NAUKI PODSTAWOWE
CZASOPISMO TECHNICZNE
TECHNICAL TRANSACTIONS
FUNDAMENTAL SCIENCES
WYDAWNICTWO POLITECHNIKI KRAKOWSKIEJ
1-NP/2009
ZESZYT 8
ROK 106
ISSUE 8
YEAR 106

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EXTENTIONS OF NOT DENSELY DEFINED EVOLUTION PROBLEMS TO DENSELY DEFINED ONES

ROZSZERZANIE NIEGĘSTO OKREŚLONYCH PROBLEMÓW EWOLUCYJNYCH DO GĘSTO OKREŚLONYCH

A b s t r a c t

Evolution equations appear in many practical problems specially the ones connected with mathematical physics. The operator may be differential as well as integral. Thus it may either bounded or unbounded. At the same time densely or not densely defined. Domains may change together with t or be independent of t. The choice of the space in which $A(t)$ is considered can cause domains to be dense or not. We present here how some problems with not densely defined operators may comes down to densely defined one.

Keywords: *operator*, *evolution problem*, *extrapolation*

S t r e s z c z e n i e

Równania ewolucyjne pojawiają się w wielu praktycznych problemach, szczególnie w zagadnieniach fizyki matematycznej. Występujący w nich operator jest operatorem różniczkowym lub całkowym. Może więc być ograniczony lub nieograniczony. Może być gęsto albo niegęsto określony, o dziedzinie zależnej lub niezależnej od t. Wybór przestrzeni, w której rozważamy $A(t)$ ma wpływ na gęstość dziedziny. W artykule pokazano jak pewne problemy z niegęsto określonymi operatorami można sprowadzić do problemów gęsto określonych.

Słowa kluczowe: *operator*, *problem ewolucyjny*, *ekstrapolacja*

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1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space and let A be a linear operator from X to X.

By $\mathscr{D}(A)$, $\mathcal{C}(X)$, $\rho(A)$ are denoted the domain of A, space of all closed linear operators from X to X, resolvent set of A, respectively. If $\lambda \in \rho(A)$ then $(A - \lambda)$ is invertible and the inverse operator is denoted by $R(\lambda, A)$.

Let Ω be a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$. By $\mathscr{C}^k(\Omega)$, $\mathscr{C}^k(\bar{\Omega})$, $\mathscr{C}_0^k(\bar{\Omega})$ are denoted the space of functions of $\mathscr{C}^k$ class in Ω, the space of function $u \in \mathscr{C}^k(\Omega)$ which partial derivatives of order $\leq k$ can be continuously extended to $(\bar{\Omega})$, the space of functions $u \in \mathscr{C}^k(\Omega)$ with compact supports, respectively.

Our considerations are connected with evolution problems of the first and second order that is evolution problems of the form

$$\begin{cases} \dfrac{du}{dt} = A(t)u + f(t,u) \\ u(0) = u_0, \ u_0 \in X \end{cases} \quad \text{or} \quad \begin{cases} \dfrac{d^2u}{dt^2} = A(t)u + f(t,u), \\ u(0) = u_0, \ u'(0) = u_1, \quad u_0, u_1 \in X, \end{cases} \tag{1}$$

where $(A(t))_{t\in[0,T]}$ is a family of closed operators from X to X and $f:[0,T]\times X \to X$ is a given function.

By solution of (1) we mean a classical solution that is a function $u:[0,T] \to X$ such that

(i) $u \in \mathscr{C}^0([0,T])$ and it is differentiable in $(0,T]$ (or $u \in \mathscr{C}^1([0,T])$ and it is two times differentiable in $(0,T]$),

(ii) $u(t) \in \mathscr{D}(A(t))$ for $t \in [0,T]$,

(iii) $u(0) = u_0$ (or $u(0) = u_0$, $u'(0) = u_1$), $u_0, u_1 \in X$,

(iv) $\dfrac{du}{dt} = A(t)u(t) + f(t,u(t))$ (or $\dfrac{d^2u}{dt^2} = A(t)u(t) + f(t,u(t))$) for $t \in (0,T]$.

For example, an evolution problem of the first order has been considered in [5]. The problems of order two with a densely or not densely defined family $(A(t))_{t\in[0,T]}$ of operators and dependent on t its domains has been considered in [7–9].

2. Evolution equation with *A* independent of *t*

In this part we assume that $(A(t))_{t\in[0,T]}$ is a family of operators independent of t that is $A(t) = A$ for all $t \in [0,T]$. We start with the case of densely defined operator A and next we show how to reduce not densely defined case to densely defined one.

Definition 1. *An operator $A \in \mathcal{C}(X)$ is called a Hille-Yosida operator if there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$ and*

$$\left\| R(\lambda, A)^n \right\| \leq \frac{M}{(\lambda - \beta)^n} \quad \text{for} \quad \lambda > \omega, n = 1, 2, \ldots$$

1. Densely defined operators. By the Hille-Yosida theorem (see e.g. [3]), A generate a $\mathcal{C}_0$ – semigroup $U(t),\ t \geq 0$ if and only if it is a densely defined Hille-Yosida operator. This fact can be used to prove theorems on existence, uniqueness of solutions of the problem (1) with densely defined and independent of t operator A.

Theorem 2 (see [6]). *If X is a reflexive Banach space, A a densely defined Hille-Yosida operator,* $u_0 \in \mathscr{D}(A)$ *and f satisfies the Lipshitz condition in* $[0,T] \times u\ ([0,T])$ *then problem* (1) (*with* $A(t) = A$) *has exactly one solution u satisfying the integral equation*

$$u(t) = U(t)u_0 + \int_0^t U(t-s)\, f(s,u(s))\, ds, \tag{2}$$

where $U(t),\ t \geq 0$ *is a semigroup generated by A.*

2. Not densely defined operators. Suppose now that $A \in \mathcal{C}(X)$ is not densely defined. Then $X_0 := \overline{\mathscr{D}(A)}$ is a proper subspace of the Banach space X.

Let A_0 be the part of A in X_0 that is

$$\mathscr{D}(A_0) = \{u \in \mathscr{D}(A) : Au \in X_0\},$$

$$A_0(u) = Au \quad \text{for} \quad u \in D(A_0).$$

Then any solution of the problem

$$\begin{cases} \dfrac{du}{dt} = A_0 u + f(t,u), \quad t \in [0,T], \\ u(0) = u_0, \quad u_0 \in X_0, \end{cases} \tag{3}$$

is a solution of the problem (1) (with $A(t) = A_0$). When A_0 is densely defined, the problems with a not densely defined operator A can be reduced to a problem with an operator which is densely defined.

Proposition 3. *Let A be a closed linear operator from X to X and* A_0 *the part of A in the subspace* $X_0 := \overline{\mathscr{D}(A)}$. *Then*

(i) A_0 *is closed*,

(ii) *If A is a Hille-Yosida operator then* A_0 *is densely defined in* X_0 *and also a Hille-Yosida operator.*

Proof. Let Γ_A be the graph of A. Then the set

$$\Gamma_{A_0} = (X_0 \times X_0) \cap \Gamma_A$$

is the graph of A_0 and it is closed subset of $X_0 \times X_0$, because Γ_A is closed, and so A_0 is a closed operator.

For the proof of (ii) see e.g. to the proof of Theorem 3.1.10 (point (i)) in [3]. □

Remark 1. *Assuming that* A_0 *is the part in* X_0 *of a Hille-Yosida operator A one can extend, for example, Theorem 2 to the case of not densely defined operators. Of course to obtain a theorem on existence or uniqueness for solutions one must put stronger assumptions then for densely defined one.*

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth enough boundary $\partial\Omega$. The space $X = \mathscr{C}^0(\overline{\Omega})$ of continuous functions in $\overline{\Omega}$ with the norm[1]

$$\|u\| = \sup\{|u(x)| : x \in \Omega\} \quad \text{for} \quad u \in X. \tag{4}$$

is a Banach space. The set

$$X_0 := \{u \in X : u(x) = 0 \quad \text{for} \quad x \in \partial\Omega\}$$

is a closed subspace of *X*.

Let $X_b = \mathscr{C}^0(\partial\Omega)$ be the space of continuous functions on the boundary $\partial\Omega$ of Ω. The space X_b, with "sup-norm" as defined as in (4) is also a Banach space and the mapping

$$b : X \ni \upsilon \to \upsilon_b := \upsilon\big|_{\partial\Omega} \in X_b \tag{5}$$

is a continuous linear mapping and onto.

For each *n*-tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, let

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$$

be the partial derivative of order $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and let

$$P(D) = \sum_{|\alpha| \le 2} a_\alpha(x) D^\alpha \tag{6}$$

be a partial differential operator of order 2 with coefficients $a_\alpha : \overline{\Omega} \to \mathbb{R}$ of class $\mathscr{C}^\infty$.

Consider the operator

$$A : X \supset \mathscr{D}(A) \ni u \to P(D)u \in X \tag{7}$$

with domain $\mathscr{D}(A) = X_0 \cap \mathscr{C}^2(\overline{\Omega})$ and note that *A* is not densely defined in *X*.

Assumptions 4. *We shall use the following assumptions*:

Ass 1. *For any* $f \in X_b$ *there exists exactly one function* $u = \mathscr{B}(f) \in \mathscr{C}^2(\overline{\Omega})$ *such that* $P(D)u = 0$ *and* $u_b = f$*. Furthermore, we assume that* $\mathscr{B} : X_b \to X$ *is continuous.*

Ass 2. $0 \in \rho(A)$.

One of the examples of operators *P*(*D*) satisfying Assumptions 4 is the Laplace operator Δ.

Lemma 5. *If* A_0 *is the part of A in* $X_0 = \overline{\mathscr{D}(A)}$ *and Assumptions 4 are satisfied then* A_0 *is densely defined in* X_0 *and* $0 \in \rho(A_0)$.

Proof. Since $A(\mathscr{C}_0^\infty(\Omega)) = \mathscr{C}_0^\infty(\Omega) \subset X_0$ and $\mathscr{C}_0^\infty(\Omega)$ is dense in X_0, A_0 is densely defined in X_0. To prove that $0 \in \rho(A_0)$ it is enough to observe that $R(0, A_0) = R(0, A)\big|_{X_0}$. □

[1] Often called "sup-norm".

Example 1. Let $\Omega = (0,1) \subset \mathbb{R}$. We give an example in the space $X = \mathscr{C}^0(\overline{\Omega})$. Consider the operator

$$A : X \supset \mathscr{D}(A) \ni u \to u'' \in X \tag{8}$$

with domain

$$\mathscr{D}(A) = X \cap \{u \in \mathscr{C}^2(\overline{\Omega}) : u(0) = u(1) = 0\}. \tag{9}$$

Operator A is not densely defined in X and $B : X \to \mathscr{D}(A)$ given by

$$B\upsilon(x) = \int_0^x \left(\int_0^\tau \upsilon(t)dt \right) d\tau - x\int_0^1 \left(\int_0^\tau \upsilon(t)dt \right) d\tau \tag{10}$$

is the inverse of A. Since B is bounded, A is closed.

Let A_0 be the part of A in $X_0 = \overline{\mathscr{D}(A)}$. Since $A(\mathscr{C}_0^\infty(\Omega)) = \mathscr{C}_0^\infty(\Omega) \subset X_0$ and $\mathscr{C}_0^\infty(\Omega)$ is dense in X_0, A_0 is densely defined in X_0. The operator $B_0 := B\big|_{X_0}$ is the inverse of A_0. Thus $0 \in \rho(A)$.

Remark 2. *If we consider the operator A* (*defined by* (8)) *as an operator with domain* $\mathscr{D}(A) = H_0^1(\Omega) \cap H^2(\Omega) \subset L^2(\Omega)$, *we get an operator which is closed and densely defined in* $L^2(\Omega)$.

3. Extensions to densely defined operator. One of the methods is presented in [3]. It consists in construction of extrapolation space X^{-1} associated with an operator A which is assumed to be closed with non-empty resolvent set $\rho(A)$. We present here only a brief description of construction (for details see [3]).

Since A is closed, the graph G_A of A is a closed linear subspace of $X \times X$. Then the quotient space

$$X^{-1} := (X \times X)/G_A$$

with the quotient norm is a Banach space.

The class of element (u, υ) mod G_A will be denoted by $[u, \upsilon]$. The mapping

$$i : X \ni u \to iu := [0, u] \in X^{-1}$$

is a bounded one to one mapping of X onto the linear subspace iX of X^{-1} which allow us to identify X with iX which is a dense subspace of X^{-1} if and only if A is densely defined.

We define a linear operator A^{-1} by

$$A^{-1} : \mathscr{D}(A^{-1}) := iX \ni [0, u] \to [-u, 0] \in X^{-1}. \tag{11}$$

Since, for $u \in \mathscr{D}(A)$, $iAu = [0, Au] = [-u, 0]$, we can see A^{-1} as extension of A to the whole space X. The subspace $\mathscr{D}(A^{-1})$ is a dense subspace of $X_{-1} := \overline{iX}$, where the closure of iX is taken in X^{-1}. Thus A^{-1} is densely defined if and only if $\mathscr{D}(A)$ is a dense subspace of X.

Let A_{-1} be the part of A^{-1} in X_{-1} and let A_0 be the part of A in $X_0 = \overline{\mathscr{D}(A)}$.

Proposition 6. (see [3], Proposition 3.1.9). *Let A be a closed operator with* $\lambda \in \rho(A)$. *Then*

(i) $\mathscr{D}(A_{-1}) = X_0$ *and* $A_{-1} - \lambda : X_0 \to X_{-1}$ *is an isomorphism,*

(ii) *A is the part of* A_{-1} *in X. If* $\lambda \in \rho(A)$, *then* $\lambda \in \rho(A_{-1})$ *and* $R(\lambda, A) = R(\lambda, A_{-1})\big|_X$.

For each $\lambda \in \rho(A)$ the mapping

$$\left|\cdot\right|_\lambda : X^{-1} \ni [u, \upsilon] \to \left\| AR(\lambda, A)u - R(\lambda, A)\upsilon \right\| \tag{12}$$

defines an equivalent norm on X^{-1}. Moreover $\lambda \in \rho(A^{-1})$ and the mapping $A^{-1} - \lambda$ is an isomorphism of *X* onto X^{-1} (see [3], Proposition 3.1.1).

Example 2. Suppose that *A* defined by (8) satisfies Assumptions 4. Let, for $\upsilon \in X$, $\varphi_1(\upsilon)$, $\varphi_2(\upsilon)$ and φ be defined by

$$\varphi_2(\upsilon) = \mathscr{B}(\upsilon_b),$$

$$\varphi_1(\upsilon) = \upsilon - \varphi_2(\upsilon),$$

$$\varphi : X \ni \upsilon \to \varphi_1(\upsilon) \oplus \varphi_2(\upsilon) \in \tilde{X} := X_0 \oplus \mathscr{B}(X_b),$$

where $\oplus$ is a direct sum of Banach spaces.

The mapping φ is an isomorphism of Banach spaces *X* and $\tilde{X}$. Indeed, by Assumptions 4 (Ass 1) φ_2 is continuous. Thus φ is continuous, because also φ_1 as a sum of identity and continuous map is continuous too. Continuity of the inverse of φ is obvious. Thus we may identify $\upsilon \in X$ with $\varphi(\upsilon) \in \tilde{X}$ and $u \in X_0$ with

$$\varphi(u) = u \oplus 0 \in \tilde{X}_0 := \varphi(X_0) = X_0 \oplus \{0\}.$$

Since *A* is a closed operator with non-empty resolvent set $\rho(A)$ we may construct the extrapolation space X^{-1} associated with *A*. According to (11), for $\upsilon \in X$ we have

$$\begin{aligned}[0, A^{-1}\upsilon] &= [0, A^{-1}(\varphi_1(\upsilon) + (\varphi_2(\upsilon))] = [0, A(\varphi_1(\upsilon))] + [0, A^{-1}(\varphi_2(\upsilon))] = \\ &= [-\varphi_1(\upsilon), 0] + [-\varphi_2(\upsilon), 0] = [-(\varphi_1(\upsilon) + \varphi_2(\upsilon)), 0] \in [X_0 \oplus \mathscr{B}(X_b), 0].\end{aligned}$$

Therefore $\mathscr{D}(A^{-1})$ consists of two parts. One is $X_0 = \overline{\mathscr{D}(A)}$ and second one consists of classes $[0, \upsilon]$ corresponding to $\upsilon \in \mathscr{B}(X_b)$. Since *A* is not densely defined in *X* and $[X_0 \oplus \mathscr{B}(X_b), 0] = X^{-1}$, the range of A^{-1} is equal to X^{-1} and A^{-1} is not densely defined in X^{-1}. Although we need to have a densely defined extension of *A*.

Let A_{-1} be the part of A^{-1} in X_{-1}. Since, by Lemma 5, the part A_0 of *A* in X_0 is densely defined in X_0 and $0 \in \rho(A_0)$, A_{-1} is densely defined in X_{-1} and (by Proposition 6) $\mathscr{D}(A_{-1}) = X_0$. To prove it one can use the same method as in the proof of part (ii) of Theorem 3.1.10 in [3].

Note that $[u, 0] = [0, \upsilon]$ if and only if $u \in \mathscr{D}(A)$ and $\upsilon = Au$ and the norm on $[X, 0]$ is equivalent to the norm given by (12). Thus $[X_0, 0]$ is the closure of $[\mathscr{D}(A), 0]$ and $\mathscr{B}(X_b) \cap \mathscr{D}(A_{-1}) = \{0\}$, because $\mathscr{B}(X_b) \cap X_0 = \{0\}$.

3. The case of operators dependent on *t*

1. Operators with isomorphic domains. One of possible ways of reduction of some problems concerning operators

$$(A(t))_{t\in[0,T]} \text{ with domains } \mathscr{D}_t = \mathscr{D}(A(t)) \subset X$$

depending on t is to find a sufficiently regular (with respect to $t \in [0, T]$) family $(\Psi_t)_{t\in[0,T]}$ of automorphisms of the Banach space X such that $\Psi_t(\mathscr{D}_t) = \mathscr{D}$, where $\mathscr{D}$ is a fixed linear subspace of X, for example $\mathscr{D} = \mathscr{D}_0$.

Suppose now that the described above family $(\Psi_t)_{t\in[0,T]}$ has been constructed and that $u(t)$ is a solution of the problem 1.

Since $u(t) \in \mathscr{D}_t$, there exists $\upsilon(t) \in \mathscr{D}$ such that $\Psi_t(\upsilon(t)) = u(t)$. We have

$$\frac{du}{dt} = \frac{d\Psi_t}{dt}\upsilon(t) + \Psi_t\frac{d\upsilon}{dt}$$

and after a standard calculation we obtain

$$\frac{d\upsilon}{dt} = \underbrace{\left(\Phi_t A(t)\psi_t - \frac{d\psi_t}{dt}\right)}_{B(t)}\upsilon(t) + \Phi_t \underbrace{f(t, \psi_t(\upsilon(t))}_{F(t,\,\upsilon,\,(t))},$$

where Φ_t is the inverse to ψ_t operator. Thus υ is a classical solution of the evolution equation

$$\frac{d\upsilon}{dt} = B(t)\upsilon + F(t, \upsilon(t))$$

with the family $(B_t = B(t))_{t\in\mathcal{T}}$ of operators having domains independent of t.

In general, the domain of a differential operator is determined by some boundary conditions. Thus it would be useful to find an effective construction of a family Ψ_t using the boundary conditions only. Such a construction for a family of elliptic operators of order two with the Robin boundary condition with a parameter (i.e. $\partial u/\partial n + a(x,t)u = 0$ on $\partial\Omega$) is presented in [10]. We present it here for convenience of the reader.

Example 3. *Let Ω be a bounded domain in $\mathbb{R}^n$ with boundary $S = \partial\Omega$ of class $\mathcal{C}^{k+1}, \mathcal{T} = [0,T]$ and let $a : \bar{\Omega} \times \mathcal{T} \to \mathbb{R}$ be a function of class $\mathcal{C}^k$ nonvanishing on S. The sets*

$$\mathscr{D}_t = \left\{ u \in \mathcal{L}^2(\Omega) : u \in H^2(\Omega) \quad \text{and} \quad \frac{\partial u}{\partial n} + a(x,t)u = 0 \quad \text{on} \quad \partial\Omega \right\},$$

$$\mathscr{D} = \left\{ u \in \mathcal{L}^2(\Omega) : u \in H^2(\Omega) \quad \text{and} \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \right\}$$

are dense linear subspaces of $\mathcal{L}^2(\Omega)$, where n is the interior unit normal vector field on S.

Let $\eta : \overline{\Omega} \times \mathcal{T} \to \mathbb{R}$ *be a function of class* $\mathcal{C}^k$ *such that*

$$\frac{1}{2} \le \eta_t(x) = \eta(x,t) \quad \text{for} \quad x \in \overline{\Omega}, t \in \mathcal{T},$$

$$\eta_t(x) = 1 \quad \text{and} \quad \frac{\partial \eta_t(x)}{\partial n} = a(x,t) \quad \text{for} \quad x \in \partial\Omega, t \in \mathcal{T}.$$

The function η *can be constructed in the following way. We consider S as the retract of class* $\mathcal{C}^k$ (*for* $\varepsilon > 0$ *small enough*) *of the open* ε*-tube*

$$\mathrm{TUB}^{\varepsilon}(S) = \{x + \tau n(x) : x \in S, \ |\tau| < \varepsilon\}.$$

Then we take a function h_ε *of class* $\mathcal{C}^\infty$ *in* $\mathbb{R}^n$ *satisfying the following conditions*:

$$h_\varepsilon(x) = 0 \quad \text{for} \quad x \in \mathbb{R}^n \setminus \mathrm{TUB}^{\varepsilon}(S),$$
$$h_\varepsilon(x) = 1 \quad \text{for} \quad x \in \mathrm{TUB}^{\varepsilon/2}(S),$$
$$h_\varepsilon(x) \in G \quad \text{for} \quad x \in \mathbb{R}^n.$$

The function

$$f_\varepsilon : \mathrm{TUB}^{\varepsilon}(S) \ni x + \tau n(x) \to a(t,x)\tau \in \mathbb{R}$$

is of class $\mathcal{C}^k$ *and, for* ε *small enough, the function* $\eta = h_\varepsilon f_\varepsilon + 1$ *is one we have been looking for.*

Let $\Phi_t : \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega)$ *be given by*

$$\Phi_t(u) = \eta_t \cdot u \quad \text{for} \quad u \in \mathcal{L}^2(\Omega), \ t \in [0,T]$$

and let $\Psi_t = \Phi_t^{-1}$. *One can verify that*

(i) $\Phi_t \in \mathrm{Aut}(\mathcal{L}^2(\Omega))$,

(ii) $\Phi_t(\mathcal{D}_t) = \mathcal{D}$ *and* $\Psi_t(\mathcal{D}) = \mathcal{D}_t$,

(iii) *the mapping* $\mathcal{T} \ni t \to \Phi_t \in \mathcal{B}(\mathcal{L}^2(\Omega))$ *is of class* $\mathcal{C}^k$. *Thus the mapping* $\mathcal{T} \ni t \to \Psi_t \in \mathcal{B}(\mathcal{L}^2(\Omega))$ *is also of class* $\mathcal{C}^k$.

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