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Electric circuit analysis by means of optimization criteria Part I — the simple circuits

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Abstract. One of the main problems of electrical power quality is to ensure a constant power flux from the supply system to the receiver, keeping in the same time the undisturbed wave form of the current and voltage signals. Distortion of signals are caused by nonlinear or time varying receivers, voltage changes or power losses in a supply system. The wave-form of the voltage of the source may also be deformed. This study seeks the optimal current and voltage wave-form by means of an optimization criteria. The optimization problem is defined in Hilbert space and the special functionals are minimized. The source inner impedance operator is linear and time-varying. Some examples of calculations are presented.

Keywords: electrical circuits, power theory, optimization in Hilbert spaces.

1. Introduction

The transmission of electric power from the source to the load is performed in one loop circuit, and despite its simplicity there are many unsolved problems related to it. They result from the fact that in the energy transporting process there are two signals involved the voltage and current, and they can produce the same power with various, sometimes unpredictable, wave forms. Applying some quality criteria we can find the optimal wave shape of the voltage and current signals. The power quality discipline usually deals with this crucial problem. Let us consider a simple one loop circuit depicted in Fig. 1.

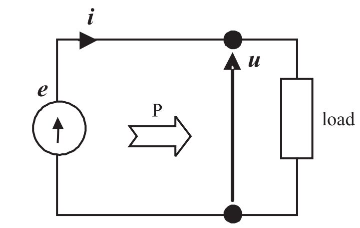

Fig. 1. The power transfer circuit

Such a circuit serves to transport power from the source $\boldsymbol{e}$ to the receiver. The problem, which seems to be easy, is to find such a signal $\boldsymbol{i}$ to assure a given prescribed power P and it can be solved only by means of optimization technique.

The source power equation

$$P = (e, i) \tag{1}$$

obviously leads to an abundance of solutions since there is an infinite amount of such signals $\boldsymbol{e}$ and $\boldsymbol{i}$ which lead to the same dot product, equal to active power P. In order to choose the unique solution from their infinite set, we must put one more optimum condition:

$$||i||^2 = (i, i) \to \min \tag{2}$$

The task (1)–(2) has only one solution which we get by minimizing the Lagrange functional

$$f(i, \lambda) = (\boldsymbol{i}, \boldsymbol{i}) + \lambda[P - (\boldsymbol{e}, \boldsymbol{i})] \to \min \tag{3}$$

where λ is real.

Indeed, calculating the variation of the functional (3) caused by the variation of current δi we get:

$$\begin{aligned} f(i + \delta i, \lambda) - f(i, \lambda) & \\ &= 2(i, \delta i) - (\lambda e, \delta i) + (2\lambda Ri, \delta i) + (\delta i, \delta i) \\ &= (2i - \lambda e, \delta i) + (\delta i, \delta i). \end{aligned} \tag{4}$$

From the previous equation it follows that the sufficient and necessary condition for the minimum (3) exists

$$2\boldsymbol{i} - \lambda \boldsymbol{e} = 0$$

or

$$i_\lambda = 0.5\lambda \boldsymbol{e}. \tag{5}$$

Now it is necessary to choose one i_λ from the abundance of solutions according to the assumption (1) i.e.

$$P = (\boldsymbol{e}, 0.5\lambda \boldsymbol{e}) = 0.5\lambda(\boldsymbol{e}, \boldsymbol{e}) = 0.5\lambda||\boldsymbol{e}||^2$$

thus

$$\lambda = \frac{2P}{||e||^2}.$$

So the optimal current signal meeting condition (1)–(2) is

$$\boldsymbol{i} =: i_{opt} = \frac{P}{||e||^2} e. \tag{6}$$

2. One loop circuit power distribution

In more complex problems the source has a passive linear inner impedance determined by the $\boldsymbol{Z}$ operator.

In such a circuit the relation between u and $\boldsymbol{i}$ is determined by

$$U = e - Zi$$

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which we can use to calculate the balance of the circuit power

$$P = (u, i) = (e, i) - (Zi, i).$$

If we notice that $(Zi, i) = (i, iZ^*)$, where '*' stands for conjugating operation, we get

$$P = (\boldsymbol{u}, \boldsymbol{i}) = (\boldsymbol{e}, \boldsymbol{i}) - (\boldsymbol{Ri}, \boldsymbol{i}) \tag{7}$$

where

$$\boldsymbol{R} = 0.5(\boldsymbol{Z} + \boldsymbol{Z}^*) \tag{8}$$

is a positively defined linear operator.

The power balance consists of three power flux: $(\boldsymbol{e}, \boldsymbol{i})$ – the power of the source, $\Delta P_Z = (\boldsymbol{Ri}, \boldsymbol{i})$ – the power losses in the source, P – the developed or absorbed power. The $(\boldsymbol{Ri}, \boldsymbol{i})$ is positive for any current signal (is positively defined) whereas the two others can be both positive or negative– it is depicted in Fig. 2.

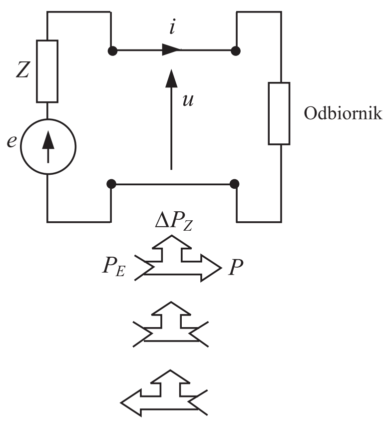

Fig. 2. Power distribution

The Eq. (7) with the unknown current vector also has infinitely many solutions. But the optimization task

$$\begin{aligned} &(i, i) \to \min \\ &(\boldsymbol{e}, \boldsymbol{i}) - (\boldsymbol{Ri}, \boldsymbol{i}) = P \end{aligned} \tag{9}$$

as it is easy to prove, will have the unique solution $\boldsymbol{i}$.

We can find it by minimizing the functional

$$f(\boldsymbol{i}, \boldsymbol{\lambda}) = (\boldsymbol{i}, \boldsymbol{i}) + \lambda[(\boldsymbol{Ri}, \boldsymbol{i}) - (\boldsymbol{e}, \boldsymbol{i}) + P] \to \min \tag{10}$$

which difference

$$\begin{aligned} \delta f(\boldsymbol{i}, \lambda) &= f(\boldsymbol{i} + \delta \boldsymbol{i}, \lambda) - f(\boldsymbol{i}, \lambda) \\ &= (2\boldsymbol{i}, \delta \boldsymbol{i}) + (2\lambda \boldsymbol{Ri}, \delta \boldsymbol{i}) - (\lambda \boldsymbol{e}, \delta \boldsymbol{i}) + (\delta \boldsymbol{i}, \delta \boldsymbol{i}) \\ &\quad + (\lambda \boldsymbol{R} \delta \boldsymbol{i}, \delta \boldsymbol{i}) \\ &= (2(\boldsymbol{i} + \lambda \boldsymbol{Ri}) - \lambda \boldsymbol{e}, \delta \boldsymbol{i}) + (\delta \boldsymbol{i} + \lambda \boldsymbol{R} \delta \boldsymbol{i}, \delta \boldsymbol{i}) \end{aligned} \tag{11}$$

is positive for all $\delta \boldsymbol{i}$. It will be true when the current signal $\boldsymbol{i}$ meets

$$2(\boldsymbol{i} + \lambda \boldsymbol{Ri}) - \lambda \boldsymbol{e} = 0$$

or

$$(\mathbf{1} + \lambda \boldsymbol{R})\boldsymbol{i} = 0.5\lambda \boldsymbol{e} \tag{12}$$

where: $\mathbf{1}$ denotes an identity operator, 0 — zero signal.

The Eq. (12) includes $(\mathbf{1} + \lambda \boldsymbol{R})$ operator with a λ parameter. The same operator appears in a quadratic form forming the second part of the increment (11).

For λ that $(\mathbf{1} + \lambda \boldsymbol{R})$ operator is positively defined, there exists a solution of (12) and simultaneously it is the minimum point of (10).

The λ for which the solution of (12) $\boldsymbol{i}_\lambda$ exists also determines the function

$$F(\lambda) = (\boldsymbol{e}, \boldsymbol{i}_\lambda) - (\boldsymbol{Ri}_\lambda, \boldsymbol{i}_\lambda \tag{13}$$

But there are also such values of λ for which $F(\lambda)$ is undefined. To find them we must use the operator spectrum of $\boldsymbol{R}$.

The operator spectrum of $\boldsymbol{R}$ is the set of λ such that the $(\mathbf{1} + \lambda \boldsymbol{R})$ operator is non-reversible. It means that the equation

$$(\lambda \mathbf{1} - \boldsymbol{R})x = 0$$

has a non-zero solution. Hence multiplying this equation by the scalar x we get:

$$\lambda ||\boldsymbol{x}||^2 = (\boldsymbol{Rx}, \boldsymbol{x}) \tag{14}$$

then the operator $\boldsymbol{R}$ spectrum consists uniquely of the positive real numbers, because the $\boldsymbol{R}$ is a self-adjoint, positively defined operator. It results from (14) that the spectrum is a non-empty closed and bounded set because the $\boldsymbol{R}$ is bounded.

In order to study $F(\lambda)$ function we must calculate the derivative of $\boldsymbol{i}_\lambda$ signal (the $\boldsymbol{i}'_\lambda$ exists in the determinate set of $F(\lambda)$), by differentiate equation (12) in the λ direction

$$(\mathbf{1} + \lambda \boldsymbol{R})\boldsymbol{i}'_\lambda = 0.5(\boldsymbol{e} - 2\boldsymbol{Ri}_\lambda) \tag{15}$$

From (13) follows the derivative formula for $F(\lambda)$:

$$\begin{aligned} F'(\lambda) &= (\boldsymbol{e}, \boldsymbol{i}'_\lambda) - (2\boldsymbol{Ri}_\lambda, \boldsymbol{i}'_\lambda) = (\boldsymbol{e} - 2\boldsymbol{Ri}_\lambda, \boldsymbol{i}'_\lambda) \\ &= 2((\mathbf{1} + \lambda \boldsymbol{R})\boldsymbol{i}'_\lambda, \boldsymbol{i}'_\lambda) \end{aligned}$$

the resulting quadratic form is positively or negatively definite, only when $(\mathbf{1} + \lambda \boldsymbol{R})$ is positively or negatively definite i.e. when the functional (10) has maximum or minimum value. It follows that the $F(\lambda)$ function has two branches: ascending and decreasing.

For the prescribed power $P < P_{\max}$ drawn from the source the equation

$$F(\lambda) = P \tag{16}$$

has two solutions of λ multipliers for which the functional (10) has respectively: a minimum – for λ from the ascending branch of $F(\lambda)$ and maximum from decreasing one. The function $F(\lambda)$ with the power fluxes for appropriate λ ranges is depicted in Fig. 3. It entails the existence of two extreme source currents: $\boldsymbol{i}_{\min}$ and $\boldsymbol{i}_{\max}$, which minimize or maximize respectively the functional (2). From the practical point of view more important is the $\boldsymbol{i}_{\min}$ current.

The $F(\lambda)$ function shows the relation between the delivered power of the source and the λ coefficient and it can be called a *power characteristics of the source.*

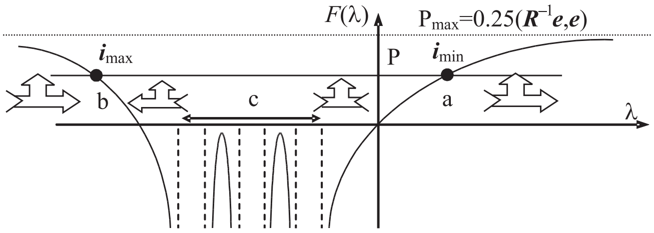

Fig. 3. The power characteristics of the source Monotonic branch a) ascending, b) decreasing, c) singular

The more detailed examination of the $F(\lambda)$ function shows that for $\lambda = 0 \rightarrow (\boldsymbol{i}_\lambda = 0)$, $F(0) = 0$, oraz $(\boldsymbol{i}'_\lambda = 0.5\boldsymbol{e})$, $F'(0) = 0.5||\boldsymbol{e}||^2$ the function is zero and has a positive derivation.

Whereas for $\lambda \pm \infty \rightarrow (\boldsymbol{i}_\lambda = 0.5\boldsymbol{R}^{-1}\boldsymbol{e})$,

$$F(\lambda) = 0.25(\boldsymbol{R}^{-1}\boldsymbol{e}, \boldsymbol{e}) = P_{\max}$$

which also results from the limits of the functional (10)

$$f(\boldsymbol{i}, \lambda \rightarrow \infty) \quad [(\boldsymbol{Ri}, \boldsymbol{i}) - (\boldsymbol{e}, \boldsymbol{i}) + P] \rightarrow \min$$

or

$$-f(\boldsymbol{i}, \lambda \rightarrow \infty) \quad -[(\boldsymbol{Ri}, \boldsymbol{i}) - (\boldsymbol{e}, \boldsymbol{i}) + P] \rightarrow \max$$

and using the variation method we get

$$\boldsymbol{Ri} = 0.5e.$$

It is equivalent to the solution of the so called receiver matching problem according to the maximum power given and it is discussed in the literature for various particular current spaces and impedance operators of the source. For the lossless source with the zero loss operator $\boldsymbol{R} : \boldsymbol{R} = 0$, Eq. (12) gives

$$\boldsymbol{i}_\lambda = 0.5\lambda\boldsymbol{e}$$

and

$$F(\lambda) = 0.5||\boldsymbol{e}||^2\lambda$$

which means that the power characteristics of the source is linear and unlimited. The lossy source has always the power characteristics limited by the maximum power $P_{\max}$.

Example 1. Let us find the power characteristic of the source with the time varying inner impedance (Fig. 4).

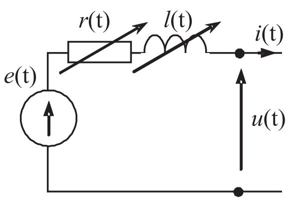

Fig. 4. The equivalent circuit diagram of the source with the time varying resistor-inductance inner impedance

In this case

$$Z = \left[r(t) + l(t)\frac{d}{dt} \right]$$

and it affects a current signal according to the formula

$$Zi(t) = \left[r(t)i(t) + l(t)\frac{di}{dt} \right].$$

In order to define the conjugated operator $\boldsymbol{Z}^*$ we calculate the quadratic form

$$(\boldsymbol{Zi}, \boldsymbol{i}) = \int_{-\infty}^{\infty} r(t)[i(t)]^2 dt + \infty{-}\infty^{\infty} l(t)\frac{di}{dt}i(t)dt.$$

After applying the Liouville's transformation in integration by parts of the second component we get

$$(\boldsymbol{Zi}, \boldsymbol{i}) = \int_{-\infty}^{\infty} r(t)[i(t)]^2 dt - \int_{-\infty}^{\infty} \frac{d[l(t)i(t)]}{dt} i(t)dt = (Z^*\boldsymbol{i}, \boldsymbol{i})$$

of which results the conjugated operator of the form

$$\boldsymbol{Z}^* = \left[r(t) - l(t)\frac{d}{dt} - \frac{dl(t)}{dt} \right].$$

Therefore the loss operator has the form

$$\boldsymbol{R} = 0.5(\boldsymbol{Z} + \boldsymbol{Z}^*) = \left[r(t) - 0.5\frac{dl(t)}{dt} \right]. \qquad (17)$$

The operator equation to calculate the optimal current signal (12) has now the algebraic form

$$\left(1 + \lambda \left[r(t) - 0.5\frac{dl(t)}{dt} \right] \right) i(t) = 0.5\lambda e(t).$$

Thus the power characteristics of the source $F(\lambda)$ is

$$F(\lambda) = \frac{\lambda}{2} \int_{-\infty}^{\infty} \frac{1 + 0.5\rho(t)}{[1 + \lambda\rho(t)]^2} [e(t)]^2 dt$$

where $\rho(t) = r(t) - 0.5\frac{dL(t)}{dt}$.

Analyzing the foregoing denominator of the integral we find the singularities for $\lambda < 0$ when for the positively defined two terminal networks is

$$r(t) - 0.5\frac{dL(t)}{dt} > 0$$

for any t.

Example 2. In the case of a stationary circuit the operator (8) is defined with the single variable function of s

$$\boldsymbol{R}(s) = 0.5(\boldsymbol{Z}(s) + \boldsymbol{Z}(-s))$$

where $\boldsymbol{Z}(s)$ is an ordinary operator function of the source inner impedance, and $\boldsymbol{Z}(-s)$ is its conjugated operator. It follows from the Liouville's transformation of a single difference operator

$$\int_{-\infty}^{\infty} (s\boldsymbol{x})(t)\boldsymbol{y}(t)dt = \int_{\infty}^{\infty} \boldsymbol{x}(t)[(-1)s\boldsymbol{y}]dt.$$

Thus for the conjugated operator the following formula is in force

$$s^* = s^{-1}.$$

The operator Eq . (12) has then the following form

$$[\boldsymbol{1} + \lambda\boldsymbol{R}(s)]\boldsymbol{I}(s) = 0.5\lambda\boldsymbol{E}(s) \qquad (18)$$

where $\boldsymbol{I}(s)$, $\boldsymbol{E}(s)$ stand for the two-sided Laplace transformation of the source current and voltage.

Equation (18) instantly gives the solution

$$I_\lambda(s) = \frac{0.5\lambda E(s)}{1+\lambda R(s)}.$$

The power characteristics of the source (13) is defined by the Parsevall's formula:

$$\begin{aligned}
&F(\lambda) = (\boldsymbol{e}, \boldsymbol{i}_\lambda) - (\boldsymbol{R}\boldsymbol{i}_\lambda, \boldsymbol{i}_\lambda)\\
&= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} E(j\omega)I_\lambda(-j\omega)d(j\omega)\\
&\quad - \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} R(j\omega)I_\lambda(j\omega)I_\lambda(-j\omega)d(j\omega)\\
&= \frac{1}{2\pi j}\int_{Re(s)=0} E(s)I_\lambda(-s)ds\\
&\quad - \frac{1}{2\pi j}\int_{Re(s)=0} R(s)I_\lambda(s)I_\lambda(-s)ds\\
&= \frac{1}{2\pi j}\int_{Re(s)=0}\left(\frac{2}{1+\lambda R(s)} - \frac{\lambda R(s)}{[1+\lambda R(s)]^2}\right)E(s)E(-s)ds\\
&= \frac{1}{8\pi j}\int_{Re(s)=0}\left(\frac{2+\lambda R(s)}{[1+\lambda R(s)]^2}\right)E(s)E(-s)ds.
\end{aligned}$$

From the Jordan lemma the integrals along the imaginary axis are substituted by the integrals along the contour including the right or left half plain i.e. the contour consists of an imaginary axis and an appropriate semi circle with the (0,0) center point and a radius approaching to infinity. These contours are marked as $\subset$ and $\supset$. Thus

$$F(\lambda) = \frac{1}{8\pi j}\int_{\subset\vee\supset}\left(\frac{2+\lambda R(s)}{[1+\lambda R(s)]^2}\right)E(s)E(-s)ds. \qquad (19)$$

The formula (19) is suitable for a rational function in integral, when we can use the Cauchy integral formula.

The derivative of $F'(\lambda)$ has the form

$$F'(\lambda) = \frac{1}{4\pi j}\int_{\subset\vee\supset}\left(\frac{E(s)E(-s)}{[1+\lambda R(s)]^3}\right)ds.$$

The root λ_* of the Eq. (16) for the prescribed power $P \in (-\infty, P_{\max})$ is on the ascending branch of $F(\lambda)$ and determines a particular case of the current signal i.e. the optimal current. The root can be found both graphically or analytically using the Newton's method.

$$\lambda_{k+1} = \lambda_k + \frac{P - F(\lambda_k)}{F'(\lambda_k)} = \Gamma(\lambda_k). \qquad (20)$$

This procedure is always convergent for $\lambda > 0$

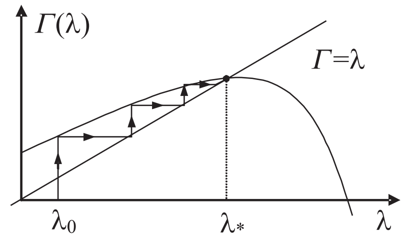

Fig. 5. $\Gamma(\lambda)$ function subsequent and consecutive iteration points

It results from the shape of the $\Gamma(\lambda)$ function – its graph and consecutive iteration points are shown in Fig. 5 [1, 2].

3. Power analysis of the circuit by means of other evaluation functionals

In order to calculate the optimal source current other practical functionals, besides (9), can be used. Introducing the prescribed reference signal $\boldsymbol{i}_0$ we minimize the functional of the current deviation

$$\Delta\boldsymbol{i} = \boldsymbol{i} - \boldsymbol{i}_0.$$

We can create optimization task in the form

$$\begin{aligned}&(\Delta\boldsymbol{i}, \Delta\boldsymbol{i}) \to \min\\ &(\boldsymbol{e}, \boldsymbol{i}) - (\boldsymbol{R}\boldsymbol{i}, \boldsymbol{i}) = P\end{aligned} \qquad (21)$$

The task (21) has a similar solution to the task (9) – considered in the preceding chapter. By analogy we can prove that the minimum condition of (21) result from equation

$$(\boldsymbol{1} + \lambda\boldsymbol{R})\boldsymbol{i} = 0.5\lambda\boldsymbol{e} + \boldsymbol{i}_0. \qquad (22)$$

The power characteristics of the source (13) is similar to the previous one from the Section 2.

From the practical point of view it is important to minimize the voltage deviation on the source terminals referred to the prescribed signal $\boldsymbol{u}_0$:

$$\Delta\boldsymbol{u} = \boldsymbol{u}_0 - \boldsymbol{u}.$$

It leads to the following minimizing problem:

$$\begin{aligned}&(\Delta\boldsymbol{u}, \Delta\boldsymbol{u}) \to \min\\ &(\boldsymbol{e}, \boldsymbol{i}) - (\boldsymbol{R}\boldsymbol{i}, \boldsymbol{i}) = P.\end{aligned} \qquad (23)$$

If we note that

$$\Delta\boldsymbol{u} = \boldsymbol{Z}\boldsymbol{i} - \Delta\boldsymbol{e}$$

where: $\Delta\boldsymbol{e} = \boldsymbol{e} - \boldsymbol{u}_0$, $\boldsymbol{Z}$ – the inner source impedance operator, we can form the Lagrange functional depended on the current

$$f_\lambda(\boldsymbol{i}) = (\Delta\boldsymbol{u}, \Delta\boldsymbol{u}) + \lambda[(\boldsymbol{R}\boldsymbol{i}, \boldsymbol{i}) - (\boldsymbol{e}, \boldsymbol{i}) + P] \to \min \quad (24)$$

and its deviation (noting that $\delta(\Delta\boldsymbol{u}) = \boldsymbol{Z}\delta\boldsymbol{i}$) gives

$$\begin{aligned}\delta f_\lambda(\boldsymbol{i}) &= f_\lambda(\boldsymbol{i} + \delta\boldsymbol{i}) - f_\lambda(\boldsymbol{i})\\ &= (2\boldsymbol{Z}^*\Delta\boldsymbol{u} + 2\lambda\boldsymbol{R}\boldsymbol{i} - \lambda\boldsymbol{e}, \delta\boldsymbol{i})\\ &\quad + ((\boldsymbol{Z}^*\boldsymbol{Z} + \lambda\boldsymbol{R})\delta\boldsymbol{i}, \delta\boldsymbol{i})\ldots\end{aligned} \qquad (25)$$

From (25) results the sufficient and necessary condition for the minimum (24) to exist.

$$\boldsymbol{Z}^*\Delta\boldsymbol{u} + \lambda\boldsymbol{R}\boldsymbol{i} = 0.5\lambda\boldsymbol{e}$$

or

$$(\boldsymbol{Z}^*\boldsymbol{Z} + \lambda\boldsymbol{R})\boldsymbol{i} = 0.5\lambda\boldsymbol{e} + \boldsymbol{Z}^*\Delta\boldsymbol{e}. \quad (26)$$

The operator Eq. (26) gives the optimal current for the (24) criterion

$$\boldsymbol{i}_\lambda = (\boldsymbol{Z}^*\boldsymbol{Z} + \lambda\boldsymbol{R})^{-1}(0.5\lambda\boldsymbol{e} + \boldsymbol{Z}^*\Delta e). \quad (27)$$

The characteristics $F(\lambda)$ is defined in λ points for which the solution of (26) exists i.e. when the invert operator $(\boldsymbol{Z}^*\boldsymbol{Z} + \lambda\boldsymbol{R})^{-1}$ exists.

The function $F(\lambda)$ has a derivative

$$F'(\lambda) = (\boldsymbol{e} - 2\boldsymbol{R}\boldsymbol{i}_\lambda, \boldsymbol{i}'_\lambda)$$

where $\boldsymbol{i}'_\lambda$ is the current derivative in λ and is calculated by differencing the Eq. (26)

$$(\boldsymbol{Z}^*\boldsymbol{Z} + \lambda\boldsymbol{R})\boldsymbol{i}'_\lambda = 0.5(\boldsymbol{e} - 2\boldsymbol{R}\boldsymbol{i}'_\lambda). \quad (28)$$

From (27), (28) it results that $F'(\lambda)$ is a quadratic form

$$F'(\lambda) = (2(\boldsymbol{Z}^*\boldsymbol{Z} + \lambda\boldsymbol{R})\boldsymbol{i}'_\lambda, \boldsymbol{i}'_\lambda) \quad (29)$$

with $(\boldsymbol{Z}^*\boldsymbol{Z} + \lambda\boldsymbol{R})$ operator, which also appear in (25). The positive definition of this operator means the monotonically ascending branch of $F(\lambda)$ and the existence of the minimum of the functional (24) there. Whereas the negative definition of this operator means the monotonically decreasing branch of $F(\lambda)$ and the existence of the minimum of the functional (24) there.

The $F(\lambda)$ function has the following quality for $\lambda = 0$

$$(\boldsymbol{Z}^*\boldsymbol{Z})^{-1}\boldsymbol{Z}^*\Delta\boldsymbol{e} = \boldsymbol{Z}^{-1}\Delta\boldsymbol{e} = \boldsymbol{Y}\boldsymbol{e}$$

$$F(0) = (\boldsymbol{e}, \boldsymbol{Y}\Delta\boldsymbol{e}) - (\boldsymbol{Y}^*\boldsymbol{R}\boldsymbol{Y}\Delta\boldsymbol{e}, \Delta\boldsymbol{e})$$

$\lambda \to \infty$

$$F(\lambda) \to 0.25(\boldsymbol{R}^{-1}\boldsymbol{e}, \boldsymbol{e}) = P_{\max}$$

The graph 6 shows $F(\lambda)$ function

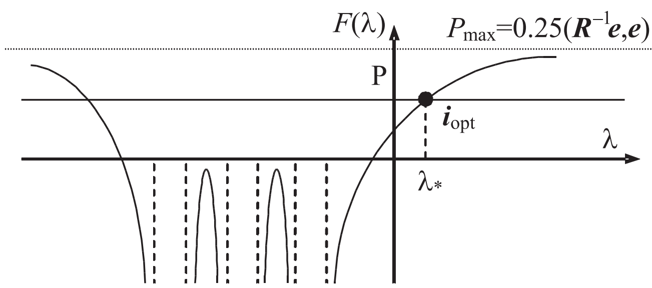

Fig. 6. The power characteristic of a source

The calculated optimal current signal (27) for (23) condition can be written in a form

$$\boldsymbol{i}_{opt} = \boldsymbol{G}_{opt}\boldsymbol{e} + \boldsymbol{j}_0 \quad (30)$$

where: $\boldsymbol{G}_{opt} = 0.5\lambda\boldsymbol{e}(\boldsymbol{Z}^*\boldsymbol{Z} + \lambda\boldsymbol{R})^{-1}$,

$$\boldsymbol{j}_0 = (\boldsymbol{Z}^*\boldsymbol{Z} + \lambda\boldsymbol{R})^{-1}\boldsymbol{Z}^*\Delta\boldsymbol{e}.$$

$\boldsymbol{G}_{opt}$ – self-adjoint, positively defined linear operator, $\boldsymbol{j}_0$ – additional current signal.

This result is depicted in Fig. 7.

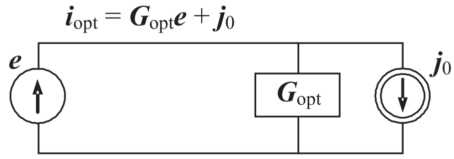

Fig. 7. The equivalent circuit diagram of the optimal current

It is worth considering the particular optimal solutions when the source is lossless i.e. when $\boldsymbol{R}$ is the zero operator. In that case the Eq. (26) is reduced to

$$\boldsymbol{Z}^*\boldsymbol{Z}\boldsymbol{i} = 0.5\lambda\boldsymbol{e} + \boldsymbol{Z}^*\Delta e \quad (31)$$

and can be solved directly

$$\boldsymbol{i}_\lambda = 0.5\lambda\boldsymbol{Y}\boldsymbol{Y}^*e + \boldsymbol{Y}\Delta\boldsymbol{e} \quad (32)$$

where $\boldsymbol{Y} = \boldsymbol{Z}^{-1}$ is the inner admittance of the source.

The power characteristics

$$F(\lambda) = 0.5\lambda(\boldsymbol{Y}\boldsymbol{Y}^*\boldsymbol{e}, \boldsymbol{e}) + (\boldsymbol{Y}^*\Delta e, e) \quad (33)$$

is then an affine function.

The solution of (16) is a point

$$\lambda_* = 2\frac{P - \varDelta P}{(YY^*e, e)} \quad (34)$$

where

$$\varDelta P = (\boldsymbol{Y}\varDelta\boldsymbol{e}, \boldsymbol{e}).$$

So the optimal current of a lossless but flexible source, giving the closest voltage to $\boldsymbol{u}_0$, is given by the formula:

$$\boldsymbol{i}_{opt} = \frac{P - \varDelta P}{(YY^*e, e)}\boldsymbol{Y}\boldsymbol{Y}^*\boldsymbol{e} + \boldsymbol{Y}\varDelta\boldsymbol{e}. \quad (35)$$

These results even get reduced for $\boldsymbol{u}_0 = \boldsymbol{e}$, when $\Delta\boldsymbol{e} = 0$ and the formulas (29)–(34) take the form

$$\boldsymbol{i}_\lambda = 0.5\lambda\boldsymbol{Y}\boldsymbol{Y}^*\boldsymbol{e}$$

The $F(\lambda)$ function is linear

$$F(\lambda) = 0.5\lambda(\boldsymbol{Y}\boldsymbol{Y}^*\boldsymbol{e}, \boldsymbol{e}). \quad (36)$$

Thus the optimal source current giving the minimal voltage drop on terminals is

$$\boldsymbol{i}_{opt} = \frac{P}{(YY^*e, e)}\boldsymbol{Y}\boldsymbol{Y}^*\boldsymbol{e}. \quad (37)$$

It is worth comparing the foregoing current form to the result (6), for the current norm criterion

$$||\boldsymbol{i}||^2 \to \min : \boldsymbol{i}_{opt} = \frac{P}{(e, e)}\boldsymbol{e}$$

which exists in the literature as a Fryze current [1, 2, 3–8, 9] and it is a particular case of the current (37), which is easy to see after substituting

$$\boldsymbol{Y}\boldsymbol{Y}^*\boldsymbol{e} = \boldsymbol{e}'.$$

Therefore the optimal current assuring the minimal inner RMS voltage drop($(\Delta\boldsymbol{u}, \Delta\boldsymbol{u}) \to \min$) is given by

$$\boldsymbol{i}_{opt} = \frac{P}{(e', e)}\boldsymbol{e}'. \quad (38)$$

Example 3. It is easy to show a definite example of the result (37). Let's take the two terminal network with a time invariant and reactive elements. Its impedance function is then an odd function

$$\boldsymbol{Z}(s) := s\boldsymbol{X}(s^2) \tag{39}$$

where $\boldsymbol{X}(s^2)$ – even rational function. Thus the loss function is

$$\begin{aligned}\boldsymbol{R}(s) &= 0.5(\boldsymbol{Z}(s) + \boldsymbol{Z}(-s)) \\ &= 0.5[s\boldsymbol{X}(s^2) + (-s)\boldsymbol{X}((s^2))] = 0.\end{aligned}$$

The (39) operator has the inversion (admittance)

$$\boldsymbol{Y}(s) = \frac{1}{sX(s^2)}$$

then

$$\boldsymbol{Y}(s)\boldsymbol{Y}^*(s) = \boldsymbol{Y}(s)\boldsymbol{Y}(-s) = \frac{-1}{s^2[X(s^2)]^2}$$

$$\begin{aligned}(\boldsymbol{Y}\boldsymbol{Y}^*\boldsymbol{e},\boldsymbol{e}) &= -\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{E(s)E(-s)}{s^2[X(s^2)]^2}ds \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|E(j\omega)|^2}{\omega^2[X(-\omega^2)]^2}d\omega. \end{aligned} \tag{40}$$

Applying (39) and (40) in (37) we get the two sided Laplace transformation of the optimal current

$$\boldsymbol{I}_{opt}(s) = \frac{-P}{\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \frac{|E(j\omega)|^2}{\omega^2[X(-\omega^2)]^2}d\omega} \cdot \frac{E(s)}{s^2[X(s^2)]^2}. \tag{41}$$

Given result (41) can be reduced e.g. for the almost periodic signals. In that case the function $F(s)$ has the values

$$Z(j\omega) = j\omega X(-\omega^2)$$

where $\omega \in \{\omega_n : n = 0, 1, 2, ...\}$ is a countable set of harmonic components making signal $e(t)$:

$$e(t) = real\left[\sum_{n=0}^{\infty}\sqrt{2}E_n e^{j\omega_n t}\right] = \sum_{n=-\infty}^{\infty}\frac{E_n}{\sqrt{2}}e^{j\omega_n t} \tag{42}$$

where:

$$E_{-n} = E_n^*, \quad \omega_{-n} = -\omega_n. \tag{43}$$

Defining the inner product of the almost periodic signals

$$(u, i) = \lim_{T\to\infty}\frac{1}{T}\int_0^T u(t)i(t)dt \tag{44}$$

and using (42) in (44) we get a particular form of the Parsevall formula:

$$\begin{aligned}(\boldsymbol{u},\boldsymbol{i}) &= \sum\sum_{n,m=-\infty}^{\infty}\frac{1}{2}U_n I_m \lim_{T\to\infty}\frac{1}{T}\int_0^T e^{j(\omega_n+\omega_m)t}dt \\ &= \frac{1}{2}\sum_{n=-\infty}^{\infty}U_n I_n^* = real\sum_{n=0}^{\infty}U_n I_n^* = real\sum_{n=0}^{\infty}U_n^* I_n\end{aligned}$$

where $I_{-n} = I_n^*$ because

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{j(\omega_n+\omega_m)t}dt = \begin{cases}1 & \text{dla } n+m=0 \\ 0 & \text{dla } n+m\neq 0\end{cases}.$$

In such a case the quadratic form (40) gives

$$(YY^*e, e) = \sum_{n=0}^{\infty}\frac{|E_n|^2}{\omega_n^2 X_n^2}$$

where: $X_n = X(-\omega_n^2)$ whereas the spectrum of the almost periodic current signal – a counterpart of the current (41) – is given by

$$I_n^{opt} = \frac{P}{\sum_{m=0}^{\infty}\frac{|E_m|^2}{\omega_m^2 X_m^2}}\frac{E_n}{\omega_n^2 X_n^2}. \tag{46}$$

Note that the forms (45) and (46) remain valid for periodic signals.

4. Power analysis of the circuit by means of multi criterion optimization tasks

In the previous chapter we have formulated some practical quality criterions concerning power transmission in a one loop circuit. In order to satisfy many individual receivers at the same time, we need to form some mixed optimization criterions to give reasonable compromises. They can be e.g. the two criterion optimization tasks like:

$$\begin{gathered}(i, i) \to \min \\ (e, i) - (Ri, \boldsymbol{i}) = P \\ (\Delta i, \Delta i) = Q\end{gathered} \tag{47}$$

or

$$\begin{gathered}(i, i) \to \min \\ (e, i) - (\boldsymbol{R}i, i) = P \\ (\Delta u, \Delta u) = Q\end{gathered} \tag{48}$$

where: $\Delta\boldsymbol{i} = \boldsymbol{i} - \boldsymbol{i}_0$, $\Delta\boldsymbol{u} = \boldsymbol{u} - \boldsymbol{u}_0$ – current and voltage deviations, $\boldsymbol{i}_0$, $\boldsymbol{u}_0$ – prescribed signals of the current and voltage.

In the condition (47) Q is the acceptable distortion measure of $\boldsymbol{i}$ regarding $\boldsymbol{i}_0$ or acceptable distortion measure of $\boldsymbol{u}$ regarding $\boldsymbol{u}_0$ (48). It is obvious that these are not the unique practical optimization conditions of the source signals. As further examples we can formulate

$$\begin{gathered}(\Delta\boldsymbol{i}, \Delta\boldsymbol{i}) \to \min \\ (\boldsymbol{e}, \boldsymbol{i}) - (\boldsymbol{R}\boldsymbol{i}, \boldsymbol{i}) = P \\ (\Delta\boldsymbol{i}, \Delta\boldsymbol{i}) = Q\end{gathered} \tag{49}$$

or

$$\begin{gathered}(\Delta\boldsymbol{i}, \Delta\boldsymbol{i}) \to \min \\ (\boldsymbol{e}, \boldsymbol{i}) - (\boldsymbol{R}\boldsymbol{i}, \boldsymbol{i}) = P \\ (\Delta\boldsymbol{u}, \Delta\boldsymbol{u}) = Q\end{gathered} \tag{50}$$

These tasks are formulated correctly and all of theme have a unique solution [60]. The task (47) is equivalent to minimize the Lagrange's functional

$$f_{\lambda,\mu}(\boldsymbol{i}) = (\boldsymbol{i},\boldsymbol{i}) + \lambda[(\boldsymbol{Ri},\boldsymbol{i}) - (\boldsymbol{e},\boldsymbol{i})] + \mu(\boldsymbol{\Delta i},\boldsymbol{\Delta i}) \to \min \tag{51}$$

which difference in the minimum point is

$$\begin{aligned} f_{\lambda,\mu}(\boldsymbol{i}) &= f_{\lambda,\mu}(\boldsymbol{i}+\delta\boldsymbol{i}) - f_{\lambda,\mu}(\boldsymbol{i}) \\ &= (2\boldsymbol{i} + \lambda\boldsymbol{Ri} + 2\mu\boldsymbol{\Delta i} - \lambda\boldsymbol{e}, \delta\boldsymbol{i}) \\ &\quad + ((1+\mu 1 + \lambda\boldsymbol{R})\delta\boldsymbol{i}, \delta\boldsymbol{i}) > 0 \end{aligned} \tag{52}$$

for any variation of the current signal δi. Thus we get the necessary minimum condition

$$(1+\mu 1+\lambda\boldsymbol{R})\boldsymbol{i} = 0.5\lambda\boldsymbol{e} + \mu\boldsymbol{i}_0. \tag{53}$$

For the positively defined operator

$$A_{\lambda,\mu} =: 1 + \mu 1 + \lambda R \tag{54}$$

the equation has the solution $i_{\lambda,\mu}$. Then the vector function is defined

$$\boldsymbol{F}(\lambda,\mu) = \begin{bmatrix} F_\lambda(\lambda,\mu) \\ F_\mu(\lambda,\mu)\end{bmatrix} = \begin{bmatrix} (e, i_{\lambda,\mu}) - (Ri_{\lambda,\mu}, i_{\lambda,\mu}) \\ (i_{\lambda,\mu} - i_0, i_{\lambda,\mu} - i_0)\end{bmatrix}. \tag{55}$$

The first component of the foregoing function is the source active power and the second is the deviation norm of the current. This function can be called a power-distortion characteristics.

The search of λ_* and μ_* which determine the optimal current

$$i_{opt} := i_{\lambda_*,\mu_*} = A^{-1}_{\lambda_*,\mu_*}(0.5\lambda_*\boldsymbol{e} + \mu_*\boldsymbol{i}_0) \tag{56}$$

is performed using the following set of equations

$$\boldsymbol{F}(\lambda,\mu) = \begin{bmatrix} P \\ Q \end{bmatrix}. \tag{57}$$

It is the counterpart of the power Eq. (16), which appear in the one criterion optimization task. It is proved [2] that there exist such P and Q that above equation has the unique solution.

Example 4. The optimization equation which solves the problem (47) has form (53) but for lossless source it get reduced to

$$(1+\mu)i = 0.5\lambda e + \mu i_0 \tag{58}$$

and it can be solved directly

$$i = 0.5\frac{\lambda}{1+\mu}e - \frac{1}{1+\mu}i_0 + i_0 \tag{59}$$

therefore

$$\Delta i = \frac{0.5\lambda}{1+\mu}e - \frac{1}{1+\mu}i_0. \tag{60}$$

The component functions in (55) gives the form

$$\begin{aligned} F_\lambda(\lambda,\mu) &= 0.5\frac{\|e\|^2}{1+\mu}\lambda - \frac{1}{1+\mu}(e,i_0) + (e,i_0) \\ F_\mu(\lambda,\mu) &= 0.25\frac{\|e\|^2\lambda^2 - 4\lambda(e,i_0) + 4\|i_0\|^2}{(1+\mu)^2}. \end{aligned} \tag{61}$$

It turns out that for the lossless source the set of Eq. (57) can be unequivocally solved to λ and μ. It takes the following form

$$\begin{aligned} 0.5\|e\|^2\lambda - (e,i_0) &= (1+\mu)(e,\Delta i) \\ \|e\|^2\lambda^2 - 4(e,i_0)\lambda + 4\|i_0\|^2 &= 4\|\Delta i\|^2(1+\mu)^2. \end{aligned} \tag{62}$$

The solution of Eq. (62) is [2]

$$\begin{aligned} \lambda &= \frac{2}{\|e\|^2}\left[(e,i_0) \pm (e,\Delta i)\frac{K_0}{K}\right] \\ \mu &= \pm\frac{K_0}{K} - 1 \end{aligned} \tag{63}$$

where K_0 and K meet the Schwarz inequality

$$K^2 = \|e\|^2\|\Delta i\|^2 - (e,\Delta i)^2 > 0 \tag{64}$$

$$K_0^2 = \|e\|^2\|i_0\|^2 - (e,i_0)^2 > 0. \tag{65}$$

These coefficients have a characteristic form. They stand for the residual nonnative powers in the geometric differences between the active and apparent powers. They can be associated with the reactive powers of the source signals $\boldsymbol{\Delta i}$ and $\boldsymbol{i}_0$ respectively.

The minimum point demands that

$$(1+\mu) > 0$$

and it assures a positive definition of the operator (54). Then the solution (63) of (62) is unequivocal

$$\begin{aligned} \lambda_* &= \frac{2}{\|e\|^2}\left[(e,i_0) + (e,\Delta i)\frac{K_0}{K}\right] \\ \mu_* &= +\frac{K_0}{K} - 1 \end{aligned} \tag{66}$$

The optimal current is given by

$$\begin{aligned} i_{opt} &= \frac{1}{1+\mu_*}(0.5\lambda_* e - i_0) + i_0 \\ &= \frac{K}{K_0}0.5\lambda_* e + \left(1 - \frac{K}{K_0}\right)i_0 \\ &= \left[\left(\frac{K}{K_0} - 1\right)(e,i_0) + P\right]\frac{e}{\|e\|^2} + \left(1-\frac{K}{K_0}\right)i_0. \end{aligned}$$

Finally it will be convenient to rearrange it to the form

$$i_{opt} = \left[P - \left(1-\frac{K}{K_0}\right)P_0\right]\frac{e}{\|e\|^2} + \left(1-\frac{K}{K_0}\right)i_0 \tag{67}$$

where:

$P_0 = (e,i_0)$ – power of the ideal source giving the reference signal i_0;

$K_0 = \sqrt{\|e\|^2\|i_0\|^2 - P_0^2}$ – remaining (reactive) power of the ideal source;

$$K = \sqrt{\beta^2\|e\|^2\|i_0\|^2 - (P-P_0)^2} \tag{68}$$

$\beta^2 = \frac{Q}{\|i_0\|^2} = \frac{\|i-i_0\|^2}{\|i_0\|^2}$ – relative coefficient of the distorted current. (69)

From the (68) results a certain existence condition of the minimization solution

$$\beta^2\|e\|^2\|i_0\|^2 - (P-P_0)^2 \geqslant 0$$

or

$$\beta \geqslant \frac{|P-P_0|}{\|e\|\|i_0\|} \tag{70}$$

From (69) and (70) results one more form of the condition

$$Q \geqslant \frac{(P-P_0)^2}{\|e\|^2}. \tag{71}$$

Inequality (71) can be treated as an existence condition of the solution for the lossless source with the arbitrary assumed signals $\boldsymbol{e}$, $\boldsymbol{i}$ and scalars P, Q. The assumed relative distortion value cannot be too small.

The signal (67) is then the minimal RMS current with a prescribed distortion level, giving the prescribed source power P.

This result can be treated as one more generalization of the so called Fryze current (6) with an additional condition defining the signal distortion level.

Example 5. We will consider the minimal RMS current problem of the source with the prescribed voltage distortion (48). The appropriate solution of the optimization equation has the form

$$(1+\mu Z^*Z+\lambda R)i^\lambda = 0.5\lambda e + \mu Z^*\Delta e \tag{72}$$

where $\Delta\boldsymbol{e} = \boldsymbol{e} - \boldsymbol{u}_0$, $\boldsymbol{u}_0$ – prescribed voltage signals.

In a particular case of the lossless source and for $\boldsymbol{u}_0 = \boldsymbol{e}$ (the prescribed voltage equals to the open state voltage) we get the problem of finding the minimal RMS current having the prescribed RMS voltage drop in the source. Thus the Eq. (72) reduces to

$$(1+\mu Z^*Z)i = 0.5\lambda e. \tag{73}$$

This problem is a little more complex then in the Example 4. In the foregoing equation there is μZ^*Z operator instead of the μ scalar.

The Eq. (73) has the solution

$$i_{\lambda\mu} = 0.5\lambda(1+\mu Z^*Z)^{-1}e = 0.5\lambda K(\mu)e$$

where

$$K(\mu) = (1+\mu Z^*Z)^{-1} \tag{74}$$

is a linear self-adjoint operator, positively defined for $\mu > 0$. For $\mu = 0$ is

$$K(0) = 1 \qquad \text{(identityoperator)}$$

thus:

$$\begin{aligned} F_\lambda(\lambda,\mu) &= 0.5\lambda(K(\mu)e,e) \\ F_\mu(\lambda,\mu) &= (0.5\lambda)^2(Z^*ZK(\mu)e,K(\mu)e) \end{aligned} \tag{75}$$

From the equation

$$F_\lambda(\lambda,\mu) = P \tag{76}$$

we get

$$\lambda_*(\mu) = \frac{2P}{(K(\mu)e,e)},$$

from this results the function

$$\mu \to F_\mu[\lambda_*(\mu),\mu] = P^2\frac{(K(\mu)Z^*ZK(\mu)e,e)}{(K(\mu)e,e)^2}. \tag{78}$$

A more detailed examination of the function (78) shows that

$$F_\mu[\lambda_*(0),0] = P^2\frac{(Z^*Ze,e)}{\|e\|^4} \tag{79}$$

for $\mu\to\infty$:

$$\begin{aligned} K(\mu) &\to \mu^{-1}YY^* \\ \lambda_*(\mu) &\to \frac{2P}{(YY^*e,e)}\mu \\ F_\mu[\lambda_*(\mu),\mu] &\to \frac{P^2}{(YY^*e,e)}. \end{aligned} \tag{80}$$

Thus we get the function increase

$$\begin{aligned} &F_\mu[\lambda_*(0),0] - F_\mu[\lambda_*(\mu),\mu]_{\mu\to\infty} \\ &\quad = \frac{(Z^*Ze,e)(YY^*e,e)-(e,e)^2}{(YY^*e,e)}\left(\frac{P}{\|e\|^2}\right)^2 > 0. \end{aligned} \tag{81}$$

The proof of the inequality (81) will be carried out for the convolution operator. In a general situation the use of the spectral theory is needed. If the sequence $\{E_n\}_{n=0,\pm1,\pm2,\ldots}$ stands for the appropriate voltage harmonics of the source and $\{b_n\}_{n=0,\pm1,\pm2,\ldots}$ the module components of the $\boldsymbol{Z}$ spectrum then

$$\begin{aligned} &(Z^*Ze,e)(YY^*e,e)-(e,e)^2 \\ &\quad = \left(\sum_n b_n^2|E_n|^2\right)\left(\sum_m \frac{1}{b_m^2}|E_m|^2\right) \\ &\quad\quad - \left(\sum_n |E_n|^2\right)\left(\sum_m |E_m|^2\right) \\ &\quad = \sum_{n>m}\sum a_{nm}|E_n|^2|E_m|^2 \end{aligned}$$

where:

$$a_{nm} = \frac{b_n^2}{b_m^2} + \frac{b_m^2}{b_n^2} - 2 = \left(\frac{b_n^2-b_m^2}{b_nb_m}\right) > 0.$$

This completes the proof.

Then for Q between

$$\frac{P^2}{(YY^*e,e)} < Q < P^2\frac{(Z^*Ze,e)}{\|e\|^4} \tag{82}$$

there exist a unique root μ_* which is calculated from

$$P^2\frac{(K(\mu)Z^*ZK(\mu)e,e)}{(K(\mu)e,e)} = Q. \tag{83}$$

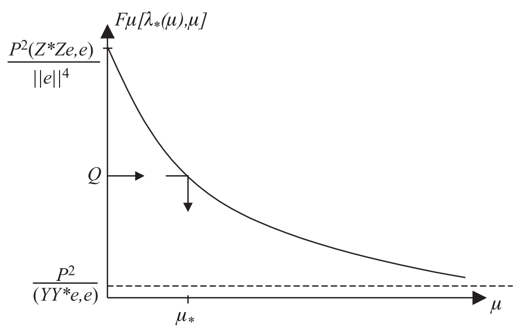

Fig. 8. The root determining process for the equation set (83)

Therefore the second coordinate of the optimum point (λ_*, μ_*) – the root of the power-distortion equations (56) is

$$\lambda_* = \frac{2P}{(K(\mu_*)e, e)}. \tag{84}$$

To complete the $F_\mu[\lambda_*(\mu), \mu]$ investigation it should be proved that it is monotonic and unequivocal, what is carried out in [2].

In a periodic steady state μ_* is the root of the convolution case of Eq. (83)

$$P^2 \frac{\sum\limits_n \left(\frac{X_n}{1+\mu X_n^2}\right)^2 |E_n|^2}{\sum\limits_{n,m}\sum \frac{|E_n|^2|E_m|^2}{(1+\mu X_n^2)(1+\mu X_m^2)}} = Q \tag{85}$$

Then the second coordinate of the optimum point (λ_*, μ_*) is:

$$\lambda_* = \frac{2P}{\sum\limits_n \frac{|E_n|^2}{1+\mu_* X_n^2}}. \tag{86}$$

Where X_n – stands for the spectrum of the inner source reactance (see ex. 3). The frequency distribution of the optimal current is given by

$$I_n^{opt} = \frac{0.5\lambda_*}{1+\mu_* X_n^2} E_n. \tag{87}$$

5. Conclusion

The afore presented results are connected with the problem of matching the load to the source.

This issue is presented in many disciplines connected not only with electrical energy transportation. It consists in the appropriate choice of the load to obtain the maximum power from the source. In the electrical domain this problem is reduced to the choice of an appropriate current signal which maximize source power functional i.e.

$$(\boldsymbol{e}, \boldsymbol{i}) - (\boldsymbol{Ri}, \boldsymbol{i}) \to \max.$$

Its solution is well known and leads to the operator equation [1, 2, 7–10]:

$$\boldsymbol{Ri} = 0.5e.$$

This is used to determine the power balance equation

$$P_{\max} = (\boldsymbol{e}, \boldsymbol{i}_{dop}) - 0.5(\boldsymbol{e}, \boldsymbol{i}_{dop})$$

where

$$\boldsymbol{i}_{dop} = 0.5\boldsymbol{R}^{-1}\boldsymbol{e}$$

is the matching current signal.

From the foregoing power balance it results that exactly one half of source power is lost. This fact makes such matching method extremely impractical. Nevertheless, the knowledge of possible maximum source power is useful.

It is possible to formulate a lot of more useful matching problems. In the case of e.g. stiff sources – keeping invariable voltage signal regardless of the current, there exists the unique current signal giving prescribed source power P.

This result is of a great practical importance and is known as the Fryze current. In the present article this signal is defined in the (6). However the problem becomes more sophisticated when we take in consideration the inner impedance $\boldsymbol{Z}$ operator of the source. It causes the source becomes lossy and flexible. Nevertheless we can still get a great variety of appropriate optimal currents using the minimum criteria, concerning the power balance of the circuit or shape restriction of signals. These results can be applied to improve power quality when transmitting it to many individual receivers. Obviously, to get the optimal currents, some modification of the load is needed. It is carried out by compensator circuits but this is the problem which is not under the consideration in this article.

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