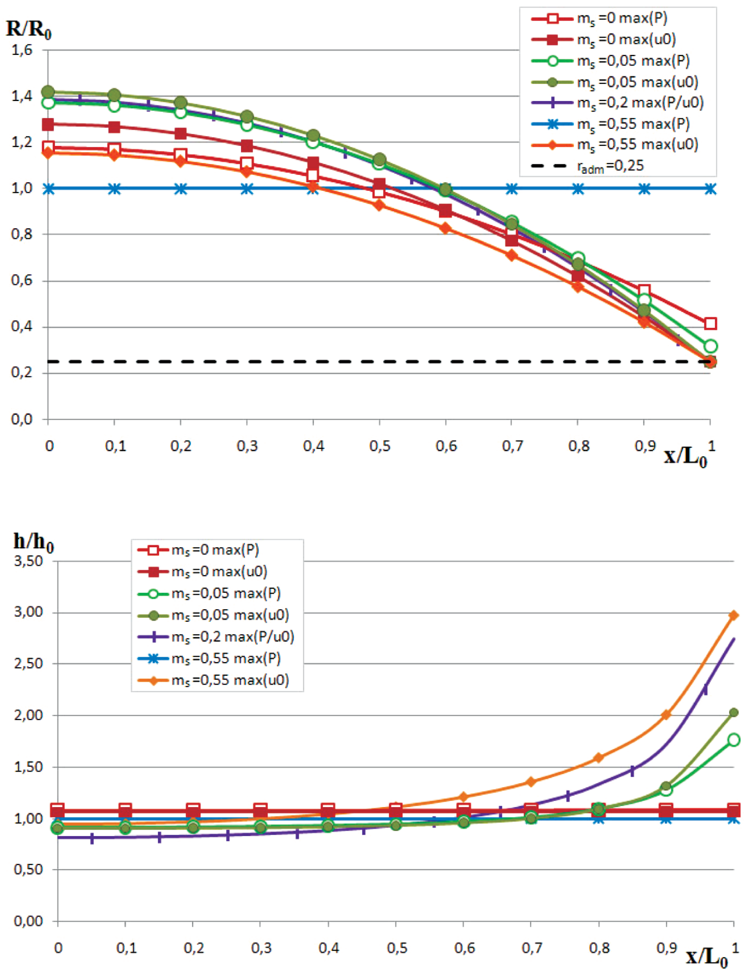

Fig. 7. Comparison of optimal uniform stability columns for both optimization problems for μ = 0.05 and r_{adm} = 0.25

Rys. 7. Porównanie optymalnych kolumn równomiernej stateczności dla obu przypadków optymalizacji dla μ = 0,05 i r_{adm} = 0,25

7. Concluding remarks

The results of optimization of columns for a combined state of loading show that there can be considerable differences in the optimal shapes of columns obtained for loading controlled by displacements and for the classical problem. The differences depend on a length of column but first of all on applied value of m_s. In the considered range of loadings and the length parameter μ, the optimal solutions are found always at the boundary of admissible region defined by the geometrical constraints for both considered objective functions. It should be underlined here that the optimal column can take form of a reference cylindrical column, specially for larger values of m_s and smaller values of μ and more restrictive geometrical constraints. The profit of optimization, measured by the increase of critical displacements, related to the reference cylindrical column is quite large. Most of the profit is connected with the optimal shape of the middle surface. The influence of a variable wall thickness is much lower.

APPENDIX

CYLINDRICAL REFERENCE COLUMN

For a proper comparison of the optimal and reference columns both structures have to satisfy the same constraints. For a reference cylindrical column with given R_0 and L_0 the geometrical constraints (28)–(30) are automatically fulfilled whereas the thickness H_0 for this structure should be taken in such a way that both global (1) and local (5) buckling constraints are satisfied simultaneously. It means that for different states of loadings the wall thickness H_0 can have different values.

According to Bažant and Cedolin [5] and Ziegler [28] the critical loading of a simply supported column simultaneously subjected to a twisting moment M_s and a compressive force P can be written as follows

$$\frac{j_g P_{\mathrm{cr}}}{P_{\mathrm{cr}}^0}+\left(\frac{j_g M_{\mathrm{scr}}}{M_{\mathrm{cr}}^0}\right)^2=1 \tag{A1}$$

where

$$P_{\mathrm{cr}}^0=\frac{\pi EI}{4L_0^2} \tag{A2}$$

is the critical axial force for a pure compression and

$$M_{cr}^0=\frac{\pi EI}{L_0} \tag{A3}$$

is the critical twisting moment for a pure torsion. Taking the axial force as a main loading and substituting (14) into (A1) the critical force for a reference column is equal to

$$p_{\mathrm{cyl}}=\frac{4}{j_1\sqrt{3(1-\nu^2)}\mu^4 m_s^2}\left(\sqrt{1+m_s^2\mu^2\frac{\pi^2}{4}}-1\right)\frac{H_0}{R_0} \tag{A4}$$

where dimensionless force p is defined as follows

$$p = \frac{2Pj_g}{\pi E R_0^2 \mu^4 j_1 \sqrt{3(1-\nu^2)}} \tag{A5}$$

and

$$\mu = R_0 / L_0 \tag{A6}$$

is a length parameter.

On the other hand, when any torsion ($S \neq 0$) is taken into account then the local condition of stability (5), which defines the critical loading parameter, is not valid for a cylindrical shell. Therefore, to describe critical force for cylindrical shell, we have to apply the equivalent local stability condition based on the Papkovich theorem [23], namely

$$\frac{j_1\sigma}{\sigma_{cr}^0} + \left(\frac{j_1\tau}{\tau_{cr}^0}\right)^2 = 1 \tag{A7}$$

where

$$\sigma_{cr} = \frac{E}{\sqrt{3(1-\nu^2)}} \frac{H_0}{R_0}, \qquad \tau_{cr} = \frac{E}{3\sqrt{2}(1-\nu^2)^{3/4}} \left(\frac{H_0}{R_0}\right)^{3/2} \tag{A8}$$

and

$$\sigma = \frac{j_g P}{2\pi R_0 H_0}, \qquad \tau = \frac{j_g M_s}{2\pi R_0^2 H_0} \tag{A9}$$

Substituting (A8) and (A9) into (A7) we can obtain critical force for cylindrical shell

$$p_{\mathrm{cyl}} = \frac{1}{9 j_1^2 \mu^4 m_s^2 (1-\nu^2)^{3/2}} \left(\frac{H_0}{R_0}\right)^3 \left(\sqrt{1 + 24 m_s^2 (1-\nu)^{1/2} \left(\frac{R_0}{H_0}\right)} - 1\right) \tag{A10}$$

Comparing (A4) and (A10) we obtain algebraic equation

$$\frac{\sqrt{3}}{9 j_1 (1-\nu^2)} \left(\frac{H_0}{R_0}\right)^2 \left(\sqrt{1 + 24 m_s^2 (1-\nu^2)^{1/2} \frac{R_0}{H_0}} - 1\right) = 4\left(\sqrt{1 + m_s^2 \mu^2 \frac{\pi^2}{4}} - 1\right) \tag{A11}$$

which, via the ratio $h_0 = H_0 / R_0$ (dimensionless thickness), describes the constant wall thickness H_0 of a reference column. The formula (A11) holds for the limit cases. For a pure axial compression, substituting $m = 0$ into (A11), we have

$$\frac{H_0}{R_0} = h_0 = \frac{j_1 \sqrt{3(1-\nu^2)} \mu^2}{2} \frac{\pi^2}{4} = \frac{j_1 \sqrt{3(1-\nu^2)} \mu^2}{2} p_{\mathrm{cyl}}^{1/2} \tag{A12}$$

where $p_{\mathrm{cyl}}^{1/2} = \frac{\pi^2}{4}$. For a pure twisting, utilizing (25) and substituting $P = 0$, we obtain

$$\frac{H_0}{R_0} = h_0 = \left[\frac{3\sqrt{2}}{2}\pi\mu j_1(1-\nu^2)^{3/4}\right]^{2/3} \tag{A13}$$

and

$$m_{scyl}^{1/2} = 2^{7/6}3^{1/2}\pi^{5/6} j_1^{1/6}\mu^{1/3} \tag{A14}$$

The relations (A11)–(A13), describing a wall thickness of a reference cylindrical column, hold only for thin-walled columns because the local stability condition (A7) used to obtain those relations is valid only for thin shells. It can be seen from (A11)–(A13) that dimensionless thickness H_0/R_0, besides the state of loading, clearly depends on the length parameter $\mu = R_0/L_0$. Hence, for assumed state of loading (assumed m_s) the minimal length of a column (maximal value of the ratio $\mu = R_0/L_0$) should be chosen in such a way that it satisfies the condition of slenderness. For example, assuming that the limit thickness equals to $H_0/R_0 = 0.05$ then maximal $\mu \cong 0.157$ for a pure compression and $\mu \cong 3.6\times10^{-3}$ for a pure twisting. It means that reference column and optimal column satisfying simultaneously the global and local buckling conditions should be rather long structures (structures with small value of $\mu = R_0/L_0$). Now, for a reference cylindrical column we can calculate the axial compressive displacement U_0^{cyl} and the angle of torsion θ_0^{cyl} for any state of loading using (19) and (20), respectively. Utilizing (19), (A4) and (A5) we can obtain the axial displacement of a reference column

$$\frac{U_0^{\text{cyl}}}{L_0} = u_0^{\text{cyl}} = \frac{2}{m_s^2}\left(\sqrt{1+m_s^2\mu^2\frac{\pi^2}{4}}-1\right) \tag{A15}$$

which for a pure compression ($m_s = 0$) leads to

$$\frac{U_0^{\text{cyl}}}{L_0} = u_0^{\text{cyl}} = \mu^2\frac{\pi^2}{4} = \mu^2 p_{\text{cyl}}^{1/2} \tag{A16}$$

whereas for a pure torsion ($m_s \to \infty$) the axial displacement $U_0^{\text{cyl}} = 0$. On the other hand, utilizing (20), (14), (A4) and (A5), the angle of torsion for a reference column can be written as follows

$$\theta_0^{cyl} = \frac{4(1+\nu)}{\mu\, m_s}\left(\sqrt{1+m_s^2\mu^2\frac{\pi^2}{4}}-1\right) \tag{A17}$$

For a pure compression ($m_s = 0$) the formulae (A17) leads to $\theta_0^{\text{cyl}} = 0$ whereas for a pure torsion ($m_s \to \infty$) one can obtain

$$\theta_0^{\text{cyl}} = 2\pi(1+\nu) \tag{A18}$$

The formulae (A14) and (A17) are applied to define the objective function (26).

References

[1] A x e l r a d E.I., *On local buckling of thin shells*, International Journal of Nonlinear Mechanics, 24, 1985, 249-259.

[2] B a r s k i M., *Optimal design of shells against buckling subjected to combined loadings*,Structural and Multidisciplinary Optimization, 31, 2006, 211-222.

[3] B a r s k i M., K r u ż e l e c k i J., *Optimal design of shells against buckling by means of the simulated annealing method*, Structural and Multidisciplinary Optimization, 29, 2005, 61-62.

[4] B a r s k i M., K r u ż e l e c k i J., *Optimal design of shells against buckling under overall bending and external pressure*, Thin-Walled Structures, 43, 2005, 1677-1698.

[5] B a ž a n t Z.P., C e d o l i n L., *Stability of structures*, Dover Publications, 2003.

[6] B o c h e n e k B., K r u ż e l e c k i J., *Optimization of structural stability. Modern problems* (in Polish), Wydawnictwo Politechniki Krakowskiej, Kraków 2007.

[7] Feigen M., *Minimum weight of a tapered round thin-walled column*, J.Appl. Mech., 19 (3), 1952, 375-380.

[8] G a j e w s k i A., Ż y c z k o w s k i M., *Optimal structural design under stability constraints*, Kluwer Academic Publishers, Dordrecht, Boston, London 1988.

[9] H u a n g N.C., S h e u C.Y., *Optimal design of an elastic column of thin-walled cross--section*, J. Appl. Mech., E35 (2), 1968, 285-288.

[10] J a n i c z e k R., *Axially compressed thin-walled tube of the minimal weight* (in Polish), Zeszyty Naukowe Politechniki Częstochowskiej, Nauki Podstawowe, 6, 1964, 51-72.

[11] K i c i a k P., *Basis of modelling of curves and surfaces* (in Polish), Wyd. Naukowo--Techniczne, Warszawa 2000.

[12] K i r k p a t r i c k S., G e l l a t C.D. Jr., V e c c h i M.P, *Optimization by simulated annealing*, Science 220, 1983, 671-680.

[13] K r i s t e L., *Beitrag zum Problem des "Tragwerks-Mindestgewichts"*, Z. Flugwiss, 8 (12), 1960, 352-359.

[14] K r u ż e l e c k i J., S m a ś P., *Optimal design of columns for buckling under loadings controlled by displacements and forces*, 4th World Congress of Structural and Multidisciplinary Optimization, Gengdong Cheng, Yuanxian Gu, Shutian Liu, Yuefang Wang, Liaoning (Eds.), Electronic Press, Dalian, China 2001 (CD-ROM only).

[15] K r u ż e l e c k i J., S m a ś P., *Optimum design of simply supported columns for buckling under loading controlled by displacements*, Engineering Optimization, 36 (6), 2004, 645-658.

[16] K r u ż e l e c k i J., S m a ś P., *Optimal annular plates with respect to their stability under thermal loadings*, Structural and Multidisciplinary Optimization, 32, 2006, 229-241.

[17] K r u ż e l e c k i J., S t a w i a r s k i A., *Optimal design of thin-walled columns for buckling under loadings controlled by displacements*, 8th World Congress of Structural and Multidisciplinary Optimization, Lisbon, Portugal 2009 (CD-ROM only).

[18] K r u ż e l e c k i J., S t a w i a r s k i A., *Optimal design of thin-walled columns for buckling under loadings controlled by displacements*, Structural and Multidisciplinary Optimization, 42, 2010, 305-314.

[19] K r u ż e l e c k i J., T r z e c i a k P., *Optimal design of axially symmetrical shells under hydrostatic pressure with respect to their stability*, Structural and Multidisciplinary Optimization, 19, 2000, 148-154.

[20] Krzyś W., *Optimum design of thin-walled closed cross-section columns*, Bull. Acad. Sci. Techn., 21 (9), 1973, 653-663.

[21] van Laarhoven P.J.M., Aarts E.H.L., *Simulated annealing – theory and applications*, Kulwer Academic Publishers, Dordrecht, Boston, Londyn 1992.

[22] Masters T., *Practical Neural Network Recipes in C++*, Academic Press, 1993.

[23] Papkovitch P.F., *Theory of structural design of ships* (in Russian), Part.2, Gos. Soyuz. Izd. Sudostr. Promyshl., Leningrad 1941.

[24] Shirshov V.P., *Local buckling of shells* (in Russian), Proc. II Vses. Konf. Teori Plastin i Obol., 1962, 314-317.

[25] Smaś P., *Design of clamped columns for maximizing the axial displacement at buckling,* Structural and Multidisciplinary Optimization, 33, 2007, 229-241.

[26] Volkersen O., *Ein Beitrag zum optimalen Bemessen von axial gedrückten Rohren und Integralplatten*, VDI-Zeitschrift, 108 (26), 1966, 1281-1284.

[27] Wlasow W.S., *Allgemeine Schalentheorie und ihre Anwendung in der Technik*, Akademie-Verlag, Berlin 1958 (translated from Russian, Moskva-Leningrad 1949).

[28] Ziegler H., *Principles of structural stability*, Birkhäuser, Stuttgart, Germany 1977 (second ed.).

[29] Życzkowski M., Krużelecki J., Trzeciak P., *Optimal design of rotationally symmetric shells for buckling under thermal loadings*, J. Theor. Appl. Mech., 39 (2), 2001, 443-455.

[30] Życzkowski M., Krużelecki J., *Optimal design of shells with respect to their stability. IUTAM Symposium on Optimization in Structural Design*, A. Sawczuk, Z. Mróz, (Eds.), Springer, Berlin, Heidelberg, New York 1975, 229-247.