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INFINITE IMPLICIT SYSTEM OF WEAK PARABOLIC FUNCTIONAL-DIFFERENTIAL INEQUALITIES AND STRONG MAXIMUM PRINCIPLE FOR THIS INFINITE SYSTEM

NIESKOŃCZONY UWIKŁANY UKŁAD SŁABYCH PARABOLICZNYCH FUNKCJONALNO-RÓŻNICZKOWYCH NIERÓWNOŚCI I MOCNA ZASADA MAKSIMUM DLA TEGO NIESKOŃCZONEGO UKŁADU

A b s t r a c t

The purpose of the paper is to prove a theorem on infinite implicit system of weak parabolic functional-differential inequalities and a theorem on strong maximum principle for infinite implicit system of parabolic functional-differential inequalities.

Keywords: *infinite parabolic system, implicit system, theorem on weak inequalities, strong maximum principle*

S t r e s z c z e n i e

Celem niniejszego artykułu jest udowodnienie twierdzenia o nieskończonym uwikłanym układzie słabych parabolicznych nierówności funkcjonalno-różniczkowych i twierdzenia o mocnej zasadzie maksimum dla nieskończonego uwikłanego układu parabolicznych nierówności funkcjonalno-różniczkowych.

Słowa kluczowe: *nieskończone układy paraboliczne, układy uwikłane, twierdzenie o słabych nierównościach, mocna zasada maksimum*

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1. Introduction

In this paper we consider an infinite implicit system of non-linear parabolic functional--differential inequalities of the form

$$F_i(x,t,u^i(x,t),u_t^i(x,t),u_x^i(x,t),u_{xx}^i(x,t),u) \geq F_i(x,t,v^i(x,t),v_t^i(x,t),v_x^i(x,t),v_{xx}^i(x,t),v) \quad (i\in\mathbb{N}) \tag{1}$$

for $(x,t)=(x_1,...,x_n,t)\in D$, where $D\subset\mathbb{R}^n\times(t_0,t_0+T]$ is a relatively arbitrary set more general than the cylindrical domain $D_0\times(t_0,t_0+T]\subset\mathbb{R}^{\mathbf{n+1}}$.

The symbol w (=u *or* v) denotes the mapping

$$w:\mathbb{N}\times\tilde{D}\ni(i,x,t)\longrightarrow w^i(x,t)\in\mathbb{R}$$

where $\tilde{D}$ is an arbitrary set such that

$$\overline{D}\subset\tilde{D}\subset\mathbb{R}^n\times(-\infty,t_0+T];$$

F_i $(i\in\mathbb{N})$ are functionals of $w, w_x^i(x,t)=\mathrm{grad}_x w^i(x,t)$ and $w_{xx}^i(x,t)$ denote the matrices of second order derivatives with respect to x of $w^i(x,t)$ $(i\in\mathbb{N})$.

In the paper we prove Theorem 3.1 on infinite implicit system (1) of weak parabolic functional-differential inequalities and Theorem 4.1 on strong maximum principle for infinite implicit system (1).

Some results obtained are based on [1, 2, 5, 12–16, 18]. Some infinite and finite, parabolic and hyperbolic systems were considered by [3, 4, 7–11, 19].

Infinite parabolic systems have physical application. For this purpose please see the publication [17, 6].

2. Preliminaries

We shall use the following notations

$$\mathbb{R}=(-\infty,\infty),\ \mathbb{N}=\{1,2,...\},\ x=(x_1,...,x_n)\in\mathbb{R}^n\ (n\in\mathbb{N})$$

Let t_0 be a real finite number and let $0<T<\infty$.

A set $D\subset\{(x,t):x\in\mathbb{R}^n, t_0<t\leq t_0+T\}$ is called a set of type (P) if:

a) the projection of the interior of set D on the t-axis is the interval (t_0,t_0+T),

b) for every $(\tilde{x},\tilde{t})\in D$ there exists a positive number $r=r(\tilde{x},\tilde{t})$ such that

$$\{(x,t):\sum_{i=1}^{n}(x_i-\tilde{x}_i)^2+(t-\tilde{t})^2<r,\ t<\tilde{t}\}\subset D$$

c) all the boundary points $(\tilde{x},\tilde{t})$ of D for which there is a positive number $r = r(\tilde{x},\tilde{t})$ such that

$$\{(x,t): \sum_{i=1}^{n}(x_i - \tilde{x}_i)^2 + (t-\tilde{t})^2 < r,\, t \le \tilde{t}\} \subset D$$

belong to D.

Let $\tilde{D}$ be an arbitrary set such that

$$\bar{D} \subset \tilde{D} \subset \mathbb{R}^n \times (-\infty, t_0 + T]$$

and let

$$\partial_p D := \tilde{D} \setminus D \tag{2}$$

For an arbitrary fixed point $(\tilde{x},\tilde{t}) \in D$, we denote by $S^-(\tilde{x},\tilde{t})$ the set of points $(x,t) \in D$ that can be joined to $(\tilde{x},\tilde{t})$ by a polygonal line contained in D along which the t-coordinate is weakly increasing from (x,t) to $(\tilde{x},\tilde{t})$.

Let $Z_\infty(\tilde{D})$ denote the linear space of mappings

$$w : \mathbb{N} \times \tilde{D} \ni (i,x,t) \to w^i(x,t) \in \mathbb{R}$$

where functions

$$w^i : \tilde{D} \ni (x,t) \to w^i(x,t) \in \mathbb{R}$$

are continuous in $\bar{D}$ and

$$\sup\{\left|w^i(x,t)\right| : (x,t) \in \tilde{D},\, i \in \mathbb{N}\} < \infty$$

For $w, \tilde{w} \in Z_\infty(\tilde{D})$ we write $w \le \tilde{w}$ in the sense $w^i \le \tilde{w}^i\, (i \in \mathbb{N})$.

In the set of mappings w belonging to $Z_\infty(\tilde{D})$ we define the functional $[\cdot]_{t,\infty}$ by the formula

$$[w]_{t,\infty} = \sup\{0, w^i(x,\tilde{t}) : (x,\tilde{t}) \in \tilde{D},\quad \tilde{t} \le t,\ i \in \mathbb{N}\} \tag{3}$$

where $t \le t_0 + T$.

In particular

$$[w]_{t_0+T,\infty} = \sup\{0, w^i(x,t) : (x,t) \in \tilde{D},\quad i \in \mathbb{N}\}$$

By $Z_\infty^{2,1}(\tilde{D})$ we denote the linear subspace of $Z_\infty(\tilde{D})$. A mapping w belongs to $Z_\infty^{2,1}(\tilde{D})$ if $w_t^i, w_x^i = (w_{x_1}^i, \cdots, w_{x_n}^i), w_{xx}^i = [w_{x_j x_k}^i]_{n\times n}$ $(i \in \mathbb{N})$ are continuous in D.

By $M_{n\times n}(\mathbb{R})$ we denote the space of real square symmetric matrices $r = [r_{jk}]_{n\times n}$.

For $r \in M_{n\times n}(\mathbb{R})$ we write $r \ge 0$ if $\sum_{j,k=1}^{n} r_{jk}\lambda_j\lambda_k \ge 0$ for all $(\lambda_1,\ldots,\lambda_n) \in \mathbb{R}^n$.

Let the mappings

$$F_i : D\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times M_{n\times n}(\mathbb{R})\times Z_\infty(\tilde{D}) \ni (x,t,z,p,q,r,w)$$
$$\to F_i(x,t,z,p,q,r,w)\in\mathbb{R} \quad (i\in\mathbb{N})$$

be given and let for an arbitrary function $w\in Z_\infty^{2,1}(\tilde{D})$

$$F_i[x,t,w] := F_i(x,t,w^i(x,t),w_t^i(x,t),w_x^i(x,t),w_{xx}^i(x,t),w), \quad (x,t)\in D \quad (i\in\mathbb{N}) \tag{4}$$

For a given subset $E\subset D$ and a given mapping $w\in Z_\infty^{2,1}(\tilde{D})$, and a fixed index $i\in\mathbb{N}$ the function F_i is called uniformly parabolic with respect to w in E if there is a constant $\kappa>0$ (depending on E) such that for any two matrices $r=[r_{jk}]\in M_{n\times n}(\mathbb{R})$, $\tilde{r}=[\tilde{r}_{jk}]\in M_{n\times n}(\mathbb{R})$ and for $(x,t)\in E$ we have

$$r\le\tilde{r}\Rightarrow F_i(x,t,w^i(x,t),w_t^i(x,t),w_x^i(x,t),\tilde{r},w)$$
$$-F_i(x,t,w^i(x,t),w_t^i(x,t),w_x^i(x,t),r,w)\ge\kappa\sum_{j=1}^n(\tilde{r}_{jj}-r_{jj}) \tag{5}$$

If (5) is satisfied for $\kappa=0$ and $r=w_{xx}^i(x,t)$, where $(x,t)\in D$, and for $\tilde{r}=w_{xx}^i(x,t)+\hat{r}$, where $(x,t)\in E$ and $\hat{r}\ge 0$, then F_i is called parabolic with respect to w in E.

Two functions $u,v\in Z_\infty^{2,1}(\tilde{D})$ are called solutions of the system

$$F_i[x,t,u]\ge F_i[x,t,v] \quad (i\in\mathbb{N}) \tag{6}$$

in D, if they satisfy (6) for $(x,t)\in D$.

3. Theorem on infinite implicit system of weak parabolic functional-differential inequalities

Theorem 3.1. Assume that:

1^0 $D\subset\mathbb{R}^n\times(t_0,t_0+T]$ is a set of type (P) and F_i $(i\in\mathbb{N})$ are the functions from Section 2.

2^0 There is a constant $L_0>0$ such that

$$F_i(x,t,z,p,q,r,w)-F_i(x,t,z,\tilde{p},q,r,w)\le L_0(\tilde{p}-p)$$

for $(x,t)\in D$, $z\in\mathbb{R}$, $p>\tilde{p}$, $q\in\mathbb{R}^n$, $r\in M_{n\times n}(\mathbb{R})$, $w\in Z_\infty(\tilde{D})$.

3^0 There is a constant $L>0$ such that

$$F_i(x,t,z,p,q,r,w)-F_i(x,t,\tilde{z},p,q,r,w)$$

$$\le L((z-\tilde{z})+|x|\sum_{j=1}^n|q_j-\tilde{q}_j|+|x|^2\sum_{j,k=1}^n|r_{jk}-\tilde{r}_{jk}|+[w-\tilde{w}]_{t,\infty}) \quad (i\in\mathbb{N})$$

for

(i) $(x,t)\in D, |x|>L,\ z\geq\tilde{z},\ p\in\mathbb{R},\ q,\tilde{q}\in\mathbb{R}^n,\ r,\tilde{r}\in M_{n\times n}(\mathbb{R}),\ w,\tilde{w}\in Z_\infty(\tilde{D})$

and for

(ii) $(x,t)\in D, |x|\leq L,\ z\geq\tilde{z},\ p\in\mathbb{R},\ q=\tilde{q},\ r=\tilde{r},\ w,\tilde{w}\in Z_\infty(\tilde{D})$

4^0 The functions $u,v\in Z_\infty^{2,1}(\tilde{D})$ satisfy the inequality

$$u(x,t)\leq v(x,t) \quad \text{for} \quad (x,t)\in\partial_p D \tag{7}$$

5^0 F_i $(i\in\mathbb{N})$ are parabolic with respect to *u* in *D*.
6^0 The functions *u* and *v* are solutions of system (6) in *D*.
Then

$$u(x,t)\leq v(x,t) \quad \text{for} \quad (x,t)\in\tilde{D} \tag{8}$$

Proof. Firstly we shall show that for *h*, defined by the formula

$$h:=\min\left\{T, L_0L^{-1}\ln\tfrac{3}{2}\right\} \tag{9}$$

we have

$$[u-v]_{t_0+h,\infty}\leq 0 \tag{10}$$

For this purpose fix index $j\in\mathbb{N}$ and put

$$F(x,t,z,p,q,r):=F_j(x,t,z,p,q,r,u) \tag{11}$$

Let

$$D_h:=\left\{(x,t):(x,t)\in D, t_0<t\leq t_{0+h}\right\}$$

Introduce the function η by the formula

$$\eta(x,t)=v^j(x,t)+[u-v]_{t_0+h,\infty}\left[\exp\left(\tfrac{L}{L_0}(t-t_0)\right)-1\right] \tag{12}$$

and observe that, by (7) and the definition of the functional $[\cdot]_{t,\infty}$, we have the inequality

$$u^j(x,t)\leq\eta(x,t) \quad \text{for} \quad (x,t)\in(\tilde{D}\setminus D)\cap\partial D_h \tag{13}$$

From (11), assumption 6^0, (12), assumptions 2^0 and 3^0, and again (12), we obtain

$$\begin{aligned}&F(x,t,u^j(x,t),u_t^j(x,t),u_x^j(x,t),u_{xx}^j(x,t))\\&-F(x,t,\eta(x,t),\eta_t(x,t),\eta_x(x,t),\eta_{xx}(x,t))\end{aligned} \tag{14}$$

$$\begin{aligned}&=F_j(x,t,u^j(x,t),u_t^j(x,t),u_x^j(x,t),u_{xx}^j(x,t),u)\\&-F_j(x,t,\eta(x,t),\eta_t(x,t),\eta_x(x,t),\eta_{xx}(x,t),u)\end{aligned}$$

$$\geq F_j(x,t,v^j(x,t),v_t^j(x,t),v_x^j(x,t),v_{xx}^j(x,t),v)$$
$$-F_j(x,t,\eta(x,t),\eta_t(x,t),\eta_x(x,t),\eta_{xx}(x,t),u)$$

$$= F_j(x,t,v^j(x,t),v_t^j(x,t),\eta_x(x,t),\eta_{xx}(x,t),v)$$
$$-F_j(x,t,\eta(x,t),\eta_t(x,t),\eta_x(x,t),\eta_{xx}(x,t),u)$$

$$= F_j(x,t,v^j(x,t),v_t^i(x,t),\eta_x(x,t),\eta_{xx}(x,t),v)$$
$$-F_j(x,t,v^j(x,t),\eta_t(x,t),\eta_x(x,t),\eta_{xx}(x,t),v)$$
$$+F_j(x,t,v^j(x,t),\eta_t(x,t),\eta_x(x,t),\eta_{xx}(x,t),v)$$
$$-F_j(x,t,\eta(x,t),\eta_t(x,t),\eta_x(x,t),\eta_{xx}(x,t),u)$$

$$\geq L_0 \tfrac{L}{L_0}[u-v]_{t_0+h,\infty} \exp\left(\tfrac{L}{L_0}(t-t_0)\right)$$
$$-L[u-v]_{t_0+h,\infty}[\exp\left(\tfrac{L}{L_0}(t-t_0)\right)-1]-L[u-v]_{t_0+h,\infty}$$
$$\geq 0 \quad \text{for} \quad (x,t)\in D_h$$

because

$$\eta_t(x,t)=v_t^j(x,t)+\tfrac{L}{L_0}[u-v]_{t_0+h,\infty}\exp\left(\tfrac{L}{L_0}(t-t_0)\right)$$

By (14), (13), assumptions 2^0, 3^0 and 5^0, we have, from Theorem 2.1 in [5], that

$$u^j(x,t)\leq \eta(x,t) \quad \text{for} \quad (x,t)\in \bar{D}_h \tag{15}$$

From (15) and (12)

$$u^j(x,t)-v^j(x,t)\leq [u-v]_{t_0+h,\infty}[\exp\left(\tfrac{L}{L_0}(t-t_0)\right)-1] \tag{16}$$

By (9), we have

$$e^{LL_0^{-1(t-t_0)}}-1\leq e^{LL_0^{-1}h}-1 \tag{17}$$

$$\leq e^{LL_0^{-1}L_0L^{-1}\ln\frac{3}{2}}-1=e^{\ln\frac{3}{2}}-1=\tfrac{1}{2} \quad \text{for} \quad t\in(t_0,t_0+h]$$

and consequently, from (16) and (17),

$$u^j(x,t)-v^j(x,t)\leq \tfrac{1}{2}[u-v]_{t_0+h,\infty} \quad \text{for} \quad (x,t)\in \bar{D}_h \tag{18}$$

Since (18) holds for an arbitrary fixed index $j\in\mathbb{N}$ then, by (3), (7) and (18)

$$[u-v]_{t_0+h,\infty}\leq \tfrac{1}{2}[u-v]_{t_0+h,\infty} \tag{19}$$

Consequently, (10) is satisfied.

If $h = T$ then the proof is complete.

For

$$h = L_0 L^{-1} \ln \tfrac{3}{2} < T$$

there are numbers $s \in \mathbb{N}$ and $\tilde{h} \in [0, L_0 L^{-1} \ln \tfrac{3}{2})$ such that

$$t_0 + T = t_0 + s\, L_0 L^{-1} \ln \tfrac{3}{2} + \tilde{h}$$

If $s = 1$ then the above argument implies that (10) holds.

If $s = 2$ then substituting $t_0 + h$ in the place of t_0 and $\hat{D} = D \setminus D_h$ in the place of D, we have, by (7) and (10)

$$u(x,t) \le v(x,t) \in \tilde{D} \setminus \hat{D}$$

We see that the above argument holds in the set $\hat{D}$ and, consequently, we get

$$[u - v]_{t_0 + 2h, \infty} \le 0$$

If $s > 2$ then, by the suitably modified argument for $s = 1$ and $s = 2$, we have

$$[u - v]_{t_0 + sh, \infty} \le 0$$

In the case if $\tilde{h} = 0$, and so if

$$t_0 + sh = t_0 + T$$

the proof of Theorem 3.1 is complete.

To prove, finally, the proof of our theorem it is enough to show that

$$[u - v]_{t_0 + sh + \tilde{h}, \infty} \le 0$$

where $\tilde{h} \in (0, L_0 L^{-1} \ln \tfrac{3}{2})$. But, in this case, the proof is the modification of the proofs in cases $s = 1$ and $s > 2$. In particular for $\tilde{h} \in (0, L_0 L^{-1} \ln \tfrac{3}{2})$ the following inequality holds

$$e^{L L_0^{-1} \tilde{h}} - 1 < \tfrac{1}{2}$$

Then, for

$$h = L_0 L^{-1} \ln \tfrac{3}{2} < T$$

the proof is complete.

4. Theorem on strong maximum principle for infinite implicit system of parabolic functional-differential inequalities

Theorem 4.1. Let assumptions 1^0, 2^0, 4^0, 5^0 and 6^0 of Theorem 3.1 be satisfied. Replace assumption 3^0 of Theorem 3.1 by a stronger assumption:

7^0 There is a constant $L > 0$ such that

$$F_i(x,t,z,p,q,r,w) - F_i(x,t,\tilde{z},p,\tilde{q},\tilde{r},\tilde{w})$$
$$\le L(|z-\tilde{z}| + |x| \sum_{j=1}^{n} |q_j - \tilde{q}_j| + |x|^2 \sum_{j,k=1}^{n} |r_{jk} - \tilde{r}_{jk}| + [w-\tilde{w}]_{t,\infty}) \quad (i \in \mathbb{N})$$

for all $(x,t) \in D, \quad z,\tilde{z} \in \mathbb{R}, \quad p \in \mathbb{R}, \quad q,\tilde{q} \in \mathbb{R}^n, \quad r,\tilde{r} \in M_{n\times n}(\mathbb{R}), \quad w,\tilde{w} \in Z_\infty(\tilde{D}).$

Suppose finally that:

8^0 There is a constant $L_* > 0$ such that

$$F_i(x,t,z,p,q,r,w) - F_i(x,t,z,\tilde{p},q,r,w) \le L_* |p - \tilde{p}| \quad (i \in \mathbb{N})$$

for all $(x,t) \in D, \quad z \in \mathbb{R}, \quad p,\tilde{p} \in \mathbb{R}, \quad q \in \mathbb{R}^n, \quad r \in M_{n\times n}(\mathbb{R}), \quad w \in Z_\infty(\tilde{D}).$

9^0 F_i $(i \in \mathbb{N})$ are uniformly parabolic with respect to v any compact subset of D.

Under those assumptions inequality (8) is satisfied and, moreover, if for some point $(\tilde{x},\tilde{t}) \in D$ and some index $j \in \mathbb{N}$ the inequality

$$u^j(\tilde{x},\tilde{t}) = v^j(\tilde{x},\tilde{t})$$

is satisfied then

$$u^j(x,t) = v^j(x,t) \quad \text{for} \quad (x,t) \in S^-(\tilde{x},\tilde{t}) \tag{20}$$

Proof. Inequality (8) is a consequence of Theorem 3.1. Now, suppose that the second assertion of Theorem 4.1 is not true. Then there exist two points $(x^*,t^*) \in S^-(\tilde{x},\tilde{t})$ and $(x^{**},t^{**}) \in S^-(\tilde{x},\tilde{t})$, where $t^* < t^{**}$, such that the segment joining them is contained in D and

$$u^j(x^{**},t^{**}) = v^j(x^{**},t^{**}) \tag{21}$$

and

$$u^j(x^*,t^*) < v^j(x^*,t^*) \tag{22}$$

Denoting by $|\cdot|$ the Euclidean norm and putting

$$\xi = \frac{x^{**} - x^*}{t^{**} - t^*}$$

we introduce the functions ψ and φ by the formulas

$$\psi(x,t) = \delta^2 - |x - x^* - \xi(t - t^*)|^2$$
$$\varphi(x,t) = e^{-\alpha t}\psi^2(x,t)$$

δ and α being positive constants to be determined later.

Consider the oblique cylinder

$$Q = \{(x,t) : \psi > 0, t^* < t \le t^{**}\}$$

with (x^*, t^*) and (x^{**}, t^{**}) as the centers of the bases. We choose δ so small that $\bar{Q} \subset D$ and that on the lower base B of Q the inequality

$$u^j(x,t^*) < v^j(x,t^*), \quad (x,t^*) \in B \tag{23}$$

is satisfied.

Put

$$F(x,t,z,p,q,r) = F_j(x,t,z,p,q,r,u) \tag{24}$$

Now, setting

$$w^\varepsilon(x,t) = v^j(x,t) - \varepsilon\varphi(x,t), \quad \varepsilon > 0$$

we will prove that

$$\begin{aligned} &F(x,t,u^j(x,t),u_t^j(x,t),u_x^j(x,t),u_{xx}^j(x,t)) \\ &\ge F(x,t,w^\varepsilon(x,t),w_t^\varepsilon(x,t),w_x^\varepsilon(x,t),w_{xx}^\varepsilon(x,t)), \quad (x,t) \in Q \end{aligned} \tag{25}$$

provided that α is chosen large enough.

To this effect, we substitute $v^j = w^\varepsilon + \varepsilon\varphi$ in (6) and we obtain

$$\begin{aligned} &F_j(x,t,u^j(x,t),u_t^j(x,t),u_x^j(x,t),u_{xx}^j(x,t),u) \\ &\ge F_j(x,t,v^j(x,t),w_t^\varepsilon(x,t)+\varepsilon\varphi_t,v_x^j(x,t),w_{xx}^\varepsilon(x,t)+\varepsilon\varphi_{xx},v), \quad (x,t) \in D \end{aligned} \tag{26}$$

A direct computation gives

$$\varphi_t = -\alpha e^{-\alpha t}\psi^2 + 4e^{-\alpha t}\psi\xi \circ (x - x^* - \xi(t-t^*)) \tag{27}$$

$$\varphi_x = -4e^{-\alpha t}\psi(x - x^* - \xi(t-t^*)) \tag{28}$$

$$\varphi_{xx} = -4e^{-\alpha t}\psi I + r \tag{29}$$

where I is the identity matrix and r is the matrix with the elements

$$8e^{-\alpha t}(x_k - x_k^* - \xi_k(t-t^*))(x_l - x_l^* - \xi_l(t-t^*))$$

Since $\varepsilon r \ge 0$, it follows from (26) and (29), by assumption 9^0 (see (5)) that for $(x,t) \in Q$

$$\begin{aligned} &F_j(x,t,u^j(x,t),u_t^j(x,t),u_x^j(x,t),u_{xx}^j(x,t),u) \\ &\ge F_j(x,t,v^i(x,t),w_t^\varepsilon(x,t)+\varepsilon\varphi_t,v_x^j(x,t),w_{xx}^\varepsilon(x,t)-4\varepsilon e^{-\alpha t}\psi I,v) \\ &+8\varepsilon\kappa e^{-\alpha t}\left|x - x^* - \xi(t-t^*)\right|^2 \end{aligned}$$

The last inequality and assumption 7^0 together with (8) imply that

$$\begin{gathered}F_j(x,t,u^j(x,t),u_t^j(x,t),u_x^j(x,t),u_{xx}^j(x,t),u)\\ \geq F_j(x,t,w^\varepsilon(x,t)+\varepsilon\varphi,w_t^\varepsilon(x,t)+\varepsilon\varphi_t,w_x^\varepsilon(x,t)+\varepsilon\varphi_x\\ w_{xx}^\varepsilon(x,t)-4\varepsilon e^{-\alpha t}\psi I,u)\\ +8\varepsilon\kappa e^{-\alpha t}\left|x-x^*-\xi(t-t^*)\right|^2,\quad (x,t)\in Q\end{gathered}$$

From assumption 2^0 and formula (27) it follows that

$$\begin{gathered}F_j(x,t,u^j(x,t),u_t^j(x,t),u_x^j(x,t),u_{xx}^j(x,t),u)\\ \geq F_j(x,t,w^\varepsilon(x,t)+\varepsilon\varphi,w_t^\varepsilon(x,t)+4\varepsilon e^{-\alpha t}\psi\varphi\circ(x-x^*-\xi(t-t^*))\\ w_x^\varepsilon(x,t)+\varepsilon\varphi_x,w_{xx}^\varepsilon(x,t)-4\varepsilon e^{-\alpha t}\psi I,u)\\ +8\varepsilon\kappa e^{-\alpha t}\left|x-x^*-\xi(t-t^*)\right|^2+L_0\varepsilon\alpha e^{-\alpha t}\psi^2,\quad (x,t)\in Q\end{gathered}$$

Now, by applying assumptions 7^0 and 8^0, the definition of function φ, (28) and (24), we get from the above inequality the following estimate

$$\begin{aligned}&F(x,t,u^j(x,t),u_t^j(x,t),u_x^j(x,t),u_{xx}^j(x,t)\\ &\geq F(x,t,w^\varepsilon(x,t),w_t^\varepsilon(x,t),w_x^\varepsilon(x,t),w_{xx}^\varepsilon(x,t))\\ &+\varepsilon g(x,t),\ (x,t)\in Q\end{aligned}\tag{30}$$

where

$$\begin{aligned}g(x,t)&=(L_0\alpha-L)e^{-\alpha t}\psi^2+\{8\kappa\left|x-x^*-\xi(t-t^*)\right|^2\\ &-4\psi[nL\delta(\delta+\left|x^*\right|+\left|x^{**}-x^*\right|)\\ &+nL(\delta+\left|x^*\right|+\left|x^{**}-x^*\right|)^2+L_*\delta|\xi|]\}e^{-\alpha t}\end{aligned}$$

The expression in the braces tends uniformly in Q to $8\kappa\delta^2>0$ as ψ tends to 0. Hence, there is a μ such that $\delta>\mu>0$ and $g(x,t)>0$ for $(x,t)\in Q$ with $0<\psi(x,t)<\mu$. On the other hand, for $(x,t)\in Q$ such that $\delta\geq\psi(x,t)\geq\mu>0$, we can choose $\alpha>0$ so large as to make $(L_0\alpha-L)\psi^2$ larger than the absolute value of the expression in braces which is bounded independently of α. In this way, α being chosen sufficiently large, we can make $g(x,t)>0$ in Q, and thus, by (30), we get (25). As a consequence of (23) we can choose $\varepsilon\in(0,1]$ so small that

$$u^j(x,t^*)\leq w^\varepsilon(x,t^*),\quad (x,t^*)\in B\tag{31}$$

Finally, observe that, by (8), on the side surface of Q we have

$$u^j(x,t) \le w^\varepsilon(x,t) \tag{32}$$

since $w^\varepsilon(x,t) = v^j(x,t)$ there.

According to assumption 5^0 we verify that the function F is parabolic with respect to u^j. This last remark together with assumptions 1^0, 2^0, 7^0, and formulas (25), (31), (32) shows that for functions u^j, w^ε and F all the assumptions of Theorem 2.1 in [5] are satisfied in Q with $m=1$. Therefore

$$u^j(x,t) \le w^\varepsilon(x,t) = v^j(x,t) - \varepsilon\varphi(x,t)$$

in $\bar{Q}$ and in particular, since $\varphi(x^{**},t^{**}) > 0$

$$u^j(x^{**},t^{**}) < v^j(x^{**},t^{**})$$

what contradicts (21).

The proof of Theorem 4.1 is complete.

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