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FUNCTIONAL DIFFERENTIAL EQUATIONS OF SECOND ORDER

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A b s t r a c t

In this article, we present some results on the existence and uniqueness of the solutions of boundary value problem for functional differential equations of second order.

Keywords: *functional differential equations, lower and upper solution, existence and uniqueness, monotone iterative method*

S t r e s z c z e n i e

Artykuł zawiera wyniki dotyczące istnienia i jednoznaczności rozwiązań zagadnienia brzegowego dla równań różniczkowo-funkcjonalnych drugiego rzędu.

Słowa kluczowe: *równania różniczkowo-funkcjonalne, dolne i górne rozwiązanie, istnienie i jednoznaczność, metoda monotonicznej iteracji*

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1. Introduction

In this paper we discuss the boundary value problem for functional differential equations of second order

$$-x''(t) = f(t, x(t), x(t-\tau_1), x_t), \quad t \in J = [0,1], \tag{1}$$

$$x_0 = \phi, \quad x(1) = B, \tag{2}$$

where: $f : J \times \mathbb{R} \times \mathbb{R} \times C([-\tau, 0], \mathbb{R}) \to \mathbb{R}$, $\tau \geq \tau_1 > 0$ are given numbers, $x_t \in C([-\tau, 0], \mathbb{R})$ is defined by $x_t(s) = x(t+s)$, $s \in [-\tau, 0]$, $\phi \in C([-\tau, 0], \mathbb{R})$. Condition $x_0 = \phi$ implies that $x(s) = \phi(s)$, $s \in [-\tau, 0]$ and $B \in \mathbb{R}$.

Functional differential equations constitute a very interesting and important class of equations and have been studied by a number of authors. We refer the reader to the papers [1–6] and the references therein. This paper is inspired by [3].

2. Preliminaries

In this section, we introduce notations, definitions and preliminary facts which are used throughout this paper.

$C([-\tau, 0], \mathbb{R})$ is the Banach space of all continuous functions from $[-\tau, 0]$ into $\mathbb{R}$ with the norm

$$\| x \|_0 = \sup\{| x(t) | : t \in [-\tau, 0]\}.$$

$C(J, \mathbb{R})$ is the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\| x \| = \sup\{| x(t) | : t \in J\}, \quad J = [0,1].$$

Definition 2.1. *A function x is a solution for equations (1), (2) if* $x \in C([-\tau, 1], \mathbb{R}) \cap C^2(J, \mathbb{R})$ *and x satisfies equations (1), (2).*

We need the following auxiliary result:

Lemma 2.1. *If* $f \in C(J \times \mathbb{R} \times \mathbb{R} \times C([-\tau, 0], \mathbb{R}), \mathbb{R})$, *then* $x \in C([-\tau, 1], \mathbb{R}) \cap C^2(J, \mathbb{R})$ *is a solution of (1), (2) with* $x_0 = \phi$ *if and only if* $x \in C([-\tau, 1], \mathbb{R})$ *is a solution of the following integral equation*

$$\begin{cases} x(t) = \int_0^1 G(t,s) f(s, x(s), x(s-\tau_1), x_s) ds + (B - \phi(0))t + \phi(0), & t \in J, \\ x_0 = \phi, \end{cases} \tag{3}$$

where

$$G(t,s) = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1, \\ t(1-s), & 0 \leq t \leq s \leq 1 \end{cases}$$

is the Green's function of the problem.

3. Existence results

Theorem 3.1. *Assume*

(H1) $f \in C(J\times\mathbb{R}\times\mathbb{R}\times C([-\tau,0],\mathbb{R}),\mathbb{R})$ *and there exist* $M>0,\ b>0,\ c>0,\ d>0$ *such that for any* $(t,x,v,w)\in J\times\mathbb{R}\times\mathbb{R}\times C([-\tau,0],\mathbb{R})$

$$|f(t,x,v,w)| \le M + b|x| + c|v| + d\|w\|_0,$$

(H2) *there exists* $R>0$ *such that*

$$\frac{1}{8}(M+bR+(c+d)\max\{\|\phi\|_0, R\}) + |B-\phi(0)| + |\phi(0)| < R.$$

Then the equation (3) has at least a solution in $C(J,\mathbb{R})$ *with* $x_0=\phi$.

Proof. The proof will be given in several steps. First, we shall show, that the operator $A: C(J,\mathbb{R}) \to C(J,\mathbb{R})$ defined by

$$(Ax)(t) = \int_0^1 G(t,s) f(s, x(s), x(s-\tau_1), x_s)ds + (B-\phi(0))t + \phi(0), \tag{4}$$

where $t\in J$, $x_s(r) = x(s+r) = \phi(s+r)$ for $r\in[-\tau,0]$ and $s+r\le 0$, is completely continuous.

Step 1. *A* is continuous.

Now let $\{x_n\}$ be a sequence such that $\{x_n\}\subseteq C(J,\mathbb{R})$, $x_n\to x$ in $C(J,\mathbb{R})$. Then

$$|(Ax_n)(t) - (Ax)(t)| \le \int_0^1 |G(t,s)|\,|f(s,x_n(s),x_n(s-\tau_1),x_{n,s}) - f(s,x(s),x(s-\tau_1),x_s)|\,ds.$$

Since the function *f* is continuous and by virtue of the Lebesgue dominated convergence theorem, we have

$$\|Ax_n - Ax\| \le \sup_{(t,s)\in J\times J} |G(t,s)| \int_0^1 |f(s,x_n(s),x_n(s-\tau_1),x_{n,s}) - f(s,x(s),x(s-\tau_1),x_s)|\,ds \to 0$$

as $n\to\infty$.

Step 2. *A* maps bounded sets into bounded sets in $C(J,\mathbb{R})$.

Let $q>0$ and let

$$D = \{x\in C(J,\mathbb{R}): \|x\|\le q\}$$

be a bounded set. We show that there exists a positive constant l such that for each $x\in D$, we have $\|Ax\|\le l$. For each $x\in D$, $t\in J$ using H1 we have

$$\begin{aligned}|(Ax)(t)| &\le \int_0^1 |G(t,s)|\,|f(s,x(s),x(s-\tau_1),x_s)|\,ds + |B-\phi(0)|\,|t| + |\phi(0)|\\ &\le \int_0^1 |G(t,s)|(M + b|x(s)| + c|x(s-\tau_1)| + d\|x_s\|_0)\,ds + |B-\phi(0)|\,|t| + |\phi(0)|.\end{aligned}$$

This, together with

$$|x(s)| \le \|x\|,$$

$$|x(s-\tau_1)| \le \max\{\|\phi\|_0, \|x\|\},$$

$$\|x_s\|_0 \le \max\{\|\phi\|_0, \|x\|\}$$

implies

$$|(Ax)(t)| \le (M + bq + (c+d)\max\{\|\phi\|_0, q\}) \sup_{(t,s)\in J\times J} |G(t,s)| + 2|\phi(0)| + |B| := l.$$

Therefore

$$\|Ax\| \le l.$$

Step 3. A maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.

Let r_1, $r_2 \in J$, $r_1 < r_2$, $D = \{x \in C(J,\mathbb{R}) : \|x\| \le q\}$ is a bounded set of $C(J,\mathbb{R})$ and $x \in D$. Using H1 we obtain

$$\begin{aligned} |(Ax)(r_2) - (Ax)(r_1)| &\le \int_0^1 |G(r_2,s) - G(r_1,s)|\,|f(s, x(s), x(s-\tau_1), x_s)|\,ds + |B - \phi(0)|\,|r_2 - r_1| \\ &\le \int_0^1 |G(r_2,s) - G(r_1,s)|(M + bq + (c+d)\max\{\|\phi\|_0, q\})ds \\ &\quad + |B - \phi(0)|\,|r_2 - r_1| \to 0 \quad \text{as} \quad r_2 - r_1 \to 0. \end{aligned}$$

As a consequence of Steps 1 to 3 together with the Arzela–Ascoli theorem we can conclude that A is completely continuous operator.

Step 4. Let $C_R = \{x \in C(J,\mathbb{R}) : \|x\| \le R\}$, where R is the constant from H2. Then C_R is convex, closed and bounded. We show that if $x \in C_R$ then $Ax \in C_R$.

For each $x \in C_R$, $t \in J$ using H1, the inequality

$$\int_0^1 |G(t,s)|ds \le \frac{1}{8}, \quad t \in [0,1]$$

and the Assumption H2 we have

$$\begin{aligned} |(Ax)(t)| &\le \int_0^1 |G(t,s)|\,|f(s, x(s), x(s-\tau_1), x_s)|\,ds + |B - \phi(0)|\,|t| + |\phi(0)| \\ &\le \int_0^1 |G(t,s)|(M + b|x(s)| + c|x(s-\tau_1)| + d\|x_s\|_0)\,ds + |B - \phi(0)|\,|t| + |\phi(0)| \\ &\le (M + bR + (c+d)\max\{\|\phi\|_0, R\}) \sup_{t\in[0,1]} \int_0^1 |G(t,s)|\,ds + |B - \phi(0)| + |\phi(0)| \\ &\le \frac{1}{8}(M + bR + (c+d)\max\{\|\phi\|_0, R\}) + |B - \phi(0)| + |\phi(0)| < R. \end{aligned}$$

Hence

$$\|Ax\| \le R.$$

Then, the Tychonoff fixed point theorem implies that A has at least one fixed point. The proof is complete.

Example. Consider the following problem

$$-x''(t) = 2 + \frac{\sin t}{2}\sin(x(t)) - \frac{\cos^2 t}{3}\cdot x(t-\frac{1}{2}) + \int_{-1}^{0} x_t(s)ds,\ t \in [0,1], \tag{5}$$

$$x(s) = 0, \quad s \in [-1,0], \quad x(1) = 1. \tag{6}$$

The right-hand side of (5) is of the form

$$f(t,x,v,w) = 2 + \frac{1}{2}\sin t \cdot \sin x - \frac{1}{3}\cos^2 t \cdot v + \int_{-1}^{0} w(s)ds,$$

where $t \in [0,1],\ x,\ v \in \mathbb{R},\ w \in C([-1,0],\mathbb{R}).$

We have

$$|f(t,x,v,w)| \le 2 + \frac{1}{2}|\sin t|\cdot|\sin x| + \frac{1}{3}\cos^2 t\cdot|v| + \int_{-1}^{0}|w(s)|ds \le 2 + \frac{1}{2}|x| + \frac{1}{3}|v| + \|w\|_0,$$

which shows that the Assumption H1 is satisfied $(M = 2,\ b = \frac{1}{2},\ c = \frac{1}{3},\ d = 1).$ Now we check the Assumption H2. The inequality from H2 is of the from

$$\frac{1}{8}\left(2 + \frac{R}{2} + \left(\frac{1}{3}+1\right)\max\{0,R\}\right) + 1 < R.$$

Condition $R > 0$ implies

$$\left(2 + \frac{11}{6}R\right)\frac{1}{8} + 1 < R.$$

Solving this inequality we get $R > \frac{60}{37}$. There exists $R > 0$ such that Assumption H2 is satisfied. By Theorem 3.1 the problem (5)–(6) has at least one solution.

Theorem 3.2. *Assume* $f \in C(J\times\mathbb{R}\times\mathbb{R}\times C([-\tau,0],\mathbb{R}),\mathbb{R})$ *and there exist* $a,b,c \in C(J,\mathbb{R}^+)$ *such that* $N = \max_{t\in J}(a(t)+b(t)+c(t)) < 8$ *and for any* $x,y,w,z \in \mathbb{R}$, $\varphi,\psi \in C([-\tau,0],\mathbb{R})$

$$|f(t,x,w,\varphi) - f(t,y,z,\psi)| \le a(t)|x-y| + b(t)|w-z| + c(t)\|\varphi-\psi\|_0.$$

Then the problem (1), (2) has a unique solution $x \in C(J,\mathbb{R})$ *with* $x_0 = \phi$.

Proof. Transform the problem (1), (2) into a fixed point problem. Consider the operator A defined by (4). For $x, y \in C(J, \mathbb{R})$ we have

$$|(Ax)(t)-(Ay)(t)| \le \int_0^1 |G(t,s)||f(s,x(s),x(s-\tau_1),x_s)-f(s,y(s),y(s-\tau_1),y_s)|ds$$

$$\le \int_0^1 |G(t,s)|(a(s)|x(s)-y(s)|+b(s)|x(s-\tau_1)-y(s-\tau_1)|$$

$$+c(s)\|x_s-y_s\|_0)ds.$$

Since

$$|x(s)-y(s)| \le \max_{r\in[0,s]} |x(r)-y(r)|,$$

$$|x(s-\tau_1)-y(s-\tau_1)| \le \max_{r\in[0,s]} |x(r)-y(r)|,$$

$$\|x_s-y_s\|_0 = \sup\{|x_s(t)-y_s(t)|, t\in[-\tau,0]\} = \sup\{|x(s+t)-y(s+t)|, t\in[-\tau,0]\}$$
$$\le \max_{r\in[0,s]} |x(r)-y(r)|,$$

we have

$$|(Ax)(t)-(Ay)(t)| \le N\int_0^1 |G(t,s)| \max_{r\in[0,s]} |x(r)-y(r)|ds \le \frac{N}{8}\|x-y\|.$$

Then

$$\|Ax-Ay\| \le \frac{N}{8}\|x-y\|.$$

Consequently, A has a unique fixed point $x\in C(J,\mathbb{R})$.

Corollary 3.1. *Assume that* $f\in C(J\times\mathbb{R}\times\mathbb{R}\times C([-\tau,0],\mathbb{R}),\mathbb{R})$ *and there exist constants* $M,N,K\ge 0$, *such that* $M+N+\tau K<8$ *and for any* $x,y,w,z\in\mathbb{R}$, $\varphi,\psi\in C([-\tau,0],\mathbb{R})$

$$|f(t,x,w,\varphi)-f(t,y,z,\psi)| \le M|x-y|+N|w-z|+K\int_{-\tau}^0 |\varphi(s)-\psi(s)|ds.$$

Then problem (1)–(2) has a unique solution $x\in C(J,\mathbb{R})$ *with* $x_0=\phi$.

4. Monotone method

In this section we will assume that f satisfy the one-sided Lipschitz condition.

Definition 4.1. *We say that a function* $v\in C([-\tau,1],\mathbb{R})\cap C^2(J,\mathbb{R})$ *is a lower solution of (1)–(2) if*

$$-v''(t) \le f(t,v(t),v(t-\tau_1),v_t), \quad t\in J,$$

$$v_0\le\phi, \quad v(1)\le B.$$

Analogously, we say that $w \in C([-\tau,1],\mathbb{R}) \cap C^2(J,\mathbb{R})$ *is an upper solution of (1), (2) if*

$$-w''(t) \geq f(t, w(t), w(t-\tau_1), w_t), \quad t \in J,$$

$$w_0 \geq \phi, \quad w(1) \geq B.$$

We introduce the following assumptions

(B1) $f \in C(J \times \mathbb{R} \times \mathbb{R} \times C([-\tau,0],\mathbb{R}),\mathbb{R})$.

(B2) $v_0, w_0 \in C([-\tau,1],\mathbb{R}) \cap C^2(J,\mathbb{R})$ are respectively lower and upper solution of (1)–(2) and $v_0(t) \leq w_0(t)$, $t \in J$.

(B3) There exist $M, N, K \geq 0$ such that $M + N + K\tau \in [0,8)$ and

$$f(t, x(t), x(t-\tau_1), x_t) - f(t, y(t), y(t-\tau_1), y_t) \geq M(x(t) - y(t)) + N(x(t-\tau_1) - y(t-\tau_1)) + K\int_{-\tau}^{0} (x_t(s) - y_t(s))ds$$

for $t \in J, v_0(t) \leq y(t) \leq x(t) \leq w_0(t), v_0(t-\tau_1) \leq y(t-\tau_1) \leq x(t-\tau_1) \leq w_0(t-\tau_1)$ and $v_{0,t} \leq y_t \leq x_t \leq w_{0,t}$ on $[-\tau,0]$.

For $v, w \in C([-\tau,1],\mathbb{R}) \cap C^2(J,\mathbb{R})$ such that $v(t) \leq w(t)$, $t \in [-\tau,1]$ we define the set

$$[v,w] = \{u \in C([-\tau,1],\mathbb{R}) \cap C^2(J,\mathbb{R}) : v(t) \leq u(t) \leq w(t), t \in [-\tau,1]\}.$$

We are interested in solutions of some linear problems.

Lemma 4.1. *Let the assumptions B1–B3 hold. Let* v, w *be lower and upper solutions of (1), (2), respectively,* $v(t) \leq w(t)$, $t \in J$ *and* $[v,w] \subset [v_0, w_0]$. *Then the problems*

$$\begin{cases} -x''(t) = F_1(t, x(t), x(t-\tau_1), x_t), & t \in J, \\ x_0 = \phi, \quad x(1) = B, \end{cases} \tag{7}$$

where

$$F_1(t, x(t), x(t-\tau_1), x_t) = f(t, v(t), v(t-\tau_1), v_t) + M(x(t) - v(t)) + N(x(t-\tau_1) - v(t-\tau_1)) + K\int_{-\tau}^{0} (x_t(s) - v_t(s))ds$$

and

$$\begin{cases} -y''(t) = F_2(t, y(t), y(t-\tau_1), y_t), & t \in J, \\ y_0 = \phi, \quad y(1) = B, \end{cases} \tag{8}$$

where

$$F_2(t, y(t), y(t-\tau_1), y_t) = f(t, w(t), w(t-\tau_1), w_t) + M(y(t) - w(t)) + N(y(t-\tau_1) - w(t-\tau_1)) + K\int_{-\tau}^{0} (y_t(s) - w_t(s))ds$$

have their unique solutions x, y such that $v(t) \le x(t) \le y(t) \le w(t),\ t \in J$. *The solutions x, y are respectively lower and upper solutions to problem (1), (2).*

Proof. Theorem 3.1 guarantees the existence of a unique solution $x \in C([-\tau,1],\mathbb{R}) \cap C^2(J,\mathbb{R})$ of (7) and the unique solution $y \in C([-\tau,1],\mathbb{R}) \cap C^2(J,\mathbb{R})$ of (8). We will prove that

$$v(t) \le x(t) \le y(t) \le w(t),\ t \in J.$$

Note that problems (7) and (8) are equivalent to the following integral equations

$$\begin{cases} x(t) = \int_0^1 G(t,s)F_1(t, x(s), x(s-\tau_1), x_s)ds + (B - \phi(0))t + \phi(0), \quad t \in J, \\ x_0 = \phi, \quad x(1) = B \end{cases}$$

and

$$\begin{cases} y(t) = \int_0^1 G(t,s)F_2(t, y(s), y(s-\tau_1), y_s)ds + (B - \phi(0))t + \phi(0), \quad t \in J, \\ y_0 = \phi, \quad y(1) = B. \end{cases}$$

Consider operators $A_1, A_2 : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ defined by

$$(A_1x)(t) = \int_0^1 G(t,s)F_1(t, x(s), x(s-\tau_1), x_s)ds + (B - \phi(0))t + \phi(0),$$

$$(A_2y)(t) = \int_0^1 G(t,s)F_2(t, y(s), y(s-\tau_1), y_s)ds + (B - \phi(0))t + \phi(0),$$

for $t \in J$, where $x_s(r) = x(s+r) = \phi(s+r)$, $y_s(r) = y(s+r) = \phi(s+r)$ for $s + r \le 0$.

Notice that operators A_1, A_2 are monotone nonincreasing i.e. for $\eta_1, \eta_2 \in C(J,\mathbb{R})$, $\eta_1 \le \eta_2$ on *J* implies

$$A_1\eta_1 \le A_1\eta_2 \quad \text{on} \quad J \tag{9}$$

and

$$A_2\eta_1 \le A_2\eta_2 \quad \text{on} \quad J. \tag{10}$$

A solution *x* of (7) is a fixed point of A_1 and $x = \lim_{n\to\infty} x_n$, where x_1 is arbitrary function belonging to $C(J,\mathbb{R})$, $x_{n+1} = A_1x_n$, $n \ge 1$. Analogously, a solution *y* of (8) is a fixed point of A_2 and $y = \lim_{n\to\infty} y_n$, where y_1 is an arbitrary function belonging to $C(J,\mathbb{R})$, $y_{n+1} = A_2y_n$, $n \ge 1$.

Consider sequences

$$x_1 = v,$$

$$x_{n+1} = A_1x_n, \quad n \ge 1$$

and

$$y_1 = w,$$

$$y_{n+1} = A_2 y_n, \quad n \geq 1.$$

Obviously $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y.$

Notice that

$$F_1(t, v(t), v(t-\tau_1), v_t) = f(t, v(t), v(t-\tau_1), v_t) \geq -v''(t), \quad t \in J.$$

Integration by parts gives (see [3])

$$\begin{aligned} x_2(t) = (A_1 v)(t) &= \int_0^1 G(t,s) F_1(s, v(s), v(s-\tau_1), v_s) ds + (B - \phi(0))t + \phi(0) \\ &\geq -\int_0^1 G(t,s) v''(s) ds + (B - \phi(0))t + \phi(0) = \int_0^t s(t-1) v''(s) ds \\ &+ \int_t^1 t(s-1) v''(s) ds + (B - \phi(0))t + \phi(0) = v(t) + (1-t)[\phi(0) - v(0)] + t[B - v(1)] \\ &\geq v(t) = x_1(t), \quad t \in J. \end{aligned}$$

Assume that for some $n \in \mathbb{N}$

$$x_n(t) \leq x_{n+1}(t), \quad t \in J.$$

By (9) we have

$$x_{n+1}(t) = (A_1 x_n)(t) \leq (A_1 x_{n+1})(t) = x_{n+2}(t), \quad t \in J.$$

By induction, it proves that $\{x_n\}_{n\in\mathbb{N}}$ is a nondecreasing sequence i.e.

$$v(t) = x_1(t) \leq x_2(t) \leq \ldots \leq x_n(t) \leq \ldots, \quad t \in J.$$

Analogously notice that

$$F_2(t, w(t), w(t-\tau_1), w_t) = f(t, w(t), w(t-\tau_1), w_t) \leq -w''(t), \quad t \in J.$$

Integration by parts gives (see [3])

$$\begin{aligned} y_2(t) = (A_2 w)(t) &= \int_0^1 G(t,s) F_2(s, w(s), w(s-\tau_1), w_s) ds + (B - \phi(0))t + \phi(0) \\ &\leq -\int_0^1 G(t,s) w''(s) ds + (B - \phi(0))t + \phi(0) \leq \int_0^t s(t-1) w''(s) ds \\ &+ \int_t^1 t(s-1) w''(s) ds + (B - \phi(0))t + \phi(0) = w(t) + (1-t)[\phi(0) - w(0)] + t[B - w(1)] \\ &\leq w(t) = y_1(t), \quad t \in J. \end{aligned}$$

Assume that

$$y_{n+1}(t) \leq y_n(t), \quad t \in J.$$

By (10) we have

$$y_{n+2}(t) = (A_2 y_{n+1})(t) \le (A_2 y_n)(t) = y_{n+1}(t), \quad t \in J.$$

By induction, it proves that $\{y_n\}_{n\in\mathbb{N}}$ is a nonincreasing sequence i.e.

$$\ldots \le y_n(t) \le \ldots \le y_2(t) \le y_1(t) = w(t), \quad t \in J.$$

Now we are going to prove that

$$x_n(t) \le y_n(t), \quad t \in J, \quad n \in \mathbb{N}. \tag{11}$$

Note that

$$v(t) = x_1(t) \le y_1(t) = w(t), \quad t \in J.$$

Assume that for some $n \in \mathbb{N}$

$$x_n(t) \le y_n(t), \quad t \in J. \tag{12}$$

We will show that

$$x_{n+1}(t) \le y_{n+1}(t), \quad t \in J.$$

Note that

$$\begin{aligned} F_1(t, x_n, x_n(t-\tau_1), x_{n,t}) = {} & f(t, v(t), v(t-\tau_1), v_t) + M(x_n(t) - v(t)) \\ & + N(x_n(t-\tau_1) - v(t-\tau_1)) + K\int_{-\tau}^{0} (x_{n,t}(s) - v_t(s))ds \\ & + f(t, w(t), w(t-\tau_1), w_t) - f(t, w(t), w(t-\tau_1), w_t), \quad t \in J. \end{aligned} \tag{13}$$

Using B3 we obtain

$$\begin{aligned} f(t, v(t), v(t-\tau_1), v_t) - f(t, w(t), w(t-\tau_1), w_t) \le {} & -M(w(t) - v(t)) \\ & - N(w(t-\tau_1) - v(t-\tau_1)) - K\int_{-\tau}^{0} (w_t(s) - v_t(s))ds \end{aligned} \tag{14}$$

for $t \in J$.

From (13), (14) and (12) we derive

$$\begin{aligned} F_1(t, x_n(t), x_n(t-\tau_1), x_{n,t}) = {} & f(t, w(t), w(t-\tau_1), w_t) + M(x_n(t) - w(t)) \\ & + N(x_n(t-\tau_1) - w(t-\tau_1)) + K\int_{-\tau}^{0} (x_{n,t}(s) - w_t(s))\,ds \\ & \le f(t, w(t), w(t-\tau_1), w_t) + M(y_n(t) - w(t)) \\ & + N(y_n(t-\tau_1) - w(t-\tau_1)) + K\int_{-\tau}^{0} (y_{n,t}(s) - w_t(s))\,ds \\ & = F_2(t, y_n(t), y_n(t-\tau_1), y_{n,t}), \quad t \in J. \end{aligned}$$

Hence

$$x_{n+1}(t)=\int_0^1 G(t,s)F_1(s,x_n(s),x_n(s-\tau_1),v_{n,s})ds+(B-\phi(0))t+\phi(0)$$
$$\le\int_0^1 G(t,s)F_2(s,y_n(s),y_n(s-\tau_1),w_{n,s})ds+(B-\phi(0))t+\phi(0)$$
$$=y_{n+1}(t),\quad t\in J.$$

By induction, it proves (11). We have proved that sequences $\{x_n\}$, $\{y_n\}$ satisfy the property

$$v(t)=x_1(t)\le x_2(t)\le\ldots\le x_n(t)\le y_n(t)\le\ldots\le y_2(t)\le y_1(t)=w(t),\quad t\in J.$$

This implies that

$$v(t)\le x(t)\le y(t)\le w(t),\quad t\in J,$$

where $\lim_{n\to\infty} x_n=x$ and $\lim_{n\to\infty} y_n=y$, x, y are the unique solutions of (7), (8), respectively. Since

$$-x''(t)=f(t,v(t),v(t-\tau_1),v_t)+M(x(t)-v(t))+N(x(t-\tau_1)-v(t-\tau_1))$$
$$+K\int_{-\tau}^0(x_t(s)-v_t(s))ds+f(t,x(t),x(t-\tau_1),x_t)-f(t,x(t),x(t-\tau_1),x_t),\quad t\in J$$

and by B3 we obtain

$$f(t,v(t),v(t-\tau_1),v_t)-f(t,x(t),x(t-\tau_1),x_t)\le -M(x(t)-v(t))$$
$$-N(x(t-\tau_1)-v(t-\tau_1))-K\int_{-\tau}^0(x_t(s)-v_t(s))\,ds$$

thus

$$-x''(t)\le f(t,x(t),x(t-\tau_1),x_t),\quad t\in J,$$

which means that x is a lower solution to problem (1), (2). Analogously

$$-y''(t)=f(t,w(t),w(t-\tau_1),w_t)+M(y(t)-w(t))+N(y(t-\tau_1)-w(t-\tau_1))$$
$$+K\int_{-\tau}^0(y_t(s)-w_t(s))ds+f(t,y(t),y(t-\tau_1),y_t)-f(t,y(t),y(t-\tau_1),y_t),\quad t\in J,$$

by B3 we obtain

$$f(t,w(t),w(t-\tau_1),w_t)-f(t,y(t),y(t-\tau_1),y_t)\ge M(w(t)-y(t))$$
$$+N(w(t-\tau_1)-y(t-\tau_1))+K\int_{-\tau}^0(w_t(s)-y_t(s))\,ds$$

thus

$$-y''(t)\ge f(t,y(t),y(t-\tau_1),y_t),\quad t\in J,$$

which means that y is an upper solution to problem (1), (2). This completes the proof of the lemma.

Theorem 4.1. *Assume that B1–B3 hold. Then there exist monotone sequences* $\{v_n\}$, $\{w_n\}$ *which converge uniformly to the solutions* r, ρ, *respectively, of the system (1), (2). Furthermore we have*

$$v_1(t) \le \ldots \le v_n(t) \le r(t) \le \rho(t) \le w_n(t) \le \ldots \le w_1(t), \quad t \in J.$$

Proof. We construct sequences $\{v_n\}$, $\{w_n\}$ defining $v_1 = v_0$, $w_1 = w_0$ and for $n > 1$, v_{n+1} and w_{n+1} are the solutions of

$$\begin{cases} -v''_{n+1}(t) = f(t, v_n(t), v_n(t-\tau_1), v_{n,t}) + M(v_{n+1}(t) - v_n(t)) + N(v_{n+1}(t-\tau_1) - v_n(t-\tau_1)) \\ \qquad + K\int_{-\tau}^{0} (v_{n+1,t}(s) - v_{n,t}(s))\, ds, \quad t \in J, \\ v_{n+1,0} = \phi, \quad v_{n+1}(1) = B \end{cases}$$

and

$$\begin{cases} -w''_{n+1}(t) = f(t, w_n(t), w_n(t-\tau_1), w_{n,t}) + M(w_{n+1}(t) - w_n(t)) + N(w_{n+1}(t-\tau_1) - w_n(t-\tau_1)) \\ \qquad + K\int_{-\tau}^{0} (w_{n+1,t}(s) - w_{n,t}(s))\, ds, \quad t \in J, \\ w_{n+1,0} = \phi, \quad w_{n+1}(1) = B. \end{cases}$$

By Lemma 4.1 the existence and uniqueness of solutions v_2, w_2 are guaranteed. The functions v_2, w_2 are lower and upper solutions of problem (1), (2) and

$$v_0(t) = v_1(t) \le v_2(t) \le w_2(t) \le w_1(t) = w_0(t), \quad t \in J.$$

Note that, again by Lemma 4.1, functions v_3, w_3 are well defined. They are lower and upper solutions of problem (1)–(2) and

$$v_2(t) \le v_3(t) \le w_3(t) \le w_2(t), \quad t \in J.$$

Analogously, let us assume that

$$v_0(t) = v_1(t) \le \ldots \le v_k(t) \le w_k(t) \le \ldots \le w_1(t) = w_0(t), \quad t \in J$$

and v_k, w_k are lower and upper solutions of problem (1), (2) for some $k \ge 1$. By Lemma 4.1, functions v_{k+1}, w_{k+1} are well defined. They are lower and upper solutions of problem (1), (2) and

$$v_k(t) \le v_{k+1}(t) \le w_{k+1}(t) \le w_k(t), \quad t \in J.$$

In fact, it can be shown by induction that

$$v_0(t) = v_1(t) \le \ldots \le v_n(t) \le w_n(t) \le \ldots \le w_1(t) = w_0(t), \quad t \in J$$

for all $n \in \mathbb{N}$. By standard arguments, $\{v_n\}$, $\{w_n\}$ converge uniformly and monotonically to the functions r and ρ being solutions of (1), (2) and

$$v_0(t) = v_1(t) \le \ldots \le v_n(t) \le r(t) \le \rho(t) \le w_n(t) \le \ldots \le w_1(t) = w_0(t), \quad t \in J.$$

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