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NONLOCAL CAUCHY PROBLEM IN SUN-REFLEXIVE SPACE

NIELOKALNY PROBLEM CAUCHY'EGO W PRZESTRZENI SUN-REFLEKSYWNEJ

A b s t r a c t

The aim of this paper is study the existence and uniqueness of mild and classical solution of the Cauchy problem for a linear and a semilinear evolution equation in Banach space with nonlocal condition. These theorems are proved by using the sun-reflexive Banach space relative to an operator $A(0)$.

Keywords: *evolution problem*, *mild solution*, *classical solution*, *nonlocal condition*, *sun--reflexive space*

S t r e s z c z e n i e

Celem niniejszego artykułu jest zbadanie istnienia i jednoznaczności całkowego i klasycznego rozwiązania zagadnienia Cauchy'ego dla liniowego i semiliniowego równania ewolucyjnego w przestrzeni Banacha z nielokalnym warunkiem początkowym. Przedstawione w artykule twierdzenia zostały udowodnione przy założeniu sun-refleksywności przestrzeni Banacha względem operatora $A(0)$.

Słowa kluczowe: *problem ewolucyjny*, *rozwiązanie całkowe*, *rozwiązanie klasyczne*, *nielokalny warunek początkowy*, *przestrzeń sun-refleksywna*

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1. Preliminaries

Let $(X, \|\cdot\|)$ be a Banach space and suppose that $A : X \supset D(A) \to X$ is a linear closed operator, where $D(A)$ domain of A is dense in X. For such an operator A we may define the adjoint operator A^* in the space X^*. However, the domain $D(A^*)$ of A^* need not be dense in X^* and because of this fact we cannot define the second adjoint operator A^{**} on X^{**}. To avoid this situation one may use so called $\odot$ – adjointness or "sun-adjointness". For details and proofs see, e.g. [2], Chap. 14.

Let $\rho(A)$ denotes the resolvent set and $R(\lambda, A) := (A - \lambda \mathbf{I})^{-1}$, $\lambda \in \rho(A)$ be the resolvent of A.

Definition 1.1. We shall call a linear closed operator $A : X \supset D(A) \to X$ a *sun-operator* if

(i) $\overline{D(A)} = X$,

(ii) $\|R(\lambda, A)\| = O(1/\lambda)$ as $\lambda \to \infty$.

For a given sun-operator A, the space $X^{\odot} := \overline{D(A^*)} \subset X^*$ will be called the *sun-adjoint space of X relative to A.*

We denote by $A^{\odot}$ restriction of A^* with domain

$$D(A^{\odot}) := \{\, x^* \in D(A^*) : A^* x^* \in X^{\odot} \}.$$

We collect some facts concerning this "sun-adjointness" in the following proposition:

Proposition 1.2. ([2], Chap. 14).

(i) For arbitrary X and unbounded A, the sun-adjoint space may very well be a proper subspace of X^*.

(ii) If

$$\forall x \in X \quad \| x \|_1 := \sup \{ \| x^{\odot}(x) \| : x^{\odot} \in X^{\odot}, \| x^{\odot} \| \le 1 \}$$

then $\|\cdot\|_1$ is a norm in X which is equivalent to the original norm $\|\cdot\|$ in X; in fact

$$\forall x \in X \ \|x\|_1 \le \|x\| \le M \|x\|$$

where $M := \liminf_{\lambda \to \infty} \|\lambda R(\lambda, A)\|$.

(iii) $A^{\odot}$ is a sun-operator.

(iv) The generator A of a C_0 – semigroup is a sun-operator.

From this proposition it follows that we can repeat the above procedure with A and X replaced by $A^{\odot}$ and $X^{\odot}$. We define

$$X^{\odot\odot} := (X^{\odot})^{\odot}.$$

It is sun – adjoint space of $X^{\odot}$ relative to $A^{\odot}$.

Now we recall the following definition:

Definition 1.3. We assume that X is renormed with the norm $\|\cdot\|_1$.
Then the space X is called *"sun-reflexive" relative to A* if

$$j(X) = X^{\odot\odot},$$

where j: $X \to X^{**}$ is the canonical embedding of X into X^{**}.

Remark 1.4. In particular, X is sun-reflexive whenever it is reflexive in the usual sense. Furthermore, a non-reflexive space may be sun-reflexive relative to certain sun--operators.

In the sequel we shall need the following theorem:

Theorem 1.5. ([3], Th. 1.52). Let $g : \Delta_T = \{(t,s) : 0 \le s \le t \le T\} \to X$ and suppose that:

(i) for almost all $s \in [0,T]$ the function $[s,T] \ni t \to g(t,s)$ is continuous,

(ii) for each $t \in [0,T]$, $g(t,\cdot)$ is summable over $[0,t]$,

(iii) there exists a function $\varphi \in L^1(0,T;[0,\infty))$ such that for $(t,s) \in \Delta_T$, $\| g(t,s) \| \le \varphi(s)$.

Then the function $G : [0,T] \ni t \to \int_0^t g(t,s)ds \in X$ is continuous.

2. The linear case

Let (X, $\|\cdot\|$) be a Banach space. We shall consider the following nonlocal linear problem

$$\begin{cases} u'(t) = A(t)u(t) + g(t), t \in (0,T] \\ u(0) = u_0 + h(u) \in X, \end{cases} \tag{1}$$

where $A(t) : X \supset D \to X$, $t \in [0,T]$ is a linear closed operator with domain $D[A(t)] = D$ dependent on t and dense in X; $g : [0,T] \to X$, h: $C([0,T], X) \to X$ are given functions and u: $[0,T] \to X$ is an unknown function.

We shall give examples of function h. For details see [1], Chap. I.5.

Examples 2.1.

(i) $h(u) := \sum_{j=1}^{k} a_j u(t_j)$, where $u \in X$, $a_j \in \mathbf{R}$, $j = 1,...,k$, $0 \le t_1 \le \ldots \le t_k \le T$, $k \in \mathbf{N}$.

(ii) $h(u) := \sum_{j=1}^{k} \frac{a_j}{\varepsilon_j} \int_{t_j - \varepsilon_j}^{t_j} u(s)ds$, where $u \in X$, $a_j, \varepsilon_j \in \mathbf{R}$, $j = 1,...,k$, $0 \le t_1 \le \ldots \le t_k \le T$, $t_{j-1} < t_j - \varepsilon_j$, $k \in \mathbf{N}$.

We make the following assumptions:

(A_1) For each $t \in [0,T]$, $A(t)$ is the generator of a C_0 – semigroup.

(A_2) The family $\{A(t)\}$, $t \in [0,T]$ is stable in the sense that there exist real numbers $M \geq 1$ and ω such that

$$\left\| \prod_{j=1}^{k} R(\lambda, A(t_j)) \right\| \leq M(\lambda - \omega)^{-k}$$

for all $\lambda > \omega$, $0 \leq t_1 \leq \ldots \leq t_k \leq T$, $k \in \mathbf{N}$.

(A_3) For each $x \in X$ the mapping $t \to A(t)x$ is of class $C^1([0,T], X)$.

(A_4) The resolvent set $\rho(A(t))$ does not depend on t and 0 belongs to $\rho(A(t))$.

(A_5) $\exists L > 0\ \forall t, \tau \in [0,T]\ \| g(t) - g(\tau) \| \leq L\, |t - \tau|$.

Remark 2.2. From assumption (A_1) and Proposition 1.2 (iv), it follows that $A(0)$ is the sun-operator.

Definition 2.3. A twoparameter family of bounded operators $\{U(t, s)\}$, $0 \leq s \leq t \leq T$ on X is called *an evolution system of problem*

$$\begin{cases} u'(t) = A(t)u(t), t \in (0,T] \\ u(0) = u_0 \in X \end{cases} \tag{2}$$

if the following two conditions are satisfied

(i) $U(s, s) = \mathbf{I}$, $s \in [0,T]$,

$$U(t,r)U(r,s) = U(t,s), \quad 0 \leq s \leq r \leq t \leq T,$$

(ii) $(t,s) \to U(t,s)$ is strongly continuous for $0 \leq s \leq t \leq T$.

It is known that following theorem is true:

Theorem 2.4 ([4], Th. 5.4.8). Let assumptions (A_1), (A_2), (A_3), (A_4) hold. Then there exists a unique evolution system of (2) $\{U(t, s)\}$, $0 \leq s \leq t \leq T$ satisfying

(i) $\|U(t,s)\|_{X \to X} \leq M \exp\{\omega(t-s)\}$, $0 \leq s \leq t \leq T$,

(ii) $\dfrac{\partial}{\partial t} U(t,s)x = A(t)U(t,s)x, \quad x \in D, \quad 0 \leq s \leq t \leq T,$

(iii) $\dfrac{\partial}{\partial s} U(t,s)x = -U(t,s)A(s)x, \quad x \in D, \quad 0 \leq s \leq t \leq T,$

(iv) $U(t,s)D \subset D$, $0 \leq s \leq t \leq T$,

(v) for $x \in D$ the mapping $(t,s) \to U(t,s)x$ is continuous in $Y := (D, \|\cdot\|_{D[A(0)]})$ for $0 \leq s \leq t \leq T$, where

$$\|x\|_{D[A(0)]} := \|x\| + \|A(0)x\|, \quad x \in D.$$

Now, we recall the following definition:

Definition 2.5. A function u: $[0,T] \to X$ is *a classical solution of the nonlocal linear problem* (1) if u is continuous on $[0,T]$, continuously differentiable on $(0,T]$, $u(t) \in D$ for $t \in [0,T]$ and (1) is satisfied.

Let u be a classical solution of (1) and assumptions (A_1), (A_2), (A_3), (A_4), (A_5) hold. Then the function

$$s \to U(t,s)u(s), \quad s \in (0,t), \quad t \in [0,T]$$

is differentiable and

$$\frac{\partial}{\partial s}[U(t,s)u(s)] = -U(t,s)A(s)u(s) + U(t,s)u'(s) = \\ = -U(t,s)A(s)u(s) + U(t,s)[A(s)u(s) + g(s)] = U(t,s)g(s). \tag{3}$$

Furthermore, from (A_5) it follows that $g \in L^1([0,T],X)$. Since $\{U(t, s)\}$, $0 \le s \le t \le T$ is family of bounded operators strongly continuous, it is uniformly bounded ([6], Th. 1.31). Thus

$$\exists C > 0 \ \|U(t,s)\|_{X \to X} \le C.$$

Consequently, the function

$$s \to U(t,s)g(s), \quad s \in (0,t), \quad t \in [0,T]$$

is integrable and integrating (3) from 0 to t yields

$$U(t,t)u(t) - U(t,0)u(0) = \int_0^t U(t,s)g(s)ds.$$

This implies

$$u(t) = U(t,0)u_0 + U(t,0)h(u) + \int_0^t U(t,s)g(s)ds. \tag{4}$$

Suppose that the function h: $C([0,T],X) \to X$ satisfies the following assumption: (A_6) there exists $K > 0$ such that $0 < CK < 1$ and

$$\forall u, w \in C([0,T],X) \ \ \|h(u) - h(w)\| \le K \|u - w\|_{C([0,T],X)}.$$

Applying Theorem 1.5 we obtain that the right-hand side of (4) is a continuous function on $[0,T]$. It is natural consider it as a generalized solution of (1) even if it is not differentiable and does not strictly satisfy the equation in the sense Definition 2.5. Motivated by this remark we make the following definition:

Definition 2.6. A function $u \in C([0,T],X)$ such that

$$u(t) = U(t,0)u_0 + U(t,0)h(u) + \int_0^t U(t,s)g(s)ds \tag{5}$$

is called *the "mild solution" of nonlocal linear problem* (1).

Theorem 2.7. Under assumptions (A_1), (A_2), (A_3), (A_4), (A_5), (A_6) the problem (1) has exactly one "mild solution".

Proof. Let

$$(Gu)(t) := U(t,0)u_0 + U(t,0)h(u) + \int_0^t U(t,s)g(s)ds, \quad u \in C([0,T],X)\,, \; t \in [0,T].$$

The operator G is a mapping from $C([0,T],X)$ into itself.
From assumptions it follows

$$\|(Gu)(t) - (Gw)(t)\| = \|U(t,0)[h(u) - h(w)]\| \le CK\|u - w\|_{C([0,T],X)}.$$

Consequently, G is a contraction. By Banach's contraction principle, this implies that (1) has exactly one "mild solution". □

Using the same method as in [5] (Th. 2.1) we prove the main theorem in this section.

Theorem 2.8. Let (X, || · ||) be a sun-reflexive Banach space relative to $A(0)$. Assume (A_1), (A_2), (A_3), (A_4), (A_5), (A_6) and let $u_0 \in D$ and $h(u) \in D$. Then the nonlocal linear problem (1) has exactly one classical solution u such that

$$u(t) = U(t,0)u_0 + U(t,0)h(u) + \int_0^t U(t,s)g(s)ds, \qquad t \in [0,T]$$

3. The semilinear case

Let (X, || · ||) be a Banach space. In this section we consider the nonlocal semilinear problem

$$\begin{cases} u'(t) = A(t)u(t) + f(t,u(t)), t \in (0,T] \\ u(0) = u_0 + h(u) \in X, \end{cases} \tag{6}$$

where $A(t) : X \supset D \to X$, $t \in [0,T]$ is a linear closed operator with domain $D[A(t)] = D$ dependent on t and dense in X; $f : [0,T] \times X \to X, h : C([0,T],X) \to X$ are given functions and u: $[0,T] \to X$ is an unknown function.

The classical solution of the problem (6) is defined analogously to the classical solution of problem (1).

Let u be a classical solution of (6), assumptions (A_1), (A_2), (A_3), (A_4), (A_5) hold and $f : [0,T] \times X \to X$ be continuous. Then as in Section 2

$$u(t) = U(t,0)u_0 + U(t,0)h(u) + \int_0^t U(t,s)f(s,u(s))ds \tag{7}$$

and the right-hand side of (7) is a continuous function on $[0,T]$. Therefore, we define "mild solution" of (6) analogously to the problem (1).

Theorem 3.1. Let assumptions (A_1), (A_2), (A_3), (A_4), (A_6) hold.

If $f:[0,T]\times X\to X$ is continuous, there exists $\overline{L}>0$ such that $0<C(K+\overline{L}T)<1$ and

$$\forall t\in[0,T]\quad \forall x,y\in X\quad \left\|f(t,x)-f(t,y)\right\|\le \overline{L}\left\|u-w\right\|,$$

then the problem (6) has exactly one "mild solution".

Proof. By analogy to the proof of Theorem 2.7 we define the mapping G: $C([0,T],X)\to C([0,T],X)$

$$(Gu)(t):=U(t,0)u_0+U(t,0)h(u)+\int_0^t U(t,s)f(s,u(s))ds,\qquad u\in C([0,T],X),\qquad t\in[0,T].$$

Since

$$\left\|(Gu)(t)-(Gw)(t)\right\|\le C(K+\overline{L}T)\left\|u-w\right\|_{C([0,T],X)},$$

where C and K as in the proof of Theorem 2.7, G is a contraction. By Banach's contraction principle, this implies that (6) has exactly one "mild solution". □

Applying the same method as in the proof of [5] (Th. 2.2) we prove the following theorem:

Theorem 3.3. Let $(X, \|\cdot\|)$ be a sun-reflexive Banach space relative to $A(0)$.

Assume (A_1), (A_2), (A_3), (A_4), (A_6). If $u_0\in D$, $h(u)\in D$ and $f:[0,T]\times X\to X$ satisfy the Lipschitz condition with constant $\overline{L}>0$ then the nonlocal semilinear problem (6) has the unique classical solution u such that

$$u(t)=U(t,0)u_0+U(t,0)h(u)+\int_0^t U(t,s)f(s,u(s))ds,\qquad t\in[0,T].$$

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