CZASOPISMO TECHNICZNE M
WYDAWNICTWO POLITECHNIKI KRAKOWSKIEJ

z. 3-M/2008
ISSN 0011-4561
ISSN 1897-6328

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TO THE QUESTION OF VECTOR FIELD CANONIC REPER IN FOUR-DIMENSIONAL EQUIAFFINE SPACE

ZNAJDOWANIE KANONICZNEGO REPERA POLA WEKTOROWEGO W CZTEROWYMIAROWEJ PRZESTRZENI AFINICZNEJ

A b s t r a c t

In four-dimensional equiaffine space the set of geometric objects of vector field has been built, the invariant straight has been defined. Canonic reper has been built for a particular case.

Keywords: *vector field*, *canonic reper*, *4D space*

S t r e s z c z e n i e

W czterowymiarowej przestrzeni afinicznej został zbudowany zestaw obiektów geometrycznych oraz zdefiniowana stała. Reper kanoniczny został zbudowany dla szczególnych przypadków.

Słowa kluczowe: *pole wektorowe*, *reper kanoniczny*, *czterowymiarowa przestrzeń*

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1. Differential equations of consistent geometric objects of vector field in A_4

Let`s consider four-dimensional equiaffine space A_4, related to moving reper

$$d\vec{A} = \omega^{\alpha}\vec{e}_{\alpha} \tag{1}$$

$$d\vec{e^{\alpha}} = \omega^{\alpha}_{\beta}\vec{e}_{\beta}$$

Equations of space A_4 structure are taken into consideration are following

$$D\omega^{\alpha} = \left[\omega^{\beta}\omega^{\alpha}_{\beta}\right], \quad D\omega^{\alpha}_{\beta} = \left[\omega^{\gamma}_{\beta}\omega^{\alpha}_{\gamma}\right], \quad \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \ldots = 1{,}4 \quad \omega^{\alpha}_{\alpha} = 0 \tag{2}$$

Definition: Vector element of A_4 space is called a set which consists of point A and vector a, for this one point A is beginning. Vector element will be define as (A, $\vec{a}$). In this case point A will be called as the beginning of vector element.

It is obvious that a $\vec{a}$ є V_4 and A є A_4

Definition: vector field in A_4 is the set in which every point of A_4 space is coordinated in some way definite vector $\vec{a}$ with beginning in this point.

In all vector elements' set vector fields define a kind of semi-polytype.

Remark, that vector field can be done as in whole space A_4 but also in a separate its region.

In future we'll late into account that beginning of vector coincides in A point then **δA** = 0 forms ω^{α} are the main (central).

We'll express vector $\vec{a}$ through vectors of basis $\vec{e}^{\alpha}$ in the form

$$\vec{a} = a^{\alpha}\vec{e}_{\alpha} \tag{3}$$

Coordinates of vector $\vec{a}$ will satisfy differential equations

$$da^{\alpha} + a^{\beta}\omega^{\alpha}_{\beta} = a^{\alpha}_{\beta}\omega^{\beta} \tag{4}$$

Continuing equations (4) we receive

$$da^{\alpha}_{\beta} = a^{\alpha}_{j}\omega^{\gamma}_{\beta} - a^{\chi}_{\beta}\omega^{\alpha}_{j} + a^{\alpha}_{\beta j}\omega^{\gamma} \tag{5}$$

where $a^{\alpha}_{\beta j} = a^{\alpha}_{j\beta}$

Equation (5) can be presented in form

$$\begin{aligned}
&da^{i}_{j} - a^{i}_{k}\omega^{k}_{j} - a^{i}_{4}\omega^{4}_{j} + a^{k}_{j}\omega^{i}_{k} + a^{4}_{j}\omega^{i}_{4} = a^{i}_{j\beta}\omega^{2}\\
&da^{i}_{4} - a^{i}_{k}\omega^{k}_{4} - a^{i}_{4}\omega^{4}_{4} + a^{k}_{4}\omega^{i}_{k} + a^{4}_{4}\omega^{i}_{4} = a^{i}_{4\alpha}\omega^{2}\\
&da^{4}_{j} - a^{4}_{j}\omega^{j}_{i} - a^{4}_{4}\omega^{4}_{i} + a^{j}_{i}\omega^{4}_{j} + a^{4}_{i}\omega^{4}_{4} = a^{4}_{i\alpha}\omega^{2}\\
&da^{4}_{4} - a^{4}_{j}\omega^{j}_{4} - a^{i}_{4}\omega^{4}_{j} = a^{4}_{4\alpha}\omega^{2}, \ \left(i, j, k = \overline{1,3}\right)
\end{aligned} \tag{6}$$

In fixation of main parameters differential equations (4) and (5) take correspondently form

$$\delta a^{\alpha} + a^{\beta}\pi^{\alpha}_{\beta} = 0 \tag{4'}$$

$$\delta a^{\alpha}_{\beta} = a^{\alpha}_{j}\pi^{\gamma}_{\beta} - a^{\gamma}_{\beta}\pi^{\alpha}_{j} = 0 \tag{5'}$$

From the previous data it is clear, that fundamental object of the first order $\{a^{\alpha}\}$ is tensor and fundamental object of the second order $\{a^{\alpha}, a^{\alpha}_{\beta}\}$ consists of two tensors a^{α} and a^{α}_{β}.

Continuing differential equations (5) we'll receive a set of fundamental objects $\{a^{\alpha}, a^{\alpha}_{\beta}, a^{\alpha}_{\beta j}, a^{\alpha}_{\beta j\delta}, \ldots\}$, which lies in basis of differential geometry of vector field in four--dimensional equiaffine space A_4.

2. Some fields of invariant geometric objects are united with vector field

Let's find differential equations of some invariant geometric objects joined to vector field.

2.1. Field of points

Let's consider point $P(X^{\alpha})$in affine space A_4. When $\vec{P}$ – radius – vector of this point in this case related to affine reper $(\vec{A}, \vec{e}_{\alpha})$ it can be done with the following phrase

$$\vec{P} = \vec{A} + x^{\alpha}\vec{e}_{\alpha} \tag{7}$$

Differentiating (7) and taking into account equation of structure, we'll receive

$$dx^{\alpha} + x^{\beta}\omega^{\alpha}_{\beta} = x^{\alpha}_{\beta}\omega^{\beta} \tag{8}$$

or in fixation of main parameters

$$\delta x^{\alpha} + x^{\beta}\pi^{\alpha}_{\beta} = 0 \tag{9}$$

2.2. Field of straights

Straight, which cross the A point with directed vector $R = \nu^{\alpha}e_{\alpha}$ define as $l = [A, R]$

Conditions of invariantness of straight will be

$$\delta R = QR\ ,\ dQ = 0 \tag{10}$$

From the previous data

$$\delta\nu^{\alpha} + \nu^{\beta}\pi^{\alpha}_{\beta} = Q\nu^{2} \tag{11}$$

or

$$\begin{aligned} \delta\nu^{i} + \nu^{i}\pi^{i}_{j} + \nu^{n}\pi^{i}_{n} &= Q\nu^{i} \\ \delta\nu^{n} + \nu^{i}\pi^{n}_{i} + \nu^{n}\pi^{n}_{n} &= Q\nu^{n} \end{aligned} \tag{12}$$

Writing down correspondence (12) we'll have

$$
\begin{aligned}
&d\nu^1 + \nu^1\omega_1^1 + \nu^2\omega_2^1 + \nu^3\omega_3^1 + \nu^4\omega_4^1 = Q\nu^1 \\
&d\nu^2 + \nu^1\omega_1^2 + \nu^2\omega_2^2 + \nu^3\omega_3^2 + \nu^4\omega_4^2 = Q\nu^2 \\
&d\nu^3 + \nu^1\omega_1^3 + \nu^2\omega_2^3 + \nu^3\omega_3^3 + \nu^4\omega_4^3 = Q\nu^3 \\
&d\nu^4 + \nu^1\omega_1^4 + \nu^2\omega_2^4 + \nu^3\omega_3^4 + \nu^4\omega_4^4 = Q\nu^4
\end{aligned}
\qquad (12)'
$$

Sometimes it is convenient to promote norm of vector $\vec{R}$ in which $\nu^n = 1$. Then

$$Q = \omega_n^n + \nu^i \omega_i^n \qquad (13)$$

Putting (13) in the first equation (12) we'll have

$$\delta\nu^i + \nu^j\pi_j^i - \nu^i\pi_n^n - \nu^j\nu^k\pi_k^n + \pi_n^i = 0 \qquad (14)$$

Thus, differential equations of invariantness of straight will have a form

$$d\nu^i + \nu^j\omega_j^i - \nu^i\omega_n^n - \nu^i\nu^k\omega_k^n + \omega_n^i = \nu^i_{\ \alpha}\omega^\alpha \qquad (15)$$

Let's consider values

$$N^\alpha = a_\beta^\alpha a^\beta \qquad (16)$$

If differential equations consider values (16) have the form $dN^\alpha + N^\alpha\omega_j^\beta = N_\beta^\alpha\omega^\beta$ according to (11) they define invariant direction and pair [A, N], where $N^\alpha = N^\alpha e_\alpha$ defines invariant straight joined to vector field.

3. Main directions and main curves of vector field

All the previous investigations were fulfilled in reper of zero order. It permitted to formulate them in the most general form. Thus, such investigation does not permit to find out close invariant models connected with vector field. In order to find them, we'll introduce **definition**: The main directions of vector field are the directions for which differential of vector field is collinear to direction of beginning of vector field displacement.

Thus, for vector field

$$d\vec{a} = \lambda d\vec{A} \qquad (17)$$

Taking into account (3) we'll have equation for calculation of main directions

$$\left(a_\beta^\alpha - \lambda\delta_\beta^\alpha\right)\omega^\beta = 0 \qquad (18)$$

or in detailed form

$$
\begin{aligned}
(a_1^1-\lambda)\omega^1+a_2^1\omega^2+a_3^1\omega^3+a_4^1\omega^4&=0\\
a_1^2\omega^1+(a_2^2-\lambda)\omega^2+a_3^2\omega^3+a_4^2\omega^4&=0\\
a_1^3\omega^1+a_2^3\omega^2+(a_3^3-\lambda)\omega^3+a_4^3\omega^4&=0\\
a_1^4\omega^1+a_2^4\omega^2+a_3^4\omega^3+(a_4^4-\lambda)\omega^4&=0
\end{aligned}
\tag{19}
$$

Let's consider homogenious system of four equations with four unknown. In case this system has non-zero solution it is necessary and sufficiently, that

$$
\begin{vmatrix}
a_1^1-\lambda & a_2^1 & a_3^1 & a_4^1\\
a_1^2 & a_2^2-\lambda & a_3^2 & a_4^2\\
a_1^3 & a_2^3 & a_3^3-\lambda & a_4^3\\
a_1^4 & a_2^4 & a_3^4 & a_4^4-\lambda
\end{vmatrix}=0
\tag{20}
$$

Let's analyze only case when the roots of equation (20) are different. We'll indicate them as $\boldsymbol{\lambda_1}$, $\boldsymbol{\lambda_2}$, $\boldsymbol{\lambda_3}$, $\boldsymbol{\lambda_4}$ and name the main curves. Putting them turn by turn in (18) receive, that it is satisfy with the following directions

$$
\begin{aligned}
&1.\ \omega^2=\omega^3=\omega^4=0\\
&2.\ \omega^1=\omega^3=\omega^4=0\\
&3.\ \omega^1=\omega^2=\omega^4=0\\
&4.\ \omega^1=\omega^2=\omega^3=0
\end{aligned}
\tag{21}
$$

In every case the displacement direction of vector field beginning coinsides with correspondent basis vector that is directed to the main direction.

It is easy to be sure in the fact that straights cross the beginning of vector field parallel to basis vectors, have directed vectors which satisfy equation (12). From equation (12) considering (21) we'll have

$$
\begin{aligned}
d\lambda_\alpha&=a_{1\alpha}^1\omega^\alpha\\
\omega_\beta^\alpha&=\frac{a_{\beta\gamma}^\alpha\omega^\gamma}{\lambda_\beta-\lambda_\alpha}
\end{aligned}
\tag{22}
$$

$$\alpha\neq\beta$$

In case reper $(\vec{e}_\alpha)$ becomes canonic

$$
w_\alpha^\alpha=a_\beta^\alpha\omega^\beta,\quad \alpha,\beta=\overline{1,4}
\tag{23}
$$

Besides these formula satisfy conditions of equiaffinness

$$\begin{aligned} a_{11}^1 + a_{21}^2 + a_{31}^3 + a_{41}^4 &= 0 \\ a_{12}^1 + a_{22}^2 + a_{32}^3 + a_{42}^4 &= 0 \\ a_{13}^1 + a_{23}^2 + a_{33}^3 + a_{43}^4 &= 0 \\ a_{14}^1 + a_{24}^2 + a_{34}^3 + a_{44}^4 &= 0 \end{aligned} \tag{24}$$

Thus, for vector field in space A which has four different main curveness canonic reper invariant to it has been built.

The received results are transformed into spaces of free size.

References

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