## OPTIMIZATION OF SKI JUMPING INRUN PROFILE

## OPTYMALIZACJA PROFILU NAJAZDU SKOCZNI NARCIARSKIEJ


#### Abstract

The paper deals with the problem of modelling the inrun profile of a ski jumping hill with a variable radius of curvature. It presents the possibility of reducing the normal reaction of the track acting on a ski jumper in the immediate neighbourhood of the take-off track while maintaining the inclination angles of the starting segment of the track and the take-off track which are applied in the present day constructions. The problem of a further decrease of the normal reaction resulting from the increase of the inclination angle of the starting section has also been discussed. Finally, the profiles of the inrun which do not contain the inflexion point have been determined.


Keywords: inrun profile of ski jumping hill, nonlinear boundary-value problem

## Streszczenie

Niniejszy artykuł dotyczy zagadnienia projektowania najazdu skoczni narciarskiej o zmiennym promieniu krzywizny. W artykule pokazano możliwość obniżenia reakcji normalnej toru w bezpośrednim sąsiedztwie z progiem, przy zachowaniu stosowanych w obecnych rozwiązaniach kątów nachylenia części startowej i progu. Zbadano również wpływ zwiększenia kąta $\beta$ na dalsze obniżenie reakcji normalnej. Na koniec wyznaczono łagodne pod względem dynamicznym profile najazdu niezawierające punktów przegięcia.

Słowa kluczowe: profil najazdu skoczni narciarskiej, nieliniowe zagadnienie brzegowe

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## 1. Introduction

The inrun profiles of ski jumping hills designed and constructed nowadays [1] consist of two segments ( $A C$ and $D E$ ) and a circular arc $C D$ that is tangent to them (Fig. 1).


Fig. 1. The profile of a ski jumping inrun in Zakopane (Wielka Krokiew) Rys. 1. Profil najazdu Wielkiej Krokwi w Zakopanem

Segment $A C$ constitutes a rectilinear part of the inrun, is inclined at an angle $\beta=35^{\circ}$. It includes the segment $A B$ in the length of which the starting gate is situated. The arc $C D$ forms a curvilinear part of the inrun. Its task is to reduce the inclination angle of the tangent towards the track. The segment $D E$ plays the role of a take-off track for the ski jumper and it is inclined at an angle $\alpha=10,5^{\circ}$. The coordinates of point $A$ and the angles $\alpha$ and $\beta$ clearly define the length of the segment $A C$ and the radius of curvature of the arc $C D$.

A normal reaction of the curvilinear part of the track is caused by the weight of the ski jumper and the normal inertia force, which is inversely proportional to the radius of curvature. This reaction is a measure of the load put on the ski jumper's legs. In the case of the ski jumping hills used nowadays a normal reaction of the curvilinear part of the track exceeds the weight of the ski jumper at the point bordering on the take-off track by over $70 \%$. Its value can be reduced by changing the profile of the inrun of the jumping hill. It is important to achieve the highest possible value of the radius of curvature at the point bordering on the take-off track. The reduction of the normal inertia force will make it easier for the ski jumper to take off and perform a good jump.

The increase of the radius of curvature at the point bordering on the take-off track can be obtained by replacing the rectilinear segment $B C$ and the circular arc $C D$ with one curvilinear segment $B D$. This exchange can be carried through in many ways. It is important that the normal reaction force should rise as the ski jumper approaches the take--off track until reaching its maximum at point $D$. It turns out that it is possible to find various profiles that meet this requirement. From among the possible solutions the profile for which the value of the normal reaction force at the point bordering on the take-off track is the lowest should be chosen.

In the present paper the profile of the inrun is achieved by means of solving a nonlinear second order differential equation with four boundary conditions. Such an approach allows for the elimination of a big inclination angle of the starting segment and the obtainment of the required variability of the normal reaction force. From among various possible solutions we choose such a one for which the maximum of the reaction at the point bordering on the take-off track is the lowest and, at the same time, the ski jumper does not lose contact with the track during the descent.

## 2. The dependency of the normal reaction on the inrun profile

The rectilinear segment $B C$ and the circular arc $C D$ can be replaced with one curvilinear segment $B D$ which is tangent to the take-off track $D E$ and the starting segment $A B$ (Fig. 2).


Fig. 2. A graphic illustration of the problem
Rys. 2. Graficzna ilustracja zagadnienia
The normal reaction and the radius of curvature of the curvilinear segment $B D$ are described with the following formulae

$$
\begin{align*}
& N=m g \cos \gamma \pm \frac{m v^{2}}{r}  \tag{1}\\
& r= \pm \frac{\left\{1+\left[y^{\prime}(x)\right]^{2}\right\}^{\frac{3}{2}}}{y^{\prime \prime}(x)} \tag{2}
\end{align*}
$$

where the sign " + " corresponds to a downward concavity of a plane curve, the sign " - " corresponds to an upward concavity and the symbols used stand for:
$N$ - normal reaction of the track,
$m$ - mass of the ski jumper,
$g$ - gravitational acceleration,
$\gamma$ - inclination angle of the tangent line to the inrun,
$v$ - velocity of the ski jumper,
$r$ - radius of curvature of the track,
$y(x)$ - profile of the inrun.
By applying the geometrical interpretation of the derivative and the principle of the conservation of mechanical energy the following formulas can be written down

$$
\begin{align*}
& \cos \gamma=\frac{1}{\left\{1+\left[y^{\prime}(x)\right]^{2}\right\}^{\frac{1}{2}}}  \tag{3}\\
& v^{2}=2 g\left[h+h_{0}-y(x)\right] \tag{4}
\end{align*}
$$

where:
$h$ - signifies the height of point $B$,
$h_{0}$ - the height of the starting gate over point $B$.
By inserting Eq. (2), (3), (4) into Eq. (1), regardless of the type of concavity, a straightforward relation between the normal reaction $N$ and the profile of the inrun $y(x)$ and its derivatives is achieved

$$
\begin{equation*}
N(x)=m g \frac{1}{\left\{1+\left[y^{\prime}(x)\right]^{2}\right\}^{\frac{1}{2}}}+2 m g \frac{h+h_{0}-y(x)}{\left\{1+\left[y^{\prime}(x)\right]^{2}\right\}^{\frac{3}{2}}} y^{\prime \prime}(x) \tag{5}
\end{equation*}
$$

In order to facilitate the calculations, nondimensional variables have been introduced. The reference variables are: distance $d$ (Fig. 2) - for the parameters described with length units and weight of the ski jumper $G$ - for the normal reaction $N$. After the introduction of the nondimensional variables marked with the letters of the Greek alphabet, Eq. (5) takes the following form

$$
\begin{equation*}
v(\xi)=\frac{1}{\left\{1+\left[\psi^{\prime}(\xi)\right]^{2}\right\}^{\frac{1}{2}}}+2 \frac{\eta+\eta_{0}-\psi(\xi)}{\left\{1+\left[\psi^{\prime}(\xi)\right]^{2}\right\}^{\frac{3}{2}}} \psi^{\prime \prime}(\xi) \tag{6}
\end{equation*}
$$

## 3. Formulation of the problem and the method of solution

By eliminating $\psi^{\prime \prime}(\xi)$ from Eq. (6) we obtain a differential equation with respect to $\psi(\xi)$ which contains an unknown function $v(\xi)$ that describes the nondimensional normal reaction

$$
\begin{equation*}
\psi^{\prime \prime}(\xi)=\frac{v(\xi)\left\{1+\left[\psi^{\prime}(\xi)\right]^{2}\right\}^{\frac{3}{2}}+\left[\psi^{\prime}(\xi)\right]^{2}+1}{2\left[\eta+\eta_{0}-\psi(\xi)\right]} \tag{7}
\end{equation*}
$$

The sought profile of the inrun has to cross points $B$ and $D$ and it must be tangent to the take-off track $D E$ and the starting segment $A B$ which are inclined, respectively, at angles $\alpha$ and $\beta$ to the horizontal. The solution of the differential Eq. (7) should then fulfill the following boundary conditions

$$
\left\{\begin{array}{l}
\psi(0)=0  \tag{8}\\
\psi^{\prime}(0)=\operatorname{tg} \alpha \\
\psi(1)=\eta \\
\psi^{\prime}(1)=\operatorname{tg} \beta
\end{array}\right.
$$

For arbitrary function $v(\xi)$ the solution of Eq. (7) can fulfill two of the four conditions (8) at the most. By selecting in a particular way the form of the function $v(\xi)$ a solution of Eq. (7) that fulfills all the boundary conditions (8) can be found.

Function $v(\xi)$ should increase as the ski jumper approaches the take-off track until reaching its maximum at the point bordering on the take-off track. In addition, this function should contain two control parameters so that, through their proper selection, the solution sought fulfills all the boundary conditions.

For example, the following even polynomial functions possess the demanded properties of the normal reaction

$$
\begin{equation*}
v_{i}(\xi)=-a_{i} \xi^{2 i}+b_{i} \quad, \quad i=1,2 \ldots \tag{9}
\end{equation*}
$$

The functions listed above impose solely the variability of the normal reaction without a numerical description of its value at any point. While setting initially the values of the
control parameters $a_{i}$ and $b_{i}$, it is necessary to solve Eq. (7) regarding it as an initial-value problem which has been defined with the first two conditions (8). By selecting in a proper way the value of the control parameters $a_{i}$ and $b_{i}$, a solution that fulfills all the boundary conditions (8) is found for the subsequent exponents $i$.

## 4. A comparison between the characteristics of inrun profiles with given dynamic properties of the track and the characteristics of a typical ski jumping hill

Four subsequent polynomial functions $v_{i}(\xi)$ described with formula (9) have been considered in the calculations. The values of control parameters $a_{i}$ i $b_{i}$ which guarantee the fulfillment of all the boundary conditions (8) by the solution of Eq. (7) have been set for each of them. Figure 3 presents the obtained profiles of the inrun with a variable radius of curvature together with the profile of Wielka Krokiew.


Fig. 3. Inrun profiles determined for four subsequent exponents $i$ against the background of Wielka Krokiew inrun profile
Rys. 3. Profile najazdu wyznaczone dla czterech kolejnych wykładników $i$ na tle najazdu Wielkiej Krokwi

It is possible to observe in Fig. 3 the points of inflexion which appear in the neighbourhood of point $B$. At these points the radius of curvature tends to infinity. Figure 4 shows the graphs of the radii of curvature of the solutions obtained together with the graph of the radius of curvature of the profile of a typical ski jumping hill.


Fig. 4. Radii of curvature of the inrun profiles determined for four subsequent exponents $i$ against the background of the radius of curvature of Wielka Krokiew inrun profile
Rys. 4. Promienie krzywizny profilów najazdu, wyznaczonych dla czterech kolejnych wykładników $i$ na tle promienia krzywizny Wielkiej Krokwi

Together with the increase of exponent $i$, the value of the radius of curvature at the point bordering on the take-off track increases as well and the inflexion point approaches point $B$. The fact that the upper part of the curvilinear segment $B D$ is concave upward brings about the danger of the ski jumper losing contact with the track. The graphs of the normal reaction of the track shown in Fig. 5 ilustrate and explain the problem of the ski jumper's contact with the track.


Fig. 5. Normal reactions of the inrun tracks determined for four subsequent exponents $i$ against the background of the reaction of the inrun track of Wielka Krokiew
Rys. 5. Reakcje normalne torów najazdu wyznaczonych
dla czterech kolejnych wykładników $i$ na tle reakcji toru najazdu Wielkiej Krokwi

For $i=1,2,3$ the normal reaction of the track is positive at each point of the inrun, while for $i=4$ the ski jumper loses contact with the track immediately after crossing point $B$ because the reaction is negative. A continuing growth of the exponent $i$ increases the negative values of the normal reaction of the track in the immediate neighbourhood of point $B$. Summing up, every function describing a nondimensional normal reaction with formula (9) reduces its value at the point bordering on the take-off track but only the first three functions guarantee the ski jumper's retaining constant contact with the track.

## 5. The influence of the starting segment angle inclination of the track on the inrun profiles characteristics with given track dynamic properties

The inclination angle of the starting segment is usually determined by the degree of the inclination of the slope. This angle can be increased, especially in the case of an artificially constructed inrun. The following figures present the influence of the angle $\beta$, with the exponent $i=2$, on: the profile of the inrun, the value of the radius of curvature and the normal reaction of the track.

The increase of the angle $\beta$ has little effect on the profile of the inrun, especially on its lower part (Fig. 6). Together with the increase of the angle $\beta$, the inflexion point moves towards point $B$ (Fig. 7). The increase of the inclination angle of the starting section slightly lowers the value of the normal track reaction at the point bordering on the take-off track (Fig. 8).


Fig. 6. Profiles of the inrun determined for three different angles $\beta$
Rys. 6. Profile najazdu wyznaczone dla trzech różnych katów $\beta$


Fig. 7. Radii of curvature of the inrun profiles and normal reactions of the inrun tracks determined for three different angles $\beta$
Rys. 7. Promienie krzywizny profilów najazdu oraz reakcje normalne torów najazdu wyznaczonych dla trzech różnych kątów $\beta$

## 6. Inrun profiles with no inflexion points

The inflexion point which can be seen in the majority of the inrun profiles with a variable radius of curvature results from a relatively small inclination angle $\beta$ of the starting segment. This point can be eliminated by means of a proper increase of that angle.

The following figures show: the profiles of the inrun, the radii of curvature and the normal reactions of the track for the solutions in which the inflexion point coincides with point $B\left(\beta=41^{\circ} 80^{\prime}\right.$ for $i=1, \beta=45^{\circ} 37^{\prime}$ for $i=2, \beta=47^{\circ} 87^{\prime}$ for $i=3, \beta=49^{\circ} 68^{\prime}$ for $i=4$ ).


Fig. 8. Inrun profiles in which the inflexion point coincides with point $B$ Rys. 8. Profile najazdu, w których punkt przegięcia pokrywa się z punktem $B$

The increase of the exponent $i$ causes the increase of the value of the angle $\beta$ (Tab. 1, Fig. 9) which guarantees the coincidence of the inflexion point with point $B$. Together with the increase of the exponent $i$, the radius of the curvature of the track at the point bordering



Fig. 9. Radii of curvature of the inrun profiles and normal reactions of the inrun tracks for the solutions in which the inflexion point coincides with point $B$
Rys. 9. Promienie krzywizny profilów najazdu oraz reakcje normalne torów najazdu w przypadku, gdy punkt przegięcia pokrywa się z punktem $B$
on the take-off track increases as well (Fig. 10), while the value of the normal reaction of the track at that point decreases (Fig. 11).

## 7. Conclusions

Replacing the rectilinear section $B C$ and the circular arc $C D$ (Fig. 1) with one curvilinear section $B D$ (Fig. 2) allowed to increase the radius of curvature at point $D$. The profiles of the inrun with a variable radius of curvature presented in the paper fulfill the requirement regarding the normal reaction of the track which should increase as the ski jumper approaches the take-off track until reaching its maximum at point $D$. The nonlinear differential equation presented in the paper together with a set of boundary conditions makes it possible to design the inrun profiles of ski jumping hills which are characterized by a decreased reaction of the track for various inclination angles of the starting section $A B$. The reduction of the inertia forces which affect the ski jumper can contribute to performing longer jumps that will in addition be better from the technical point of view.

## References

[1] Neufert E., Bauentwurfslehre, Vieweg Verlag, Wiesbaden 2002.
[2] Palej R., Struk R., The inrun profile of a ski jumping hill with lowered normal reaction of the track (in Polish), Czasopismo Techniczne 6-M/2003, Cracow 2003.
[3] Palej R., Struk R., Modeling of the inrun profile of a ski jumping hill (in Polish), Registration number of the invention project: P-361249, 2003.
[4] Palej R., Struk R., Optimization of ski jumping inrun profile (in Polish), Czasopismo Techniczne 5-M/2004, Cracow 2004.
[5] Palej R., Struk R., The problem of plane curve of a constant normal reaction (in Polish), Czasopismo Techniczne 14-M/2005, Cracow 2005.


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