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DEVELOPMENT OF GRADIENT-ENHANCED DAMAGE-PLASTICITY FORMULATIONS FOR LARGE DEFORMATIONS

SFORMUŁOWANIA GRADIENTOWEJ TEORII USZKODZENIA I PLASTYCZNOŚCI W DUŻYCH DEFORMACJACH

Abstract

The development of gradient-enhanced coupled elastic-plastic damage theory accounting for finite deformations is presented. The formulation of strain energy function used in hyper-elasticity, the description of damage using an evolution equation for a scalar damage parameter as well as a yield function and flow rule are discussed. The selection of a non-local parameter and the formulation of a corresponding averaging equation are analysed. The issue of material or spatial averaging and the corresponding definition of the internal length parameter in the averaging equation is also considered.

Keywords: non-local damage-plasticty, finite strains

Streszczenie

W niniejszym artykule zaprezentowano sformułowanie gradientowej teorii uszkodenia i plastyczności w dużych deformacjach. Określono postać funkcji energii odkształcenia dla hypersprężystości, opis rozwoju uszkodzenia z wykorzystaniem równań ewolucji skalarnego parametru uszkodzenia, jak również postać warunku plastyczności i funkcji płynięcia. Przeanalizowano dwie alternatywne definicje nielokalnego parametru oraz odpowiadające im równania uśredniające. Podjęto również problem wyboru sposobu uśredniania oraz określenia wewnętrznego parametru długości, związany z zastosowaniem opisu materialnego lub przestrzennego.

Słowa kluczowe: gradientowa teoria uszkodzenia, plastyczność, duże deformacje

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1. Introduction

The formulation of gradient-enhanced coupled elastic-plastic damage theories accounting for finite deformations remains an open issue. The differences in existing proposals for the classical elastic-plastic damage models concern the formulation of the strain energy function W used in hyper-elasticity, the description of damage using an evolution equation for the scalar damage parameter d as well as the choice of a yield function and yield condition [10]. Generally, two types of damage descriptions are found in the literature. According to the first one damage causes the degradation of elastic properties of the material (elastic damage). In the second one the plastic damage variable is introduced which reduces the yield strength of the material (ductile damage) [3].

The introduction of gradient-enhancement requires the selection of a non-local parameter and the formulation of a corresponding averaging equation. Within elastic damage models there is an energy gradient formulation with a non-local stored energy $\overline{W}$ serving as an independent variable, and a damage gradient formulation with damage parameter d serving as an independent variable and its gradient $\mathbf{D}$ entering the free energy function [6]. There are also gradient enhanced theories with non-local damage parameter $\overline{d}$ [1]. In the case of ductile damage models, a kinematic non-local variable $\overline{z}$ is introduced, having its local kinematic counterpart, e.g. the equivalent plastic strain [3].

In the paper we focus on the non-local elastic damage models. The issue of the material or spatial averaging as well as the corresponding definition of the length scale-related parameter c present in the averaging equation is raised [9]. As a result, the development of an original framework of gradient-enhanced damage-plasticity for large deformations is outlined.

2. Basic kinematics at finite strains

In the large strain formulation the deformation gradient $\mathbf{F}$ is used as a basic kinematic quantity. With the use of the polar decomposition of $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$ [8] the following deformation and strain measures are defined:

– the right and left Cauchy-Green tensors $\mathbf{C}$ and $\mathbf{b}$, where $\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^T\mathbf{F}$ is defined in the reference (material) configuration $\mathcal{B}_0$ and $\mathbf{b} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^T$ is defined in the current configuration $\mathcal{B}$. They reduce to identity tensor $\mathbf{I}$ in the case of a rigid body movement.

– The right and left Cauchy-Green strain tensors $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$ and $\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{b}^{-1})$.

– The whole family of strain measures defined with use of symmetric and positive definite $\mathbf{U}$ and $\mathbf{V}$ tensors, among which the most important is the logarithmic strain measure $\ln\mathbf{V} = \frac{1}{2}\ln\mathbf{b}$ defined in the current configuration. All these strain measures reduce to $\mathbf{0}$ in the case of a rigid body motion.

The change of volume of material element is described by Jacobian J, therefore the set of isochoric (volume preserving) deformations are characterized by the Jacobian

$$J = \det\mathbf{F} = \det\mathbf{U} = \det\mathbf{V} = \exp(\mathrm{tr}(\ln\mathbf{V})) = 1 \Longleftrightarrow \mathrm{tr}(\ln\mathbf{V}) = \frac{1}{2}\mathrm{tr}(\ln\mathbf{b}) = 0.$$

The last property is one of the reasons why the logarithmic strain measure is so convenient for applications and is applied in the developed model.

In small strain theory the decomposition of small strain tensor into the deviatoric part and the hydrostatic part is used which enables us to decompose the deformation into the change of shape and the change of volume. When such a decomposition for the logarithmic strain measure $\ln\mathbf{b}$ is performed, one obtains as output a tensor called the isochoric Finger tensor $\bar{\mathbf{b}}$ [10] or the isochoric left Cauchy-Green tensor [4] specified as follows

$$\bar{\mathbf{b}} = \exp\left[(\ln\mathbf{b})^{dev}\right] = J^{-2/3}\mathbf{b}.$$

In the large strain elasto-plasticity the deformation gradient is decomposed multiplicatively into an elastic part $\mathbf{F}_e$ and a plastic part $\mathbf{F}_p$

$$\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$$

In the formulations of coupled elastic-plastic damage theories most frequently the elastic left Cauchy-Green tensor $\mathbf{b}_e$ and the inverse right plastic Cauchy-Green tensor $\mathbf{C}_p^{-1}$ specified as

$$\mathbf{b}_e = \mathbf{F}_e\mathbf{F}_e^T, \quad \mathbf{C}_p^{-1} = \mathbf{F}_p^{-1}\mathbf{F}_p^{-T}$$

are used. Defining geometrically the *pull-back* and the *push-forward* operations

$$(\cdot)^{pull-back} = \mathbf{F}^{-1}(\cdot)\mathbf{F}^{-T}, \quad (\cdot)^{push-forward} = \mathbf{F}(\cdot)\mathbf{F}^T$$

one may notice that $\mathbf{b}_e$ and $\mathbf{C}_p^{-1}$ are related via the above operations. In view of these relations the material time derivative of the elastic left Cauchy-Green tensor reads

$$\dot{\mathbf{b}}_e = \mathcal{L}_v(\mathbf{b}_e) + \mathbf{l}\mathbf{b}_e + \mathbf{b}_e\mathbf{l}^T$$

where $\mathcal{L}_v(\mathbf{b}_e)$ is the Lie derivative (the Oldroyd derivative, objective one) of the elastic left Cauchy-Green tensor and $\mathbf{l} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ is the spatial velocity gradient. It is easy to check that

$$\mathbf{b}_e = \mathbf{F}\mathbf{C}_p^{-1}\mathbf{F}^T \Longrightarrow \mathcal{L}_v(\mathbf{b}_e) = \mathbf{F}\dot{\mathbf{C}}_p^{-1}\mathbf{F}^T$$

3. Free energy definition and dissipation inequality

The Helmholtz free energy in the coupled elastic-plastic damage is in a general way defined as [1], [7]

$$\Psi = \Psi(\mathbf{C}, \mathbf{C}_p, d, q) = \Psi(\mathbf{C}_e, d, q) \tag{1}$$

where d is the scalar damage variable describing the degradation of the material and q is the strain-like internal variable describing the state of the material at the micro-level induced by the dislocation pile-ups and other micro defects. The Clausius-Duhem inequality (the dissipation inequality) in the spatial and material descriptions has the corresponding forms

$$\mathcal{D}_{int} = \boldsymbol{\tau} : \mathbf{d} - \dot{\Psi} \geq 0, \quad \mathcal{D}_{int} = \frac{1}{2}\mathbf{S} : \dot{\mathbf{C}} - \dot{\Psi} \geq 0 \tag{2}$$

where $\mathbf{S}$ is the symmetric second Piola-Kirchhoff stress that is the pull-back of the Kirchhoff stress $\boldsymbol{\tau} = J\boldsymbol{\sigma}$.

3.1. Hyper-elasticity

In the case of hyper-elastic material ($\mathbf{F}=\mathbf{F}_e$) free energy function is restricted to the strain energy $\Psi = W = W(\mathbf{F}) = W(\mathbf{C}) = W(\mathbf{b})$. The considered process is reversible, therefore the dissipation inequality leads to the constitutive relation in the form

$$\mathbf{S} = 2\partial_{\mathbf{C}} W \quad \text{or} \quad \tau = (\partial_{\mathbf{F}} W)\mathbf{F}^T \tag{3}$$

Note that when the considered material is isotropic then it can also be written in the form

$$\mathbf{S} = 2[(\partial_{\mathrm{tr}\mathbf{C}} W)\mathbf{I} + 2(\partial_{\mathrm{tr}\mathbf{C}^2} W)\mathbf{C} + 3(\partial_{\mathrm{tr}\mathbf{C}^3} W)\mathbf{C}^2]$$

which trough the definition of the II Piola-Kirchhoff stress as a pull-back of the Kirchhoff stress and the equality of eigenvalues of $\mathbf{C}$ and $\mathbf{b}$ tensors leads to

$$\tau = \mathbf{F}\mathbf{S}\mathbf{F}^T = 2[(\partial_{\mathrm{tr}\mathbf{b}} W)\mathbf{I} + 2(\partial_{\mathrm{tr}\mathbf{b}^2} W)\mathbf{b} + 3(\partial_{\mathrm{tr}\mathbf{b}^3} W)\mathbf{b}^2]\mathbf{b} = 2\partial_{\mathbf{b}} W \mathbf{b}$$

It is easy to notice that the principal directions of τ and $\mathbf{b}$ (also other strain measures defined in the current configuration, e.g $\mathbf{V}$ or $\ln\mathbf{b}$) are the same, therefore, denoting by τ_A and λ_A the principal values of the Kirchhoff stress and the stretches (principal values of $\mathbf{V}$ or $\mathbf{U}$), respectively, we may write the equation (3) in the form (no summation over A if not indicated)

$$\tau_A = (\partial_{\lambda_A} W)\lambda_A = (\partial_{\ln\lambda_A} W) \Longrightarrow \tau = \sum_{A=1}^{3} (\partial_{\varepsilon_A} W)\mathbf{n}_A \otimes \mathbf{n}_A$$

where $\varepsilon_A = \ln\lambda_A$ and $\mathbf{n}_A$ denote the principal directions of τ.

Depending on the considered material different forms of strain energy functions have been proposed. In the presented formulation the definition corresponding to the linear Hooke's law in small strain theory has been applied (see also [10], [3], [2], [1]), that is

$$W = \Psi^e = \hat{\Psi}^{macro}(J) + \bar{\Psi}^{macro}(\bar{\mathbf{b}}) = \frac{1}{2}K(\ln J)^2 + \frac{\mu}{4}\left[\ln\bar{\mathbf{b}}\right] : \left[\ln\bar{\mathbf{b}}\right] \tag{4}$$

The relation for the principal stress τ_A takes the usual form

$$\tau_A = 3K\theta + 2\mu e_A$$

where $\theta = 1/3(\varepsilon_1+\varepsilon_2+\varepsilon_3) = 1/3\ln J$ is the hydrostatic part of $\ln\mathbf{V}$ and the eigenvalues of the deviatoric part are $e_A^2 = (\varepsilon_A-\theta)^2 = (\ln(\lambda_A/\sqrt[3]{J}))^2$. In the literature one may also find Neo-Hookean formulation of strain energy [6] or the proposal of Blitz and Ko [9] used in the considered context.

3.2. Damage

Now, damage is added to the constitutive description. It is assumed that the free energy Ψ is a function of $\mathbf{F}$ and scalar damage variable d (isotropic damage), that is $\Psi = \Psi(\mathbf{F}, d)$. Following [9], [6], [1] the function Ψ is formulated as follows

$$\Psi(\mathbf{F}, d) = (1-d)W(\mathbf{F}) = (1-d)W(\mathbf{C}) \tag{5}$$

where for strain energy $W(\mathbf{F})$ the relation (4) is used. Here, parameter d is a measure of elastic properties degradation. Let us note that in [10] multiplier $(1-d)$ is added only in front of the isochoric part of strain energy $\bar{\psi}^{macro}$, therefore in this case damage is 'deviatoric' (isochoric) - it leads to the degradation of 'deviatoric' elastic properties.

The dissipation inequality takes the form

$$\mathcal{D}_{int} = \frac{1}{2}\left(\mathbf{S} - 2(1-d)\partial_{\mathbf{C}}W\right) : \dot{\mathbf{C}} - \partial_d \Psi \dot{d} \geq 0$$

which renders the constitutive equation for the II Piola-Kirchhoff stress

$$\mathbf{S} = 2(1-d)\partial_{\mathbf{C}}W \Leftrightarrow \tau = (1-d)\partial_{\mathbf{F}}W\mathbf{F}^T$$

The effective stresses may be introduced with the use of the above relations according to $\widetilde{\mathbf{a}} = \mathbf{a}/(1-d)$ and $\mathbf{a}$ is any stress measure used above. Then, $Y = -\partial_d \Psi$ conjugated to damage variable d is introduced as the driving force for damage evolution, so

$$\mathcal{D}_{int} = Y\dot{d} \geq 0 \quad \text{and} \quad Y = W(\mathbf{F})$$

Therefore, a damage condition together with an associated damage evolution law are motivated as

$$\Phi(W;d) = \phi(W) - d \leq 0, \quad \text{with} \quad \dot{d} = \dot{\kappa}\partial_W \phi \tag{6}$$

Function Φ may be identified as dissipation potential and the damage evolution law together with the set of Kuhn-Tucker (K.-T.) conditions

$$\Phi(W;d) \leq 0 \quad \text{and} \quad \dot{\kappa} \geq 0 \quad \text{and} \quad \dot{\kappa}\Phi(W;d) = 0$$

follow from the postulate of maximum dissipation. The damage consistency condition for loading allows for a closed-form update of the damage parameter

$$\dot{W} = \dot{\kappa} \geq 0 \Rightarrow d = \phi(\kappa) \quad \text{and} \quad \kappa = \max_{-\infty < s < t}(W(s), \kappa_0)$$

where κ_0 the initial damage threshold.

Now, the damage evolution law has to be specified. In computations the following laws have been tested:

– the exponential rule (compare [9])

$$\phi(\kappa) = 1 - \exp(\eta[\kappa_0 - \kappa]) \tag{7}$$

where η is a material parameter. It is easy to see that with $\kappa \to \infty$ we obtain $d \to 1$ for positive η,

– the Lemaitre rule [5]

$$\dot{d} = \frac{\dot{\kappa}}{1-d}\left(\frac{-Y}{W_0}\right)^n \tag{8}$$

where W_0 is a material parameter,

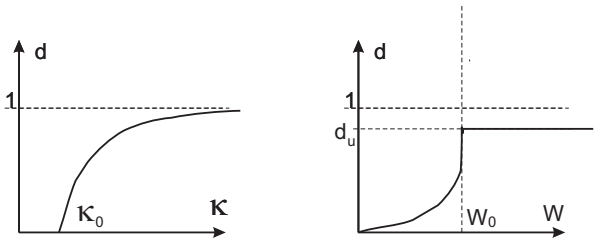

Fig. 1. Evolution rules for damage parameter: exponential (left), Lemaitre (right)
Rys. 1. Funkcje ewolucji parametru uszkodzenia: eksponencjalna (po lewej), Lemaitre'a (po prawej)

– the evolution rule [11]

$$\dot{d} = \dot{\kappa}(1-d)\left(\frac{Y}{Y_0}\right) \Rightarrow d(Y) = 1 - \exp\left[\frac{Y_0}{2}\left(1 - \frac{Y^2}{Y_0^2}\right)\right] \tag{9}$$

where Y_0 is a material parameter ($d(Y_0) = 0$) and the identity $\dot{\kappa} = \dot{Y}$ has been used.

The Lemaitre rule resulted from the specification of the dissipation potential $\Phi(W;d)$ as a power function of $W = Y$. In [5] it is additionally assumed that if $d = d_c$ failure of the volume element occurs. This evolution law was also applied in [1] and [10]. In [1] the critical value d_c is introduced such that if $d \geq d_c$ then $d = d_u$ where d_u is a residual damage value (close to 1).

3.3. Plasticity

Now let us consider the elastic-plastic material (for a moment we neglect damage). The free energy is given as follows (compare [10], [3])

$$\Psi = \Psi(\mathbf{b}_e, q),$$

where q is the internal variable describing plasticity. The elastic domain is determined through the yield condition formulated in the stress space

$$\phi = \phi(\tau, h) \leq 0$$

where h is the scalar hardening variable (the thermodynamic force) conjugated to q via dissipation inequality. The yield function ϕ serves as a dissipation potential. The dissipation inequality, together with the postulate of maximum dissipation, then leads to an associated flow rule accompanied by the evolution equation for the internal variable

$$\pounds_v(\mathbf{b}_e) = -2\dot{\lambda}\partial_\tau\phi\mathbf{b}_e, \quad \dot{q} = \dot{\lambda}\partial_h\phi \tag{10}$$

where λ is the plastic multiplier. The Kuhn-Tucker conditions are fulfilled

$$\dot{\lambda} \geq 0, \quad \phi \leq 0, \quad \dot{\lambda}\phi = 0$$

In the proposed formulation the Huber-Mises yield function is applied that is

$$\phi(\tau, h) = \|\tau^d\| - \sqrt{\frac{2}{3}}\tau_Y \leq 0, \quad \tau_Y = \tau_Y^0 + h, \quad \|\tau^d\| = \sqrt{\tau^d : \tau^d} \tag{11}$$

where τ^d is the deviator of τ and τ_Y is the yield stress. The associated flow rule and the evolution equation read

$$\frac{1}{2}\mathcal{L}_v(\mathbf{b}_e)\mathbf{b}_e^{-1} = -\dot{\lambda}\frac{\tau^d}{\|\tau^d\|} = \dot{\lambda}\mathbf{n}, \quad \dot{q} = \dot{\lambda}\sqrt{\frac{2}{3}} \tag{12}$$

so q may be related to the equivalent plastic strain [3].

Let us note that in [1] the non-associated flow rule was assumed - the yield function was not the dissipation potential. In [2] the general form of an isotropic yield function is used specified by three stress invariants and as particular examples Huber-Mises condition and the Prager rule are studied.

3.4. Coupled elastic-plastic damage

In the case of elasto-plasticity coupled with damage the free energy is proposed in the decomposed form (compare [1], [10])

$$\Psi(\mathbf{b}_e, d, q) = (1-d)\Psi(\mathbf{b}_e) + \Psi(q) \tag{13}$$

The first term in (13) accounts for elasticity and damage, while the second term accounts for plasticity. Also, the dissipation potential is postulated in a decoupled form $\Phi(\tau, h, r; d) = \tilde{\Phi}(\tilde{\tau}, h) + \tilde{\tilde{\Phi}}(r)$. Damage and plasticity are coupled via the plastic part of the dissipation potential which in the considered associative case is equal to the yield function (11) that depends on the effective Kirchhoff stress $\tilde{\tau}$ instead of τ while the plastic multiplier is equal to the history variable, $\dot{\lambda} = \dot{\kappa}$, in the damage evolution laws (7)-(9).

In [7] damage was introduced by *the modification of the power of internal forces* specified in the reference configuration. Free energy is there postulated as $\Psi = \Psi(\mathbf{C}_e, \hat{d}, q)$, where $\hat{d} = 1 - d$. In this formulation damage enters both, the power of internal forces and the free energy.

Another philosophy for formulating elasto-plasticity coupled with damage is discussed in [4] and [3] where the plastic damage parameter ω_p enters the yield function degrading the value of a yield stress, $\tilde{\tau}_Y = (1-\omega_p)\tau_Y$. The damage variable ω_p is coupled to the deformation history of the material through a history variable κ. The variable κ may be taken as the ultimate value of a kinematic variable z. The coupling between κ and z is mathematically achieved by an additional set of K.-T. conditions for plastic degradation. In the paper this kinematic variable is equal to the equivalent plastic strain, i.e. $z = \varepsilon_p$ [4], or is a function of the plastic volume change ratio and equivalent plastic strain $z = z(J_p, \varepsilon_p)$ [3]. Note that the formulation of the yield condition in terms of the effective Kirchhoff stress may be easily recovered by dividing the yield condition by $1 - \omega_p$.

4. Gradient enhancement

4.1. Gradient enhanced damage

Let us start with the gradient-enhanced damage without plasticity. Non-local or higher-order gradient continuum theories are motivated by micro-defect interactions. In the case of damage model described by free energy of the form (5), higher gradients are introduced in

the damage condition and evolution law (6) by a modification of driving force Y towards quasi-nonlocal quantity $\bar{Y}$ [9], [6]

$$\bar{Y} = Y - \mathrm{Div}\mathbf{Y} \Rightarrow \quad \kappa = \max_{-\infty<s<t}(\bar{Y}(s), \kappa_0) \tag{14}$$

where a damage flux $\mathbf{Y}$ is introduced, and the non-local quantity $\bar{Y}$ enters the damage condition as well as the update of history variable κ. Following [6] two approximations are considered to determine the damage flux $\mathbf{Y}$:

– Energy Gradient Formulation (EGF) where the modified driving force $\bar{Y}$ is identified with the Non-local Stored Energy (NSE) $\bar{W}$ and it serves as an independent variable. Averaging equation is derived from the Taylor expansion of fully non-local integral definition of a weight function in the material description

$$Y = \bar{Y} - \mathrm{Div}(c\nabla_X \bar{Y}) + ... \tag{15}$$

and restricting to the first two terms the damage flux may be connected with the material gradient of $\bar{Y}$

$$\mathbf{Y} = -c\mathbf{H} = -c\nabla_X \bar{Y} \tag{16}$$

– Damage Gradient Formulation (DGF) adopting a thermodynamically based strategy where the damage field d serves as an independent variable. The free energy function additionally depends on the material gradient of damage field $\mathbf{D}$

$$\Psi(d, \mathbf{F}, \mathbf{D}) = (1-d)W(\mathbf{F}) + \Psi^{grd}(\mathbf{D})$$

and with use of the dissipation inequality it may be shown that the damage flux $\mathbf{Y}$ is the force conjugated to the gradient damage field

$$\mathbf{Y} = -\partial_{\mathbf{D}}\Psi^{grd}$$

Satisfying the requirement that d and $\mathbf{D}$ and their rates must be compatible and proposing the dissipation as $\mathcal{D}_{int} = \bar{Y}\dot{d}$ in $\mathcal{B}_0$ the averaging equation (14) is obtained. It is also concluded that *the only driving force conjugated to damage field d is identified as the quasi-nonlocal energy release rate* $\bar{Y}$. The gradient part of the free energy is assumed in the simplified form

$$\Psi^{grd} = \frac{1}{2}c\|\mathbf{D}\|^2 \Rightarrow \mathbf{Y} = -c\mathbf{D} = -c\nabla_X d, \quad \bar{Y} = Y + \mathrm{Div}(c\nabla_X d) \tag{17}$$

In the above equations c is the length scale-related parameter.

The results of simulations presented in [9] show that DGF in contrast to EGF shows shrinkage of the localized deformation band upon further loading into a crack line, which correlates with experiments. EGF based on NSE renders in general a non-symmetric stiffness matrix while it is symmetric in the case of DGF. Due to so-called active set search the computational effort is slightly increased in DGF.

4.2. Gradient enhanced coupled elastic-plastic damage

In the developed model of the gradient-enhanced damage-plasticity the EGF and DGF strategies for the introduction of non-local damage can be adopted. In both cases the local driving force for damage Y is simply replaced by its non-local counterpart $\bar{Y}$ for the computation of the internal variable $\kappa = \lambda$ in the coupled damage-plasticity. In the case of DGF the free energy function (13) is extended and takes the form

$$\Psi(\mathbf{b}_e, \mathbf{D}, d, q) = (1-d)\Psi(\mathbf{b}_e) + \Psi(q) + \Psi^{grd}(\mathbf{D})$$

In the first case $\bar{Y} = \bar{W}$ is an independent variable while in the second case d is treated as an independent field with additional averaging equation (14) rewritten as

$$Y = \bar{Y} - \mathrm{Div}(c\nabla_X \bar{Y}) \tag{18}$$

for EDF and

$$\bar{Y} = Y + \mathrm{Div}(c\nabla_X d) \tag{19}$$

for DGF.

In literature, apart from the EGF and DGF formulations presented in [9, 6], other proposals for the definition of the non-local quantity in the context of gradient-enhanced damage-plasticity at large deformations include:

– non-local (non-constitutive) damage variable $\bar{d}$ with the corresponding averaging equation [1],

– non-local kinematic variable $\bar{z}$ originating in its local counterpart e.g. equivalent plastic strain [3]. Note that in the case of this formulation the damage degrades only the plastic properties of the material, therefore the formulation of free energy (13) seems to be inapplicable here.

For gradient-enhanced elasto-plasticity coupled with damage there also exists the proposal of Nedjar [7] who extended the power of internal forces and the gradient of damage field enters the formulation directly without defining the non-local quantity.

4.3. Material or spatial averaging

The essential ingredient of the gradient enhanced damage is an additional equation for the determination of the non-local quantity, see (14), (18), (19). In finite strain implementation it is important if the Lagrange (material), Euler (spatial) or mixed averaging is performed. The issue is discussed in detail in [9] on the example of EDF. The first two options are the most important. Averaging equations have the form:

– Lagrange (material) averaging

$$\bar{W} - c\mathrm{Div}(\nabla_X \bar{W}) = W, \quad \bar{W} - c\frac{\partial^2 \bar{W}}{\partial X_I \partial X_I} = W$$

In this case the domain of non-local influence is convected with the deformation (the same material particle within relative distance Ξ to the material point is weighted by the same factor $G(\Xi)$ throughout the deformation process). Moreover, c is specified in $\mathcal{B}_0$.

– Euler (spatial) averaging

$$\bar{w} - c\,\mathrm{div}(\nabla_x \bar{w}) = w, \quad \bar{w} - c\frac{\partial^2 \bar{w}}{\partial x_i \partial x_i} = w$$

In this case throughout the deformation process different particles within relative distance ξ to the material particles currently at the spatial point $\mathbf{x}$ are weighted by the same factor $g(\xi)$. The domain of the influence specified by c is spatially fixed.

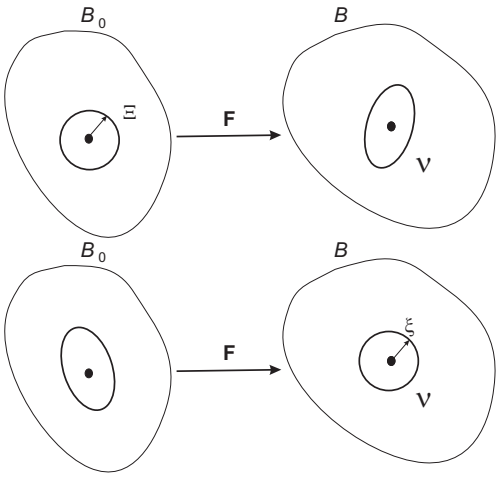

Fig. 2. Material (upper) and spatial (lower) averaging – change of domain of influence
Rys. 2. Uśrednianie materialne (u góry) i przestrzenne (u dołu) – zmiana obszaru wpływu

Note that W and $\bar{W}$ are defined per unit volume in reference configuration $\mathcal{B}_0$ thus alternatively $w = W/J$ and $\bar{w} = \bar{W}/J$ are introduced per unit volume of the deformed configuration $\mathcal{B}$. Examples studied in [9] indicate that only Lagrange averaging procedure inherits all the desirable features encountered in the small strain case. It is also the easiest formulation for implementation. All analysed papers on the gradient enhanced damage [6, 4, 1] use the material averaging except [3] where the spatial averaging is employed. In the last case the author motivates his choice by the possibility to write and solve the entire set of equations in the current configuration. He also claims that the analysis performed in the context of softening elasto-plasticity at finite strains indicated that the difference between the two approaches is limited.

5. Summary

The framework of gradient-enhanced damage-plasticity model for large deformations has been outlined. For coupled elastic-plastic damage theory the free energy Ψ in the form (13) is applied. The strain energy describing elasticity $\Psi(\mathbf{b}_e)$ in the form (4) is used. It corresponds to the classical linear Hooke's law for the small strain formulation and uses the logarithmic strain measure $(\ln \mathbf{b}_e)$ which in the case of isotropy leads to a convenient numerical algorithm

for integration. The Huber-Mises yield function (11) and the associative flow rule (12) are the natural first choice for the plasticity description. Three rules (7)-(9) for the evolution law for damage parameter *d* have been selected for implementation.

Two non-local formulations have been presented in Sec. 4. As concerns the averaging equation it seems that the material averaging is generally accepted as a better choice, though, when formulating equations in the current configuration the material Laplacian must be pushed forward to this configuration.

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