THE

ELASTICITY AND RESISTANCE

OF THE

MATERIALS OF ENGINEERING.

BY

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PREFACE.

THIS work has been the outgrowth of lectures on the elasticity and resistance of materials, given by the author to succeeding classes of students in the department of civil engineering at the Rensselaer Polytechnic Institute. Although those lectures, as given, form the basis of the work, they have, of course, been considerably elaborated and extended, so as to cover many details of the subject which it would be impossible to include in any ordinary technical course of study, but which, at the same time, are necessary to a complete and philosophical treatment.

The first, or "Rational," part of this work, is intended to furnish an analytical or rational basis for the "Technical" or practical development contained in Part II. It will undoubtedly impress a great number, and perhaps all engineers in active practice, that it is unnecessary to the proper treatment of such a subject. Indeed, a very considerably extended experience in iron and steel constructions places the author himself in position to fully appreciate the weight of such a criticism at the first glance. But it may be contended, and he thinks must be admitted, that the present advanced state of engineering as a profession implies the existence of something that may be called the "natural philosophy" of engineering. In other words, the engineer of the present time must meet the increased and increasing demands upon him in some one or more specialty, not only by the aid of sound common sense and a well-trained judgment, but also by a systematic knowledge of so much of natural philosophy as is involved in practical engineering operations. The ideal simplicity of stresses and strains in a perfectly isotropic body, and the clearness of action of "external forces" applied at any "point" or distributed over any "surface" according to some known and well-defined law, are not, it is evident, the things the technical student will encounter in his practice as an engineer. He will find few, if any, of the ideal conditions realized, and the difficulties constantly confronting him will be those involving modifications of the analytical or mathematical results based upon ideal quantities and conditions. Nevertheless, it is certainly true that in engineering practice he deals with precisely the same *quantities* as in the natural philosophy of engineering, but in different amounts and with far different and vastly more complicated conditions. And it is equally true that a correct knowledge of the consequent modifications, both in kind and amount must be based not only upon a correct recognition of the actual circumstances into which the ideal conditions transmute themselves in engineering works, *i.e.*, upon sound practical knowledge, but also upon a thorough comprehension of the things involved, in the abstract, and the laws governing their actions and relations. In other words, but in essentials the same, an engineer's preparation for active practice must consist both of that philosophical training in what is largely ideal, and which he acquires in the technical school, and of the purely practical training of the first few years of his professional life.

The first, or "Rational," part of this work is, then, designed for few others than technical students, although there are engineers whose tastes induce or circumstances require investigations in connection with the elasticity and resistance of materials. The writer would esteem himself fortunate if the mathematical portion of the book should find favor with such individuals and be useful to them.

In Part II. the mathematical results obtained in Part I. are subjected to the test of experiment. By the aid of experimental results in a great variety of material, empirical coefficients are established which involve the varied and complicated circumstances of material in actual use. The formulæ, which otherwise express ideal conditions only, are thus rendered of the greatest practical value; in fact they constitute the only reliable practical formulæ in use by engineers.

All the experimental results are, of course, compilations only, but they have been taken in all cases from what are believed to be trustworthy sources, and it has been the intention to give credit to the experimenter in every case. It may appear that too great a profusion of experimental results has been introduced. But it has been the aim of the author, even at the risk of being tedious, to represent truly and completely the great variety of both quantitative and qualitative phenomena exhibited by material under test; to show not only the variation in products of different mills but the variation in different products of the same mill; to exhibit the variations due to difference in size, shape, relative dimensions and condition of specimens; to show that specimens apparently identically the same may even give considerable diversity in results and to prove the difference between the finished member and its component parts, as well as to indicate the direction in which further investigations may most profitably be prosecuted. A few groups of tests are not sufficient to the attainment of such a series of results.

In the course of the preparation of the MSS. the author found it necessary to reduce a very great amount of experimental quantities from the crude shape of a mere record of tests to a useful condition, and to change many others from one unit to another. These numerical operations involved much labor, and although they were performed with great care and repeated in almost every instance, it is very probable that errors have crept in, though it is believed that there are few, if any, of importance. The writer will feel indebted to any one who will discover them. In all cases, unless otherwise specifically stated, the ultimate resistance, elastic limit and coefficient of elasticity are expressed in pounds per square inch of original area of section.

In a few of the tables of Art. 32 the "strains," *i.e.*, amounts of stretch, are given as decimal fractions (hundredths) of original length, while the otherwise uniform method of expression is by means of whole numbers giving per cents. of original dimensions. This diversity is unintentional and due to the fact that a part of the MSS. was a portion of that used in lectures.

The distinction between "stress" and "strain" conflicts, so far as the latter word is concerned, with ordinary usage. But some distinction is absolutely necessary, and that used has had a long existence, and is at least consistent with the etymology of the words. There certainly can be no way of filling the hiatus caused by the absence of a word to concisely express changes of shape or dimensions, without some inconvenience, and that followed will probably cause as little as any.

W. H. B.

RENSSELAER POLYTECHNIC INSTITUTE,
1883.

PREFACE TO SECOND EDITION.

THE present edition of this work is the result of a careful revision and extension of the first edition. Since the issue of the latter, very considerable developments have been made in constructions of iron and steel, particularly in those of steel; civil engineers are also pushing their investigations in timber, cement, cement mortar, building stones, bricks, etc., with energy and corresponding success. Abstract results in pure engineering science are constantly finding their applications in the practical operations of the engineer; while experimental results with members built for actual use in structures are continually furnishing bases for new inductions of the greatest practical or technical value. It has been the design to bring the present volume into such a condition as to be quite abreast of these material advances. Considerable old matter has been canceled and new matter supplied, and Addenda to many Articles have been written. For convenience of reference it is believed well to state that new matter and Addenda will be found in or added to Arts. 20, 21, 24, 32, 34, 42, 45, 46, 51, 57, 65, 66, 67, 70, 73, 76, 78, 84, 85, 86, 87, 89, 90, 91 and Addenda at the end of the book. These additions are entirely in the domain of engineering practice and contain valuable practical data.

W. H. B.

PHŒNIXVILLE, PA.,
Sept., 1887.

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PART I.—RATIONAL.

CHAPTER I.

General Theory of Elasticity in Amorphous Solid Bodies.

Art. 1.—General Statements.

THE molecules of all solid bodies known in nature are more or less free to move toward, or from, or among each other. Resistances are offered to such motions, which vary according to the circumstances under which they take place, and the nature of the body. This property of resistance is termed the "elasticity" of the body.

The summation of the displacements of the molecules of a body, for a given point, is called the "*distortion*" or "*strain*" at the point considered. The *force* by which the molecules of a body resist a strain, at any point, is called the "*stress*" at that point. This distinction between *stress* and *strain* is fundamental and important.

Stresses are developed, and strains caused, by the application of force to the exterior surface of the material. These stresses and strains vary in character according to the method of application of the external forces. Each stress, however, is accompanied by its own characteristic strain and no other. Thus, there are shearing stresses and shearing strains, tensile stresses and tensile strains, compressive stresses and compressive strains. Usually a number of different stresses with their corresponding strains are coexistent at any point in a body subjected to the action of external forces.

It is a matter of experience that strains always vary continuously and in the same direction with the corresponding stresses. Consequently the stresses are continuously increasing functions of the strains, and any stress may be represented by a series composed of the ascending powers (commencing with the first) of the strains multiplied by proper coefficients. When, as is usually the case, the displacements are very small, the terms of the series whose indices are greater than unity are exceedingly small compared with the first term, whose index is unity. Those terms may consequently be omitted without essentially changing the value of the expression. Hence follows what is ordinarily termed Hooke's law:

The ratio between stresses and corresponding strains, for a given material, is constant.

This law is susceptible of very simple algebraic representation. As the generality of the equation will not be affected, *intensities* of stresses and distortions or strains per linear unit, only, will be considered.

Let p' represent the intensity of any stress, and l' the strain per unit of length, or, in other words, the rate of strain. If E' is a constant coefficient, Hooke's law will be given by the following equation:

$$p' = E'l'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the intensity of stress varies from point to point of a body, Hooke's law may be expressed by the following differential equation:

$$\frac{dp'}{dl'} = E' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If p' and l' are rectangular co-ordinates, Eqs. (1) and (2) are evidently the equations of a straight line passing through the origin of co-ordinates. It will hereafter be seen that the line under consideration is essentially straight for very small strains only.

Art. 2.—Coefficients of Elasticity.

In general, the coefficient E' in Eq. (1) of the preceding Art., is called the "coefficient of elasticity," or, sometimes, "modulus of elasticity." The coefficient of elasticity varies both with the kind of material and kind of stress. It simply expresses *the ratio between stress and strain.*

The characteristic strain of a tensile stress is evidently an *increase* of the linear dimensions of the body in the direction of action of the external forces.

Let this increase per unit of length be represented by l, while p and E represent, respectively, the corresponding intensity and coefficient. Eq. (1) of the preceding Art. then becomes:

$$p = El, \quad \text{or,} \quad E = \frac{p}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

E is then the coefficient of elasticity for tension.

The characteristic strain for a compressive stress is evidently a *decrease* in the linear dimensions of the body in the direction of action of the external forces. Let l_1 represent this decrease per unit of length, p_1 the intensity of compressive stress, and E_1 the corresponding coefficient. Hence:

$$p_1 = E_1 l_1, \quad \text{or,} \quad E_1 = \frac{p_1}{l_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

E_1, consequently, is the coefficient of elasticity for compression.

The characteristic strain for a shearing stress may be determined by considering the effect which it produces on the layers of the body parallel to its plane of action.

In Fig. 1 let $ABCD$ represent one face of a cube, another of whose faces is fixed along AD. If a shear acts in the face BC, whose plane is normal to the plane of the paper, all layers of the cube parallel to the plane of the shearing stress, *i.e.*, BC, will slide over each other, so that the faces AB and DC will take the positions AE and DF. The amount of distortion or strain per unit of length will be represented by the angle $EAB = \varphi$. If the strain is small there may be written φ, $\sin \varphi$ or $\tan \varphi$ indifferently.

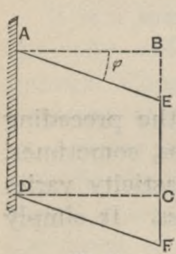


Fig. 1

Representing, therefore, the intensity of shear, coefficient and strain by S, G and φ, respectively, Eq. (1) of Art. 1 becomes:

$$S = G\varphi, \quad \text{or,} \quad G = \frac{S}{\varphi} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It will be seen hereafter that there are certain limits of stress within which Eqs. (1), (2) and (3) are essentially true, but beyond which they do not hold ; this limit is called the "limit of elasticity," and is not in general a well defined point.

Art. 3.—Lateral Strains.

If a body, like that shown in Fig. 1, be subjected to tension, all of its oblique cross sections, such as FE and GH, will sustain shearing stresses in consequence of the components of the tension tangential to those oblique sections. These tangential stresses will cause the oblique sections, in both directions, to slide over each other. Consequently *the normal cross sections of the body will be decreased;* and if the normal

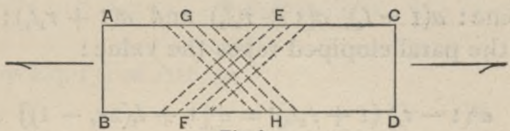


Fig. 1

cross sections of the body are made less, its capacity of resistance to the external forces acting on AB and CD will be correspondingly diminished.

If the body is subjected to compression, oblique sections of the body will be subjected to shears, but in directions *opposite* to those existing in the previous case. The effect of such shears will be an *increase* of the lateral dimensions of the body and a corresponding increase in its capacity of resistance.

These changes in the lateral dimensions of the body are termed "lateral strains"; they always accompany direct strains of tension and compression.

It is to be observed that lateral strains *decrease* a body's resistance to tension, but *increase* its resistance to compression. Also, that if they are prevented, both kinds of resistance are *increased.*

Consider a cube, each of whose edges is a, in a body subjected to tension. Let r represent the ratio between the lateral and direct strains, and let it be supposed to be the same in all directions. If l, as in Art. 2, represents the direct strain, the edges of the cube will become, by the tension: $a(1 + l)$, $a(1 - rl)$ and $a(1 - rl)$. Consequently the volume of the resulting parallelopiped will be:

$$a^3(1 + l)(1 - rl)^2 = a^3[1 + l(1 - 2r)] \quad . \quad . \quad . \quad (1)$$

if powers of l higher than the first be omitted. *With* r *between* 0 *and* ½, *there will be an increase of volume*, but not otherwise.

If the body is subjected to compression, the edges of the cube become: $a(1 - l_1)$, $a(1 + r_1 l_1)$ and $a(1 + r_1 l_1)$; while the volume of the parallelopiped takes the value:

$$a^3(1 - l_1)(1 + r_1 l_1)^2 = a^3[1 + l_1(2r_1 - 1)] \quad . \quad . \quad . \quad (2)$$

As before, the higher powers of l_1 are omitted. If the volume of the cube is decreased, r_1 must be found between 0 and ½.

Art. 4.—Relation between the Coefficients of Elasticity for Shearing and Direct Stress in a Homogeneous Body.

A body is said to be homogeneous when its elasticity, of a given kind, is the same in all directions.

Let Fig. 1 represent a body subjected to tension parallel to *CD*. That oblique section on which the shear has the greatest intensity will make an angle of 45° with either of those faces whose traces are *CD* or *BD*; for if α is the angle which *any* oblique section makes with *BD*, P the total tension on *BD*, and A' the area of the latter surface, the total shear on any section whose area is $A' \sec \alpha$, will be $P \sin \alpha$. Hence the intensity of shear is:

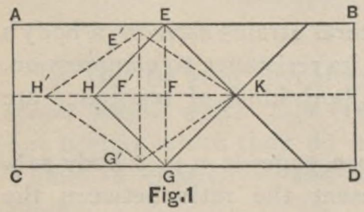


Fig.1

$$\frac{P \sin \alpha}{A' \sec \alpha} = \frac{P}{A'} \sin \alpha \cos \alpha. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The second member of Eq. (1) evidently has its greatest value for $\alpha = 45°$. Hence, if the tensile intensity on BD is represented by $\frac{P}{A'} = p$, the greatest intensity of shear will be:

$$S = \frac{p}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Then by Eq. (3) of Art. 2:

$$\varphi = \frac{p}{2\,G} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In Fig. 1 EK and KG are perpendicular to each other, while they make angles of 45° with either AB or CD. After stress, the cube $EKGH$ is distorted to the oblique parallelopiped $E'KG'H'$. Consequently $EKGH$ and $E'KG'H'$ correspond to $ABCD$ and $AEFD$, respectively, of Fig. 1, Art. 2. The angular difference $EKG - E'KG'$ is then equal to φ; and $EKE' = GKG' = \frac{\varphi}{2}$. Also $E'KF' = 45° - \frac{\varphi}{2}$.

Using, then, the notation of the preceding Arts., there will result, nearly:

$$tan\left(45° - \frac{\varphi}{2}\right) = \frac{1 - rl}{1 + l} = 1 - l(1 + r); \quad . \quad . \quad (4)$$

remembering that $F'K = FK(1 + l)$; and that

$$E'F' = FK(1 - rl).$$

From a trigonometrical formula, there is obtained, very nearly:

$$tan\left(45° - \frac{\varphi}{2}\right) = \frac{tan\,45° - tan\,\frac{\varphi}{2}}{tan\,45° + tan\,\frac{\varphi}{2}} = \frac{1 - \frac{\varphi}{2}}{1 + \frac{\varphi}{2}} = 1 - \varphi. \quad . \quad (5)$$

From Eqs. (4) and (5):

$$\varphi = l(1 + r) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Substituting from Eq. (3), as well as from Eq. (1) of Art. 2:

$$G = \frac{E}{2(1 + r)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

It has already been seen in the preceding Art. that r must be found between 0 and $\frac{1}{2}$, consequently *the coefficient of elasticity for shearing lies between the values of* $\frac{1}{3}$ *and* $\frac{1}{2}$ *of that of the coefficient of elasticity for tension.*

This result is approximately verified by experiment.

Since precisely the same form of result is obtained by treating compressive stress, instead of tensile, there will be found, by equating the two values of G:

$$\frac{E}{1 + r} = \frac{E_1}{1 + r_1} \quad \text{or,} \quad \frac{E_1}{E} = \frac{1 + r_1}{1 + r} \quad . \quad . \quad . \quad . \quad (8)$$

It is clear, from the conditions assumed and operations involved, that the relations shown by Eqs. (7) and (8) can only be approximate.

Art. 5.—Expressions for Tangential and Direct Stresses in Terms of the Rates of Strains at any point of a Homogeneous Body.

Let any portion of material, perfectly homogeneous, be subjected to any state of stress whatever. At any point as O, Fig. 1, let there be assumed any three rectangular co-ordinate planes; then consider any small rectangular parallelopiped whose faces are parallel to those planes. Finally let the stresses on the three faces nearest the origin be resolved into components normal and parallel to their planes of action, whose directions are parallel to the co-ordinate axis.

The intensities of these tangential and normal components will be represented in the usual manner, *i.e.*, p_{xy} signifies a tangential intensity on a plane normal to the axis of X (plane ZY), whose direction is parallel to the axis of Y, while p_{xx} signifies the intensity of a normal stress on a plane normal to the axis of X (plane ZY) and in the direction of the axis of X. Two unlike subscripts, therefore, indicate a tangential stress, while two of the same kind signify a normal stress.

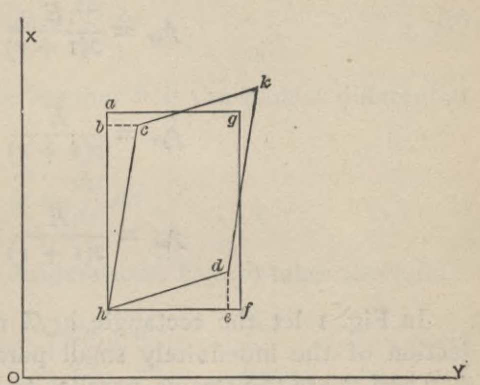


Fig.1

From Eq. (3) of Art. 2 and Eq. (7) of Art. 4, there is at once deduced:

$$S = \frac{E}{2(1 + r)} \varphi = G\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now when the material is subjected to stress the lines bounding the faces of the parallelopiped will no longer be at right angles to each other. It has already been shown in Art. 2 that the angular changes of the lines, from right angles, are the characteristic shearing strains, which, multiplied by G, give the shearing intensities.

Let φ_1 be the change of angle of the boundary lines parallel to X and Y.

Let φ_2 be the change of angle of the boundary lines parallel to Y and Z.

Let φ_3 be the change of angle of the boundary lines parallel to Z and X.

Eq. (1) will then give the following three equations:

$$p_{xy} = \frac{E}{2(1 + r)} \varphi_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$p_{yz} = \frac{E}{2(1 + r)} \varphi_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$p_{zx} = \frac{E}{2(1 + r)} \varphi_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In Fig. 1 let the rectangle *agfh* represent the right projection of the indefinitely small parallelopiped $dx\ dy\ dz$. If u, v and w are the strains, parallel to the axis of x, y and z, of the original point h, the rates of variation of strain $\frac{du}{dx}, \frac{dv}{dy}, \frac{dw}{dz}$, etc., may be considered constant throughout this parallelopiped; consequently the rectangular faces will change to oblique parallelograms. The oblique parallelogram *dhck*, whose diagonals may or may not coincide with those of *agfh*, therefore, may represent the strained condition of the latter figure.

Then, by Art. 2, the difference between *dhc* and the right angle at h will represent the strain φ_1. But, from Fig. 1, φ_1 has the following value:

$$\varphi_1 = dhe + bhc \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

But the limiting values of the angles in the second member are coincident with their tangents; hence:

$$\varphi_1 = \frac{dc}{dy} + \frac{bc}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

But, again, *de* is the distortion parallel to *OX found by moving parallel to OY, only;* hence it is a partial differential of *u*, or, it has the value:

$$de = \frac{du}{dy}\,dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In precisely the same manner *bc* is the partial differential of *v* in respect to *x*, or:

$$bc = \frac{dv}{dx}\,dx.$$

By the aid of these considerations, Eq. (6) takes the form:

$$\varphi_1 = \frac{du}{dy} + \frac{dv}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If *XY* be changed to *XZ*, and then to *ZX*, there may be at once written by the aid of Eq. (8):

$$\varphi_2 = \frac{dv}{dz} + \frac{dw}{dy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\varphi_3 = \frac{dw}{dx} + \frac{du}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Eqs. (2), (3) and (4) now take the following form:

$$p_{xy} = G\left(\frac{du}{dy} + \frac{dv}{dx}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$p_{yz} = G\left(\frac{dv}{dz} + \frac{dw}{dy}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$p_{zx} = G\left(\frac{dw}{dx} + \frac{du}{dz}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The direct stresses are next to be given in terms of the displacements u, v and w. Again, let the rectangular parallelopiped $dx\ dy\ dz$ be considered. Eq. (1), of Art. 1, shows that the strain per unit of length is found by dividing the intensity of stress by the coefficient of elasticity, *if a single stress only exists.* But in the present instance, any state of stress whatever is supposed. Consequently the strain caused by p_{xx}, for example, acting alone must be combined with the lateral strains induced by p_{yy} and p_{zz}. Denoting the actual rates of strain along the axes of X, Y and Z by l_1, l_2 and l_3, therefore, the following equations may be at once written by the aid of the principles given in Art. 3 :

$$\frac{p_{xx}}{E} = l_1 + \left(p_{yy} + p_{zz}\right)\frac{r}{E} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$\frac{p_{yy}}{E} = l_2 + \left(p_{xx} + p_{zz}\right)\frac{r}{E} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$\frac{p_{zz}}{E} = l_3 + \left(p_{yy} + p_{xx}\right)\frac{r}{E} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Eliminating between these three equations :

$$p_{xx} = \frac{E}{1 + r}\left[l_1 + \frac{r}{1 - 2r}(l_1 + l_2 + l_3)\right] \quad . \quad . \quad . \quad (17)$$

$$p_{yy} = \frac{E}{1 + r}\left[l_2 + \frac{r}{1 - 2r}(l_1 + l_2 + l_3)\right] \quad . \quad . \quad . \quad (18)$$

$$p_{zz} = \frac{E}{1 + r}\left[l_3 + \frac{r}{1 - 2r}(l_1 + l_2 + l_3)\right] \quad . \quad . \quad . \quad (19)$$

But if u, v and w are the actual strains at the point where these stresses exist, the rates of strain l_1, l_2 and l_3 will evidently be equal to $\frac{du}{dx}$, $\frac{dv}{dy}$ and $\frac{dw}{dz}$, respectively. The volume of the parallelopiped will be changed by those strains to

$$dx(1 + l_1)dy(1 + l_2)dz(1 + l_3) = dx\,dy\,dz(1 + l_1 + l_2 + l_3),$$

if powers of l_1, l_2 and l_3 above the first be omitted. The quantity $(l_1 + l_2 + l_3)$ is, then, *the rate of variation of volume*, or *the amount of variation of volume for a cubic unit.* If there be put

$$\theta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}, \quad \text{and} \quad G = \frac{E}{2(1 + r)},$$

Eqs. (17), (18) and (19) will take the forms:

$$p_{xx} = \frac{2Gr}{1 - 2r}\,\theta + 2G\frac{du}{dx} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$p_{yy} = \frac{2Gr}{1 - 2r}\,\theta + 2G\frac{dv}{dy} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$$p_{zz} = \frac{2Gr}{1 - 2r}\,\theta + 2G\frac{dw}{dz} \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The form in which Eqs. (14), (15) and (16) are written, shows that if p_{xx}, p_{yy} or p_{zz}, is positive, the stress is tension, and compression if it is negative. Consequently a positive value for any of the intensities in Eqs. (20), (21) or (22) will indicate a tensile stress, while a negative value will show the stress to be compressive.

The Eqs. (14) to (19), together with the elimination involved, also show that the coefficients of elasticity for tension and compression have been taken equal to each other, and that the ratio r is the same for tensile and compressive strains.

Further, in Eqs. (11), (12) and (13), it has been assumed that G is the same for all planes.

Hence Eqs. (11), (12), (13), (20), (21) and (22) *apply only to bodies perfectly homogeneous in all directions.*

It is to be observed that the co-ordinate axes have been taken perfectly arbitrarily.

Art. 6.—General Equations of Internal Motion and Equilibrium.

In establishing the general equations of motion and equilibrium, the principles of dynamics and statics are to be applied to the forces which act upon the parallelopiped represented in Fig. 1, the edges of which are dx, dy and dz. The notation to be used for the intensities of the stresses acting on the different faces will be the same as that used in the preceding Article.

Let the stresses which act on the faces nearest the origin be considered negative, while those which act on the other three faces are taken as positive.

The stresses which act in the direction of the axis of X are the following:

On the face normal to X, nearest to O; $-p_{xx}\,dy\,dz$.

" " " farthest from O; $\left(p_{xx} + \frac{dp_{xx}}{dx}dx\right)dy\,dz$.

" " $dy\,dx$ nearest to O; $-p_{zx}dy\,dx$.

" " " farthest from O; $\left(p_{zx} + \frac{dp_{zx}}{dz}dz\right)dy\,dx$.

" " $dz\,dx$ nearest to O; $-p_{yx}dz\,dx$.

" " " farthest from O; $\left(p_{yx} + \frac{dp_{yx}}{dy}dy\right)dz\,dx$.

The differential coefficients of the intensities are the rates of variation of those intensities for each unit of the variable, which, multiplied by the differentials of the variables, give the amounts of variation for the different edges of the parallelopiped.

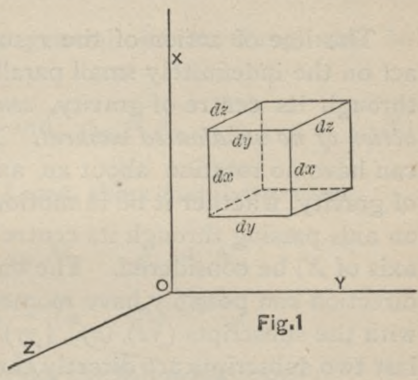


Fig.1

Let X_0 be the external force acting in the direction of X on a unit of volume at the point considered; then $X_0\ dx\ dy\ dz$ will be the amount of external force acting on the parallelopiped.

These constitute all the forces acting on the parallelopiped inthe direction of the axis of X, and their sum, if unbalanced, must be equal to $m\frac{d^2u}{dt^2}dx\,dy\,dz$; in which m is the mass or inertia of a unit of volume, and dt the differential of the time. Forming such an equation, therefore, and dropping the common factor $dx\,dy\,dz$, there will result:

$$\frac{dp_{xx}}{dx} + \frac{dp_{yx}}{dy} + \frac{dp_{zx}}{dz} + X_0 = m\frac{d^2u}{dt^2} \quad . \quad . \quad . \quad . \quad (1)$$

Changing x to y, y to z, and z to x, Eq. (1) will become:

$$\frac{dp_{xy}}{dx} + \frac{dp_{xy}}{dy} + \frac{dp_{zy}}{dz} + Y_0 = m\frac{d^2v}{dt^2} \quad . \quad . \quad . \quad . \quad (2)$$

Again, in Eq. (1), changing x to z, z to y, and y to x:

$$\frac{dp_{xz}}{dx} + \frac{dp_{yz}}{dy} + \frac{dp_{zz}}{dz} + Z_0 = m\frac{d^2w}{dt^2} \quad . \quad . \quad . \quad . \quad (3)$$

The line of action of the resultant of all the forces which act on the indefinitely small parallelopiped, at its limit, passes through its centre of gravity, *consequently it is subjected to the action of no unbalanced moment.* The parallelopiped, therefore, can have no rotation about an axis passing through its centre of gravity, whether it be in motion or equilibrium. Hence, let an axis passing through its centre of gravity and parallel to the axis of X, be considered. The only stresses, which, from their direction can possibly have moments about that axis, are those with the subscripts (yz), (zy), (yy), or (zz). But those with the last two subscripts act directly through the centre of the parallelopiped, consequently their moments are zero. The stresses $\frac{dp_{yz}}{dy}dy \,.\, dx\,dz$ and $\frac{dp_{zy}}{dz}dz \,.\, dx\,dy$ are two of six forces whose resultant is directly opposed to the resultant of those three forces which represent the increase of the intensities of the normal, or direct, stresses on three of the faces of the parallelopiped; these, therefore, have no moments about the assumed axis. The only stresses remaining are those whose intensities are p_{zy} and p_{yz}. The resultant moment, which must be equal to zero, then, has the following value:

$$p_{yz}dx\,dz \,.\, dy + p_{zy}dx\,dy \,.\, dz = 0 \quad . \quad . \quad . \quad . \quad (4)$$

$$\therefore \; p_{yz} = -p_{zy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence the two intensities are equal to each other.

The negative sign in Eq. (5) simply indicates that their *moments* have opposite signs or directions; consequently, that the shears themselves, on adjacent faces, act toward or from the edge between those faces. In Eqs. (1), (2) and (3), the tangential stresses, or shears, are all to be affected by the same sign, since direct, or normal, stresses only can have different signs.

The Eq. (5) is perfectly general, hence there may be written :

$$p_{xy} = p_{yx}, \text{ and } p_{zx} = p_{xz} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Adopting the notation of Lamé, there may be written :

$$p_{xx} = N_1, \quad p_{yy} = N_2, \quad p_{zz} = N_3,$$
$$p_{zy} = T_1, \quad p_{xz} = T_2, \quad p_{xy} = T_3,$$

by which Eqs. (1), (2) and (3) take the following forms :

$$\frac{dN_1}{dx} + \frac{dT_3}{dy} + \frac{dT_2}{dz} + X_0 = m\frac{d^2u}{dt^2} \quad . \quad . \quad . \quad . \quad (7)$$

$$\frac{dT_3}{dx} + \frac{dN_2}{dy} + \frac{dT_1}{dz} + Y_0 = m\frac{d^2v}{dt^2} \quad . \quad . \quad . \quad . \quad (8)$$

$$\frac{dT_2}{dx} + \frac{dT_1}{dy} + \frac{dN_3}{dz} + Z_0 = m\frac{d^2w}{dt^2} \quad . \quad . \quad . \quad . \quad (9)$$

The equations (11), (12), (13), (20), (21) and (22) of the preceding Art. are really kinematical in nature; in order that the principles of dynamics may hold, they must satisfy Eqs. (7), (8) and (9). As the latter stand, by themselves, they are applicable to rigid bodies as well as elastic ones; but when the values of N and T, in terms of the strains u, v and w, have been inserted they are restricted, in their use, to elastic bodies only. With those values so inserted, they form the equations on which are based the mathematical theory of sound and light vibrations, as well as those of elastic rods, membranes, etc.

In general, they are the equations of motion which the different parts of the body can have in reference to each other, in consequence of the elastic nature of the material of which the body is composed.

If all parts of the body are in equilibrium under the action of the internal stresses, the rates of variation of the strains $\frac{d^2u}{dt^2}$, $\frac{d^2v}{dt^2}$ and $\frac{d^2w}{dt^2}$, will each be equal to zero. Hence, Eqs. (7), (8) and (9) will take the forms:

$$\frac{dN_1}{dx} + \frac{dT_3}{dy} + \frac{dT_2}{dz} + X_0 = 0 \quad . \quad . \quad . \quad . \quad (10)$$

$$\frac{dT_3}{dx} + \frac{dN_2}{dy} + \frac{dT_1}{dz} + Y_0 = 0 \quad . \quad . \quad . \quad . \quad (11)$$

$$\frac{dT_2}{dx} + \frac{dT_1}{dy} + \frac{dN_3}{dz} + Z_0 = 0 \quad . \quad . \quad . \quad . \quad (12)$$

These are the general equations of equilibrium. As they stand, they apply to a rigid body. For an elastic body, the values of N and T from the preceding Art., in terms of the strains u, v and w, must satisfy these equations.

The Eqs. (10), (11) and (12) express the three conditions of equilibrium that the sums of the forces acting on the small parallelopiped, taken in three rectangular co-ordinate directions, must each be equal to zero. The other three conditions, indicating that the three component moments about the same co-ordinate axes must each be equal to zero, are fulfilled by Eqs. (5) and (6). The latter conditions really eliminate three of the nine unknown stresses. The remaining six consequently appear in both the equations of motion and equilibrium.

The equations (7) to (12), inclusive, belong to the interior of the body. At the exterior surface, only a portion of the small parallelopiped will exist, and that portion will be a tetrahedron, the base of which forms a part of the exterior surface of the body, and is acted upon by external forces. Let $\frac{da}{2}$ be the area of the base of this tetrahedron, and let p, q and r be the angles which a normal to it forms with the three axes of X, Y and Z, respectively. Then will

$$da \cos p = dy\, dz, \quad da \cos q = dz\, dx, \quad \text{and} \quad da \cos r = dx\, dy.$$

Let P be the known intensity of the external force acting on da, and let π, χ and ρ be the angles which its direction makes with the co-ordinate axes. Then there will result:

$$X_0 = P da \,.\, \cos \pi, \quad Y_0 = P da \,.\, \cos \chi \quad \text{and} \quad Z_0 = P da \,.\, \cos \rho.$$

The origin is now supposed to be so taken that the apex of the tetrahedron is located between it and the base; hence that part of the parallelopiped in which acted the stresses involving the derivatives, or differential coefficients, is wanting; consequently those stresses are also wanting.

The sums of the forces, then, which act on the tetrahedron, in the co-ordinate directions, are the following:

$$-(N_1\, dy\, dz + T_3\, dz\, dx + T_2\, dy\, dx) + Pda \cos \pi = 0,$$

$$-(T_3\, dz\, dy + N_2\, dz\, dx + T_1\, dy\, dx) + Pda \cos \chi = 0,$$

$$-(T_2\, dz\, dy + T_1\, dz\, dx + N_3\, dy\, dx) + Pda \cos \rho = 0.$$

Substituting from above:

$$N_1 \cos p + T_3 \cos q + T_2 \cos r = P \cos \pi \quad . \quad . \quad (13)$$

$$T_3 \cos p + N_2 \cos q + T_1 \cos r = P \cos \chi \quad . \quad . \quad (14)$$

$$T_2 \cos p + T_1 \cos q + N_3 \cos r = P \cos \rho \quad . \quad . \quad (15)$$

These equations must always be satisfied at the exterior surface of the body; and since the external forces must always be known, in order that a problem may be determinate, they will serve to determine constants which arise from the integration of the general equations of motion and equilibrium.

Art. 7.—Equations of Motion and Equilibrium in Semi-Polar Co-ordinates.

For many purposes it is convenient to have the conditions of motion and equilibrium expressed in either semi-polar or polar co-ordinates; the first form of such expression will be given in this Article.

The general analytical method of transformation of co-ordinates may be applied to the equations of the preceding Article, but the direct treatment of an indefinitely small portion of the material, limited by co-ordinate surfaces, possesses many advantages. In Fig. 1 are shown both the small portion of material and the co-ordinates, semi-polar as well as rectangular. The angle made by a plane normal to ZY, and containing OX, with the plane XY is represented by φ; the distance of any point from OX, measured parallel to ZY, is called r; the third co-ordinate, normal to r and φ, is the co-ordinate x, as before. It is important to observe that the co-ordinates x, r and φ, at any point, are *rectangular*.

The indefinitely small portion of material to be considered will, as shown in Fig. 1, be limited by the edges dx, dr and $r\,d\varphi$. The faces $dx\,dr$ are inclined to each other at the angle $d\varphi$.

The intensities of the normal stresses in the directions of X and r will be indicated by N_1 and R, respectively. The remainder of the notation will be of the same general character as that in the preceding Article; *i.e.*, T_{xr} will represent a shear on the face $dr \,.\, r\, d\varphi$ in the direction of r, while $N_{\phi\phi}$ is a normal stress, in the direction of φ, on the face $dx\, dr$.

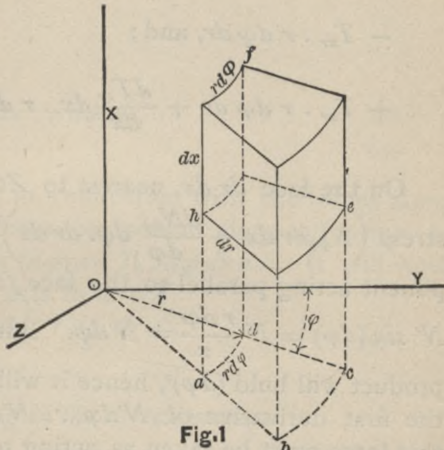


Fig. 1

The strains or displacements, in the directions of x, r and φ, will be represented by u, ρ and w; consequently the unbalanced forces in those directions, per unit of mass, will be:

$$m\frac{d^2u}{dt^2},\ m\frac{d^2\rho}{dt^2}\ \text{and}\ m\frac{d^2w}{dt^2}\quad .\ .\ .\ .\ .\ .\quad (1)$$

Those forces acting on the faces hf, fe, and he, will be considered negative; those acting on the other faces, positive.

Forces acting in the direction of r.

$- R \,.\, r\, d\varphi\, dx$, and;

$$+ Rr\, d\varphi\, dx + \left(\frac{d(Rr)}{dr}dr = r\frac{dR}{dr}dr + R\, dr\right)d\varphi\, dx.$$

$- T_{\phi r} dr\, dx$, and;

$$+ T_{\phi r} dr\, dx + \frac{dT_{\phi r}}{d\varphi} d\varphi.\, dr\, dx.$$

$$- T_{xr} \,.\, r\, d\varphi\, dr, \text{ and };$$

$$+ T_{xr} \,.\, r\, d\varphi\, dr + \frac{dT_{xr}}{dx} dx \,.\, r\, d\varphi\, dr.$$

On the face $dr\, dx$, nearest to ZOX, there acts the normal stress $\left(N_{\phi\phi} dr\, dx + \frac{dN_{\phi\phi}}{d\varphi} d\varphi.\, dr\, dx\right) = N'$. Now N' has a component acting parallel to the face fe and toward OX, equal to $N' \sin(d\varphi) = N' \frac{r\, d\varphi}{r} = N' d\varphi$. But the second term of this product will hold $(d\varphi)^2$, hence it will disappear, at the limit, in the first derivative of $N' d\varphi$ $\therefore$ $N' d\varphi = N_{\phi\phi} d\varphi\, dr\, dx$. Since this force must be taken as acting toward OX, it acts with the normal forces on hf, and, consequently, must be given the negative sign.

If R_0 is the external force acting on a unit of volume, another force (external) acting along r will be $R_0 \,.\, r\, d\varphi\, dr\, dx$.

The sum of all these forces will be equal to

$$m \,.\, r\, d\varphi\, dr\, dx \,.\, \frac{d^2\rho}{dt^2}.$$

Forces acting in the direction of φ.

$$- N_{\phi\phi}\, dr\, dx, \text{ and };$$

$$+ N_{\phi\phi}\, dr\, dx + \frac{dN_{\phi\phi}}{d\varphi} d\varphi \,.\, dr\, dx.$$

$$- T_{r\phi} \,.\, r\, d\varphi\, dx, \text{ and };$$

$$+ T_{r\phi} \, . \, r \, d\varphi \, dx + \left(\frac{d(r T_{r\phi})}{dr} dr = r \frac{dT_{r\phi}}{dr} dr + T_{r\phi} dr \right) d\varphi \, dx.$$

$- T_{x\phi} \, . \, r \, d\varphi \, dr$, and ;

$$+ T_{x\phi} \, . \, r \, d\varphi \, dr + \frac{dT_{x\phi}}{dx} dx \, . \, r \, d\varphi \, dr.$$

As in the case of $N_{\phi\phi}$, in connection with the forces along r, so the force $T_{\phi r} \, dr \, dx$ has a component along φ (normal to *fe*) equal to $T_{\phi r} \, dr \, dx \, . \, sin \, (d\varphi) = T_{\phi r} \, d\varphi \, dr \, dx$. It will have a positive sign, because it acts from OX.

The external force is, $\Phi_0 \, . \, r \, d\varphi \, dr \, dx$.

Forces acting in the direction of x.

$- N_1 \, . \, r \, d\varphi \, dr$, and ;

$$+ N_1 r \, d\varphi \, dr + \frac{dN_1}{dx} dx \, . \, r \, d\varphi \, dr.$$

$- T_{rx} \, . \, dx \, r \, d\varphi$, and

$$+ T_{rx} \, . \, dx \, r \, d\varphi + \left(\frac{d(r T_{rx})}{dr} dr = r \frac{dT_{rx}}{dr} dr + T_{rx} dr \right) dx \, d\varphi.$$

$- T_{\phi x} dx \, dr$, and ;

$$+ T_{\phi x} dx \, dr + \frac{dT_{\phi x}}{d\varphi} d\varphi \, . \, dx \, dr.$$

The external force is, $X_0 \, . \, r \, d\varphi \, dx \, dr$.

Putting each of these three sums equal to the proper rates of variation of momentum, and dropping the common factor, $r\,d\varphi\,dx\,dr$:

$$\frac{dN_1}{dx} + \frac{dT_{rx}}{dr} + \frac{dT_{\phi x}}{r\,d\varphi} + \frac{T_{rx}}{r} + X_0 = m\frac{d^2u}{dt^2} \quad (2)$$

$$\frac{dT_{xr}}{dx} + \frac{dR}{dr} + \frac{dT_{\phi r}}{r\,d\varphi} + \frac{R - N_{\phi\phi}}{r} + R_0 = m\frac{d^2\rho}{dt^2} \quad (3)$$

$$\frac{dT_{x\phi}}{dx} + \frac{dT_{r\phi}}{dr} + \frac{dN_{\phi\phi}}{r\,d\varphi} + \frac{T_{r\phi} + T_{r\phi}}{r} + \Phi_0 = m\frac{d^2w}{dt^2} \quad (4)$$

These are the general equations of motion (vibration) in terms of semi-polar co-ordinates; if the second members are made equal to zero, they become equations of equilibrium. Eqs. (2), (3), and (4) are not dependent upon the nature of the body.

Since x, r, and φ are rectangular, it at once follows that:

$$T_{rx} = T_{xr},\ T_{r\phi} = T_{\phi r},\ \text{and}\ T_{x\phi} = T_{\phi x}. \quad . \quad . \quad . \quad (5)$$

In order that Eqs. (2), (3), and (4) may be restricted to elastic bodies, it is necessary to express the six intensities of stresses involved, in terms of the rates of variation of the strains in the rectangular co-ordinate directions of x, r, and φ. Since these co-ordinates are rectangular, the Eqs. (11), (12), (13), (20), (21), and (22) of Article 5, may be made applicable to the present case by some very simple changes dependent upon the nature of semi-polar co-ordinates.

For the present purpose the strains in the co-ordinate directions of x, y, and z will be represented by u', v', and w'. Since the axis of x remains the same in the two systems, evidently:

$$\frac{du'}{dx} = \frac{du}{dx}.$$

From Fig. 1 it is clear that the axis of y corresponds exactly to the co-ordinate direction r; hence:

$$\frac{dv'}{dy} = \frac{d\rho}{dr}.$$

From the same Fig. it is seen that the axis of z corresponds to φ, or $r\varphi$. But the total differential, dw', must be considered as made up of two parts; consequently the rate of variation $\frac{dw'}{dz}$ will consist of two parts also. If there is no distortion in the direction of r, or if the distance of a molecule from the origin remains the same, one part will be $\frac{dw}{d(r\varphi)} = \frac{dw}{r\,d\varphi}$. If, however, a unit's length of material be removed from the distance r to $r + \rho$ from the centre O, Fig. 1, while φ remains constant, its length will be changed from 1 to $1 \,.\left(1 + \frac{\rho}{r}\right)$, in which ρ may be implicitly positive or negative. Consequently there will result:

$$\frac{dw'}{dz} = \frac{dw}{r\,d\varphi} + \frac{\rho}{r}.$$

For the reasons already given, there follow:

$$\frac{du'}{dy} = \frac{du}{dr} \quad \text{and} \quad \frac{dv'}{dx} = \frac{d\rho}{dx}.$$

In Fig. 2 let dc be the side of a distorted small portion of the material, the original position of which was $d'e$. Od is the distance r from the origin, $ad = dr$ and $ac = dw$, while $dd' = w$. The angular change in position of dc is $\frac{ac}{ad} = \frac{dw}{dr}$; but an

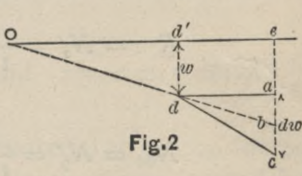


Fig. 2

amount equal to $\frac{ab}{ad} = \frac{w}{r}$ is due to the movement of r, and is not a movement of *dc* relatively to the material immediately adjacent to *d*.

Hence:

$$\frac{dw'}{dy} = \frac{dw}{dr} - \frac{w}{r}, \text{ also } \frac{dv'}{dz} = \frac{d\rho}{r\,d\varphi}.$$

There only remain the following two, which may be at once written:

$$\frac{dw'}{dx} = \frac{dw}{dx} \quad \text{and} \quad \frac{du'}{dz} = \frac{du}{r\,d\varphi}.$$

The rate of variation of volume takes the following form in terms of the new co-ordinates:

$$\theta = \frac{du'}{dx} + \frac{dv'}{dy} + \frac{dw'}{dz} = \frac{du}{dx} + \frac{d\rho}{dr} + \frac{dw}{r\,d\varphi} + \frac{\rho}{r} \quad . \quad . \quad (6)$$

Accenting the intensities which belong to the rectangular system x, y, z, the Eqs. (11), (12), (13), (20), (21) and (22), of Art. 5, take the following form:

$$N_1 = N_1' = \frac{2G\mathfrak{r}}{1 - 2\mathfrak{r}}\theta + 2G\frac{du}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$R = N_2' = \frac{2G\mathfrak{r}}{1 - 2\mathfrak{r}}\theta + 2G\frac{d\rho}{dr} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$N_{\phi\phi} = N_3' = \frac{2G\mathfrak{r}}{1 - 2\mathfrak{r}}\theta + 2G\left(\frac{dw}{r\,d\varphi} + \frac{\rho}{r}\right) \quad . \quad . \quad . \quad (9)$$

$$T_{xr} = T_3' = G\left(\frac{du}{dr} + \frac{d\rho}{dx}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$T_{r\phi} = T_1' = G\left(\frac{d\rho}{r\,d\varphi} + \frac{dw}{dr} - \frac{w}{r}\right) \quad . \quad . \quad . \quad . \quad (11)$$

$$T_{\phi x} = T_2' = G\left(\frac{dw}{dx} + \frac{du}{r\,d\varphi}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If these values are introduced in Eqs. (2), (3) and (4), those equations will be restricted in application to bodies of homogeneous elasticity only.

The notation r is used to indicate that the r involved is the ratio of lateral to direct strain, and that it has no relation whatever to the co-ordinate r.

The limiting equations of condition, (13), (14) and (15) of Art. 6, remain the same, except for the changes of notation, shown in Eqs. (7) to (12), for the intensities N and T.

Art. 8.—Equations of Motion and Equilibrium in Polar Co-ordinates.

The relation, in space, existing between the polar and rectangular systems of co-ordinates is shown in Fig. 1. The angle φ is measured in the plane ZY and from that of XY; while ψ is measured normal to ZY in a plane which contains OX. The analytical relation existing between the two systems is, then, the following:

$$x = r \sin\psi, \quad y = r \cos\psi \cos\varphi, \quad \text{and} \quad z = r \cos\psi \sin\varphi.$$

The indefinitely small portion of material to be considered is $a\,h\,c\,d$. It is limited by the co-ordinate planes located by

φ and ψ, and concentric spherical surfaces with radii r and $r + dr$. The directions r, φ and ψ, at any point, are rectangular; hence, the sums of the forces acting on the small portion of the material, taken in these directions, must be found and put equal to

$$m\frac{d^2\rho}{dt^2},\quad m\frac{d^2\eta}{dt^2},\quad \text{and} \quad m\frac{d^2\omega}{dt^2},$$

in which expressions, ρ, η and ω represent the strains in the direction of r, φ and ψ respectively.

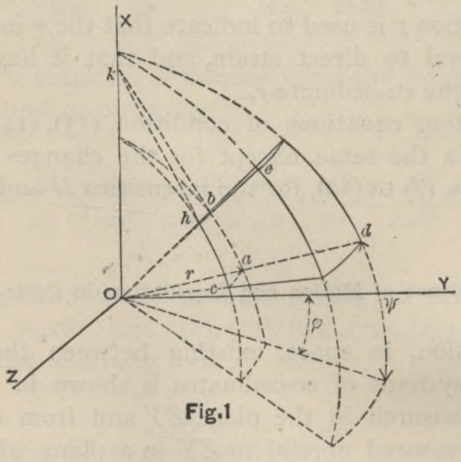


Fig.1

Those forces which act on the faces ah, bd and cd will be considered negative, and those which act on the other faces positive.

The notation will remain the same as in the preceding Articles, except that the three normal stresses will be indicated by N_r, N_ϕ and N_ψ.

Forces acting along r.

$$- N_r \, . \, r \, d\psi \, r \cos\psi \, d\varphi.$$

$$+ N_r \, . \, r^2 \cos\psi \, d\psi \, d\varphi + \left(\frac{d(N_r \, r^2)}{dr} \, dr = r^2 \frac{dN_r}{dr} \, dr + 2r \, N_r \, dr \right)$$
$$\cos\psi \, d\psi \, d\varphi.$$

$$- T_{\phi r} \, . \, r \, d\psi \, dr.$$

$$+ T_{\phi r} \, . \, r \, d\psi \, dr + \frac{d \, T_{\phi r}}{d\varphi} \, d\varphi \, . \, r \, d\psi \, dr.$$

$$- T_{\psi r} \, . \, r \cos\psi \, d\varphi \, dr.$$

$$+ T_{\psi r} \, . \, r \cos\psi \, d\varphi \, dr + \left(\frac{d(T_{\psi r} \cos\psi)}{d\psi} \, d\psi = \cos\psi \, \frac{dT_{\psi r}}{d\psi} \, d\psi \right.$$
$$\left. - \, T_{\psi r} \sin\psi \, d\psi \right) r \, d\varphi \, dr.$$

$$- N_\phi \, . \, r \, d\psi \, dr \, . \sin aOc = - N_\phi \, . \, r \, d\psi \, dr \, . \cos\psi \, d\varphi;$$
on face ce.

$$- N_\psi \, . \, r \cos\psi \, d\varphi \, dr \, . \sin aOb = - N_\psi \, . \, r \cos\psi \, d\varphi \, dr \, . \, d\psi;$$
on face be.

Forces acting along φ.

$$- T_{r\phi} \, . \, r \cos\psi \, d\varphi \, r \, d\psi.$$

$$+ T_{r\phi} \, . \, r^2 \cos\psi \, d\varphi \, d\psi + \left(\frac{d(T_{r\phi} \, r^2)}{dr} \, dr = r^2 \frac{dT_{r\phi}}{dr} \, dr + 2r \, T_{r\phi} \, dr \right)$$
$$\cos\psi \, d\psi \, d\varphi.$$

$$- N_\phi \,.\, r\, d\psi\, dr.$$

$$+ N_\phi \,.\, r\, d\psi\, dr + \frac{dN_\phi}{d\varphi} d\varphi\, r\, d\psi\, dr.$$

$$- T_{\psi\phi} \,.\, r \cos\psi\, d\varphi\, dr.$$

$$+ T_{\psi\phi} \cos\psi \,.\, r\, d\varphi\, dr + \left(\frac{d(T_{\psi\phi}\cos\psi)}{d\psi} d\psi = \cos\psi \frac{dT_{\psi\phi}}{d\psi} d\psi \right.$$
$$\left. - T_{\psi\phi} \sin\psi\, d\psi\right) r\, d\varphi\, dr.$$

$$+ T_{\phi r}\, r\, d\psi\, dr \,.\, \cos\psi\, d\varphi \text{ ; } \textit{on face ce.}$$

$$- T_{\phi\psi}\, r\, d\psi\, dr \left(\sin akc = \frac{r \cos\psi\, d\varphi}{r \cot\psi}\right) = - T_{\phi\psi}\, r\, d\psi\, dr \,.\, \sin\psi\, d\varphi;$$
$$\textit{on face cc.}$$

The lines *ak* and *ck* are drawn normal to *Oc* and *Oa*.

Forces acting along ψ.

$$- T_{r\psi} \,.\, r \cos\psi\, d\varphi \,.\, r\, d\psi.$$

$$+ T_{r\psi}\, r^2 \cos\psi\, d\varphi\, d\psi + \left(\frac{d(T_{r\psi}\, r^2)}{dr} dr = r^2 \frac{dT_{r\psi}}{dr} ar + 2r\, T_{r\psi}\, dr\right)$$
$$\cos\psi\, d\psi\, d\varphi.$$

$$- T_{\phi\psi} \,.\, r\, d\psi\, dr.$$

$$+ T_{\phi\psi}\, r\, d\psi\, dr + \frac{dT_{\phi\psi}}{d\varphi} d\varphi \,.\, r\, d\psi\, dr.$$

$$- N_\psi \,.\, r \cos\psi\, d\varphi\, dr.$$

$$+ N_\psi \,.\, r \cos\psi \, d\varphi \, dr + \left(\frac{d(N_\psi \cos\psi)}{d\psi} d\psi = \cos\psi \frac{dN_\psi}{d\psi} d\psi \right.$$

$$\left. - N_\psi \sin\psi \, d\psi\right) r \, d\varphi \, dr.$$

$+ T_{\psi r} \,.\, r \cos\psi \, d\varphi \, dr \,.\, d\psi$; *on face be.*

$+ N_\phi \,.\, r \, d\psi \, dr \,.\, \sin akc = + N_\phi \, r \, d\psi \, dr \,.\, \sin\psi \, d\varphi$; *on face ce.*

The volume of the indefinitely small portion of the material is (omitting second powers of indefinitely small quantities):

$$r \cos\psi \, d\varphi \,.\, r \, d\psi \,.\, dr = \Delta V;$$

and its mass is m multiplied by this small volume. The latter may be made a common factor in each of the three sums to be taken.

The external forces acting in the directions R, φ and ψ will be represented by:

$$R_0 \Delta V, \quad \Phi_0 \Delta V \quad \text{and} \quad \Psi_0 \Delta V,$$

respectively.

Taking each of the three sums, already mentioned, and dropping the common factor ΔV, there will result:

$$\frac{dN_r}{dr} + \frac{dT_{\phi r}}{r \cos\psi \,.\, d\varphi} + \frac{dT_{\psi r}}{r \, d\psi} + \frac{2N_r - N_\phi - N_\psi - T_{\psi r} \tan\psi}{r}$$

$$+ R_0 = m \frac{d^2\rho}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{dT_{r\phi}}{dr} + \frac{dN_\phi}{r \cos\psi \,.\, d\varphi} + \frac{dT_{\psi\phi}}{r \, d\psi}$$

$$+ \frac{2T_{r\phi} + T_{\phi r} - T_{\psi\phi} \tan\psi - T_{\phi\psi} \tan\psi}{r} + \Phi_0 = m \frac{d^2\eta}{dt^2} \quad (2)$$

$$\frac{dT_{r\psi}}{dr} + \frac{dT_{\phi\psi}}{r \cos\psi\, d\varphi} + \frac{dN_\psi}{r\, d\psi}$$
$$+ \frac{2T_{r\psi} + T_{\psi r} - N_\psi \tan\psi + N_\phi \tan\psi}{r} + \Psi_\circ = m\frac{d^2\omega}{dt^2} \quad (3)$$

Since r, φ and ψ are rectangular at any point:

$$T_{\phi r} = T_{r\phi}, \quad T_{r\psi} = T_{\psi r} \quad \text{and} \quad T_{\psi\phi} = T_{\phi\psi}.$$

Hence:

$$\frac{2T_{r\phi} + T_{\phi r} - \tan\psi(T_{\psi\phi} + T_{\phi\psi})}{r} = \frac{3T_{r\phi} - 2\tan\psi \,.\, T_{\psi\phi}}{r}.$$

$$\frac{2T_{r\psi} + T_{\psi r} - \tan\psi(N_\psi - N_\phi)}{r} = \frac{3T_{r\psi} - \tan\psi(N_\psi - N_\phi)}{r}.$$

These relations somewhat simplify the first members of Eqs. (2) and (3).

Eqs. (1), (2) and (3) are entirely independent of the nature of the material; also, they apply to the case of equilibrium, if the second members are made equal to zero.

The rectangular rates of strain, at any point, in terms of r, φ and ψ are next to be found. As in the preceding Art., the rates of strain in the rectangular directions of r, φ and ψ will be indicated by:

$$\frac{dv'}{dy}, \frac{dw'}{dz}, \frac{du'}{dx}, \frac{dv'}{dx}, \frac{du'}{dy}, \text{ etc.}$$

Remembering the reasoning in connection with the value of $\frac{dw'}{dz}$, in the preceding Art., and attentively considering Fig. 1, there may at once be written:

$$\frac{du'}{dx'} = \frac{d\omega}{r\,d\psi} + \frac{\rho}{r}.$$

In Fig. 1, if $ac = 1$ and $ab = \omega$, while $ak = r \cot . \psi$ (ak is perpendicular to aO), the difference in length between ac and bh will be:

$$-\frac{\omega}{r \cot \psi} = -\frac{\omega \tan \psi}{r}.$$

This expression is negative because a *decrease* in length takes place in consequence of a movement in the *positive* direction of $r\psi$.

Again, a consideration of Fig. 1, and the reasoning connected with the equation above, will give:

$$\frac{dw'}{dz'} = \frac{d\eta}{r \cos \psi\, d\varphi} + \frac{\rho}{r} - \frac{\omega \tan \psi}{r}.$$

Without explanation there may at once be written:

$$\frac{dv'}{dy'} = \frac{d\rho}{dr}.$$

Fig. 1 of this, and Fig. 2 of the preceding Art. give:

$$\frac{du'}{dy'} = \frac{d\omega}{dr} - \frac{\omega}{r}, \quad \text{and} \quad \frac{dv'}{dx'} = \frac{d\rho}{r\,d\psi}.$$

These are to be used in the expression for $T_{\psi r}$. Precisely the same Figs. and method give:

$$\frac{dv'}{dz'} = \frac{d\rho}{r \cos \psi\, d\varphi}, \quad \text{and} \quad \frac{dw'}{dy'} = \frac{d\eta}{dr} - \frac{\eta}{r};$$

which are to be used in finding $T_{\phi r}$.

The expression for $\frac{dw'}{dx'}$ will be composed of the *sum* of two parts. In Fig. 2, *ab* is the original position of $r\,d\psi$, and after the strain η exists it takes the position *ec*. Consequently *ac* (equal and parallel to *bd* and perpendicular to *ak*) represents the strain η, while *ed* represents $d\eta$. Since, also, *fc* is perpendicular to *ck*, the strains of the kind η change the right angle *fck* to the angle *fce*; or the angle *eck* is equal to

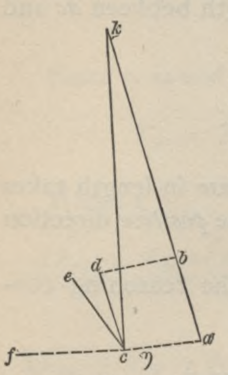


Fig. 2

$$\frac{dw'}{dx'} = ecd + dck = \frac{ed}{dc} + \frac{ca}{ak}$$

$$= \frac{d\eta}{r\,d\psi} + \frac{\eta}{r \cot \psi}.$$

In Fig. 2, the points *a*, *b* and *k* are identical with the points similarly lettered in Fig. 1. The expression for $\frac{du'}{dz'}$ may be at once written from Fig. 1. There may, then, finally be written:

$$\frac{dw'}{dx'} = \frac{d\eta}{r\,d\psi} + \frac{\eta \tan \psi}{r}, \quad \text{and,} \quad \frac{du'}{dz'} = \frac{d\omega}{r \cos \psi\, d\varphi}.$$

These equations will give the expression for $T_{\phi\psi}$.

The value of

$$\theta = \frac{du'}{dx'} + \frac{dv'}{dy'} + \frac{dw'}{dz'}$$

now takes the following form:

$$\theta = \frac{d\rho}{dr} + \frac{d\eta}{r \cos \psi\, d\varphi} + \frac{d\omega}{r\,d\psi} + \frac{2\rho}{r} - \frac{\omega \tan \psi}{r}. \quad . \quad (4)$$

The last two terms are characteristic of the spherical co-ordinates.

The equations (20), (21), (22), (11), (12) and (13), of Art. (5), take the forms:

$$N_r = \frac{2G\mathfrak{r}}{1 - 2\mathfrak{r}}\theta + 2G\frac{d\rho}{dr} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$N_\phi = \frac{2G\mathfrak{r}}{1 - 2\mathfrak{r}}\theta + 2G\left(\frac{d\eta}{r\cos\psi\, d\varphi} + \frac{\rho}{r} - \frac{\omega\tan\psi}{r}\right) \quad (6)$$

$$N_\psi = \frac{2G\mathfrak{r}}{1 - 2\mathfrak{r}}\theta + 2G\left(\frac{d\omega}{r\,d\psi} + \frac{\rho}{r}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$T_{\phi\psi} = G\left(\frac{d\eta}{r\,d\psi} + \frac{d\omega}{r\cos\psi\, d\varphi} + \frac{\eta\tan\psi}{r}\right) \quad . \quad . \quad . \quad . \quad (8)$$

$$T_{r\psi} = G\left(\frac{d\omega}{dr} - \frac{\omega}{r} + \frac{d\rho}{r\,d\psi}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$T_{r\phi} = G\left(\frac{d\rho}{r\cos\psi\, d\varphi} + \frac{d\eta}{dr} - \frac{\eta}{r}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If these values are inserted in Eqs. (1), (2) and (3), the resulting equations will be applicable to isotropic material only.

As in the preceding Art., $\mathfrak{r}$ is used to express the ratio between direct and lateral strains, and has no relation whatever to the co-ordinate r.

It is interesting and important to observe that the equations of motion and equilibrium for elastic bodies, are only special cases of equations which are entirely independent of the nature of the material, of equations, in fact, which express the most general conditions of motion or equilibrium.

CHAPTER II.

THICK, HOLLOW CYLINDERS AND SPHERES, AND TORSION.

Art. 9.—Thick, Hollow Cylinders.

IN Fig. 1 is represented a section, taken normal to its axis, of a circular cylinder whose walls are of the appreciable thickness t. Let p and p_1 represent the interior and exterior intensities of pressures, respectively. The material will not be stressed with uniform intensity throughout the thickness t. Yet if that thickness, comparatively speaking, is small, the variation will also be small; or, in other words, the intensity of stress throughout the thickness t may be considered constant. This approximate case will first be considered.

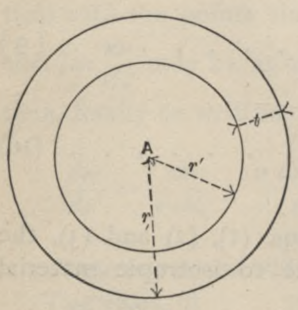


Fig. 1

The interior intensity p will be considered greater than the exterior p_1, consequently the tendency will be toward rupture along a diametral plane. If, at the same time, the ends of the cylinder are taken as closed, as will be done, a tendency to rupture through the section shown in the Fig. will exist.

The force tending to produce rupture of the latter kind will be:

$$F = \pi(pr'^2 - p_1r_1^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If N_1 represents the intensity of stress developed by this force,

$$N_1 = \frac{F}{\pi(r_1^2 - r'^2)} = \frac{pr'^2 - p_1 r_1^2}{r_1^2 - r'^2} \quad . \quad . \quad . \quad . \quad (2)$$

If the exterior pressure is zero, and if r' is nearly equal to

$$\frac{r_1 + r'}{2}:$$

$$N_1 = \frac{pr'}{2(r_1 - r')} = \frac{pr'}{2t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In this same approximate case, the tendency to split the cylinder along a diametral plane, for unit of length, will be:

$$F' = pr' - p_1 r_1.$$

If N' is the intensity of stress developed by F':

$$N' = \frac{F'}{t} = \frac{pr' - p_1 r_1}{t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

N' is thus seen to be *twice* as great as N_1 when $p_1 = 0$. If, therefore, the material has the same ultimate resistance in both directions the cylinder will fail longitudinally when the interior intensity is only *half* great enough to produce transverse rupture ; *the thickness being assumed to be very small and the exterior pressure zero.*

N_1 and N' are tensile stresses, because the interior pressure was assumed to be large compared with the exterior. If the opposite assumption were made, they would be found to be compression, while the general forms would remain exactly the same.

The preceding formulas are too loosely approximate for many cases. The exact treatment requires the use of the general equations of equilibrium, and the forms which they take in Art. 7 are particularly convenient. As in that Art., the axis of x will be taken as the axis of the cylinder.

Since all external pressure is uniform in intensity and normal in direction, no shearing stresses will exist in the material of the cylinder. This condition is expressed in the notation of Art. 7 by putting:

$$T_{\phi x} = T_{rx} = T_{r\phi} = 0.$$

Again the cylinder will be considered closed at the ends, and the force F, Eq. (1), will be assumed to develop a stress of *uniform* intensity throughout the transverse section shown in Fig. 1. This condition, in fact, is involved in that of making all the tangential stresses equal to zero.

Since this case is that of equilibrium, the equations (2), (3) and (4) of Art. 7 take the following form, after neglecting X_0, R_0 and Φ_0:

$$\frac{dN_1}{dx} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\frac{dR}{dr} + \frac{R - N_{\phi\phi}}{r} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\frac{dN_{\phi\phi}}{rd\varphi} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

These equations are next to be expressed in terms of the strains u, ρ and w.

In consequence of the manner of application of the external forces, all movements of indefinitely small portions of the material will be along the radii and axis of the cylinder. Hence:

u will be independent of r and φ;
ρ " " " " φ " x;
$w = 0$.

The rate of change, therefore, of volume will be (Eq. (6) of Art. 7):

$$\theta = \frac{du}{dx} + \frac{d\rho}{dr} + \frac{\rho}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

As ρ is independent of x, $\frac{d\theta}{dx} = \frac{d^2u}{dx^2}$; hence if the value of N_1 be taken from Eq. (7) of Art. 7 and put in Eq. (5) of this Art.:

$$\frac{dN_1}{dx} = \frac{2G\text{r}}{1 - 2\text{r}}\frac{d^2u}{dx^2} + 2G\frac{d^2u}{dx^2} = 0.$$

$$\therefore \quad \frac{d^2u}{dx^2} = 0, \quad \text{and} \quad u = ax + a'.$$

But the transverse section in which the origin is located may be considered fixed. Consequently if $x = 0$, $u = 0$ and thus $a' = 0$. The expression for u is then: $u = ax$.

The ratio $u \div x$ is the l' of Eq. (1), Art. 1; while the p' of the same equation is simply N_1 of Eq. (2), given above. Hence:

$$a = \frac{u}{x} = \frac{N_1}{E} = \frac{pr'^2 - p_1r_1^2}{E(r_1^2 - r'^2)} \quad . \quad . \quad . \quad . \quad (9)$$

Again, Eq. (8), of Art. 7, in connection with Eqs. (8) and (6) of this, gives:

$$\frac{2G\mathfrak{r}}{1 - 2\mathfrak{r}}\left(\frac{d^2\rho}{dr^2} + \frac{d\rho}{r\,dr} - \frac{\rho}{r^2}\right) + 2G\left(\frac{d^2\rho}{dr^2} + \frac{d\rho}{r\,dr} - \frac{\rho}{r^2}\right) = 0.$$

$$\therefore \quad \frac{d^2\rho}{dr^2} + \frac{d\rho}{r\,dr} - \frac{\rho}{r^2} = \frac{d^2\rho}{dr^2} + \frac{d\left(\frac{\rho}{r}\right)}{dr} = 0.$$

$$\therefore \quad \frac{d\rho}{dr} + \frac{\rho}{r} = c; \text{ or:}$$

$$r\,d\rho + \rho\,dr = d(\rho r) = cr\,dr.$$

$$\therefore \quad \rho r = \frac{cr^2}{2} + b; \text{ or, } \rho = \frac{cr}{2} + \frac{b}{r} \quad . \quad . \quad . \quad (10)$$

This value of ρ in Eqs. (8) and (9) of Art. 7 will give:

$$R = 2G\left\{\frac{\mathfrak{r}(a + c)}{1 - 2\mathfrak{r}} + \frac{c}{2} - \frac{b}{r^2}\right\} \quad . \quad . \quad . \quad (11)$$

$$N_{\phi\phi} = 2G\left\{\frac{\mathfrak{r}(a + c)}{1 - 2\mathfrak{r}} + \frac{c}{2} + \frac{b}{r^2}\right\} \quad . \quad . \quad . \quad (12)$$

At the interior surface R must be equal to the internal pressure, and at the exterior surface to the external pressure. Or, since negative signs indicate compression;

If $r = r'$ . . . . . . $R = -p$.

If $r = r_1$ . . . . . . $R = -p_1$.

Either of these equations is the simple result of applying Eqs. (13), (14) and (15) to the present case, for which,

$$\cos p = \cos r = \cos \pi = \cos \rho = 0,$$

$$\cos q = \cos \chi = 1, \quad \text{and} \quad P = -p \quad \text{or} \quad -p_1.$$

Applying Eq. (11) to the two surfaces:

$$-p = 2G\left\{\frac{\mathfrak{r}(a+c)}{1-2\mathfrak{r}} + \frac{c}{2} - \frac{b}{r'^2}\right\} \quad . \quad . \quad . \quad (13)$$

$$-p_1 = 2G\left\{\frac{\mathfrak{r}(a+c)}{1-2\mathfrak{r}} + \frac{c}{2} - \frac{b}{r_1^2}\right\} \quad . \quad . \quad . \quad (14)$$

Subtracting (14) from (13):

$$2Gb = \frac{(p_1 - p)\, r_1^2\, r'^2}{r'^2 - r_1^2}.$$

Inserting this value in Eq. (13):

$$2G\left\{\frac{\mathfrak{r}(a+c)}{1-2\mathfrak{r}} + \frac{c}{2}\right\} = \frac{p_1 r_1^2 - p r'^2}{r'^2 - r_1^2}.$$

The general expressions of R and $N_{\phi\phi}$, freed from the arbitrary constants of integration, can now be easily written by inserting these last two values in Eqs. (11) and (12). By making the insertions there will result:

$$R = \frac{p_1 r_1^2 - p r'^2}{r'^2 - r_1^2} - \frac{(p_1 - p)\, r_1^2 r'^2}{r'^2 - r_1^2} \cdot \frac{1}{r^2} \quad . \quad . \quad . \quad (15)$$

$$N_{\phi\phi} = \frac{p_1 r_1^2 - p r'^2}{r'^2 - r_1^2} + \frac{(p_1 - p)\, r_1^2 r'^2}{r'^2 - r_1^2} \cdot \frac{1}{r^2} \quad . \quad . \quad . \quad (16)$$

The stress $N_{\phi\phi}$ is a tension directed *around* the cylinder, and has been called "hoop tension." Eq. (16) shows that the hoop tension will be greatest at the interior of the cylinder. An expression for the thickness, t, of the annulus in terms of the greatest hoop tension (which will be called h) can easily be obtained from Eq. (16).

If $r = r'$ in that equation:

$$h = \frac{2p_1 r_1^2 - p(r'^2 + r_1^2)}{r'^2 - r_1^2}$$

$$\therefore \quad \frac{r_1}{r'} = \left(\frac{h + p}{2p_1 - p + h}\right)^{\frac{1}{2}}$$

$$\therefore \quad r_1 - r' = t = r' \left\{ \left(\frac{h + p}{2p_1 - p + h}\right)^{\frac{1}{2}} - 1 \right\} \quad . \quad . \quad (17)$$

Eq. (17) will enable the thickness to be so determined that the hoop tension shall not exceed any assigned limit h. If p_1 is so small in comparison with p that it may be neglected, t will become:

$$t = r' \left\{ \left(\frac{h + p}{h - p}\right)^{\frac{1}{2}} - 1 \right\} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

If p_1 is greater than p, $N_{\phi\phi}$ becomes compression, but the equations are in no manner changed.

The values of the constants b and c may easily be found from the two equations immediately preceding Eq. (15).

It is interesting to notice that the rate of change of volume, θ, is equal to $(a + c)$ and, therefore, constant for all points.

Art. 10.—Torsion in Equilibrium.

The formulas to be deduced in this Article are those first given by Saint-Venaut, but, with one or two exceptions, established in a different manner.

It will in all cases, except that of the final result for a rectangular cross section, be convenient to use those equations of Art. 7 which are given in terms of semi-polar co-ordinates.

Let Fig. 1 represent a cylindrical piece of material, with any cross section, fixed in the plane ZY, and let the origin of co-ordinates be taken at O. Let it be twisted, also, by a couple

$$P \, . \, ab = Pl,$$

the plane of which is parallel to ZY. The material will thus be subjected to no bending, but to pure torsion.

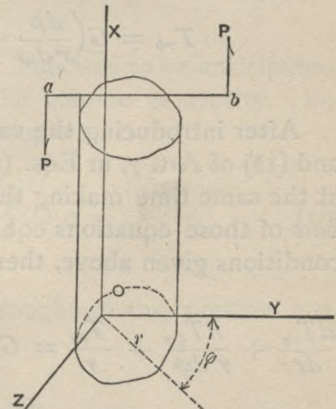


Fig. 1

The axis of the piece is supposed to be parallel to the axis of X as well as the axis of the couple. Normal sections of the piece, originally parallel to ZOY, will not remain plane after torsion takes place. But the tendency to twist any elementary portion of the piece about an axis passing through its centre and parallel to the axis of X will be very small compared with the tendency to twist it about either the axis of r or φ; consequently the first will be neglected. In the notation of Art. 7, this condition is equivalent to making $T_{r\phi} = 0$.

As the piece is acted upon by a couple only, all normal stresses will be zero.

Eqs. (7), (8), (9) and (11), of Art. 7, then become:

$$N_1 = \frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\theta + 2G\frac{du}{dx} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$R = \frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\theta + 2G\frac{d\rho}{dr} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$N_{\phi\phi} = \frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\theta + 2G\left(\frac{dw}{r\,d\varphi} + \frac{\rho}{r}\right) = 0 \quad . \quad . \quad . \quad (3)$$

$$T_{r\phi} = G\left(\frac{d\rho}{r\,d\varphi} + \frac{dw}{dr} - \frac{w}{r}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

After introducing the values of T_{rx} and $T_{\phi x}$, from Eqs. (10) and (12) of Art. 7, in Eqs. (2), (3) and (4) of the same Article, at the same time making the external forces and second members of those equations equal to zero, and bearing in mind the conditions given above, there will result:

$$\frac{dT_{rx}}{dr} + \frac{dT_{\phi x}}{r\,d\varphi} + \frac{T_{rx}}{r} = G\left(\frac{d^2u}{dr^2} + \frac{d^2\rho}{dr\,dx} + \frac{d^2w}{r\,d\varphi\,dx} + \frac{d^2u}{r^2d\varphi^2}\right.$$
$$\left. + \frac{du}{r\,dr} + \frac{d\rho}{r\,dx}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\frac{dT_{rx}}{dx} = G\left(\frac{d^2u}{dr\,dx} + \frac{d^2\rho}{dx^2}\right) = 0 \quad . \quad . \quad . \quad . \quad (6)$$

$$\frac{dT_{x\phi}}{dx} = G\left(\frac{d^2w}{dx^2} + \frac{d^2u}{r\,d\varphi\,dx}\right) = 0 \quad . \quad . \quad . \quad . \quad (7)$$

Also by Eq. (6) of Art. 7:

$$\theta = \frac{du}{dx} + \frac{d\rho}{dr} + \frac{dw}{r\,d\varphi} + \frac{\rho}{r} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The cylindrical piece of material is supposed to be of such length, that the portion to which these equations apply is not affected by the manner of application of the couple. This portion is, therefore, twisted uniformly from end to end; consequently the strain u will not vary with any change in x. Hence:

$$\frac{du}{dx} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Eq. (1) then shows that $\theta = 0$. This was to be anticipated, since a pure shear cannot change the volume or density. Because $\theta = 0$, Eqs. (2) and (3) at once give:

$$\frac{d\rho}{dr} = \frac{dw}{r\,d\varphi} + \frac{\rho}{r} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

As the torsion is uniform throughout the portion considered:

$$\frac{d\rho}{dx} = 0 = \frac{d\rho}{r\,dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Eq. (11) in connection with Eq. (10), gives:

$$\frac{d^2w}{r\,dx\,d\varphi} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Eqs. (11) and (12), in connection with Eq. (10), reduce Eq. (5) to the following form:

$$\frac{d^2u}{r^2\,d\varphi^2} + \frac{d^2u}{dr^2} + \frac{du}{r\,dr} = 0 = \frac{d^2u}{d\varphi^2} + r\frac{d\left(r\frac{du}{dr}\right)}{dr} \quad . \quad (13)$$

Both terms of the second member of Eq. (6) reduce to zero by Eqs. (9) and (11), and give no new condition. The second term of the second member of Eq. (7) is zero by Eq. (9); the remaining term therefore gives:

$$\frac{d^2w}{dx^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

As the stress is all shearing, ρ will not vary with φ. Hence:

$$\frac{d\rho}{r\,d\varphi} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Eqs. (10), (11) and (15) show that $\rho = 0$, and reduce Eq. (4) to:

$$\frac{dw}{dr} - \frac{w}{r} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Eq. (10) now becomes $\frac{dw}{r\,d\varphi} = 0$, and shows that w does not contain φ; while Eq. (14) shows that w does not contain x^2 or any higher power of x. The strain w, in connection with these conditions, is to be so determined as to satisfy Eq. (16).

If α is a constant, the following form fulfills all conditions:

$$w = \alpha r x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Eq. (17) shows that *the strain* w, *in the direction of* φ, i.e., *the angular strain at any point, varies directly as the distance* *from the axis of* X, *and, as the distance from the origin measured along that axis.* This is a direct consequence of making $T_{r\phi} = 0$.

The quantity α is evidently the *angle of torsion*, or the angle through which one end of a unit of fibre, situated at unit's distance from the axis, is twisted ; for if ;

$$r = x = 1, \quad w = \alpha.$$

An equation of condition relative to the exterior surface of the twisted piece yet remains to be determined ; and that is to be based on the supposition that no external force whatever acts on the outer surface of the piece. In Eqs. (13), (14) and (15) of Art. 6, consequently, $P = 0$. The conditions of the problem also make all the stresses except:

$$T_3 = T_{xr} \quad \text{and} \quad T_2 = T_{\phi x}$$

equal to zero, while the cylindrical character of the piece makes:

$$p = 90^\circ \quad \therefore \quad \cos p = 0.$$

If $\cos t$ be written for $\cos r$:

$$\cos t = \sin q.$$

Eq. (13), just cited, then gives:

$$T_{xr} \cos q + T_{\phi x} \sin q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

But since $\rho = 0$ and $w = \alpha r x$:

$$T_{xr} = G\frac{du}{dr} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and $$T_{x\phi} = G\left(\frac{du}{r\,d\varphi} + \alpha r\right) \quad . \quad . \quad . \quad . \quad (20)$$

Eq. (18) now becomes:

$$\frac{\frac{du}{dr}}{\frac{du}{r\,d\varphi} + \alpha r} = -\,tan\,q = -\,\frac{dr_0}{r_0\,d\varphi} \cdot \quad . \quad . \quad . \quad . \quad (21)$$

in which r_0 is the value of r for the perimeter of any normal section.

Eqs. (13) and (21) are all that are necessary and all that exist, for the determination of the strain u. Eq. (13) must be fulfilled at all points in the interior of the twisted piece, while Eq. (21) must, at the same time, hold true at all points of the exterior surface.

After u is determined, T_{xr} and $T_{x\phi}$ at once result from Eqs. (19) and (20). The resisting moment of torsion then becomes:

$$M = \int\!\!\int T_{x\phi}\, r^2\, d\varphi \,.\, dr = G\int\!\!\int \frac{du}{d\varphi} \,.\, r\, dr\, d\varphi + G\alpha I_p \quad . \quad (22)$$

In this equation $I_p = \int\!\!\int r^2 \,.\, r\, d\varphi\, dr$ is the polar moment of inertia of the normal section of the piece about the axis of X, and the double integral is to be extended over the whole section.

According to the old, or common, theory of torsion:

$$M = G\alpha I_p.$$

The third member of Eq. (22), shows, however, that such an expression is not correct unless u is equal to zero, *i.e.*, unless all normal sections remain plane while the piece is subjected to torsion. It will be seen that this is true for a circular section only.

It may sometimes be convenient to put Eq. (22) in the following form:

$$M = G\iint r\,dr\,.\,\frac{du}{d\varphi}\,d\varphi + G\alpha I_p = G\int u\,.\,r\,dr + G\alpha I_p\,. \quad (23)$$

In this equation u is to be considered as:

$$\int_0^\phi \frac{du}{d\varphi}\,d\varphi\,;$$

while the remaining integration in r is to be so made that the whole section shall be covered.

It is very important to observe that the equations of condition for the determination of u, and consequently the general values of T_{xr} and $T_{x\phi}$, are wholly independent of any considerations regarding the position of the axis of torsion, or the axis of X. It follows from this, that *the resistance of pure torsion is precisely the same wherever may be the axis about which the piece is twisted.* It is to be borne in mind, however, that, if the axis of the twisting is not the axis of the cylindrical piece, the latter will be subjected to combined bending and torsion; the bending being produced by a force sufficient to cause the piece to take the helical position necessitated by the torsion. The cylindrical axis is the straight line locus of the centres of gravity of all the normal sections.

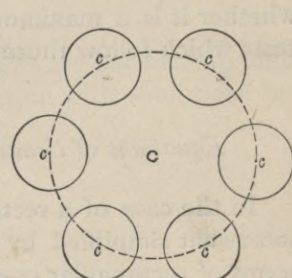


Fig. 2

If, as in Fig. 2, there are n cylinders whose centres c are all at the same distance $Cc = l$ from the centre C of twisting, or motion; and if M is the total moment of torsion of the system, while m is the moment of torsion of each cylinder about its own axis or centre c, then will $M = nm$; and each cylinder will be subject to a bending moment whose amount can be determined from the condition that the diameter of each piece lying along Cc before torsion, must pass through C after, and during, torsion, also.

Since T_{xr} and $T_{x\phi}$ act at right angles to each other, the resultant intensity of shear at any point in an originally normal section of the twisted piece will be:

$$T = \sqrt{T_{xr}^2 + T_{x\phi}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

According to the ordinary methods of the calculus, the co-ordinates of the point at which T has its greatest value must satisfy the equations:

$$\frac{dT}{d\varphi} = \frac{dT}{dr} = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$\frac{d^2T}{d\varphi^2} < 0; \quad \frac{d^2T}{dr^2} < 0; \quad \left(\frac{d^2T}{d\varphi\, dr}\right)^2 - \frac{d^2T}{d\varphi^2} \cdot \frac{d^2T}{dr^2} \leqq 0.$$

After the solution of Eqs. (25), it will usually be necessary only to inspect the resulting value of T, in order to determine whether it is a maximum or minimum, without applying the tests which follow those equations.

Equations of Condition in Rectangular Co-ordinates.

In the case of a rectangular normal section, the analysis is somewhat simplified by taking some of the quantities used in terms of rectangular co-ordinates.

In the notation of Art. 6, all stresses will be zero except

T_3 and T_2. Hence Eqs. (10), (11) and (12) of that Article reduce to:

$$\frac{dT_3}{dy} + \frac{dT_2}{dz} = 0.$$

$$\frac{dT_3}{dx} = 0.$$

$$\frac{dT_2}{dx} = 0.$$

The strains in the directions of x, y and z are, respectively, u, v and w. Introducing the values of T_3 and T_2 in the equations above, in terms of these strains, from Eqs. (11) and (13) of Art. 5; and then doing the same in reference to the conditions

$$N_1 = N_2 = N_3 = T_1 = 0:$$

the following equations will result:

$$\frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$\frac{dv}{dz} + \frac{dw}{dy} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The operations by which these results are reached are identical with those used above in connection with semi-polar coordinates, and need not be repeated.

Eq. (27) is satisfied by taking:

$$v = \alpha xz;$$
$$w = -\alpha xy;$$

in which α is the angle of torsion, as before.

Eqs. (11) and (13) of Art. 5 then give:

$$T_3 = G\left(\frac{du}{dy} + \frac{dv}{dx}\right) = G\left(\frac{du}{dy} + \alpha z\right) \quad . \quad . \quad . \quad (28)$$

$$T_2 = G\left(\frac{du}{dz} + \frac{dw}{dx}\right) = G\left(\frac{du}{dz} - \alpha y\right) \quad . \quad . \quad . \quad (29)$$

The element of a normal section is *dz dy*. Hence the moment of torsion is

$$M = \iint (T_3 z - T_2 y)\, dy\, dz.$$

$$\therefore \quad M = G\iint\left(\frac{du}{dy} z - \frac{du}{dz}\, y\right) dy\, dz + G\alpha I_p \quad . \quad (30)$$

$$\therefore \quad M = G\int (zu\, dz - yu\, dy) + G\alpha I_p \quad . \quad . \quad . \quad . \quad (31)$$

$$I_p = \iint (z^2 + y^2)\, dy\, dz$$

is the polar moment of inertia of any section about the axis of X.

The integrals are to be extended over the whole section; hence, in Eq. (31), $zu\, dz$ is to be taken as:

$$z\, dz \,.\int_{-y_0}^{+y_0} \frac{du}{dy}\, dy,$$

and $yu\, dy$ as:

$$y\, dy \int_{-z_0}^{+z_0} \frac{du}{dz}\, dz;$$

in which expressions, y_0 and z_0 are general co-ordinates of the perimeter of the normal section.

Eq. (26) is identical with Eq. (13), and can be derived from it, through a change in the independent variables, by the aid of the relations:

$$z = r \cos \varphi; \text{ and } y = r \sin \varphi.$$

Solutions of Eqs. (13) *and* (21).

It has been shown that the function u, which represents the strain parallel to the axis of the piece, must satisfy Eq. (13) [or Eq. (26)] for all points of any normal section, and Eq. (21) (or a corresponding one in rectangular co-ordinates) at all points of the perimeter; and those two are the only conditions to be satisfied.

It is shown by the ordinary operations of the calculus that an indefinite number of functions u, of r and φ, will satisfy Eq. (13); and, of these, that some are algebraic and some transcendental.

It is further shown that the various functions u which satisfy both Eqs. (13) and (21) differ only by constants.

If u is first supposed to be algebraic in character, and if c_1, c_2, c_3 etc., represent constant coefficients, the following general function will satisfy Eq. (13):

$$u = \alpha \left\{ \begin{array}{l} c_1 r \sin \varphi + c_2 r^2 \sin 2\varphi + c_3 r^3 \sin 3\varphi + \ldots \\ + c_1' r \cos \varphi + c_2' r^2 \cos 2\varphi + c_3' r^3 \cos 3\varphi + \ldots \end{array} \right\} \quad (32)$$

and the following equation, which is supposed to belong to the perimeter of a normal section only, will be found to satisfy Eq. (21):

$$\frac{r^2}{2} + c_1 r \cos \varphi + c_2 r^2 \cos 2\varphi + c_3 r^3 \cos 3\varphi + \ldots$$
$$- c_1' r \sin \varphi - c_2' r^2 \sin 2\varphi - c_3' r^3 \sin 3\varphi - \ldots = C \quad (33)$$

C is a constant which changes only with the form of section.

If $\frac{du}{dr}$ and $\frac{du}{r\,d\varphi}$ be found from Eq. (32), while $\frac{dr_0}{r_0\,d\varphi}$ be taken from Eq. (33), and if these quantities be then introduced in Eq. (21), it will be found that that equation is satisfied.

The only form of transcendental function needed, among those to which the integration of Eq. (13) or Eq. (26) leads, will be given in connection with the consideration of pieces with rectangular section, where it will be used.

Elliptical Section about its Centre.

Let a cylindrical piece of material with elliptical normal section be taken, and let a be the semi-major and b the semi-minor axis, while the angle φ is measured from a with the centre of the ellipse as the origin of co-ordinates, since the cylinder will be twisted about its own axis. The polar equation of the elliptical perimeter may take the following shape:

$$\frac{r^2}{2} + \frac{r^2}{2} \cdot \frac{b^2 - a^2}{a^2 + b^2} \cos 2\varphi = \frac{a^2 b^2}{a^2 + b^2} \quad . \quad . \quad . \quad (34)$$

By a comparison of Eqs. (33) and (34), it is seen that:

$$c_2 = \frac{b^2 - a^2}{2(a^2 + b^2)}; \text{ and } C = \frac{a^2 b^2}{a^2 + b^2};$$

and that all the other constants are zero. Hence Eq. (32) gives:

$$u = \alpha \frac{b^2 - a^2}{2(a^2 + b^2)} r^2 \sin 2\varphi = \frac{\alpha}{2} f r^2 \sin 2\varphi \quad . \quad . \quad (35)$$

The quantity represented by f is evident.

By Eqs. (19) and (20):

$$T_{xr} = G\alpha\frac{b^2 - a^2}{a^2 + b^2} r \sin 2\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

$$T_{x\phi} = G\alpha\left(\frac{b^2 - a^2}{a^2 + b^2} r \cos 2\varphi + r\right) \quad . \quad . \quad . \quad (37)$$

Since $\frac{r_0 \cdot r_0 \, d\varphi}{2} = dA$, A being the area of the ellipse, or πab, the second member of Eq. (22), by the aid of Eq. (37), may take the form:

$$M = G\alpha\int d\varphi\int_0^r \left(\frac{b^2 - a^2}{a^2 + b^2} r^3 \cos 2\varphi + r^3\right)dr.$$

$$\therefore \quad M = G\alpha\int\left(\frac{b^2 - a^2}{a^2 + b^2}\frac{r^4}{4} \cos 2\varphi + \frac{r^4}{4}\right) d\varphi.$$

Then using Eq. (34):

$$M = G\alpha\frac{a^2b^2}{a^2 + b^2}\int dA = G\alpha\frac{\pi a^3b^3}{a^2 + b^2} \quad . \quad . \quad . \quad (38)$$

If I_p is the polar moment of inertia of the ellipse (*i.e.*, about an axis normal to its plane and passing through its centre), so that

$$I_p = \frac{\pi ab(a^2 + b^2)}{4};$$

then:

$$M = G\alpha\frac{A^4}{4\pi^2 I_p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Using f in the manner shown in Eq. (35), the resultant shear at any point becomes, by Eq. (24):

$$T = G\alpha r\sqrt{f^2 + 2f\cos 2\varphi + 1}.$$

$$\therefore \quad \frac{dT}{d\varphi} = 0,$$

gives:

$$\sin 2\varphi = 0, \quad \text{or} \quad \varphi = 90^\circ \quad \text{or} \quad 0^\circ.$$

Since f is negative, T will evidently take its maximum when φ has such a value that $2f\cos 2\varphi$ is positive; or, φ must be 90°.

Hence the greatest intensity of shear will be found somewhere along the minor axis. But the preceding expression shows that T varies directly as the distance from the centre. Hence, *the greatest intensity of shear is found at the extremities of the minor axis.*

Making $\varphi = 90^\circ$ and $r = b$ in the value of T:

$$T = T_m = G\alpha b(1 - f) = G\alpha\frac{2a^2b}{a^2 + b^2} \quad . \quad . \quad . \quad (40)$$

Taking $G\alpha$ from Eq. (40) and inserting it in Eq. (38):

$$M = T_m\frac{\pi ab^2}{2} = 2T_m\frac{I_a}{b}\ ; \quad . \quad . \quad . \quad . \quad . \quad (41)$$

in which:

$$I_a = \frac{\pi ab^3}{4},$$

or the moment of inertia of the section about the major axis.

Equilateral Triangle about its Centre of Gravity.

This case is that of a cylindrical piece whose normal cross section is an equilateral triangle, and the torsion will be supposed about an axis passing through the centres of gravity of the different normal sections. The cross section is represented in Fig. 3, G being the centre of gravity as well as the origin of coordinates.

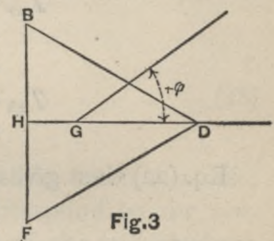


Fig.3

Let $GH = \frac{1}{2} GD = a$. Then from the known properties of such a triangle:

$$FD = DB = BF = 2a\sqrt{3}.$$

Hence, the equation for DB is; $r \sin \varphi - \dfrac{2a - r\cos\varphi}{\sqrt{3}} = 0.$

Hence, the equation for BF is; $r \cos \varphi + a = 0.$

Hence, the equation for FD is; $r \sin \varphi + \dfrac{2a - r\cos\varphi}{\sqrt{3}} = 0.$

Taking the product of these three equations, and reducing, there will result for the equation to the perimeter:

$$\frac{r^2}{2} - \frac{r^3}{6a}\cos 3\varphi = \frac{2a^2}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

Comparing this equation with Eq. (33):

$$c_3 = -\frac{1}{6a}; \quad \text{and,} \quad C = \frac{2a^2}{3}.$$

Hence:

$$u = -\alpha\frac{r^3 \sin 3\varphi}{6a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

And by Eqs. (19) and (20):

$$T_{xr} = -G\alpha\frac{r^2 \sin 3\varphi}{2a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

$$T_{x\phi} = \quad G\alpha\left(r - \frac{r^2 \cos 3\varphi}{2a}\right) \quad . \quad . \quad . \quad . \quad (45)$$

Eq. (22) then gives:

$$M = G\alpha I_p - G\alpha\int\int\frac{r^4 \cos 3\varphi}{2a}\,dr\,d\varphi.$$

$$= G\alpha I_p - G\alpha\int\frac{r^4 \sin 3\varphi}{6a}\,dr.$$

$$= G\alpha\left(I_p - \frac{6}{5}\,a^4\sqrt{3}\right) = 0.6\ G\alpha I_p = 1.8\ G\alpha\ a^4\sqrt{3}\,; \quad (46)$$

since $I_p =$ polar moment of inertia $= 3a^4\sqrt{3}$.

By Eq. (24):

$$T = G\alpha\sqrt{r^2 - \frac{r^3 \cos 3\varphi}{a} + \frac{r^4}{4a^2}} \quad . \quad . \quad . \quad . \quad (47)$$

$$\therefore \quad \frac{dT}{d\varphi} = 0, \quad \text{gives} \quad \sin 3\varphi = 0\,;$$

or $\qquad \varphi = 0°, 60°, 120°, 180°, 240°, 300°$ or $360°$.

The values 0°, 120°, 240° and 360° make:

$$\cos 3\varphi = +1\,;$$

hence, for a given value of r, these make T a minimum. The values 60°, 180° and 300° make:

$$cos\ 3\varphi = -1;$$

hence, for a given value of r these make T a maximum. Putting $cos\ 3\varphi = -1$ in Eq. (47):

$$T = G\alpha\left(r + \frac{r^2}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

This value will be the greatest possible when r is the greatest. But $\varphi = 60°$, 180° and 300°, correspond to the normal a dropped on each of the three sides of the triangle from G. Hence $r = a$, in Eq. (48), gives the greatest intensity of shear T_m, or:

$$T_m = \frac{3}{2} G\alpha a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Or, *the greatest intensity of shear exists at the middle point of each side.* Those points are the nearest of all, in the perimeter, to the axis of torsion.

The value of $G\alpha$, from Eq. (49), inserted in Eq. (46), gives:

$$M = 0.4\frac{I_p}{a}T_m = \frac{l^3 T_m}{20}; \quad . \quad . \quad . \quad . \quad . \quad (50)$$

in which $l =$ side of section $= 2a\sqrt{3}$.

Rectangular Section about an Axis passing through its Centre of Gravity.

In this case it will be necessary to consider one of the transcendental forms to which the integration of Eq. (13) [or

(26)] leads; for if the polar equation to the perimeter be formed, as was done in the preceding case, it will be found to contain r^4, to which no term in Eq. (33) corresponds.

If e is the base of the Napierian system of logarithms (numerically, $e = 2.71828$, nearly), and A any constant whatever, it is known that the general integral of the partial differential equation (13) may be expressed as follows:

$$u = Ae^{nr \cos \phi}\, e^{n'r \sin \phi}; \quad . \quad . \quad . \quad . \quad . \quad (51)$$

when $n^2 + n'^2 = 0$. For:

$$\frac{d^2u}{dr^2} + \frac{d^2u}{r^2\, d\varphi^2} + \frac{du}{r\, dr} = A(n^2 + n'^2)\, e^{nr \cos \phi}\, e^{n'r \sin \phi}.$$

But the second member of this equation is evidently equal to zero if

$$(n^2 + n'^2) = 0, \quad \text{or} \quad n' = \sqrt{-n^2}.$$

These relations make it necessary that either n or n' shall be imaginary.

It will hereafter be convenient to use the following notation for hyperbolic sines, cosines and tangents:

$$sih\ t = \frac{e^t - e^{-t}}{2}; \ coh\ t = \frac{e^t + e^{-t}}{2}; \text{ and, } tah\ t = \frac{e^t - e^{-t}}{e^t + e^{-t}}$$

By the use of Euler's exponential formula, as is well known, and remembering that $n'^2 = -n^2$, Eq. (51) may be put in the following form:

$$u = \Sigma e^{nr \cos \phi}\, [A_n \sin (nr \sin \varphi) + A'_n \cos (nr \sin \varphi)];$$

in which the sign of summation is to be extended to all possible values of A_n and A'_n. At the centre of any section for which r is zero, u must be zero also, for the axis of the piece is not shortened. This condition requires that $A'_n = 0$; u then becomes :

$$u = \Sigma e^{nr \cos \phi} A_n \sin (nr \sin \varphi).$$

The subsequent analysis will be simplified by introducing the form of the hyperbolic sine, and this may be done by adding and subtracting the same quantity to that already under the sign of summation, in such a manner that:

$$u = \Sigma[A_n \sin (nr \sin \varphi) \,.\, sih\, (nr \cos \varphi) + \tfrac{1}{2} A_n \sin (nr \sin \varphi)\, e^{-nr \cos \phi}] \quad . \quad . \quad . \quad . \quad (52)$$

Now if the product:

$$\sin (nr \sin \varphi)\, e^{-nr \cos \phi}$$

be developed in a series and multiplied by A_n, one term will consist of the quantity:

$$- r^2 \sin \varphi \cos \varphi$$

multiplied by a constant, and if:

$$\Sigma A_n \sin (nr \sin \varphi)\, e^{-nr \cos \phi}$$

be replaced by simply :

$$- \alpha r^2 \sin \varphi \cos \varphi$$

all the conditions of the problem will be found to be satisfied. This is equivalent to putting:

$$- \alpha r^2 \sin \varphi \cos \varphi$$

for a general function of $r\ sin\ \varphi$ and $r\ cos\ \varphi$. This change will give the following form to u, first used by Saint-Venant:

$$u = \Sigma A_n\ sin\ (nr\ sin\ \varphi)\ .\ sih\ (nr\ cos\ \varphi) - \alpha r^2\ sin\ \varphi\ cos\ \varphi\ . \quad (53)$$

Fig. 4 represents the cross section with C as the origin of co-ordinates, and axis. The angle φ is measured positively

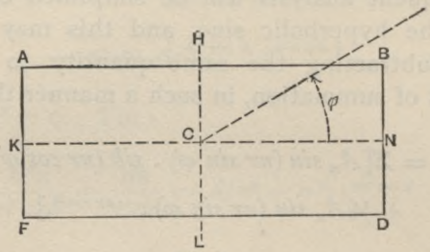


Fig. 4

from CN toward CH. At the points N, H, K and L, in the equation to the perimeter, dr_0 will be zero. Hence, at those points, by Eq. (21):

$$\frac{du}{dr} = \Sigma[A_n\ sin\ (nr\ sin\ \varphi)\ .\ n\ cos\ \varphi\ .\ coh\ (nr\ cos\ \varphi)$$
$$+ A_n\ .\ n\ sin\ \varphi\ .\ cos\ (nr\ sin\ \varphi)\ .\ sih\ (nr\ cos\ \varphi)]$$
$$- 2\alpha r\ sin\ \varphi\ cos\ \varphi = 0.$$

At the points under consideration φ has the values 0°, 90°, 180°, 270° and 360°. At the points N and K, $\varphi = 0°$ or $180°$; hence, $sin\ \varphi = 0$, and both terms of the second member of $\frac{du}{dr}$ reduce to zero, whatever may be the value of n. But at H and L, $\varphi = 90°$ and $270°$; hence, $sin\ \varphi = +1$ or -1 and $cos\ \varphi = 0$.

In order then, that $\frac{du}{dr} = 0$ at H and L, these must obtain:

$$\cos nr = \cos(-nr) = 0.$$

If $HL = c$; and, $KN = b$; then:

$$\cos\frac{nc}{2} = \cos\left(-\frac{nc}{2}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad (54)$$

If the signification of n be now somewhat changed so as to represent all possible whole numbers between 0 and ∞, Eq. (54) will be satisfied by writing:

$$\frac{2n-1}{c}\pi$$

for n, in that equation. Eq. (53) will then become:

$$u = \sum_1^\infty A_n \sin\left(\frac{2n-1}{c}\pi r \sin\varphi\right) . \operatorname{sih}\left(\frac{2n-1}{c}\pi r \cos\varphi\right)$$
$$- \alpha r^2 \sin\varphi\cos\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

The quantity A_n yet remains to be determined by the aid of Eq. (21), which expresses the condition existing at the perimeter of any section.

Now, for the portion BN of the perimeter:

$$r\cos\varphi = \frac{c}{2},$$

and $\frac{dr_0}{r_0\, d\varphi}$ will be the tangent of $(-\varphi)$; or,

$$-\frac{dr_0}{r_0\, d\varphi} = -\tan(-\varphi) = \tan\varphi.$$

Hence, Eq. (21) becomes:

$$\frac{\frac{du}{dr}}{\frac{du}{r\,d\varphi} + \alpha r} = tan\ \varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (56)$$

or:

$$\alpha r\ sin\ \varphi = \frac{du}{dr}\ cos\ \varphi - \frac{du}{r\,d\varphi}\ sin\ \varphi.$$

Substituting from Eq. (55), then making:

$$r\ cos\ \varphi = \frac{b}{2}:$$

$$r\ sin\ \varphi = \sum_{1}^{\infty} A_n \cdot \frac{2n-1}{2\alpha c}\ \pi \cdot coh\left(\frac{2n-1}{2c}\ \pi b\right)$$

$$\cdot\ sin\left(\frac{2n-1}{c}\ \pi r\ sin\ \varphi\right).$$

If $r\ sin\ \varphi$ be represented by the rectangular co-ordinate y, and another quantity by H, the above equation may be written:

$$y = H_1\ sin\ \frac{\pi y}{c} + H_2\ sin\ \frac{3\pi y}{c} + H_3\ sin\ \frac{5\pi y}{c}$$

$$+ \ldots - H_n\ sin\left(\frac{2n-1}{c}\ \pi\right) y + \ldots$$

If both sides of this equation be multiplied by

$$sin\left(\frac{2n-1}{c}\ \pi y\right) \cdot dy,$$

and if the integral then be taken between the limits 0 and $\frac{c}{2}$, it is known from the integral calculus that all terms except the n^{th} will disappear, and that:

$$H_n = \int_0^{\frac{c}{2}} y \cdot sin\left(\frac{2n-1}{c}\pi y\right) \cdot dy$$

$$\div \int_0^{\frac{c}{2}} sin^2\left(\frac{2n-1}{c}\pi y\right) \cdot dy.$$

Completing these simple integrations:

$$H_n = \left(\frac{c}{(2n-1)\pi}\right)^2 (-1)^{n-1} \cdot \frac{4}{c}.$$

Hence:

$$A_n = \frac{(-1)^{n-1}c^2}{(2n-1)^2\pi^2} \cdot \frac{4}{c} \cdot \frac{2\alpha c}{(2n-1)\pi} \cdot \frac{1}{coh\left(\frac{2n-1}{2c}\pi b\right)}.$$

If this value of A_n be put in Eq. (55), and if rectangular co-ordinates:

$$y = r\, sin\, \varphi, \quad \text{and} \quad z = r\, cos\, \varphi,$$

be introduced, that equation will become:

$$u = -\alpha zy + \left(\frac{2}{\pi}\right)^3$$

$$\cdot\, \alpha c^2 \sum_1^\infty \frac{(-1)^{n-1} sin\left(\frac{2n-1}{c}\pi y\right) \cdot sih\left(\frac{2n-1}{c}\pi z\right)}{(2n-1)^3\, coh\left(\frac{2n-1}{2c}\pi b\right)}. \quad (57)$$

This value of u placed in Eq. (31) will enable the moment of torsion to be at once written.

The limits $+y_0$ and $-y_0$ are $+\frac{c}{2}$ and $-\frac{c}{2}$; and the limits $+z_0$ and $-z_0$ are $+\frac{b}{2}$ and $-\frac{b}{2}$. Hence:

$$\Big[u\Big]_{-\frac{c}{2}}^{+\frac{c}{2}} = \alpha\, bc \left[-\frac{z}{b} + \left(\frac{2}{\pi}\right)^3 \frac{c}{b} \sum_1^\infty \frac{2\, sih\left(\frac{2n-1}{c}\,\pi z\right)}{(2n-1)^3\, coh\left(\frac{2n-1}{2c}\,\pi b\right)} \right]$$

$= Q$, for brevity.

$$-\Big[u\Big]_{-\frac{b}{2}}^{+\frac{b}{2}} = \alpha bc \left[\frac{y}{c} - \left(\frac{2}{\pi}\right)^3 \frac{c}{b} \sum_1^\infty \frac{(-1)^{n-1}.\, 2\, sih\left(\frac{2n-1}{2c}\,\pi b\right).\, sin\left(\frac{2n-1}{c}\,\pi y\right)}{(2n-1)^3\, coh\left(\frac{2n-1}{2c}\,\pi b\right)}\right] = R$$

For the next integration:

$$\int_{-\frac{b}{2}}^{+\frac{b}{2}} Qz\, dz = \alpha bc \left[-\frac{b^2}{12} + \left(\frac{2}{\pi}\right)^3 \frac{c}{b} \sum_1^\infty \frac{\frac{2bc}{(2n-1)\pi}\, .\, coh\, \frac{2n-1}{2c}\,\pi b - \frac{4c^2}{(2n-1)^2\pi^2}\, sih\left(\frac{2n-1}{2c}\,\pi b\right)}{(2n-1)^3\, coh\left(\frac{2n-1}{2c}\,\pi b\right)} \right]$$

$$\int_{-\frac{c}{2}}^{+\frac{c}{2}} Ry\,dy = \alpha bc\left[\frac{c^2}{12}\right.$$

$$\left. - \left(\frac{2}{\pi}\right)^3 \frac{c}{b} \sum_1^\infty \frac{\frac{4c^2}{(2n-1)^2\pi^2}\, sih\left(\frac{2n-1}{2c}\pi b\right)}{(2n-1)^3\, coh\left(\frac{2n-1}{2c}\pi b\right)}\right]$$

Thus the integrations indicated in Eq. (31) are completed. Hence:

$$M = G\left\{\int Qz\,dz + \int Ry\,dy + \alpha I_p\right\}.$$

Remembering that:

$$I_p = bc\left(\frac{c^2+b^2}{12}\right):$$

$$M = G\alpha\left[\frac{bc^3}{6} + \frac{16bc^3}{\pi^4}\sum_1^\infty \frac{1}{(2n-1)^4}\right.$$

$$\left. - \frac{64c^4}{\pi^5}\sum_1^\infty \frac{tah\left(\frac{2n-1}{2c}\pi b\right)}{(2n-1)^5}\right] \quad . \quad . \quad . \quad . \quad (58)$$

But it is known that:

$$\sum_1^\infty \frac{1}{(2n-1)^4} = \frac{2}{1.2.3} \cdot \frac{\pi^4}{2^5}.$$

Hence Eq. (58) becomes:

$$M = G\alpha bc^3\left[\frac{1}{3} - \frac{64}{\pi^5}\frac{c}{b}\sum_1^\infty \frac{tah\left(\frac{2n-1}{2c}\pi b\right)}{(2n-1)^5}\right]. \quad (59)$$

Since:

$$\left(\frac{1}{1} + \frac{1}{3^5} + \frac{1}{5^5} + \ldots\right)$$

$$- \left(\frac{1 - tah\,\pi}{1} + \frac{1 - tah\,3\pi}{3^5} + \frac{1 - tah\,5\pi}{5^5} + \ldots\right)$$

$$= \frac{tah\,\pi}{1} + \frac{tah\,3\pi}{3^5} + \frac{tah\,5\pi}{5^5} + \ldots;$$

and since:

$$\frac{64}{\pi^5} = 0.209137,$$

and remembering that:

$$\sum_1^\infty\left(\frac{1}{2n-1}\right)^5 = 1 + \frac{1}{3^5} + \frac{1}{5^5} + \ldots = \left(1 - \frac{1}{2^5}\right)\frac{\pi^5}{295.1215},$$

Eq. (59) becomes:

$$M = G\alpha bc^3\left[\frac{1}{3} - 0.210083\,\frac{c}{b}\right.$$

$$\left. + 0.209137\,\frac{c}{b}\left(\frac{1 - tah\frac{\pi b}{2c}}{1} + \frac{1 - tah\frac{3\pi b}{2c}}{3^5} + \ldots\right)\right] \quad (60)$$

Eq. (60) gives the value of the moment of torsion of a rectangular bar of material.

If z had been taken parallel to b, and y parallel to c, a moment of equal value would have been found, which can be at once written from Eq. (60) by writing b for c and c for b.

That moment will be:

$$M = G\alpha cb^3 \left[\frac{1}{3} - 0.210083 \frac{b}{c} \right.$$

$$\left. + 0.209137 \frac{b}{c} \left(\frac{1 - tah \frac{\pi c}{2b}}{1} + \frac{1 - tah \frac{3\pi c}{2b}}{3^5} + \ldots \right) \right] . \quad (61)$$

Eq. (60) *should be used when b is greater than c, and* Eq. (61) *when c is greater than b*, because the series in the parentheses are then very rapidly converging, and not diverging. It will never be necessary to take more than three or four terms and one, only, will ordinarily be sufficient. The following are the values of,

$$\left(1 - tah \frac{n\pi}{2} \right)$$

for a few values of n:

$$\left(1 - tah \frac{n\pi}{2} \right) = 0.083 : 0.00373 : 0.000162 : 0.000007.$$

$$n = 1 : 2 : 3 : 4$$

Square Section.

If $c = b$ either Eq. (60) or Eq. (61) gives:

$$M = G\alpha b^4 \left[\frac{1}{3} - 0.2101 + 0.209 \left(1 - tah \frac{\pi}{2} \right) \right].$$

$$\therefore \; M = 0.1406\, G\alpha b^4 = G\alpha \frac{A^4}{42.7\, I_p}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

in which A is the area $(= b^2)$ and I_p is the polar moment of inertia $\left(= \frac{b^4}{6}\right)$.

Rectangle in which $b = 2c$.

If $b = 2c$, Eq. (60) gives:

$$M = G\alpha \,.\, 2c^4 \left(\frac{1}{3} - 0.105 + 0.1046\,(1 - tah\,\pi)\right)$$

$$\therefore \; M = 0.457\, G\alpha c^4 = G\alpha \frac{A^4}{42\, I_p}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

in which A is the area $(= 2c^2)$ and $I_p =$ polar moment of inertia

$$= \frac{bc^3 + b^3c}{12} = \frac{5c^4}{6}.$$

Rectangle in which $b = 4c$.

If $b = 4c$, Eq. (60) then gives:

$$M = G\alpha bc^3 \left(\frac{1}{3} - 0.0525\right) = 1.123\, G\alpha c^4$$

$$\therefore \; M = G\alpha \frac{A^4}{40.2\, I_p}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

in which $A =$ area $= 4c^2$ and $I =$ polar moment of inertia

$$= \frac{bc^3 + b^3c}{12} = \frac{17c^4}{3}.$$

If b is greater than $2c$, it will be sufficiently near for all ordinary purposes to write :

$$M = G\alpha \frac{bc^3}{3}\left(1 - 0.63\frac{c}{b}\right) \quad . \quad . \quad . \quad . \quad . \quad (65)$$

Greatest Intensity of Shear.

There yet remains to be determined the greatest intensity of shear at any point in a section, and in searching for this quantity it will be convenient to use Eqs. (28) and (29).

It will also be well to observe that by changing z to y, y to $-z$, c to b and b to c, in Eq. (57), there may be at once written:

$$u = \alpha zy - \left(\frac{2}{\pi}\right)^3$$

$$. \alpha b^2 \sum_1^\infty \frac{(-1)^{n-1} sin\left(\frac{2n-1}{b}\pi z\right) . sih\left(\frac{2n-1}{b}\pi y\right)}{(2n-1)^3 coh\left(\frac{2n-1}{2b}\pi c\right)} \quad (66)$$

This amounts to turning the co-ordinate axes 90°.

Since the resultant shear at any point is:

$$T = \sqrt{T_2^2 + T_3^2},$$

it will be necessary to seek the maximum of

$$\left(\frac{du}{dy} + \alpha z\right)^2 + \left(\frac{du}{dz} - \alpha y\right)^2 = \frac{T^2}{G^2}.$$

The two following equations will then give the points desired:

$$\frac{d\left(\frac{T^2}{G^2}\right)}{dy} = \left(\frac{du}{dy} + \alpha z\right)\frac{d^2u}{dy^2}$$

$$+ \left(\frac{du}{dz} - \alpha y\right)\left(\frac{d^2u}{dz\,dy} - \alpha\right) = 0 \quad . \quad . \quad . \quad (67)$$

$$\frac{d\left(\frac{T^2}{G^2}\right)}{dz} = \left(\frac{du}{dy} + \alpha z\right)\left(\frac{d^2u}{dz\,dy} + \alpha\right)$$

$$+ \left(\frac{du}{dz} - \alpha y\right)\frac{d^2u}{dz^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

It is unnecessary to reproduce the complete substitutions in these two equations, but such operations show that *the points of maximum values of T are at the middle points of the sides of the rectangular sections;* omitting the evident fact that $T = 0$ at the centre. It will also be found that *the greatest intensity of shear will exist at the middle points of the greater sides.*

This result may be reached independent of any analytical test, by bearing in mind that an elongated ellipse closely approximates a rectangular section, and it has already been shown that the greatest intensity in an elliptical section is found at the extremities of the smaller axis.

By the aid of Eqs. (28), (29), (57) and (66), it will also be found that $T_3 = 0$ at the extremities of the diameter c, and $T_2 = 0$ at the extremities of the diameter b. The maximum value of T will then be:

$$T_m = -T_2 = -G\left(\frac{du}{dz} - \alpha y\right)_{\substack{z=0\\ y=\frac{c}{2}}} \quad . \quad . \quad . \quad (69)$$

By the use of Eq. (57):

$$\frac{du}{dz} - \alpha y = -2\alpha y + \left(\frac{2}{\pi}\right)^3$$

$$. \, \alpha c^2 \sum_1^\infty \frac{(-1)^{n-1} \cdot \frac{\pi}{c} \cdot \sin\left(\frac{2n-1}{c}\pi y\right) \cdot coh\left(\frac{2n-1}{c}\pi z\right)}{(2n-1)^2 \, coh\left(\frac{2n-1}{2c}\pi b\right)}$$

Putting $z = 0$ and $y = \frac{c}{2}$ in this equation, there will result:

$$T_m = G\alpha c\left[1 - \frac{8}{\pi^2}\sum_1^\infty \frac{1}{(2n-1)^2 \, coh\left(\frac{2n-1}{2c}\pi b\right)}\right]. \quad (70)$$

If b is greater than c the series appearing in this equation is very rapidly convergent, and it will never be necessary to use more than two or three terms if the section is square, and if b is four or five times c there may be written:

$$T_m = G\alpha c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

Square Section.

Making $b = c$ in Eq. (70) and making $n = 1$, 2 and 3 (*i.e.*, taking three terms of the series) there will result:

$$T_m = 0.676\, G\alpha c \quad \therefore \quad G\alpha = 1.48 \frac{T_m}{c}.$$

Inserting this value in Eq. (62):

$$M = 0.21\, b^3 T_m = \frac{1.26\, I T_m}{a} \quad . \quad . \quad . \quad . \quad (72)$$

$$\therefore \quad T_m = 0.8 \frac{M}{I} a = 5 \frac{M}{b^3} \quad . \quad . \quad . \quad . \quad (73)$$

in which:

$$I = \frac{b^4}{12} \quad \text{and} \quad a = \frac{b}{2} = \frac{c}{2}.$$

Rectangular Section; $b = 2c$.

Making $b = 2c$ in Eq. (70) and making $n = 1$, only, there will result:

$$T_m = 0.93\, G\alpha c \quad \therefore \quad G\alpha = 1.08 \frac{T_m}{c}.$$

Inserting this value in Eq. (63):

$$M = 0.49\, c^3 T_m = 1.47 \frac{I T_m}{a} \quad . \quad . \quad . \quad . \quad (74)$$

$$\therefore \quad T_m = 0.68 \frac{M}{I} a = 2 \frac{M}{c^3}; \quad . \quad . \quad . \quad . \quad (75)$$

in which:

$$I = \frac{bc^3}{12} = \frac{c^4}{6} \quad \text{and} \quad a = \frac{c}{2}.$$

Rectangular Section; $b = 4c$.

Making $b = 4c$ in Eq. (70) and making $n = 1$, only:

$$T_m = 0.997\ G\alpha c \quad \therefore \quad G\alpha = 1.003\ \frac{T_m}{c}.$$

Inserting this value in Eq. (64):

$$M = 1.126\, c^3 T_m = 1.69\ \frac{IT_m}{a} \quad . \quad . \quad . \quad . \quad (76)$$

$$\therefore \quad T_m = 0.6\ \frac{M}{I}\ a = 0.9\ \frac{M}{c^3}; \quad . \quad . \quad . \quad . \quad (77)$$

in which:

$$I = \frac{bc^3}{12} = \frac{c^4}{3} \quad \text{and} \quad a = \frac{c}{2}.$$

Circular Section about its Centre.

The torsion of a circular cylinder furnishes the simplest example of all.

If r_0 is the radius of the circular section, the polar equation of that section is:

$$\frac{r_0^2}{2} = C, \quad \text{(constant).}$$

Comparing this equation with Eq. (33), it is seen that:

$$c_1 = c_2 = c_3 = \ldots \;\; = c_1' = c_2' = \ldots \;\; = 0.$$

By Eq. (32) this gives $u = 0$. Hence, *all sections remain plane during torsion.*

Eqs. (19) and (20) then give:

$$T_{xr} = 0\,; \quad \text{and,} \quad T_{x\phi} = G\alpha r \quad . \quad . \quad . \quad . \quad (78)$$

Eq. (23) gives for the moment of torsion:

$$M = G\alpha I_p \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

or:

$$M = 0.5\,\pi r_0^4 \,.\, G\alpha = \frac{A^4 G}{4\pi^2 I_p}\alpha \quad . \quad . \quad . \quad . \quad . \quad (80)$$

In which equation, A is the area of the section and

$$I_p = \frac{\pi r_0^4}{2}\,.$$

The greatest intensity of shear in the section will be obtained by making $r = r_0$ in Eq. (78); or:

$$T_m = G\alpha r_0 \quad \therefore \quad G\alpha = \frac{T_m}{r_0} \quad . \quad . \quad . \quad . \quad . \quad (81)$$

Eq. (80) then becomes:

$$M = 0.5\,\pi r_0^3\,T_m = 2\frac{IT_m}{r_0} \quad . \quad . \quad . \quad . \quad . \quad (82)$$

$$\therefore \quad T_m = 0.64\frac{M}{r_0^3} = 0.5\frac{M}{I}\,r_0\,; \quad . \quad . \quad . \quad . \quad (83)$$

in which

$$I = \frac{\pi r_0^4}{4}.$$

It is thus seen that the circular section is the only one treated which remains plane during torsion.

General Observations.

The preceding examples will sufficiently exemplify the method to be followed in any case. Some general conclusions, however, may be drawn from a consideration of Eq. (33).

If the perimeter is symmetrical about the line from which φ is measured, then r must be the same for $+\varphi$ and $-\varphi$; hence:

$$c'_1 = c'_2 = c'_3 = \ldots \quad = 0.$$

If the perimeter is symmetrical about a line at right angles to the zero position of r, then r must be the same for:

$$\varphi = 90° + \varphi' \quad \text{and} \quad 90° - \varphi';$$

hence:

$$c_1 = c_3 = c_5 \ldots \quad = c'_2 = c'_4 = c'_6 = \ldots \quad = 0.$$

In connection with the first of these sets of results, Eq. (32) shows that *every axis of symmetry of sections represented by* Eq. (33) *will not be moved from its original position by torsion.*

If the section has two axes of symmetry passing through the origin of co-ordinates, then will all the above constants be zero, and its equation will become:

$$\frac{r^2}{2} + c_2 r^2 \cos 2\varphi + c_4 r^4 \cos 4\varphi + c_6 r^6 \cos 6\varphi + \ldots \quad = K.$$

Art. 11.—Torsional Oscillations of Circular Cylinders.

Two cases of torsional oscillations will be considered, in the first of which the cylindrical body twisted is supposed to be the only one in motion. In the second case, however, the mass of the twisted body will be neglected, and the motion of a heavy body, attached to its free end, will be considered. In both cases the section of the cylinder will be considered circular.

Since these cases are those of motion, the internal stresses are not, in general, in equilibrium; hence, equations of motion must be used, and those of Art. 7 are most convenient. Of these last, the investigations of the preceding Art. show that Eq. (4) is the only one which gives any conditions of motion in the problem under consideration.

Putting the value of:

$$T = T_{\phi x} = G\frac{dw}{dx}$$

in Eq. (4) of Art. 7, that equation may take the form:

$$\frac{d^2w}{dt^2} = \frac{G}{m}\frac{d^2w}{dx^2}; \quad \text{or,} \quad \frac{d^2w}{dt^2} - b^2\frac{d^2w}{dx^2} = 0 \quad . \quad . \quad . \quad (1)$$

For brevity, b^2 is written for $\frac{G}{m}$.

That dimension of the cross section of the body which lies in the direction of the radius will be assumed so small that w may be considered a function of x and t only. The results will then apply to small solid cylinders and all hollow ones with thin walls.

The general integral of Eq. (1), on the assumption just made, is (Booles' "Differential Equations," Chap. XV., Ex. 1):

$$w = f(x + bt) + F(x - bt);$$

in which f and F signify *any arbitrary functions whatever*. Now it is evident that all oscillations are of a periodic character, *i.e.*, at the end of certain equal intervals of time, w will have the same value. Hence since f and F are arbitrary forms, and since circular functions are periodic, there may be written:

$$w = A_n\{\sin(\alpha_n x + \alpha_n bt) + \sin(\alpha_n x - \alpha_n bt)\}$$

$$- B_n\{\cos(\alpha_n x + \alpha_n bt) - \cos(\alpha_n x - \alpha_n bt)\}\ ; \quad . \quad . \quad (2)$$

in which α_n, A_n and B_n are coefficients to be determined.

Substituting for the sines and cosines of sums and differences of angles:

$$w = 2 \sin \alpha_n x (A_n \cos \alpha_n bt + B_n \sin \alpha_n bt) \quad . \quad . \quad . \quad (3)$$

Let the origin of co-ordinates be taken at the fixed end of the piece; w must then be equal to zero, as is shown by Eq. (3). But there may be other points at which w is always equal to zero, whatever value the time t may have. These points, called *nodes*, found by putting $w = 0$; or:

$$\sin \alpha x = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This equation is satisfied by taking:

$$\alpha_n = \frac{\pi}{a}, \ \frac{2\pi}{a}, \ \frac{3\pi}{a}, \ \ldots, \ \frac{n\pi}{a};$$

and $x = a$; in which a is the length of the piece.

Hence, *at the distances :*

$$a, \frac{a}{2}, \frac{a}{3}, \ldots, \frac{a}{n}$$

from the fixed end of the piece, there will exist sections which are never distorted or moved from their positions of rest. These are called *nodes*, and one is assumed at the free end, although such an assumption is not necessary, since a is really the distance from the fixed end to the farthest *node* and not necessarily to the free end.

If, as is permissible, A_n and B_n be written for twice those quantities, the general value of w now becomes:

$$w = \sin\frac{\pi x}{a}\left(A_1 \cos\frac{\pi bt}{a} + B_1 \sin\frac{\pi bt}{a}\right)$$

$$+ \sin\frac{2\pi x}{a}\left(A_2 \cos\frac{2\pi bt}{a} + B_2 \sin\frac{2\pi bt}{a}\right)$$

$$+ \sin\frac{3\pi x}{a}\left(A_3 \cos\frac{3\pi bt}{a} + B_3 \sin\frac{3\pi bt}{a}\right)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$+ \sin\frac{n\pi x}{a}\left(A_n \cos\frac{n\pi bt}{a} + B_n \sin\frac{n\pi bt}{a}\right) \quad . \quad . \quad (5)$$

The coefficients A and B are to be determined by the ordinary procedure for such cases, Let:

$$w_1 = \varphi(x)$$

be the expression for the initial or known strain at any point, for which the time t is zero. Then if A_n is any one of the coefficients A:

$$A_n = \frac{2}{a}\int_0^a \varphi(x) \sin \frac{n\pi x}{a}\, dx \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The velocity at any point, or at any time, will be given by:

$$\frac{dw}{dt} = - \sin \frac{\pi x}{a}\left(A_1 \sin \frac{\pi bt}{a} - B_1 \cos \frac{\pi bt}{a}\right) \frac{\pi b}{a} \quad . \quad . \quad (7)$$

In the initial condition, when the time is zero, or $t = 0$, it has the given, or known, value:

$$\frac{dw_1}{dt} = \Phi(x) = \frac{\pi b}{a}\left(B_1 \sin \frac{\pi x}{a} + 2B_2 \sin \frac{2\pi x}{c}\right.$$

$$\left. + 3B_3 \sin \frac{3\pi x}{a} + \ldots\right)$$

Then, as before:

$$B_n = \frac{2}{n\pi b}\int_0^a \Phi(x) \sin \frac{n\pi x}{a}\, dx \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Thus the most general value of w is completely determined.

The intensity of shear at any place or time is given by:

$$T = G\frac{dw}{dx};$$

w being taken from Eq. (5).

The second case to be treated is that of the torsion pendulum, in which the mass of the twisted body is so inconsiderable in comparison with that of the heavy body, or bob, attached to its free end that it may be neglected.

Let M represent the mass of the pendulum bob, and k, *its radius of gyration in reference to the axis about which it is to vibrate;* then will Mk^2 be its moment of inertia about the same axis.

The unbalanced moment of torsion, with the angle of torsion α, is, by Eq. (9) of Art. 10:

$$G\alpha I_p.$$

The elementary quantity of work performed by this unbalanced couple, if β is the general expression for the angular velocity of the vibrating body, is:

$$G\alpha I_p \,.\, \beta \, dt.$$

This quantity of energy is equal in amount but opposite in sign to the indefinitely small variation of actual energy in the bob; hence:

$$G\alpha I_p\beta \, dt = - d\left(\frac{Mk_2\beta^2}{2}\right) = - Mk^2\beta \, d\beta.$$

But if a is the length of the piece twisted:

$$\beta = \frac{d(\alpha a)}{dt}, \quad \text{and} \quad d\beta = \frac{d^2(\alpha a)}{dt}.$$

$$\therefore \quad \left(\frac{GI_p}{a}\right)(\alpha a) = - Mk^2 \frac{d^2(\alpha a)}{dt^2}.$$

Multiplying this equation by $2d(\alpha a)$, and for brevity putting:

$$\left(\frac{GI_p}{a}\right) = H; \quad (Mk^2) = K;$$

then integrating and dropping the common factor a^2:

$$H\alpha^2 = -K\left(\frac{d\alpha}{dt}\right)^2 + C.$$

When $\alpha = \alpha_1$, the value of the angle of torsion at the extremity of an oscillation, the bob will come to rest and $\frac{d\alpha}{dt}$ will be zero. Hence:

$$C = H\alpha_1^2,$$

and

$$K\left(\frac{d\alpha}{dt}\right)^2 = H(\alpha_1^2 - \alpha^2).$$

$$\therefore \quad \frac{d\alpha}{\sqrt{\alpha_1^2 - \alpha^2}} = \sqrt{\frac{H}{K}} \, . \, dt.$$

$$\therefore \quad sin^{-1}\frac{\alpha}{\alpha_1} = t\sqrt{\frac{H}{K}} + (C' = 0) \; . \; . \; . \; . \quad (9)$$

$C' = 0$ because α and t can be put equal to zero together.

At the opposite extremities of a complete oscillation α will have the values:

$$(+\alpha_1) \quad \text{and} \quad (-\alpha_1).$$

Putting these values in the expression:

$$t = \sqrt{\frac{K}{H}} \, . \, sin^{-1}\frac{\alpha}{\alpha_1} \; . \; . \; . \; . \; . \; . \quad (10)$$

and taking the difference between the results thus obtained, the following interval of time for a complete oscillation will be found :

$$\tau = \pi\sqrt{\frac{K}{H}} = \pi\sqrt{\frac{Mk^2a}{GI_p}} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The time required for an oscillation is thus seen to *vary directly as the square root of the moment of inertia of the bob and the length of the piece, and inversely as the square root of the coefficient of elasticity for shearing and the polar moment of inertia of the normal section of the piece twisted.*

The number of complete oscillations per second is $\frac{1}{\tau}$. If this number is the observed quantity, the following equation will give G:

$$G = \left(\frac{1}{\tau}\right)^2 \frac{\pi^2 Mk^2a}{I_p}.$$

The formulas for this case should only be used when the mass of the cylindrical piece twisted is exceedingly small in comparison with M.

Art. 12.—Thick, Hollow Spheres.

In order to investigate the conditions of equilibrium of stress at any point within the material which forms a thick hollow sphere, it will be most convenient to use the equations of Art. 8. As in the case of a thick, hollow cylinder, the interior and exterior surfaces of the sphere are supposed to be subjected to fluid pressure.

Let r' and r_1 be the interior and exterior radii, respectively.

Let $-p$ and $-p_1$ be the interior and exterior intensities, respectively.

Since each surface is subjected to normal pressure of uniform intensity *no tangential internal stress can exist*, but normal stresses in three rectangular co-ordinate directions may and do exist. Consequently, in the notation of Art. 8,

$$T_{\phi r} = T_{\psi r} = T_{\psi\phi} = 0.$$

With a given value of r, also, a uniform state of stress will exist. *Neither N_ψ nor N_ϕ can, then, vary with φ or ψ.* By the aid of these considerations, and after omitting R_0, Φ_0, Ψ_0, and the second members, the Eqs. (1), (2) and (3) of Art. 8 reduce to:

$$\frac{dN_r}{dr} + \frac{2N_r - N_\phi - N_\psi}{r} = 0 \quad . \quad . \quad . \quad . \quad (1)$$

$$-N_\psi + N_\phi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

By Eq. (2):

$$N_\psi = N_\phi \,.$$

Eq. (1) then becomes:

$$\frac{dN_r}{dr} + 2\,\frac{N_r - N_\phi}{r} = 0 \quad . \quad . \quad . \quad . \quad (3)$$

On account of the existing condition of stress, which has just been indicated, it at once results that:

$$\eta = \omega = 0,$$

and that ρ is a function of r only.

Eqs. (4) to (10), of Art. 8, then reduce to:

$$\theta = \frac{d\rho}{dr} + \frac{2\rho}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$N_r = \frac{2G\mathfrak{r}}{1 - 2\mathfrak{r}}\theta + 2G\frac{d\rho}{dr} \quad . \quad . \quad . \quad . \quad (5)$$

$$N_\psi = N_\phi = \frac{2G\mathfrak{r}}{1 - 2\mathfrak{r}}\theta + 2G\frac{\rho}{r} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

After substitution of these quantities, Eq. (3) becomes:

$$\frac{2G\mathfrak{r}}{1 - 2\mathfrak{r}}\left(\frac{d^2\rho}{dr^2} + \frac{2rd\rho - 2\rho dr}{r^2\, dr}\right) + 2G\frac{d^2\rho}{dr^2} + 4G\frac{d\rho}{r\, dr}$$

$$- 4G\frac{\rho}{r^2} = 0.$$

or:

$$\frac{d^2\rho}{dr^2} + \frac{\left(d\frac{2\rho}{r}\right)}{dr} = 0.$$

One integration gives:

$$\frac{d\rho}{dr} + \frac{2\rho}{r} = c = \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence θ, the rate of variation of volume, is a constant quantity. Eq. (7) may take the form:

$$r\, d\rho + 2\rho\, dr = cr\, dr.$$

As it stands, this equation is not integrable, but, by inspecting its form, it is seen that r is an integrating factor. Multiplying both sides of the equation, then, by r:

$$r^2\,d\rho + 2r\rho\,dr = d(r^2\rho) = cr^2\,dr.$$

$$\therefore\quad r^2\rho = c\frac{r^3}{3} + b \qquad \therefore\quad \rho = \frac{cr}{3} + \frac{b}{r^2}\quad . \quad . \quad . \quad (8)$$

Substituting from Eqs. (7) and (8) in Eq. (5):

$$N_r = \frac{2G\mathfrak{r}}{1 - 2\mathfrak{r}}c + \frac{2Gc}{3} - \frac{4bG}{r^3}\quad . \quad . \quad . \quad . \quad (9)$$

$$= A - \frac{4bG}{r^3}.$$

It is obvious what A represents.

When r' and r_1 are put for r, N_r becomes $-p$ and $-p_1$. Hence:

$$A - \frac{4bG}{r'^3} = -p;$$

and:

$$A - \frac{4bG}{r_1^3} = -p_1.$$

These equations express the conditions involved in Eqs. (13), (14) and (15), of Art. 6.

The last equations give:

$$4Gb = \frac{(p_1 - p)\,r_1^3 r'^3}{r'^3 - r_1^3}.$$

$$\therefore \quad A = \frac{p_1 r_1^3 - p r'^3}{r'^3 - r_1^3}.$$

These quantities make it possible to express N_r and N_ϕ independently of the constants of integration, c and b, for those intensities become:

$$N_r = \frac{p_1 r_1^3 - p r'^3}{r'^3 - r_1^3} - \frac{(p_1 - p)\, r_1^3 r'^3}{r'^3 - r_1^3} \cdot \frac{1}{r^3} \quad . \quad (10)$$

$$N_\phi = N_\psi = \frac{p_1 r_1^3 - p r'^3}{r'^3 - r_1^3} + \frac{(p_1 - p)\, r_1^3 r'^3}{2(r'^3 - r_1^3)} \cdot \frac{1}{r^3} \quad . \quad (11)$$

Thus it is seen that $N_\phi = N_\psi$ has its greatest value for the interior surface; that intensity will be called h.

It is now required to find $r_1 - r' = t$ in terms of h, p and p_1.

If $r = r'$ in Eq. (11):

$$2h(r'^3 - r_1^3) = 3p_1 r_1^3 - p(2r'^3 + r_1^3).$$

Dividing this equation by r'^3 and solving:

$$\frac{r_1^3}{r'^3} = \frac{2(h + p)}{2h - p + 3p_1}.$$

$$\therefore \quad r_1 - r' = t = r' \sqrt[3]{\frac{2(h + p)}{2h - p + 3p_1}} - r' \quad . \quad . \quad (12)$$

If the intensities p and p_1 are given for any case, Eq. (12) will give such a thickness that the greatest tension h (supposing p_1 considerably less than p) shall not exceed any assigned value. If the external pressure is very small compared with the internal, p_1 may be omitted.

The values of A and $4Gb$ allow the expressions for c and b to be at once written.

If p_1 is greater than p, nothing is changed except that $N_\phi = N_\psi$ becomes negative, or compression.

CHAPTER III.

THE ENERGY OF ELASTICITY.

Art. 13.—Work Expended in Producing Strains.

THE general expressions, in rectangular co-ordinates, for the unbalanced forces which act in the three co-ordinate directions upon any indefinitely small parallelopiped of material subjected to any state of stress whatever, are given by multiplying each of Eqs. (7), (8) and (9) of Art. 6 by $(dx\ dy\ dz)$. If an indefinitely small change in the state of stress takes place, that indefinitely small parallelopiped will suffer a displacement whose rectangular components are du, dv, dw; and the amount of work performed in moving it will be found by multiplying each of the three unbalanced forces, determined as above, by each of the three small strains belonging to the same direction with the force (as in Art. 6, u, v and w are strains in the directions of x, y and z). This differential quantity of work, integrated throughout the extent of the body, will give the elementary quantity of work required for the small deformation and motion of the whole body.

The resulting equations form the foundation of investigations in elastic vibrations and resilience; they also furnish the means of reaching some general conclusions in reference to suddenly applied loads.

Let dW represent the elementary quantity of work required for the motion only, then the operations which have just been indicated will give the following expression :

$$\iiint\Big[\left(\frac{dN_1}{dx}\,dx\,dy\,dz + \frac{dT_3}{dy}\,dx\,dy\,dz + \frac{dT_2}{dz}\,dx\,dy\,dz\right)du$$

$$+\left(\frac{dT_3}{dx}\,dx\,dy\,dz + \frac{dN_2}{dy}\,dx\,dy\,dz + \frac{dT_1}{dz}\,dx\,dy\,dz\right)dv$$

$$+\left(\frac{dT_2}{dx}\,dx\,dy\,dz + \frac{dT_1}{dy}\,dx\,dy\,dz + \frac{dN_3}{dz}\,dx\,dy\,dz\right)dw$$

$$+\left(X_0\,du + Y_0\,dv + Z_0\,dw\right)dx\,dy\,dz\Big] =$$

$$\iiint m\left(du\,\frac{d^2u}{dt^2} + dv\,\frac{d^2v}{dt^2} + dw\,\frac{d^2w}{dt^2}\right)dx\,dy\,dz = dW \quad . \quad (1)$$

This equation, however, can be put in a much simpler form, and, caused to take a shape which will show at a glance the true character of each part; dx, dy and dz are differentials of independent variables, hence they are arbitrary and independent. Integrating by parts, therefore:

$$\iiint\frac{dN_1}{dx}\,dx\,.\,dy\,dz\,.\,du = \iint(N_1'\,du' - N_1''\,du'')\,dy\,dz$$

$$-\iiint N_1\,d\left(\frac{du}{dx}\right)dx\,dy\,dz\,;$$

in which the *primes* indicate the values of N_1 and u at one point of the exterior surface of the body, and the *seconds* those values for another point of the exterior surface; these points being taken at opposite extremities of *a bar of the material whose normal section is* ($dy\,dz$) *and which extends entirely through* *the body in the direction of x.* Maintaining the same notation and proceeding:

$$\iiint \frac{dT_3}{dy} dy \cdot dx\, dz \cdot du = \iint (T_3'\, du' - T_3''\, du'')\, dx\, dz$$

$$- \iiint T_3\, d\left(\frac{du}{dy}\right) dy\, dx\, dz\,.$$

$$\iiint \frac{dT_2}{dz} dz \cdot dx\, dy \cdot du = \iint (T_2'\, du' - T_2''\, du')\, dx\, dy$$

$$- \iiint T_2\, d\left(\frac{du}{dz}\right) dz\, dx\, dy.$$

But by referring to the equations which immediately precede (13), (14) and (15) of Art. 6, it will be seen that the sum of these three *double* integrals will represent *the amount of work performed on the body by the external forces acting in the direction of the axis of x.* Precisely the same general results are obtained for the directions of y and z by treating in the same manner the remaining derivatives of the internal intensities in Eq. (1). The preceding operations are typical, therefore they need not be repeated.

Again, by reference to the notation and demonstrations of Art. 5:

$$d\left(\frac{du}{dy}\right) + d\left(\frac{dv}{dx}\right) = d\varphi_1\,;$$

$$d\left(\frac{dv}{dz}\right) + d\left(\frac{dw}{dy}\right) = d\varphi_2\,;$$

$$d\left(\frac{dw}{dx}\right) + d\left(\frac{du}{dz}\right) = d\varphi_3\,;$$

$$d\left(\frac{du}{dx}\right) = dl_1; \quad d\left(\frac{dv}{dy}\right) = dl_2; \quad d\left(\frac{dw}{dz}\right) = dl_3.$$

Finally:

$$du \frac{d^2u}{dt^2} = \frac{1}{2} d\left(\frac{du}{dt}\right)^2; \ dv \frac{d^2v}{dt^2} = \frac{1}{2} d\left(\frac{dv}{dt}\right)^2;$$

$$dw \frac{d^2w}{dt^2} = \frac{1}{2} d\left(\frac{dw}{dt}\right)^2.$$

Introducing these reductions and quantities, Eq. (1) becomes:

$$\int P'da' \ (\cos \pi' \ du' + \cos \chi' \ dv' + \cos \rho' \ dw)$$

$$- \int P''da'' \ (\cos \pi'' \ du'' + \cos \chi'' \ dv'' + \cos \rho'' \ dw'')$$

$$- \iiint (N_1dl_1 + N_2dl_2 + N_3dl_3 + T_3d\varphi_1 + T_1d\varphi_2 + T_2d\varphi_3) \ dx \ dy \ dz$$

$$+ \iiint (X_0 \, du + Y_0 \, dv + Z_0 \, dw) \ dx \ dy \ dz$$

$$= \iiint m \frac{1}{2} d\left[\left(\frac{du}{dt}\right)^2 + \left(\frac{dv}{dt}\right)^2 + \left(\frac{dw}{dt}\right)^2\right] dx \ dy \ dz = dW. \quad (2)$$

Eq. (2) shows clearly the distribution of the different portions of work expended. The first two (single) integrals evidently represent the total amount of work performed by forces acting on the exterior surface of the body; it will be indicated by dW_1. If the forces P' and P'' are of the same kind (*i.e.*, both pulls or both pushes), the *algebraic* sum of any two terms of these integrals will be a *numerical* sum if they involve cosines of the same letter but of opposite signs.

The correct application of Eq. (2) depends largely upon the proper observance of the signs which should affect P', P'' and the cosines.

The first triple integral in the first member of Eq. (2), in which each intensity of stress is multiplied by the differential of its characteristic strain, and which will be indicated by dW_2, is evidently the amount of work required for the small distortion alone, of the body. The quantity within the parentheses is called the *potential energy of the elasticity of a cubic unit of material*, since, *if it be multiplied by* $(dx\ dy\ dz)$, *the product will express the amount of work that small portion of material can perform in returning to its original condition.*

This potential energy for a cubic unit is easily integrated by the aid of Eqs. (11), (12), (13), (17), (18) and (19) of Art. 5. Making the substitutions from those equations and integrating:

$$H = \int (N_1\,dl_1 + N_2\,dl_2 + N_3\,dl_3 + T_3\,d\varphi_1 + T_2\,d\varphi_3 + T_1\,d\varphi_2)$$

$$= 2G\left(\frac{l_1^2 + l_2^2 + l_3^2}{2}\right) + \frac{2Gr}{1 - 2r}\,\frac{(l_1 + l_2 + l_3)^2}{2}$$

$$+ G\left(\frac{\varphi_1^2 + \varphi_2^2 + \varphi_3^2}{2}\right).$$

H is the potential energy of a cubic unit of material for a change of state extending from the limit 0 to the strains l_1, l_2, etc.

The last triple integral in the first member of Eq. (2) expresses the work done by external forces which take hold of the mass of the body. Let it be represented by dW_3. This triple integral added to the first two single integrals, which belong to the surface of the body, will give the *total* work done by external forces.

The second member of the equation is the small variation of actual energy, which usually exists in consequence of vibrations.

Let V be the resultant velocity of the parallelopiped, then will:

$$\frac{1}{2} dV^2 = VdV = \frac{1}{2} d\left[\left(\frac{du}{dt}\right)^2 + \left(\frac{dv}{dt}\right)^2 + \left(\frac{dw}{dt}\right)^2\right].$$

By transferring dW_2, the first two members of Eq. (2) may take the form:

$$dW_1 + dW_3 = dW_2 + \iiint mV\,dV\,dx\,dy\,dz.$$

$$\therefore \quad W_1 + W_3 = W_2 + \iiiint mV\,dV\,dx\,dy\,dz \quad . \quad . \quad (3)$$

Or, *the total external work performed is equal to the work done in distorting the body added to the change of actual energy.*

This result expresses the law of the conservation of energy for the elastic bodies considered.

If the external work is nothing, the first member of Eq. (3) is zero. The actual energy will then exist in consequence of a state of vibration. Let its variable value be represented by U. Since dx, dy, and dz are arbitrary:

$$U = \iiint m \frac{V^2}{2} dx\,dy\,dz - C;$$

C representing a constant of integration. Under the circumstances assumed, then :

$$W_2 + U = C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence, *the total energy of the vibrating body* (i.e., *the sum of the actual and potential*) *will be constant.*

Art. 14.—Resilience.

The term *resilience* is applied to the quantity of work which is required to be expended in order to produce a given state of strain in a body. The analytical expression for this amount of work is obtained directly from Eq. (2) of the preceding Art.

Let the simple case of a single straight bar be considered; and let all the external forces act parallel to the axis of the bar while they take hold of the end surfaces, which are normal sections. These external forces will be considered equal to the internal stresses developed; consequently no vibrations will exist. The action of the external forces X_0, Y_0 and Z_0 will also be omitted.

Now, if the axis of x be taken parallel to the axis of the bar, and if that end of the bar to which P'' is applied be fixed, there will result from the preceding conditions:

$$\cos \pi' = \cos \pi'' = 1,$$

$$\cos \chi' = \cos \rho' = \cos \chi'' = \cos \rho'' = du'' = 0,$$

$$N_2 = N_3 = T_1 = T_2 = T_3 = 0.$$

Eq. (2) will then become:

$$\int P' da\ du' = \iiint N_1\, dl_1\, dx\, dy\, dz \quad . \quad . \quad . \quad . \quad (1)$$

But if the intensity P' is uniform and A the area of normal section, Eq. (1) becomes:

$$P'A\,du' = AN_1\,x_1\,dl_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which x_1 is the length of the bar.

From Eq. (1) of Art 1:

$$N_1 = El_1,$$

hence:

$$\int P'A\,du' = Resilience = Ax_1 E \int_0^{l_1} l_1\,dl_1 = Ax_1 E \frac{l_1^2}{2}. \quad (3)$$

The quantity:

$$El_1^2 = \frac{N_1^2}{E}$$

is called the "*Modulus of Resilience.*" This term is usually applied when N_1 is the greatest intensity allowed in the bar.

If one end of a bar, placed in a vertical position, is fixed, while a falling body whose weight is w, acts upon the other end, the height of fall may be sufficient to produce rupture. Let h be the height of fall required and $N_1 = p$ the ultimate resistance of the material of the bar. In order that rupture may take place:

$$wh = \frac{Ax_1}{2} \cdot \frac{p^2}{E} \therefore h = x_1 \frac{A}{2w} \cdot \frac{p^2}{E} \cdot \quad . \quad . \quad . \quad (4)$$

Eq. (4) shows that *the height of fall varies directly as the length of the piece.* It is virtually assumed, however, that the extension or compression is uniform throughout the length of the bar, to the instant of rupture. This, in reality, is not true, and h will not vary as rapidly as x_1. The principle established in Eq. (4) is equally true for torsion and bending.

Art. 15.—Suddenly Applied External Forces or Loads.

A very important deduction can be reached by an attentive consideration of Eq. (2) of Art. 13, if it be assumed that the external forces P' and P'' are simple and direct functions of the external strains u, v and w. In such a case the following relations will hold, in which a, b and c are constants:

$$P' \cos \pi' = au' \; ; \quad P' \cos \chi' = bv' \; ; \quad P' \cos \rho' = cw' \; ;$$

$$P'' \cos \pi'' = au'' \; ; \quad P'' \cos \chi'' = bv'' \; ; \quad P'' \cos \rho'' = cw'' \, .$$

Consequently the external work performed, omitting X_0, Y_0 and Z_0, in changing the body from a state of no stress to that indicated by the strains $\mathfrak{u}'$, $\mathfrak{v}'$, $\mathfrak{w}'$, $\mathfrak{u}''$, $\mathfrak{v}''$, $\mathfrak{w}''$, will be:

$$\int dW_1 = \int da' \left(a \frac{\mathfrak{u}'^2}{2} + b \frac{\mathfrak{v}'^2}{2} + c \frac{\mathfrak{w}'^2}{2} \right)$$

$$- \int da'' \left(a \frac{\mathfrak{u}''^2}{2} + b \frac{\mathfrak{v}''^2}{2} + c \frac{\mathfrak{w}''^2}{2} \right) = W' \; ;$$

in which equations the integrals are to be made to cover the whole extent of the surface.

If, instead of being variable, the forces P' and P'' are constant and equal to the *final* values of the preceding case (*i.e.*, equal to $a\mathfrak{u}'$, $b\mathfrak{v}'$, $c\mathfrak{w}'$, $a\mathfrak{u}''$, etc.), the external work performed in bringing the body to the final state $\mathfrak{u}'$, $\mathfrak{v}'$, etc., will be:

$$\int dW_1 = \int da' (a\mathfrak{u}'^2 + b\mathfrak{v}'^2 + c\mathfrak{w}'^2)$$

$$- \int da'' (a\mathfrak{u}''^2 + b\mathfrak{v}''^2 + c\mathfrak{w}''^2) = 2W'.$$

This last case is that of "suddenly" applied external forces or loads, while the former is that of gradual application, in which the external forces, at each instant, are equal to the internal resistances. In the case of sudden application it is seen that the amount of work expended is twice as great as in the other case; consequently when the body arrives at the state of strain indicated by $\mathfrak{u}'$, $\mathfrak{v}'$, etc., *there remains to be expended just as much work as has already been performed*, and at the instant in question *it exists in the body in the shape of actual energy.*

But if an amount of energy equal to W' will produce the strains $\mathfrak{u}'$, $\mathfrak{v}'$, etc., and if, while the force acts which performed the work, an additional amount of energy equal to W' be expended on the body, additional strains equal to $\mathfrak{u}'$, $\mathfrak{v}'$, etc., will be produced in the body.

When the body comes to rest, therefore, the external strains will be $2\mathfrak{u}'$, $2\mathfrak{v}'$, $2\mathfrak{w}'$, etc. There is then no actual energy, all is potential.

Since the external strains are $2\mathfrak{u}'$, $2\mathfrak{v}'$, etc., the external work which has been performed up to this instant will be found by putting those quantities in the place of $\mathfrak{u}'$, $\mathfrak{v}'$, etc., in the expression for W', above. That expression will then become $4W'$.

For gradually applied loads Eq. (2) of Art. 13 becomes simply:

$$W' = \iiint H \, dx \, dy \, dz;$$

in which H is the potential energy per cubic unit for the state of strain corresponding to $\mathfrak{u}'$, $\mathfrak{v}'$, $\mathfrak{w}'$, etc. But, if the loads be suddenly applied, in accordance with what has been given, the Eq. (2) of Art. 13 becomes:

$$4W' = \iiint 4H\, dx\, dy\, dz\,.$$

Now the expression for H, given in Art. 13, shows that multiplying H by 4 *is the same thing as doubling the strains:*

$$l_1,\ l_2,\ l_3,\ \varphi_1,\ \varphi_2 \text{ and } \varphi_3.$$

But by doubling the strains the intensities of stresses are doubled. Hence, *if the same loads are first applied gradually and then suddenly, the strains and stresses in the latter case will be double those in the former.* This is a very important principle in engineering practice, for it covers all cases of tension, compression, torsion and bending. It also finds many important extensions in special cases of such structures as iron and steel bridges, particularly suspension bridges. For the considerations involved in this Art. show that in all cases of sudden applications of loads, actual energy will be stored and restored during different intervals of time and, consequently, that vibrations will be initiated.

Eq. (2) of Art. 13 furnishes a most convenient and elegant point of departure for investigations in such special cases, as will be exemplified in the next Art.

Art. 16.—Longitudinal Oscillations of a Straight Bar of Uniform Section.

The complete solution of this problem will not be given, though it may be reached.

Let the bar be fixed at one end in a vertical position and let a heavy weight, W, act on the other. Also, let the axis of x be taken parallel to the axis of the bar, whose uniform normal section will be represented by A.

On account of the circumstances of application of the external forces and position of bar, the following equations of condition will exist:

$$\cos\chi' = \cos\rho' = \cos\chi'' = \cos\rho'' = du'' = N_2 = N_3$$

$$= T_1 = T_2 = T_3 = 0 = Y_0 = Z_0.$$

$$\frac{dv}{dt} = \frac{dw}{dt}$$

will be very small compared with $\frac{du}{dt}$, hence they will be omitted. P' is the heavy weight attached to the free end of the bar divided by A; consequently:

$$\cos\pi' = 1.$$

Eq. (2) of Art. 13, now reduces to:

$$\int P'da'\,du' - \iiint El_1\,dl_1\,dx\,dy\,dz + \iiint X_0\,du\,dx\,dy\,dz$$

$$= \iiint m\,\frac{1}{2}\,d\left(\frac{du}{dt}\right)^2 dx\,dy\,dz \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The integrals are to be extended throughout the whole of the bar. Since strains and stresses are uniform for any one cross section of the bar, and because $X_0 = w =$ weight of a unit of volume of the bar (the force of gravity is the only external force which acts on the mass of the bar), Eq. 1 becomes:

$$W\,du' - AE\frac{l_1^2}{2}\,dx + Awu\,dx = \frac{m}{2}\,A\left(\frac{du}{dt}\right)\,dx + C\,dx\ . \quad (2)$$

This equation (C being a constant of integration) involves the complete problem of longitudinal oscillations. Two special cases, only, however, will be treated, in which the weight of the bar is so small compared with W that it may be neglected. This condition involves the omission of :

$$Awu\,dx \quad \text{and} \quad \frac{m}{2}\,A\left(\frac{du}{dt}\right)^2 dx,$$

in Eq. (2), and *makes l_1 constant throughout the length of the bar.*

Since the equation must be homogeneous, C will represent a quantity of actual energy ; in fact, a part of *that quantity stored, at any instant, in W.*

If x_1 represents the length of the bar, C may be put equal to :

$$\frac{W}{2gx_1}\left(\frac{du'}{dt}\right)^2.$$

Also, because l_1 is constant for the whole bar :

$$l_1 = \frac{u'}{x_1}.$$

Introducing all these changes in Eq. (2) and integrating :

$$Wu' - AE\frac{u'^2}{2x_1} = \frac{W}{2g}\left(\frac{du'}{dt}\right)^2 + C'\ .\ \ .\ \ .\ \ . \quad (3)$$

If W is suddenly applied to the bar while in a state of equilibrium or rest, for which :

$$u' = \frac{du'}{dt} = 0,$$

C' will be zero, as the equation shows by such a substitution.

For this case Eq. (3) becomes, after omitting the primes:

$$dt = \sqrt{\frac{Wx_1}{AEg}} \frac{du}{\sqrt{\frac{2Wx_1}{EA} u' - u^2}}.$$

$$\therefore \quad t = \sqrt{\frac{Wx_1}{AEg}} \; ver \; sin^{-1} \frac{AEu}{Wx_1} . \quad . \quad . \quad . \quad (4)$$

The limits of the amplitude are discovered by putting:

$$\frac{du'}{dt} \text{ (the velocity) } = 0,$$

in Eq. (3), remembering that C' is also equal to zero. That operation will give:

$$u = 0 \text{ and } u = \frac{2Wx_1}{AE}.$$

Putting these values in Eq. (4), successively, and taking the difference of the results, the time occupied by one oscillation will be:

$$T = \sqrt{\frac{Wx_1}{AEg}} \; ver \; sin^{-1} \; 2 = \pi \sqrt{\frac{u_1}{g}} \quad . \quad . \quad (5)$$

in which equation:

$$u_1 = \frac{Wx_1}{AE}$$

is the strain in the bar caused by a gradual application of W.

In the second case to be treated the bar is first supposed to take a vertical position, with the weight attached to its free end, in a state of equilibrium. An external force then depresses the free end a distance u_0, *measured from its position of equilibrium.* If the force F is now removed, the weight will make excursions on each side of its position of rest.

Let u_1 represent the value of u' corresponding to the weight W alone, as in the previous case; then let:

$$u' = u_1 + u,$$

u being measured from the position of equilibrium of the weight W.

Eq. (3) will then take the form:

$$W(u_1 + u) - \frac{AE}{2x_1}(u_1 + u)^2 = \frac{W}{2g}\left(\frac{d(u_1 + u)}{dt}\right)^2 + C. \quad (6)$$

When $u = u_0$ the body comes to rest. Hence:

$$W(u_1 + u_0) - \frac{AE}{2x_1}(u_1 + u_0)^2 = C \; . \; . \; . \; . \quad (7)$$

Subtracting Eq. (7) from Eq. (6):

$$W(u - u_0) - \frac{AE}{2x_1}[2u_1(u - u_0) + u^2 - u_0^2] = \frac{W}{2g}\left(\frac{du}{dt}\right)^2. \quad (8)$$

since:

$$d(u_1 + u) = du.$$

Remembering that:

$$W(u - u_0) = \frac{AEu_1}{x_1}(u - u_0),$$

Eq. (8) may take the form :

$$dt = \sqrt{\frac{Wx_1}{AEg}} \frac{du}{\sqrt{u_0^2 - u^2}} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\therefore \quad t = \sqrt{\frac{u_1}{g}} \sin^{-1} \frac{u}{u_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Eq. (9) shows that the amplitude of a vibration is found by putting :

$$u = + u_0 \quad \text{or} \quad - u_0.$$

Putting these values in Eq. (10) and taking the difference of the results, the time of a single oscillation is found to be:

$$T = \pi \sqrt{\frac{u_1}{g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Eq. (11) is seen to be identical with Eq. (5). In this case the amplitude is $2u_0$, and the body oscillates through its position of rest. Both oscillations are completely isochronous for the same weight W.

If n is the observed number of oscillations per second, either Eq. (5) or (11) gives :

$$E = \frac{1}{T^2} \cdot \frac{\pi^2 Wx_1}{Ag} = n^2 \frac{\pi^2 Wx_1}{Ag}. \quad . \quad . \quad . \quad (12)$$

from which E may be computed, if W is very great compared with the weight of the bar or wire.

CHAPTER IV.

THEORY OF FLEXURE.

Art. 17.—General Formulæ.

IF a prismatic portion of material is either supported at both ends, or fixed at one or both ends, and subjected to the action of external forces whose directions are normal to, *and cut*, the axis of the prismatic piece, that piece is said to be subjected to "flexure." If these external forces have lines of action which are oblique to the axis of the piece, it is subjected to combined flexure and direct stress.

Again, if the piece of material is acted upon by a couple having the same axis with itself, it will be subjected to "torsion."

The most general case possible is that which combines these three, and some general equations relating to it will first be established.

The co-ordinate axis of X will be taken to coincide with the axis of the prism, and *it will be assumed that all external forces act upon its ends only*. Since no external forces act upon its lateral surface, there will be taken :

$$T_1 = N_2 = N_3 = 0;$$

retaining tne notation of Art. 6. These conditions are not strictly true for the general case, but the errors are, at most, excessively small for the cases of direct stress or flexure, or for a combination of the two. By the use of Eqs. (12), (21) and (22) of Art. 5, the conditions just given become :

$$\frac{r}{1-2r}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)+\frac{dv}{dy}=0 \quad . \quad . \quad . \quad (1)$$

$$\frac{r}{1-2r}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)+\frac{dw}{dz}=0 \quad . \quad . \quad . \quad (2)$$

$$\frac{dv}{dz}+\frac{dw}{dy}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Eqs. (1) and (2) then give :

$$\frac{dv}{dy}-\frac{dw}{dz}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In consequence of Eq. (4), Eqs. (1) and (2) give :

$$\frac{dv}{dy}=\frac{dw}{dz}=-r\frac{du}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

By the aid of Eq. (5) and the use of Eqs. (11), (13) and (20) of Art. 5, in Eqs. (10), (11) and (12) of Art. 6 (in this case $X_0 = Y_0 = Z_0 = 0$), there will result :

$$2\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\frac{d^2u}{dx\,dy}+\frac{d^2v}{dx^2}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\frac{d^2u}{dx\,dz} + \frac{d^2w}{dx^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The Eqs. (3), (5), (6), (7) and (8) are five equations of condition by which the strains u, v and w are to be determined.

Let Eq. (6) be differentiated in respect to x:

$$2\frac{d^3u}{dx^3} + \frac{d^3u}{dy^2\,dx} + \frac{d^3u}{dz^2\,dx} = 0.$$

From this equation let there be subtracted the sum of the results obtained by differentiating Eq. (7) in respect to y and (8) in respect to z:

$$2\frac{d^3u}{dx^3} - \frac{d^3v}{dx^2\,dy} - \frac{d^3w}{dx^2\,dz} = 0.$$

In this equation substitute the results obtained by differentiating Eq. (5) twice in respect to x, there will result:

$$\frac{d^3u}{dx^3} = \frac{d^2\left(\frac{du}{dx}\right)}{dx^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This result, in the equation immediately preceding Eq. (9) by the aid of Eq. (5), will give:

$$\frac{d^3v}{dx^2\,dy} = 0.$$

After differentiating Eq. (7) in respect to y, and substituting the value immediately above:

$$\frac{d^3u}{dy^2\,dx} = \frac{d\left(\frac{du}{dx}\right)}{dy^2} = 0 \; . \; . \; . \; . \; . \; . \qquad (10)$$

Eqs. (9) and (10) enable the second equation preceding Eq. (9), to give :

$$\frac{d^3u}{dz^2\,dx} = \frac{d\left(\frac{du}{dx}\right)}{dz^2} = 0 \; . \; . \; . \; . \; . \; . \qquad (11)$$

Let the results obtained by differentiating Eq. (7) in respect to z and (8) in respect to y, be added :

$$2\,\frac{d^3u}{dx\,dy\,dz} + \frac{d^3v}{dx^2\,dz} + \frac{d^3w}{dx^2\,dy} = 0.$$

The sum of the second and third terms of the first member of this equation is zero, as is shown by twice differentiating Eq. (3) in respect to x. Hence:

$$\frac{d^3u}{dy\,dz\,dx} = \frac{d\left(\frac{du}{dx}\right)}{dy\,dz} = 0 \; . \; . \; . \; . \; . \qquad (12)$$

The Eqs. (9), (10), (11) and (12) are sufficient for the determination of the form of the function $\frac{du}{dx}$, if it be assumed to be algebraic, for:

Eq. (9) shows that x^2 does not appear in it;
" (10) " " y^2 " " " "
" (11) " " z^2 " " " "
" (12) " " yz " " " "

The products xz and xy may, however, be found in the function. Hence if a, a_1, a_2, b, b_1, and b_2 are constants, there may be written :

$$\frac{du}{dx} = a + a_1 z + a_2 y + x(b + b_1 z + b_2 y) \quad . \quad . \quad (13)$$

Eq. (5) then gives :

$$\frac{dv}{dy} = \frac{dw}{dz} = -r\{a + a_1 z + a_2 y + x(b + b_1 z + b_2 y)\} \; (14)$$

Substituting from Eq. (13) in Eqs. (7) and (8) :

$$\frac{d^2v}{dx^2} = -a_2 - b_2 x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$\frac{d^2w}{dx^2} = -a_1 - b_1 x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The method of treatment of the various partial derivatives in the search for Eqs. (13) and (14) is identical with that given by Clebsch in his "*Theorie der Elasticität Fester Körper.*"

It is to be noticed that the preceding treatment has been entirely independent of the *form of cross section* or *direction of external forces.*

It is evident from Eqs. (13) and (14), that the constant a depends upon that component of the external force which acts parallel to the axis of the piece and produces tension or compression only. For, by Arts. 2 and 3, it is known that if a piece of material be subjected to direct stress only :

$$\frac{du}{dx} = a \quad \text{and} \quad \frac{dv}{dy} = \frac{dw}{dz} = -ra\,;$$

the negative sign showing that ra is opposite in kind to a, both being constant.

Again, if z and y are each equal to zero, Eq. (13) shows that :

$$\frac{du}{dx} = a + bx.$$

Hence bx is a part of the rate of strain in the direction of x which is *uniform over the whole of any normal section of the piece of material*, and it varies directly with x. But such a portion of the rate of strain can only be produced by external force, acting parallel to the axis of X, and whose intensity varies directly as x. But, in the present case such a force does not exist. Hence b must equal zero.

The Eqs. (13), (14), (15) and (16), show that a_1, b_1 and a_2, b_2 are symmetrical, so to speak, in reference to the co-ordinates z and y, while Eqs. (13) and (14) show that the normal intensity N_1 is dependent on those, and no other, constants in pure flexure, in which $a = 0$. It follows, therefore, that those two pairs of constants belong to the two cases of flexure about the two axes of Z and Y.

No direct stress N_1 can exist in torsion, which is simply a twisting or turning about the axis of X.

Since the generality of the deductions will be in no manner affected, pure flexure about the axis of Y will be considered. For this case :

$$a = a_2 = b_2 = 0 = b.$$

Making these changes in (13) and (14) :

$$\frac{du}{dx} = a_1 z + b_1 xz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$\frac{dv}{dy} = \frac{dw}{dz} = - r\frac{du}{dx} = - r\,(a_1 z + b_1 xz) \quad . \quad . \quad (18)$$

$$\therefore \quad \theta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = z\,(a_1 + b_1 x)\,(1 - 2r) \quad . \quad (19)$$

Also :

$$N_1 = \frac{2Gr}{1 - 2r}\,\theta + 2G\frac{du}{dx}.$$

$$\therefore \quad N_1 = 2G\,(r + 1)\,(a_1 + b_1 x)z = E(a_1 + b_1 x)z \quad . \quad (20)$$

since :

$$2G\,(r + 1) = E.$$

Taking the first derivative of N_1 :

$$\frac{dN_1}{dz} = E\,(a_1 + b_1 x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

This important equation gives the law of variation of the intensity of stress acting parallel to the axis of a bent beam, in the case of pure flexure produced by forces exerted at its extremity. That equation proves, that in a given normal section of the beam, whatever may be the form of the section, *the rate of variation of the normal intensity of stress is constant; the rate being taken along the direction of the external forces.*

It follows from this, that N_1 must vary directly as the distance from some particular line in the normal section considered in which its value is zero. Since the external forces F are normal to the axis of the beam and direction of N_1, and because it is necessary for equilibrium that the sum of all the forces $N_1\,dy\,dz$, for a given section, must be equal to zero, it follows that on one side of this line tension must exist and on the other, compression.

Let N represent the normal intensity of stress at the dis-

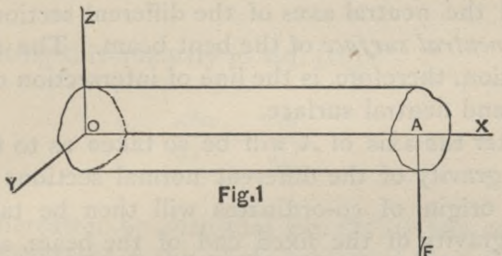


Fig. 1

tance unity from the line, b the variable width of the section parallel to y, and let $\Delta = b\,dz$. The sum of all the tensile stress in the section will be :

$$\int_0^{z'} Nz\Delta = N\int_0^{z'} z\Delta.$$

The total compressive stress will be :

$$N\int_{-z_1}^{0} z\Delta.$$

The integrals are taken between the limits 0 and the greatest value of z in each direction, so as to extend over the entire section. In order that equilibrium may exist therefore :

$$N\left\{\int_0^{z'} z\Delta + \int_{-z_1}^{0} z\Delta\right\} = 0.$$

$$\therefore \quad \int_{-z_1}^{z'} z\Delta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Eq. (22) *shows that the line of no stress must pass through the centre of gravity of the normal section.*

This line of no stress is called the *neutral axis* of the section. Regarding the whole beam, there will be a surface which will contain all the neutral axes of the different sections, and it is called the *neutral surface* of the bent beam. The neutral axis of any section, therefore, is the line of intersection of the plane of section and neutral surface.

Hereafter the axis of X will be so taken as to traverse the centres of gravity of the different normal sections before flexure. The origin of co-ordinates will then be taken at the centre of gravity of the fixed end of the beam, as shown in Fig. 1.

The value of the expression $(a_1 + b_1 x)$, in terms of the external bending moment, is yet to be determined. Consider any normal section of the beam located at the distance x from O, Fig. 1, and let $OA = l$. Also let Fig. 2 represent the section considered, in which BC is the neutral axis and d' and d_1 the distances of the most remote fibres from BC. Let moments of all the forces acting upon the portion $(l-x)$ of the beam be taken about the neutral axis BC. If, again, b is the variable width of the beam, the internal resisting moment will be:

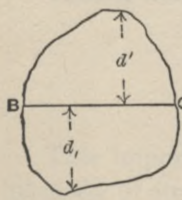


Fig. 2

$$\int_{-d_1}^{d'} N_1 bz\, dz = E(a_1 + b_1 x) \int_{-d_1}^{d'} z^2 . b\, dz.$$

But the integral expression in this equation is *the moment of inertia of the normal section about the neutral axis*, which will hereafter be represented by I. The moment of the external force, or forces, F, will be $F(l - x)$ and it will be equal, but opposite in sign, to the internal resisting moment. Hence:

$$F(l-x) = M = -E(a_1 + b_1x)I \quad . \quad . \quad . \quad (23)$$

$$\therefore \quad -(a_1 + b_1x) = \frac{M}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Substituting this quantity in Eq. (16) :

$$\frac{d^2w}{dx^2} = \frac{M}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

It will hereafter be seen that Eq. (25) is one of the most important equations in the whole subject of the "*Resistance of Materials.*"

An equation exactly similar to (25) may, of course, be written from Eq. (16); but in such an expression M will represent the external bending moment about an axis parallel to the axis of Z.

No attempt has hitherto been made to determine the complete values of u, v, and w, for the mathematical operations involved are very extended. If, however, a beam be considered whose width, parallel to the axis of Y, is indefinitely small u and w may be determined without difficulty. The conclusions reached in this manner will be applicable to any long rectangular beam without essential error.

If y is indefinitely small all terms involving it as a factor will disappear in u and w; or, *the expressions for the strains u and w will be functions of z and x only*. But making u and w functions of z and x *only* is equivalent to a restriction of lateral strains to the direction of z *only*, or, to the reduction of the direct strains one half, since direct strains and lateral strains in two directions accompany each other in the unrestricted case. Now as the lateral strain in one direction is supposed to retain the same amount as before, while the direct strain is considered only half as great, the value of their ratio for the present case will be twice as great as that used in Arts. 3 and 4. Hence $2r$ must be written for r, in order that that letter may represent the ratio for the unrestricted case, and this will be done in the following equations.

Since w and u are independent of y:

$$\frac{dw}{dy} = \frac{du}{dy} = 0, \quad \text{and} \quad T_3 = G\frac{dv}{dx}.$$

But by Eq. (14):

$$v = -2r(a_1 + b_1x)zy + f(x, z).$$

By Eq. (3), since:

$$\frac{dw}{dy} = 0;$$

$$\frac{dv}{dz} = -2r(a_1 + b_1x)y + \frac{d}{dz}f(x, z) = 0.$$

This equation, however, involves a contradiction, for it makes $f(x, z)$ equal to a function which involves y, which is impossible. Hence:

$$f(x, z) = 0.$$

Consequently:

$$\frac{dv}{dz} = -2r(a_1 + b_1x)y;$$

which is indefinitely small compared with:

$$\frac{dv}{dy} = -2r(a_1 + b_1x)z,$$

and is to be considered zero.

Because $f(x, z) = 0$:

$$\frac{dv}{dx} \doteq -\ 2rb_1zy.$$

This quantity is indefinitely small; hence:

$$T_3 = -\ 2Grb_1zy$$

is of the same magnitude.

Under the assumption made in reference to y, there may be written from Eqs. (17) and (18):

$$u = a_1xz + b_1\frac{x^2}{2}z + f'(z) \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$w = -\ r(a_1z^2 + b_1xz^2) + f(x) \quad . \quad . \quad . \quad . \quad (27)$$

Using Eq. (26) in connection with Eq. (6):

$$2b_1z = -\ \frac{d^2f'(z)}{dz^2}.$$

By two integrations:

$$f'(z) = -\ \frac{b_1z^3}{3} - c'z + c'' \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Using Eq. (27) in connection with Eq. (8):

$$\frac{d^2f(x)}{dx^2} = -\ b_1x - a_1.$$

By two integrations:

$$f(x) = -\,b_1 \frac{x^3}{6} - \frac{a_1 x^2}{2} + c_1 x + c_{11}.$$

The functions u and w now become:

$$u = a_1 xz + b_1 \frac{x^2}{2} z - \frac{b_1 z^3}{3} - c'z + c'' \quad . \quad . \quad . \quad (29)$$

$$w = -\,ra_1 z^2 - rb_1 xz^2 - b_1 \frac{x^3}{6} - \frac{a_1 x^2}{2} + c_1 x + c_{11} \quad . \quad (30)$$

The constants of integration c', c'', etc., depend upon the values of u and w, and their derivatives, for certain reference values of the co-ordinates x and z, and, also, upon the manner of application of the external forces, F, at the end of the beam, Fig. 1. The last condition is involved in the application of Eqs. (13), (14) and (15) of Art. 6.

In Fig. 1 let the beam be fixed at O. There will then result for $x = 0$ and $z = 0$:

$$\left(\frac{du}{dz} = 0\right)_{\substack{x=0\\z=0}}$$

$$(u = 0, \quad \text{and} \quad w = 0)_{\substack{z=0\\x=0}}.$$

In virtue of the last condition:

$$c'' = c_{11} = 0.$$

In consequence of the first:

$$c' = 0.$$

After inserting these values in Eqs. (29) and (30):

$$\frac{du}{dz} = a_1 x + b_1 \frac{x^2}{2} - b_1 z^2,$$

$$\frac{dw}{dx} = - r b_1 z^2 - b_1 \frac{x^2}{2} - a_1 x + c_1;$$

$$\therefore \quad T_2 = G\left(\frac{du}{dz} + \frac{dw}{dx}\right) = - G b_1 (1 + r) z^2 + G c_1 \quad . \quad (31)$$

The surface of the end of the beam, on which F is applied, is at the distance l from the origin O and parallel to the plane ZY. Also the force F has a direction parallel to the axis of Z. Using the notation of Eqs. (13), (14) and (15) of Art. 6, these conditions give :

$$\cos p = 1, \qquad \cos q = 0, \qquad \cos r = 0,$$

$$\cos \pi = 0, \qquad \cos \chi = 0, \qquad \cos \rho = 1.$$

Since for $x = l$:

$$M = F(l - x) = 0,$$

Eqs. (24) and (20) give $N_1 = 0$ for all points of the end surface. Eq. (15) is, then, the only one of those equations which is available for the determination of c_1.

That equation becomes simply :

$$T_2 = P.$$

For a given value of z, therefore, any value may be assumed for T_2. For the upper and lower surfaces of the beam let the intensity of shear be zero ; or for $z = \pm d$ let $T_2 = 0$. Hence, by Eq. (31) :

$$c_1 = b_1 (1 + r) d^2;$$

$$\therefore \quad T_2 = Gb_1(1 + r)(d^2 - z^2);$$

$$\therefore \quad T_2 = \frac{Eb_1}{2}(d^2 - z^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

The constants a_1 and b_1 still remain to be found. The only forces acting upon the portion $(l - x)$ of the beam, are F and the sum of all the shears T_2 which act in the section x. Let Δy be the indefinitely small width of the beam, which, since z is finite, is thus really made constant. The principles of equilibrium require that :

$$\int_{-d}^{+d} T_2 \,.\, \Delta y \,.\, dz = Gb_1(1 + r)\int_{-d}^{+d}(d^2 \,.\, \Delta y \,.\, dz - z^2 \,.\, \Delta y \,.\, dz) = F.$$

The first part of the integral will be $2\Delta y d^3$ and the second part will be the moment of inertia of the cross section (made rectangular by taking Δy constant) about the neutral axis. Hence :

$$2Gb_1(1 + r)I = F; \quad \text{or} \quad b_1 = \frac{F}{2G(1 + r)I} = \frac{F}{EI} \quad (33)$$

$$\therefore \quad T_2 = \frac{F}{2I}(d^2 - z^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

If $x = 0$ in Eq. (24) :

$$a_1 = -\frac{Fl}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Thus the two conditions of equilibrium are involved in the determination of a_1 and b_1. The complete values of the strains u and w are, finally :

$$u = \frac{F}{EI}\left(z\frac{x^2}{2} - \frac{z^3}{3} - xzl\right) \quad . \quad . \quad . \quad . \quad . \quad (36)$$

$$w = \frac{F}{EI}\left(lrz^2 - rxz^2 - \frac{x^3}{6} + \frac{lx^2}{2}\right) + \frac{Fd^2x}{2GI} \quad . \quad . \quad (37)$$

These results are strictly true for rectangular beams of indefinitely small width, but they may be applied to any rectangular beam fixed at one end and loaded at the other, with sufficient accuracy for the ordinary purposes of the civil engineer. It is to be remembered that the load at the end is supposed to be applied according to the law given by Eq. (34); a condition which is never realized. Hence these formulæ are better applicable to long than short beams.

The greatest value of T_2, in Eq. (34), is found at the neutral axis by making $z = 0$; for which it becomes:

$$T_2 = \frac{Fd^2}{2I} = \frac{3}{2} \cdot \frac{F}{2d} \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

$\frac{F}{2d}$ is the *mean intensity* of shear in the cross section; hence, *the greatest intensity of shear is once and a half as great as the mean.*

In Eq. (36) if $z = 0$, $u = 0$. Hence no point of the neutral surface suffers longitudinal displacement.

In Eq. (37) the last term of the second member is that part of the vertical deflection due to the shear at the neutral surface, as is shown by Eq. (38). The first term of the second member, being independent of x, is that part of the deflection which arises wholly from the deformation of the normal cross section.

The usual modification of the preceding treatment, designed to supply formulæ for the ordinary experience of the engineer, will be given in the succeeding Arts.

Art. 18.—The Common Theory of Flexure.

The "common theory" of flexure is completely expressed by Eq. (25) of Art. 17. That equation involves the condition that *no external force acts upon the exterior surface of the bar or beam.* In reality this condition is never fulfilled. External loads are applied in any manner whatever, causing normal compressive stresses to exist at any or all points of the exterior surface. *It is assumed in the common theory of flexure that the equation:*

$$\frac{d^2w}{dx^2} = \frac{M}{EI} \quad \cdots\cdots\cdots \quad (1)$$

holds true, for pure bending, whatever may be the number or manner of application of the external forces or loads.

By "pure bending" is meant the action of external forces whose directions are normal to, and cut, the axis of the beam.

As has already been seen in Art. 17, w, strictly speaking, is a function of x, y and z.

It is further assumed in the common theory of flexure that w is a function of x only.

This is equivalent to an assumption that the lateral dimensions of the piece are so small that they can have no influence on the value of w, and consequently that they will not appear in it. In other words the common theory of flexure is the theory of the flexure of pieces, one or two of whose cross dimensions are indefinitely small in comparison with their length. The neglect of this fact has led to some erroneous applications and deductions in connection with long column formulæ.

Eq. (1), taken in connection with these two important assumptions, constitutes the "Common Theory of Flexure," which is always used in engineering practice.

Since the intensity of external loading is almost invariably very small compared with the internal stress N_1, the first of the above assumptions involves very little error in all ordinary cases.

The second assumption, as was stated above, is equivalent to taking the bar or beam so small that the strain or " deflection " w is essentially the same at all points of a given cross section. With such small strains and large ratios of length to lateral dimensions as almost always occur, this assumption, also, involves no considerable error.

It is well known that if the curvature is very small, the reciprocal of the radius of curvature, in the plane zx, is represented with no essential error by $\frac{d^2w}{dx^2}$. Hence Eq. (1) may take the form :

$$\frac{EI}{\rho} = M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which ρ is the radius of curvature.

Let M' and M_1 represent two bending moments which will produce the two radii of curvature ρ' and ρ_1. Eq. (2) will then give the following :

$$\frac{EI}{\rho'} = M' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{EI}{\rho_1} = M_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence :

$$EI\left(\frac{1}{\rho_1} - \frac{1}{\rho'}\right) = M_1 - M' \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The second member shows that a bending moment:

$$M_1 - M' = M,$$

applied to a curved beam whose radius of curvature at any section is ρ', will produce a change of curvature expressed by:

$$\left(\frac{1}{\rho_1} - \frac{1}{\rho'}\right).$$

In other words: *the common theory of flexure is applicable to curved beams of slight curvature.*

In such a case $\frac{1}{\rho}$, Eq. (2), expresses the variation (increase or decrease) of curvature caused by the moment M. It is to be distinctly borne in mind, however, that Eq. (2) itself is made approximately true only by considering the curvature very small.

The limits within which the common theory is applicable to curved beams, and the degree of approximation of the application, will be shown by the following investigations, in which the longitudinal compression and extension, due to the external forces, will be neglected.

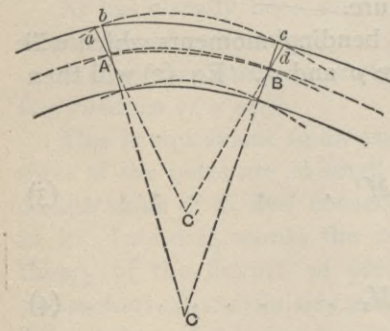


Fig. 1

In the figure let a portion of any curved beam, whose lateral dimensions are small compared with its length, be represented. Let AB represent an indefinitely short length, ds, of the neutral surface. C is the centre of curvature of ds before flexure, and C' the same point after flexure. Since the lateral dimensions are small compared with the length, if the strains are not great, any normal cross section may, without essential error, be taken as plane after flexure, and such planes passing through A and B will then contain the radii of curvature at the points A and B. Let :

$$AC' = \rho' \quad \text{and} \quad AC = \rho$$

also :

$$Aa = Ab = Bc = Bd = \text{ unity.}$$

Aa and Bd are the positions taken by Ab and Bc after flexure. The angle, before flexure, between two radii AC and BC, indefinitely near to each other, is $\frac{ds}{\rho}$; after flexure, as the figure shows, the same angle becomes $\frac{ds}{\rho'}$. Hence the change in curvature (or change of angle between consecutive radii) caused by flexure is :

$$ds\left(\frac{1}{\rho'} - \frac{1}{\rho}\right).$$

Now let the amount of shortening or lengthening of a unit of length of fibres, parallel to the neutral surface and situated at unit's distance from it, be represented by u; concisely stated, u is the *rate* of strain for any point at unit's distance from the neutral surface. In the figure, the amount of strain for $AB = ds$ is :

$$ab + cd = u\,ds.$$

But the difference between the angles $aC'd$ and bCc is :

$$(ab + cd) \div Ab = ab + cd = u\,ds.$$

But this difference is the change of curvature ; hence :

$$u = \frac{1}{\rho'} - \frac{1}{\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This relation is purely kinematical; a value for u must next be determined in terms of the bending moment M.

Under the circumstances of the case it has been seen that the longitudinal *strains* parallel to the neutral surface vary essentially directly as their distances from it (this law is the assumption that plane normal sections before flexure are also plane afterwards). The strain at any distance z from the neutral surface will then be uz. But it was shown in Art. 17 that the intensity of longitudinal stress N_1 varies directly as z; hence there may be written :

$$N_1 = Euz.$$

If b is the variable width of cross section, taken parallel to the neutral surface, the internal resisting moment of the section will be :

$$M = \int N_1 \, b \, dz \, . \, z = Eu \int bz^2 \, dz.$$

$$\therefore \quad M = EuI \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\therefore \quad u = \frac{M}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The integration is to be extended over the whole section. Then, *if the "neutral axis" is the line of intersection of the neutral surface with the normal section*, I is the moment of inertia of the normal section about the neutral axis.

Eqs. (6) and (8) then give :

$$\frac{M}{EI} = \frac{1}{\rho'} - \frac{1}{\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This equation is true, under the assumption made, for any degree of curvature whatever in the original beam.

If w and x are rectangular co-ordinates in the plane of the beam, x being the independent variable, the expressions for the reciprocals of the radius of curvature before and after flexure, are :

$$\frac{1}{\rho} = \frac{d^2w_1}{dx^2}\left(1 + \frac{dw_1^2}{dx^2}\right)^{-\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\frac{1}{\rho'} = \frac{d^2w'}{dx^2}\left(1 + \frac{dw'^2}{dx^2}\right)^{-\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

By the binomial formula :

$$\left(1 + \frac{dw_1^2}{dx^2}\right)^{-\frac{3}{2}} = 1 - \frac{3}{2}\frac{dw_1^2}{dx^2} + \frac{15}{8}\frac{dw_1^4}{dx^4} - \text{etc.};$$

and an exactly similar expression for $\frac{1}{\rho'}$. After introducing these in Eqs. (10) and (11), and supposing the deflections to be small, there may be written :

$$\frac{1}{\rho'} - \frac{1}{\rho} = \frac{d^2w'}{dx^2} - \frac{d^2w_1}{dx^2} + \frac{3}{2}\left(\frac{dw_1^2}{dx^2} - \frac{dw'^2}{dx^2}\right)\frac{d^2w'}{dx^2}$$

$$- \frac{15}{8}\left(\frac{dw_1^4}{dx^4} - \frac{dw'^4}{dx^4}\right)\frac{d^2w_1}{dx^2} - \text{etc.}$$

If, in addition to small deflections, the values of :

$$\frac{dw'}{dx}, \quad \text{and} \quad \frac{dw_1}{dx},$$

are not great, the equation just written shows that with a considerable degree of approximation :

$$\frac{1}{\rho'} - \frac{1}{\rho} = \frac{d^2w'}{dx^2} - \frac{d^2w_1}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The smaller the curvature the more nearly accurate is Eq. (12). If, as before, w is the deflection or strain normal to x :

$$w = w' - w_1$$

$$\therefore \quad d^2w = d^2w' - d^2w_1 ;$$

hence, from Eqs. (9) and (12) :

$$\frac{M}{EI} = \frac{d^2w}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Eq. (13) is exactly the same as Eq. (1) for straight beams.

These investigations show that the common theory of flexure is not strictly applicable to the general case of curved beams. In order to obtain Eq. (12) it was necessary to assume the same law for stresses and strains, in any normal section, both for curved and straight beams, which is not exactly true. It was also necessary to assume small values of

$$\frac{dw_1}{dx} \quad \text{and} \quad \frac{dw'}{dx}$$

for a close approximation. Yet the application of the common theory of flexure to curved beams, even within these restricted limits, is of the highest importance.

In Art. 22 a generalization of the common theory of flexure is given, in which the differential of the centre line of the beam is used instead of dx. The resulting formulæ are accurately applicable to curved beams of any curvature. The only assumption involved, in addition to those of the common theory, is the identity of the law of variation of stresses and strains in curved and straight beams; and that causes very little error.

One of the most important forms of Eq. (7) yet remains to be established.

Let d_1 represent the distance from the neutral axis of any normal section of the beam to that point of the section farthest from it. Let K represent the intensity of tensile or compressive stress (as the case may be) existing at this same point; K will be the greatest intensity in the section. Because the intensity of stress varies directly as the distance from the neutral axis, the intensity at distance unity from that axis will be:

$$\frac{K}{d_1}.$$

But by Art. 2, this intensity also has the value Eu. Consequently Eq. (7) becomes:

$$M = \frac{KI}{d_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

If the external moment is sufficient to break the beam, and if Eq. (14) is applied to the section at which failure begins, K is called the "Modulus of Rupture" for flexure. It is an empirical quantity.

Art. 19.—Deflection by the Common Theory of Flexure.

The common theory of flexure, as developed in the preceding Art., leads to very simple and, in nearly all ordinary cases, very closely approximate formulæ.

Let x_0 be the co-ordinate of some point at which the tangent of the inclination of the neutral surface to the axis of x is known; then, from Eq. (1) of Art. 18:

$$\frac{dw}{dx} = \int_{x_0}^{x} \frac{M}{EI} dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$\frac{dw}{dx}$ will be at once recognized as the general value of the tangent of the inclination just mentioned, or, in the case of curved beams, as approximately the difference between the tangent, before and after flexure.

Again, let x_1 represent the co-ordinate of a point at which the deflection w is known, then, from Eq. (1):

$$w = \int_{x_1}^{x} \int_{x_0}^{x} \frac{M}{EI} dx^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The points of greatest or least deflection and greatest or least inclination of neutral surface are easily found by the aid of Eqs. (1) and (2).

The point of greatest or least deflection is evidently found by putting:

$$\frac{dw}{dx} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and solving for x. Since $\frac{dw}{dx}$ is the value of the tangent of the inclination of the neutral surface, it follows that *a point of greatest or least deflection is found where the beam is horizontal.*

Again, the point at which the inclination will be greatest or least is found by the equation :

$$\frac{d\left(\frac{dw}{dx}\right)}{dx} = \frac{d^2w}{dx^2} = 0 \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

But, approximately, $\frac{d^2w}{dx^2}$ is the reciprocal of the radius of curvature ; hence, *the greatest inclination will be found at that point at which the radius of curvature becomes infinitely great*, or, *at that point at which the curvature changes from positive to negative or vice versa.* These points are called points of "contra-flexure." Since :

$$M = EI\frac{d^2w}{dx^2},$$

there is no bending at a point of contra-flexure.

The moment of the external forces, M, will always be expressed in terms of x. After the insertion of such values, Eqs. (1) and (2) may at once be integrated and (3) and (4) solved.

The coefficient of elasticity, E, is always considered a constant quantity ; hence it may always be taken outside the integral signs. In all ordinary cases, also, I is constant throughout the entire beam. In such cases, then, there will only need to be integrated the expressions :

$$\int_{x_0}^{x} M\,dx \quad \text{and} \quad \int_{x_1}^{x}\int_{x_0}^{x} M\,dx^2.$$

Before applying these formulæ to particular cases it will be necessary to consider some other matters.

Art. 20.—External Bending Moments and Shears in General.

Beams subjected to combined bending and direct stress will not be treated. Such cases are of little or no real value to the engineer, and approximate solutions, even, are only to be reached by the higher processes of analysis. In all beams, therefore, pure bending only is to be treated. A beam is said to be *non-continuous* if its extremities simply *rest* at each end of the span and *suffer no constraint whatever.*

A beam is said to be *continuous* if its length is equal to two or more spans, or if its ends, in case of one span (or more) suffer constraint.

A *cantilever* is a beam which overhangs its span; one end of which is in no manner supported. Each of the overhanging portions of an open swing bridge is a cantilever truss.

Let any beam be horizontal, and suppose it to be subjected to vertical loads. The results will evidently be applicable to any beam acted upon by loads normal to its axis. Let P be any single vertical load, and let x be any horizontal co-ordinate measured from any point as an origin. Let x_1 represent the co-ordinate, measured from the same origin, of the point of application of any load P. Finally, let it be required to determine the external bending moment M at any section, x, of the beam. The lever arm of any load P is evidently $(x - x_1)$.

Hence, for any number of forces :

$$M = \Sigma P(x - x_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The summation sign Σ refers only to x_1 and is to cover that portion of the beam on *one* side of the section x, as is evident from the manner of forming the equation.

If the origin of x is in the section considered :

$$M = - \Sigma P x_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From Eq. (1) :

$$\frac{dM}{dx} = \Sigma P = S \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now $\Sigma P = S$ is the algebraic sum of all the forces on one side of the section considered, *it is consequently the total force acting in the section tending to move one portion of the beam past the other;* it is therefore called the "*shear*" in the section. This quantity (the shear) is a most important one in the subject of the resistance of materials.

The reactions, or supporting forces, applied to the beam, are to be included both in the sum ΣP, and in the moment :

$$\Sigma P (x - x_1).$$

Eq. (3) shows that *the shear at any section is equal to the first differential coefficient of the bending moment considered as a function of x.*

The sum of all the loads on the other side of the section x would give the same *numerical* shear, but it would evidently have an opposite direction.

As is well known, the analytical condition for a maximum or minimum bending moment in a beam is :

$$\frac{dM}{dx} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From Eqs. (3) and (4) is to be deduced the following important principle: *The greatest or least bending moment in any beam is to be found in that section for which the shear is zero.*

The importance of this principle lies in the fact that in the greater portion of cases of loaded beams which come within the experience of the civil engineer, the section subjected to the greatest bending moment can thus be determined by a simple inspection of the loading.

These principles can be well illustrated by the following simple example.

Fig. 1 represents a non-continuous beam with the span l,

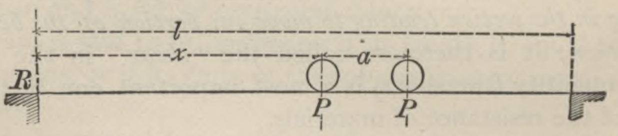


FIG. 1.

supporting two equal weights P, P. These two weights or loads are to be kept at a constant distance apart denoted by a.

It is required to find that position of the two loads which will cause the greatest bending moment to exist in the beam, and the value of that moment. The reaction R is to be found by the simple principle of the lever. Its value will therefore be:

$$R = \frac{l - \left(x + \frac{a}{2}\right)}{l} \cdot 2P \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Since the reaction R can never be equal to $2P$, ΣP, or the shear, it must be equal to zero at the point of application of one of the loads P. In searching for the greatest moment, then, it will only be necessary to find the moment about the point of application of one of the forces P. It will be most convenient to take that one nearest R.

The moment desired will be :

$$M = Rx = 2P\left(x - \frac{x^2}{l} - \frac{ax}{2l}\right) \quad . \quad . \quad . \quad . \quad (6)$$

$$\therefore \quad \frac{dM}{dx} = 0 = 2P\left(1 - \frac{2x}{l} - \frac{a}{2l}\right)$$

$$\therefore \quad x = \frac{l}{2} - \frac{a}{4}.$$

This value in Eq. (6) gives :

$$M_1 = \frac{P}{2l}\left(l - \frac{a}{2}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Since :

$$\frac{d^2M}{dx^2} = -\frac{4P}{l},$$

it appears that M_1 is a maximum.

If the load is uniformly continuous and of the intensity p, in Eqs. (1), (2) and (3) $p\,dx_1$ is to be put for P, and the sign $\int$ for Σ. Hence :

$$M = p\int(x - x_1)\,dx_1,$$

$$M = -p\int x_1\,dx_1.$$

$$\frac{dM}{dx} = p\int dx_1.$$

But since dx and dx_1 are perfectly arbitrary, they may be taken equal to each other, hence:

$$\frac{d^2M}{dx^2} = p.$$

Or, the second differential coefficient of the moment, considered as a function of x, is equal to the intensity of the continuous load.

A very important problem arises in connection with the principles discussed in this Art. It is the following:

A continuous train of any given varying or uniform density advances along a simple beam of span l. It is required to determine what position of loading will give the greatest shear at any specified section.

In Fig. 2, CD is the span l, and A is any section for which it is required to find the position of the load for the greatest transverse shear.

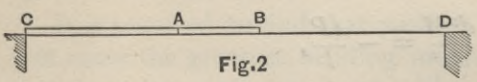


Fig.2

The load is supposed to advance continuously from C to any point B. Let R be the reaction at D, and ΣP the load between A and B. The shear S' at A will be:

$$R - \Sigma P = S' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Let R' be that part of R which is due to ΣP, and R'' that part due to the load on CA; evidently R' is less than ΣP. Then:

$$R' + R'' - \Sigma P = S'.$$

If AB carries no load, R' and ΣP disappear in the value of S. Hence :

$$R'' = S$$

is the shear for the head of the train at A. S is greater than S' because ΣP is greater than R'. But no load can be taken from AC without decreasing R''. Hence: *The greatest shear at any section will exist when the load extends from the end of the span to that section, whatever be the density of the load.*

In general, the section will divide the span into two unequal segments. The load also may approach from either direction. The greater or smaller segment, then, may be covered, and, according to the principle just established, either one of these conditions will give a maximum shear. A consideration of these conditions of loading in connection with Fig. 2, however, will show that *these greatest shears will act in opposite directions.*

When the load covers the greater segment the shear is called a *main* shear; when it covers the smaller, it is called a *counter* shear.

Addendum to Art. 20.

The position of the moving load for the greatest bending moment at any section of a non-continuous beam may be very simply determined. In Fig. 3, let $F\ G$ represent any such beam of the span l, and let any moving load whatever, as $W_1\ .\ .\ .\ W_{n'}\ .\ .\ .\ W_n$ advance from F toward G. Let C be the section at which it is desired to determine the

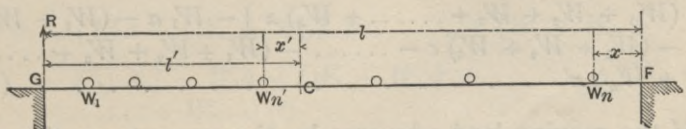


FIG. 3.

maximum bending moment, and let n' loads rest to the left of C, while n is the total number of loads on the span. Finally let x' represent the distance of $W_{n'}$ from C and to the left of that point, while x is the distance of W_n to the left of F. If a is the distance between W_1 and W_2; b the distance between W_2 and W_3; c the distance between W_3 and W_4, etc., etc., the reaction R at G will be:

$$R = \begin{cases} W_1 \dfrac{a+b+c+\ldots\ldots+x}{l} \\ + W_2 \dfrac{b+c+\ldots\ldots+x}{l} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + W_n \dfrac{x}{l} \end{cases} \quad . \quad . \quad . \quad (9)$$

The bending moment M about C will then take the value:

$$M = Rl' - \begin{cases} W_1 (a+b+c+\ldots\ldots+x') \\ + W_2 (\quad b+c+\ldots\ldots+x') \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + W_{n'} x'. \end{cases}$$

Or, after inserting the value of R from above:

$$M = \frac{l'}{l} [W_1 a + (W_1 + W_2) b + (W_1 + W_2 + W_3) c + \ldots\ldots + (W_1 + W_2 + W_3 + \ldots\ldots + W_n) x] - W_1 a - (W_1 + W_2) b - (W_1 + W_2 + W_3) c - \ldots\ldots - (W_1 + W_2 + W_3 + \ldots\ldots + W_{n'}) x' \quad . \quad . \quad . \quad (10)$$

If the moving load advances by the amount Δx, the moment becomes, since $\Delta x = \Delta x'$:

$$M' = M + \frac{l'}{l}(W_1 + W_2 + W_3 + \ldots\ldots + W_n)\,\Delta x - (W_1 + W_2$$
$$+ \ldots\ldots + W_{n'})\,\Delta x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Hence, for a maximum, the following value must never be negative:

$$M' - M = \Delta x \left\{ \frac{l'}{l}\,(W_1 + W_2 + W_3 + \ldots\ldots + W_n) - (W_1 \right.$$
$$\left. + W_2 + \ldots\ldots + W_{n'}) \right\} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Or, the desired condition for a maximum takes the form:

$$\frac{l'}{l} = \frac{W_1 + W_2 + \ldots\ldots + W_{n'}}{W_1 + W_2 + W_3 + \ldots\ldots + W_n} \quad . \quad . \quad . \quad . \quad (13)$$

It will seldom or never occur that this ratio will exactly exist if $W_{n'}$ is supposed to be a *whole* weight; hence $W_{n'}$ will usually be that part of a whole weight at C which is necessary to be taken in order that the equality (13) may hold.

It is to be observed that if the moving load is very irregular, so that there is a great and arbitrary diversity among the weights W, there may be a number of positions of the moving load which will fulfil Eq. (13), some one of which will give a value greater than any other; this is the absolute maximum desired.

From what has preceded, it follows that $W_{n'}$ may always be taken at the point C in question; hence, x' in Eq. (10) may always be taken equal to zero when that equation expresses the greatest value of the moment. The latter then becomes:

$$M = \frac{l'}{l}\,[W_1 a + (W_1 + W_2)b + \ldots\ldots + (W_1 + W_2 + \ldots\ldots$$
$$+ W_n)\,x\,] - W_1 a - (W_1 + W_2)b - \ldots\ldots - (W_1 + W_2$$
$$+ \ldots\ldots + W_{n'-1})\,(?) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

In this equation x, of course, corresponds to the position of maximum bending, while the sign (?) represents the distance between the concentrations $W_{n'-1}$ and $W_{n'}$.

It has already been shown in this Art. that for any given condition of loading the greatest bending moment in the beam will occur at that section for which the shear is zero. But if the shear is zero, the reaction R must be equal to the sum of the weights $(W_1 + W_2 + \ldots\ldots + W_{n'})$ between G and C; the latter now being the section at which the greatest moment in the span exists.

Hence, for that section, Eq. (13) will take the form:

$$\frac{l'}{l} = \frac{R}{W_1 + W_2 + W_3 + \ldots\ldots + W_n};$$

or, the centre of the gravity of the load is at the same distance from one end of the beam as the section or point of greatest bending is from the other. In other words, *the distance between the point of greatest bending for any given system of loading and the centre of gravity of the latter is bisected by the centre of span.*

If the load is uniform, therefore, it must cover the whole span.

It is to be observed that Eq. (14) is composed of the sums W_1, $W_1 + W_2$, etc., multiplied by the distances a, b, c, etc. Hence tabulations of these quantities for any given system of loading will expedite and simplify computations of actual moments.

With a given system of concentrated loads it sometimes becomes necessary to determine at what particular length of span n weights W cease to give the maximum bending moment, and $(n + 1)$ weights begin to be employed, for a special and constant fraction, $\frac{l'}{l}$, of the span. Eq. (14) gives the solution of this question at once. Let M be the moment for n weights and distance x, while M_1 is the moment for $(n + 1)$ weights corresponding to the distance x_1. Also let k be the distance between the weights or loads W_n and W_{n+1}. Then there results:

$$M_1 - M = \mathrm{o} = \frac{l'}{l}\Big[\,(k - x)\,(W_1 + W_2 + \ldots\ldots + W_n) + (W_1 + W_2 + \ldots\ldots + W_{n+1})\,x_1\Big] - (W_1 + W_2 + \ldots\ldots + W_{n'})\,?_1$$

The last term of this equation will not exist if, as is frequently the case, the maximum moment continues at the same load $W_{n'}$. Hence either:

$$x_1 = (x - k)\,\frac{W_1 + W_2 + \ldots\ldots + W_n}{W_1 + W_2 + \ldots\ldots + W_{n+1}} \quad . \ . \ . \ . \quad (15)$$

Or,

$$x_1 = (x - k)\,\frac{W_1 + W_2 + \ldots + W_{n'}}{W_1 + W_2 + \ldots + W_{n+1}} + \frac{l}{l'}\,\frac{(W_1 + W_2 + \ldots + W_{n'})\,?_1}{W_1 + W_2 + \ldots + W_{n+1}} \ . \ (16)$$

Since x_1 cannot be negative, Eq. (15) shows that $x = k$ for the condition to which it belongs. Eq. (16) gives x_1, when n loads cease to be used and $(n + 1)$ begin, if the point of maximum bending at the same time changes from $W_{n'-1}$ to $W_{n'}$.

Art. 21.—Moments and Shears in Special Cases.

Certain special cases of beams are of such common occurrence, and consequently of such importance, that a somewhat more detailed treatment than that already given may be deemed desirable. The following cases are of this character.

Case I.

Let a non-continuous beam, supporting a single weight P at any point, be considered, and let such a beam be represented in Fig. 1. If the span RR' is represented by

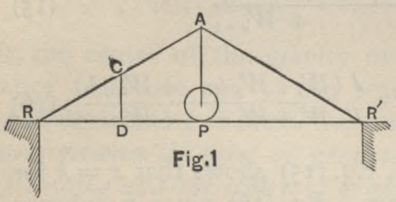


Fig. 1

$$l = a + b = RP + R'P,$$

the reactions R and R' will be :

$$R = \frac{b}{l} P, \text{ and } R' = \frac{a}{l} P \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Consequently, if x represents the distance of any section in RP from R, while x' represents the distance of any section of $R'P$ from R', the general values of the bending moments for the two segments a and b of the beam will be :

$$M = Rx, \text{ and } M' = R'x' \quad . \quad . \quad . \quad . \quad . \quad (2)$$

These two moments become equal to each other and represent *the greatest bending moment in the beam* when

$$x = a \text{ and } x' = b,$$

or, *when the section is taken at the point of application of the load P.*

Eq. (2) shows that the moments vary directly as the distances from the ends of the beam. Hence, if AP (normal to RR') is taken by any convenient scale to represent the greatest moment, $\frac{ab}{l}P$, and if RAR' is drawn, any intercept parallel to AP and lying between RAR' and RR' will represent the bending moment for the section at its foot, by the same scale. In this manner CD is the bending moment at D.

The shear is uniform for each single segment; it is evidently equal to R for RP and R' for $R'P$. It becomes zero at P, where is found the greatest bending moment.

Case II.

Again, let Fig. 2 represent the same beam shown in Fig. 1, but let the load be one of uniform intensity, p, extending from end to end of the beam. Let C be placed at the centre of the span, and let R and R', as before, represent the two reactions. Since the load is symmetrical in reference to C,

$$R = R'.$$

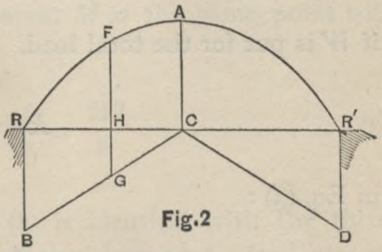


Fig. 2

For the same reason the moments and shears in one half of the beam will be exactly like those in the other; consequently, reference will be made to one half of the beam only. Let x and x_1 then be measured from R toward C. The forces acting upon the beam are R and p, the latter being uniformly continuous. Applying the formulæ of the preceding Art., the bending moment at any section x will be :

$$M = Rx - p\int_0^x (x - x_1)\,dx_1.$$

$$\therefore \quad M = Rx - \frac{px^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If l is the span, at C, M becomes :

$$M_1 = \frac{Rl}{2} - \frac{pl^2}{8} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

But because the load is uniform :

$$R = \frac{pl}{2}.$$

Hence :

$$M_1 = \frac{pl^2}{8} = \frac{Wl}{8} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

if W is put for the total load. Placing :

$$R = \frac{pl}{2},$$

in Eq. (3) :

$$M = \frac{p}{2}(lx - x^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The moments M, therefore, are proportional to the abscissæ of a parabola whose vertex is over C, and which passes through the origin of co-ordinates R. Let AC, then, normal to RR', be taken equal to M_1, and let the parabola RAR' be drawn. Intercepts, as FH, parallel to AC, will represent bending moments in the sections, as H, at their feet.

The shear at any section is :

$$S = \frac{dM}{dx} = R - px = p\left(\frac{l}{2} - x\right) \quad . \quad . \quad . \quad (7)$$

or, *it is equal to the load covering that portion of the beam between the section in question and the centre.*

Eq. (7) shows that the shear at the centre is zero; it also shows that $S = R$ at the ends of the beam. It further demonstrates that *the shear varies directly as the distance from the centre.* Hence, take RB to represent R and draw BC. The shear at any section, as H, will then be represented by the vertical intercept, as HG, included between BC and RC.

The shear being zero at the centre, the greatest bending moment will also be found at that point. This is also evident from inspection of the loading.

Eq. (2) of *Case I.*, shows that if a beam of span l carries a weight $\frac{W}{2}$ at its centre, the moment M at the same point will be :

$$M_1 = \frac{W}{4} \cdot \frac{l}{2} = \frac{Wl}{8} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The third member of Eq. (8) is identical with the third member of Eq. (5). It is shown, therefore, *that a load, concentrated at the centre of a non-continuous beam, will cause the same moment, at that centre, as double the same load uniformly distributed over the span.*

Eqs. (5) and (8) are much used in connection with the bending of ordinary non-continuous beams, whether solid or flanged: and such beams are frequently found.

Case III.

The third case to be taken, is a cantilever uniformly loaded; it is shown in Fig. 3. Let x and x_1 be measured from the free end A, and let the uniform intensity of the load be represented by p. The entire loading is uniformly continuous. Hence the principles and formulæ of Art. 20 give, for the moment about any section x:

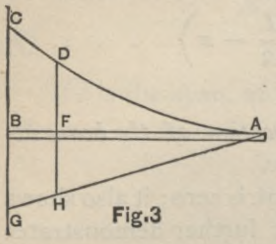


Fig.3

$$M = -p\int_0^x (x - x_1)\, dx_1 = -\frac{px^2}{2} \quad . \quad . \quad . \quad . \quad (9)$$

If $AB = l$, the moment at B is:

$$M_1 = -\frac{pl^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The negative sign is used to indicate that the *lower* side of the beam is subjected to compression. In the two preceding cases, evidently, the *upper* side is in compression.

The shear at any section is:

$$S = \frac{dM}{dx} = -px \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Hence, the shear at any section is the load between the free end and that section.

Eq. (9) shows that the moments vary as the *square* of the distance from the free end; consequently, the moment curve is a parabola with the vertex at A, and with a vertical axis. Let BC, then, represent M_1 by any convenient scale, and draw the parabola CDA. Any vertical intercept as DF will represent the moment at the section, as F, at its foot.

Again, let BG represent the shear pl, at B, then draw the straight line AG. Any vertical intercept, as HF, will then represent the shear at the corresponding section F.

Art. 22.—Recapitulation of the General Formulæ of the Common Theory of Flexure.

It is convenient for many purposes to arrange the formulæ of the Common Theory of Flexure in the most general and concise form. In this Art. the preceding general formulæ for shears, strains, resisting moments and deflections will be recapitulated and so arranged. In order to complete the generalization, the summation sign Σ will be used instead of the sign of integration.

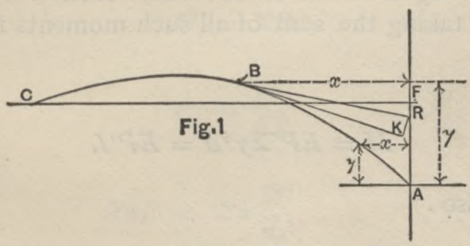


Fig. 1

In Fig. 1, let ABC represent the centre line of any bent beam; AF, a vertical line through A; CF, a horizontal line through C, while A is the section of the beam at which the deflection (vertical or horizontal) in reference to C, the bending moment, the shearing stress, etc., are to be determined. As shown in figure, let x be the horizontal co-ordinate measured from A, and y the vertical one measured from the same point; then let z be the horizontal distance from the same point to the point of application of any external vertical force P. To complete the notation, let D be the deflection desired; M_1, the moment of the external forces about A; S, the shear at A; P', the strain (extension or compression) per unit of length of a fibre parallel to the neutral surface and situated at a normal distance of unity from it; I, the general expression of the moment of inertia of a normal cross section of the beam, taken in reference to the neutral axis of that section; E, the coefficient of elasticity for the material of the beam; and M the moment of the external forces for any section, as B.

Again, let Δ be an indefinitely small portion of any normal cross section of the beam, and let y' be an ordinate normal to the neutral axis of the same section. By the "common theory" of flexure, the intensity of stress at the distance y' from the neutral surface is $(y'P'E)$. Consequently the stress developed in the portion Δ, of the section, is $EP'y'\Delta$, and the resisting moment of that stress is $EP'y'^2\Delta$.

The resisting moment of the whole section will therefore be found by taking the sum of all such moments for its whole area.

Hence:

$$M = EP'\Sigma y'^2\Delta = EP'I.$$

Hence, also .

$$P' = \frac{M}{EI}.$$

If n represents an indefinitely short portion of the neutral surface, the strain for such a length of fibre at unit's distance from that surface will be nP'.

If the beam were originally straight and horizontal, n would be equal to dx.

P' being supposed small, the effect of the strain nP' at any section, B, is to cause the end K of the tangent BK, to move vertically through the distance $nP'x$.

If BK and BR (taken equal) are the positions of the tangents before and after flexure, $nP'x$ will be the vertical distance between K and R.

By precisely the same kinematical principle, the expression $nP'y$ will be the horizontal movement of A in reference to B.

Let $\Sigma nP'x$ and $\Sigma nP'y$ represent summations extending from A to C, then will those expressions be the vertical and horizontal deflections, respectively, of A in reference to C. It is evident that these operations are perfectly general, and that x and y may be taken in any direction whatever.

The following general, but strictly approximate equations, relating to the subject of flexure, may now be written :

$$S = \Sigma P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$M_1 = \Sigma Pz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$P' = \frac{M}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\Sigma nP' = \Sigma n \frac{M}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$D = \Sigma nP'x = \Sigma \frac{nMx}{EI} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$D_h = \Sigma nP'y = \Sigma \frac{nMy}{EI} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

D_h represents horizontal deflection.

The summation ΣPz must extend from A to a point of no bending; or from A to a point at which the bending moment is M_1'. In the latter case :

$$M_1 = \Sigma Pz + M_1' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

M_1' may be positive or negative.

Art. 23.—The Theorem of Three Moments.

The object of this theorem is the determination of the relation existing between the bending moments which are found in any continuous beam at any three adjacent points of support. In the most general case to which the theorem applies, the section of the beam is supposed to be variable, the points of support are not supposed to be in the same level, and at any point, or all points, of support there may be constraint applied to the beam external to the load which it is to carry; or, what is equivalent to the last condition, the beam may not be straight at any point of support before flexure takes place.

Before establishing the theorem itself, some preliminary matters must receive attention.

If a beam is simply supported at each end, the reactions are found by dividing the applied loads according to the simple principle of the lever. If, however, either or both ends are not simply supported, the reaction, in general, is greater at one end and less at the other than would be found by the law of the lever; a portion of the reaction at one end is, as it were, transferred to the other. The transference can only be accomplished by the application of a couple to the beam, the forces of the couple being applied at the two adjacent points of support; the span, consequently, will be the lever arm of the couple. The existence of equilibrium requires the application to the beam of an equal and opposite couple. It is only necessary, however, to consider, in connection with the span AB, the one shown in Fig. 1. Further, from what has immediately preceded, it appears that the force of this couple is

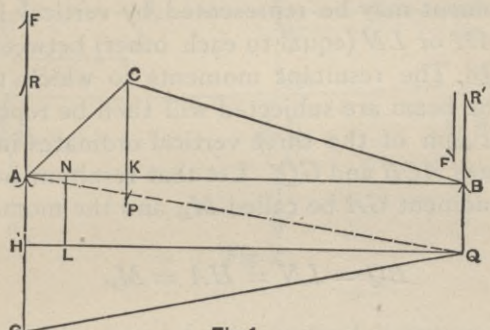


Fig. 1

equal to the difference between the actual reaction at either point of support and that found by the law of the lever. The bending caused by this couple will evidently be of an opposite kind to that existing in a beam simply supported at each end.

These results are represented graphically in Fig. 1. A and B are points of support, and AB is the beam; AR and BR' are the reactions according to the law of the lever; $RF = R'F$ is the force of the applied couple; consequently:

$$AF = AR + RF \quad \text{and} \quad BF = BR' - (R'F = RF)$$

are the reactions after the couple is applied. As is well known, lines parallel to CK, drawn in the triangle ACB, represent the bending moments at the various sections of the beam, when the reactions are AR and BR'. Finally, vertical lines parallel to AG, in the triangle QHG, will represent the bending moments caused by the force $R'F$.

In the general case there may also be applied to the beam two equal and opposite couples, having axes passing through A and B respectively. The effect of such couples will be nothing so far as the reactions are concerned, but they will cause uniform bending between A and B. This uniform or constant moment may be represented by vertical lines drawn parallel to AH or LN (equal to each other) between the lines AB and HQ. The resultant moments to which the various sections of the beam are subjected will then be represented by the *algebraic* sum of the three vertical ordinates included between the lines ACB and GQ. Let that resultant be called M.

Let the moment GA be called M_a, and the moment :

$$BQ = LN = HA = M_b.$$

Also designate the moment caused by the load P, shown by lines parallel to CK in ACB, by M_1. Then let x be any horizontal distance measured from A toward B ; l the horizontal distance AB ; and z the distance of the point of application, K, of the force P from A. With this notation there can be at once written :

$$M = M_a\left(\frac{l - x}{l}\right) + M_b\left(\frac{x}{l}\right) + M_1 \quad . \quad . \quad . \quad . \quad (1)$$

Eq. (1) is simply the general form of Eq. (2).

It is to be noticed that Fig. 1 does not show all the moments M_a, M_b and M_1 to be of the same sign, but, for convenience, they are so written in Eq. (1).

The formula which represents the theorem of three moments can now be written without difficulty. The method to be followed involves the improvements added by Prof. H. T. Eddy, and is the same as that given by him in the "American Journal of Mathematics," Vol. I., No. 1.

Fig. 2 shows a portion of a continuous beam, including two spans and three points of support. The deflections will be supposed measured from the horizontal line *NQ*. The spans

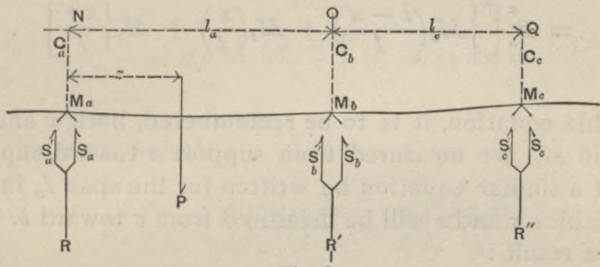


Fig. 2

are represented by l_a and l_c; the vertical distances of *NQ* from the points of support by c_a, c_b and c_c; the moments at the same points by M_a, M_b and M_c, while the letters S and R represent shears and reactions respectively.

In order to make the case general, it will be supposed that the beam is curved in a vertical plane, and has an elbow at b, before flexure, and that, at that point of support, the tangent of its inclination to a horizontal line, toward the span l_a is t, while t' represents the tangent on the other side of the same point of support; also let d and d' be the vertical distances, before bending takes place, of the points a and c, respectively, below the tangents at the point b.

A portion of the difference between c_a and c_b is due to the original inclination, whose tangent is t, and the original lack of straightness, and is not caused by the bending; that portion which is due to the bending, however, is, remembering Eq. (5), Art. 22 :

$$D = c_a - c_b - l_a t - d = \Sigma_b^a \frac{Mxn}{EI} .$$

By the aid of Eq. (1) this equation may be written :

$$E\,(c_a - c_b - l_a t - d)$$

$$= \Sigma_b^a \left[\left\{ M_a\left(\frac{l-x}{l}\right) + M_b\left(\frac{x}{l}\right) + M_1 \right\} \frac{xn}{I} \right] \quad . \quad . \quad (2)$$

In this equation, it is to be remembered, both x and z (involved in M_1) are measured from support a toward support b. Now let a similar equation be written for the span l_c, in which the variables x and z will be measured from c toward b. There will then result :

$$E\,(c_c - c_b - l_c t' - d')$$

$$= \Sigma_b^c \left[\left\{ M_c\left(\frac{l-x}{l}\right) + M_b\left(\frac{x}{l}\right) + M_1 \right\} \frac{xn}{I} \right] \quad . \quad . \quad (3)$$

When the general sign of summation is displaced by the integral sign, n becomes the differential of the axis of the beam, or ds. But ds may be represented by $u\,dx$, u being such a function of x as becomes unity if the axis of the beam is originally straight and parallel to the axis of x. The Eqs. (2) and (3) may then be reduced to simpler forms by the following methods :

In Eq. (2) put :

$$\Sigma_b^a \left(\frac{l-x}{l}\right) \frac{xn}{I} = \frac{1}{l_a}\int_b^a \frac{u\,(l_a - x)x\,dx}{I}$$

$$= \frac{x_a}{l_a}\int_b^a \frac{u\,(l_a - x)\,dx}{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Also :

$$\frac{x_a}{l_a}\int_b^a \frac{u\,(l_a - x)\,dx}{I} = \frac{i_a x_a}{l_a}\int_b^a u\,(l_a - x)\,dx \quad . \quad . \quad . \quad (5)$$

Also:

$$\frac{i_a x_a}{l_a}\int_b^a u\,(l_a - x)\,dx = \frac{i_a x_a u_a}{l_a}\int_b^a (l_a - x)\,dx = \frac{i_a x_a u_a l_a}{2} \quad (6)$$

In the same manner:

$$\Sigma_b^a \frac{x^2 n}{l_a I} = \frac{1}{l_a}\int_b^a \frac{u x^2\,dx}{I} = \frac{x_a'}{l_a}\int_b^a \frac{ux\,dx}{I} \quad . \quad . \quad . \quad (7)$$

Also:

$$\frac{x_a'}{l_a}\int_b^a \frac{ux\,dx}{I} = \frac{i_a' x_a'}{l_a}\int_b^a ux\,dx \quad . \quad . \quad . \quad . \quad . \quad (8)$$

And,

$$\frac{i_a' x_a'}{l_a}\int_b^a ux\,dx = \frac{i_a' x_a' u_a'}{l_a}\int_b^a x\,dx = \frac{i_a' x_a' u_a' l_a}{2} \quad . \quad . \quad . \quad (9)$$

Again, in the same manner:

$$\Sigma_b^a \frac{M_1 x n}{I} = i_{1a} u_{1a} \Sigma M_1 x\,\Delta x \quad . \quad . \quad . \quad . \quad (10)$$

Using Eqs. (4) to (10), Eq. (2) may be written:

$$E\,(c_c - c_b - l_a t - d) = \frac{l_a}{2}\,(M_a u_a i_a x_a + M_b u_a' i_a' x_a')$$

$$+ u_{1a} i_{1a} \Sigma_b^a M_1 x\,\Delta x \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Proceeding in precisely the same manner with the span l_c, Eq. (3) becomes :

$$E(c_c - c_b - l_c t' - d') = \frac{l_c}{2}(M_c u_c i_c x_c + M_b u'_c i''_c x'_c)$$

$$+ u_{1c} i_{1c} \Sigma_b^c M_1 x \Delta x \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The quantities x_a and x_c are to be determined by applying Eq. (4) to the span indicated by the subscript ; while u_a, i_a, u_c and i_c are to be determined by using Eqs. (5) and (6) in the same way. Similar observations apply to u'_a, i''_a, x'_a, u'_c, i''_c and x'_c, taken in connection with Eqs. (7), (8) and (9).

If I is not a continuous function of x, the various integrations of Eqs. (4), (5), (7) and (8) must give place to summations (Σ) taken between the proper limits.

Dividing Eqs. (11) and (12) by l_a and l_c, respectively, and adding the results :

$$E\left(\frac{c_a - c_b}{l_a} + \frac{c_c - c_b}{l_c} - T - \frac{d}{l_a} - \frac{d'}{l_c}\right)$$

$$= \frac{u_{,a} i_{,a}}{l_a} \Sigma_b^a M_1 x \Delta x + \frac{u_{,c} i_{,c}}{l_c} \Sigma_b^c M_1 x \Delta x$$

$$+ \tfrac{1}{2}(M_a u_a i_a x_a + M_b u'_a i''_a x'_a + M_c u_c i_c x_c + M_b u'_c i''_c x'_c) \quad . \quad (13)$$

in which $T = t + t'$.

Eq. (13) is the most general form of the theorem of three moments if E, the coefficient of elasticity, is a constant quantity. Indeed, that equation expresses, as it stands, the "theorem" for a variable coefficient of elasticity if (ie) be written instead of i; e representing a quantity determined in a manner exactly similar to that used in connection with the quantity i.

In the ordinary case of an engineer's experience $T = 0$, $d = d' = 0$, $I =$ *constant*, $u = u_a = u_c =$ *etc.*, $= c' =$ *secant of the inclination for which* $t = -t'$ *is the tangent*; consequently:

$$\dot{i}_a = \ddot{i}_a = \dot{i}_c = \ddot{i}_c = \dot{i}_{,a} = \dot{i}_{,c} = \frac{1}{I}.$$

From Eq. (4):

$$x_a = \frac{2l_a}{6}, \qquad x_c = \frac{2l_c}{6}.$$

From Eq. (7):

$$x'_a = \frac{4l_a}{6}, \qquad x'_c = \frac{4l_c}{6}.$$

The summation $\Sigma M_1 x \Delta x$ can be readily made by referring to Fig. 1.

The moment represented by CK in that figure is:

$$P\left(\frac{l-z}{l}\right) . z ;$$

consequently the moment at any point between A and K, due to P, is:

$$M_1 = P\left(\frac{l-z}{l}\right) . z . \frac{x}{z} = P\left(\frac{l-z}{l}\right)x.$$

Between K and B:

$$M_1' = \left(\frac{l-x}{l-z}\right) . CK = P\frac{z}{l}(l-x).$$

Using these quantities for the span l_a:

$$\Sigma_b^a M_1 x\,\Delta x = \int_0^z M_1 x\,dx + \int_z^{la} M_1' x\,dx = \tfrac{1}{6}P(l_a^2 - z^2)z.$$

For the span l_c, the subscript a is to be changed to c.

Introducing all these quantities Eq. (13) becomes, after providing for any number of weights, P:

$$\frac{6EI}{c'}\left(\frac{c_a - c_b}{l_a} + \frac{c_c - c_b}{l_c}\right) = M_a l_a + 2M_b(l_a + l_c) + M_c l_c$$

$$+ \frac{1}{l_a}\overset{a}{\Sigma}P(l_a^2 - z^2)z + \frac{1}{l_c}\overset{c}{\Sigma}P(l_c^2 - z^2)z \quad . \quad . \quad . \quad (14)$$

Eq. (14), with c' equal to unity, is the form in which the theorem of three moments is usually given; with c' equal to unity or not, *it applies only to a beam which is straight before flexure*, since:

$$T = t + t' = 0 = d = d'.$$

If such a beam rests on the supports a, b, and c, before bending takes place,

$$\frac{c_a - c_b}{l_a} = -\frac{c_c - c_b}{l_c},$$

and the first member of Eq. (14) becomes zero.

If, in the general case to which Eq. (13) applies, the deflections c_a, c_b, and c_c belong to the beam in a position of no bending, the first member of that equation disappears, since it is the sum of the deflections *due to bending only*, for the spans l_a, and l_c, divided by those spans, and each of those quantities is zero by the equation immediately preceding, Eq. (2). Also, if the beam or truss belonging to each span is straight between the points of support (*such points being supposed in the same level or not*), $u_a = u'_a = u_{1a} =$ *constant*, and $u_c = u'_c = u_{1c} =$ *another constant*. If, finally, I be again taken as constant, x_a and x_c, as well as $\Sigma M_1 x\ \Delta x$, will have the values found above.

From these considerations it at once follows that the second member of Eq. (14), put equal to zero, expresses the theorem of three moments for a beam or truss straight between points of support, when those points are not in the same level, but when they belong to a configuration of no bending in the beam. Such an equation, however, does not belong to a beam not straight between points of support.

The shear at either end of any span, as l_a, is the next to be found, and it can be at once written by referring to the observations made in connection with Fig. 1. It was there seen that the reaction found by the simple law of the lever is to be increased or decreased for the continuous beam, by an amount found by dividing the difference of the moments at the extremities of any span by the span itself. Referring therefore, to Fig. 2, for the shears S, there may at once be written :

$$S_a = \overset{a}{\Sigma} P \frac{l_a - z}{l_a} - \frac{M_a - M_b}{l_a} \quad . \quad . \quad . \quad . \quad (15)$$

$$S'_b = \overset{a}{\Sigma} P \frac{z}{l_a} + \frac{M_a - M_b}{l_a} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$S_b = \overset{c}{\Sigma} P \frac{z}{l_c} + \frac{M_c - M_b}{l_c} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$S_c = \overset{c}{\Sigma} P \frac{l_c - z}{l_c} - \frac{M_c - M_b}{l_c} \quad . \quad . \quad . \quad . \quad (18)$$

The negative sign is put before the fraction,

$$\frac{M_a - M_b}{l_a},$$

in Eq. (15), because in Fig. 1 the moments M_a and M_b are represented opposite in sign to that caused by P, while in Eq. (1) the three moments are given the same sign, as has already been noticed.

Eqs. (15) to (18) are so written as to make an upward reaction positive, and they may, perhaps, be more simply found by taking moments about either end of a span. For example, taking moments about the right end of l_a:

$$S_a l_a - \overset{a}{\Sigma} P(l_a - z) + M_a = M_b.$$

From this, Eq. (15) at once results. Again, moments about the left end of the same span give:

$$S_b' l_a - \overset{a}{\Sigma} Pz + M_b = M_a.$$

This equation gives Eq. (16), and the same process will give the others.

If the loading over the different spans is of uniform intensity, then, in general, $P = w\,dz$; w being the intensity. Consequently:

$$\Sigma P(l^2 - z^2)z = \int_0^l w\,(l^2 - z^2)\,z\,dz = w\,\frac{l^4}{4}.$$

In all equations, therefore, for

$$\frac{1}{l_a}\,\overset{a}{\Sigma} P(l_a^2 - z^2)\,z$$

there is to be placed the term $w_a \frac{l_a^3}{4}$; and for

$$\frac{1}{l_c} \overset{c}{\Sigma} P(l_c^2 - z^2) z$$

the term $w_c \frac{l_c^3}{4}$. The letters a and c mean, of course, that reference is made to the spans l_a and l_c.

From Fig. 2, there may at once be written :

$$R = S_a' + S_a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$R' = S_b' + S_b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$R'' = S_c' + S_c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$$\text{etc.} = \text{etc.} + \text{etc.}$$

Art. 23a.—Reactions under Continuous Beam of any Number of Spans.

The general value of the reactions at the points of support under any continuous beam have been given in Eqs. (19), (20), (21), etc., of the preceding Art. Before those equations, however, can be applied to any particular case, the values of the bending moments, which appear in the expressions S_a, S_b', S_b, etc., for the shears, must be determined. In the application of the theorem of three moments, it is invariably virtually assumed that the continuous beam before flexure is straight between the points of support, and that the latter belong to a configuration of no bending. The moment of inertia I and the coefficient of elasticity E are also assumed to be constant. This is frequently not strictly true, yet it will be assumed in what follows, since the method to be used in finding the moments is entirely independent of the assumption, and remains precisely the same whatever form for the theorem of three moments may be chosen.

Agreeably to the assumption made, Eq. (14) of the preceding Art. takes the following form, which is almost, or quite, invariably used in engineering practice:

$$M_a l_a + 2M_b(l_a + l_c) + M_c l_c = -\frac{1}{l_a}\sum^{a} P(l_a^2 - z^2)z$$

$$-\frac{1}{l_c}\sum^{c} P(l_c^2 - z^2)z \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let Fig. 1 represent a continuous beam of n spans, equal or unequal in length. At the points of support, 0, 1, 2, 3, 4, 5,

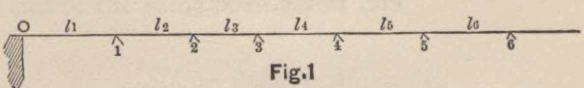


Fig. 1

etc., let the bending moments be represented by M_0, M_1, M_2, M_3, etc. The moment M_0 is always known; it is ordinarily zero, and that will be considered its value.

An examination of Fig. 1 shows that, by repeated applications of Eq. (1), the number of resulting equations of condition will be one less than the number of spans. But if the two end moments are known (here assumed to be zero), the number of unknown moments will also be one less than the number of spans. Hence the number of equations will always be sufficient for the determination of the unknown moments.

For the sake of brevity let the following notation be adopted:

$$u_1 = -\frac{1}{l_1}\overset{1}{\Sigma}P(l_1^2 - z^2)z - \frac{1}{l_2}\overset{2}{\Sigma}P(l_2^2 - z^2)z.$$

$$u_2 = -\frac{1}{l_2}\overset{2}{\Sigma}P(l_2^2 - z^2)z - \frac{1}{l_3}\overset{3}{\Sigma}P(l_3^2 - z^2)z.$$

$$u_3 = -\frac{1}{l_3}\overset{3}{\Sigma}P(l_3^2 - z^2)z - \frac{1}{l_4}\overset{4}{\Sigma}P(l_4^2 - z^2)z.$$

etc. = etc. − etc.

$$a_1 = 2(l_1 + l_2); \qquad b_1 = l_2.$$

$$a_2 = l_2; \qquad b_2 = 2(l_2 + l_3); \qquad c_2 = l_3.$$

$$b_3 = l_3; \qquad c_3 = 2(l_3 + l_4); \qquad d_3 = l_4.$$

$$c_4 = l_4; \qquad d_4 = 2(l_4 + l_5); \qquad f_4 = l_5.$$

. . . .

$$p_i = l_i; \qquad q_i = 2(l_i + l_{i+1}); \qquad s_i = l_{i+1};$$

i denoting any number of the series 1, 2, 3, 4, ... n. It is thus seen that, in general,

$$q_i = 2(p_i + s_i);$$

also that $a_2 = b_1$, $c_2 = b_3$, $d_3 = c_4$, etc. These relations can be used to simplify the final result.

By repeated applications of Eq. (1) the following n equations of condition, involving the notation given above, will result :

$$\left.\begin{array}{ll}
a_1M_1 + b_1M_2 & = u_1 \\
a_2M_1 + b_2M_2 + c_2M_3 & = u_2 \\
\quad + b_3M_2 + c_3M_3 + d_3M_4 & = u_3 \\
\quad\quad + c_4M_3 + d_4M_4 + f_4M_5 & = u_4 \\
\quad\quad\quad + d_5M_4 + f_5M_5 + g_5M_6 & = u_5 \\
\quad\quad\quad\quad \ldots\ldots\ldots & = \ldots \\
\quad\quad\quad\quad\quad \ldots\ldots & = u_n
\end{array}\right\} \quad (2)$$

The moment M_{n+1} will also be equal to zero. In consequence of this last condition it is seen that the coefficients of the Ms occupy precisely the places of the elements of a determinant of the n^{th} degree. Of the array indicating the determinant, however, there exists only the leading diagonal and one diagonal on each side of it. The determinant for n equations, or $(n + 1)$ spans, has, then, the value :

$$D = \left\{\begin{array}{l}
a_1, b_1, 0, 0, 0, 0, \ldots\ldots \\
a_2, b_2, c_2, 0, 0, 0, \ldots\ldots \\
0, b_3, c_3, d_3, 0, 0, \ldots\ldots \\
0, 0, c_4, d_4, f_4, 0, \ldots\ldots \\
0, 0, 0, d_5, f_5, g_5, \ldots\ldots \\
\quad \ldots\ldots\ldots \\
0, 0, 0, 0, 0, \ldots\ldots 0, p_n, q_n
\end{array}\right\} \quad \ldots \quad (3)$$

Also let D_i represent the value of the determinant D when the column indicated by the i^{th} letter of the series a, b, c, d, f, etc., is replaced by the column u_1, u_2, u_3, u_4, etc. If, for example, $i = 3$, the i^{th} letter is c. Hence:

$$D_3 = \left\{ \begin{array}{l} a_1, b_1, u_1, 0, 0, 0, \ldots \ldots \\ a_2, b_2, u_2, 0, 0, 0, \ldots \ldots \\ 0, b_3, u_3, d_3, 0, 0, \ldots \ldots \\ 0, 0, u_4, d_4, f_4, 0, \ldots \ldots \\ 0, 0, u_5, d_5, f_5, g_5, \ldots \ldots \\ \quad \ldots \quad \ldots \quad \ldots \\ 0, 0, u_n, 0, 0, 0, \ldots \; 0, p_n, q_n \end{array} \right\} \quad \ldots \quad (4)$$

Then, in general:

$$M_i = \frac{D_i}{D} \quad \ldots \ldots \ldots \quad (5)$$

Eq. (5) will give the value of the bending moment at any point of support, whatever may be the number of spans or the law of loading on any or all the spans.

Precisely the same formulæ are to be used if M_0 and M_n are not zero, but have definite values and are known. In such a case, however, u_1 and u_n would be replaced by:

$$u'_1 = u_1 - a_0 M_0.$$

$$u'_n = u_n - r_n M_{n+1}.$$

The same equations also hold true whatever form of the theorem of three moments may be chosen. It is only to be remembered that the values of the quantities a, b, c, etc., u_1, u_2, u_3, etc., will depend upon the choice.

If all the moments are desired, it will be most convenient to put the vertical column $u_1, u_2, u_3, \ldots u_n$ in place of the vertical column $a_1, a_2, 0, 0, \ldots 0$, in Eq. (4), and then find the resulting determinant D_1. Eq. (5) will then give the value of M_1, which, placed in the first of Eqs. (2), will enable M_2 to be at once found. M_3 will then result from the second of Eqs. (2), M_4 from the third, etc., etc.

So far as the general treatment of the question is concerned, there yet remains to be considered the expansion of the determinants D and D_i.

The expansion of the determinant D is very simple, and leads to the following results :

For two spans :

$$D = a_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

For three spans :

$$D = a_1b_2 - a_2b_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

For four spans :

$$D = a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 \quad . \quad . \quad . \quad . \quad . \quad (8)$$

For five spans :

$$D = a_1b_2c_3d_4 - a_1b_3c_2d_4 - a_2b_1c_3d_4 - a_1b_2c_4d_3 + a_2b_1c_4d_3 \quad . \quad (9)$$

For six spans :

$$D = a_1b_2c_3d_4f_5 - a_1b_3c_2d_4f_5 - a_2b_1c_3d_4f_5 - a_1b_2c_4d_3f_5 + a_2b_1c_4d_3f_5$$
$$- a_1b_2c_3d_5f_4 + a_1b_3c_2d_5f_4 + a_2b_1c_3d_5f_4 \quad . \quad . \quad . \quad (10)$$

By the observance of two or three simple rules, the determinant for $(n + 1)$ spans, or n points of support, may easily be written.

A series of numbers such as 1, 2, 3, 4, 5, 6, etc., is said to be written in its natural order. Let any permutation of this series, 2, 1, 3, 6, 5, 4, be written, in which 2 is placed before 1, 6 before 5 and 4, and 5 before 4. In this permutation, therefore, there are said to be $(1 + 2 + 1) = 4$ *inversions.*

Let (λ_n) represent any letter of the series a, b, c, d, etc., which has the subscript n; also, let $(\lambda_n)_n$ and $(\lambda_n)_{n-1}$ represent the n^{th} and $(n - 1)^{\text{th}}$ letters of the same series which have the subscripts n. In general, the letter inside the parenthesis represents the subscript figure in the determinant, and that outside, the place of the letter in the series a, b, c, d, f, etc.

The n^{th} determinant for $(n + 1)$ spans, or n points of support, will then be :

$$D_n = D_{n-1}(\lambda_n)_n + D_{n-2}(\lambda_n)_{n-1}(\lambda_{n-1})_n.$$

Now, with the notation taken, if the letters in each term of the determinant are written in their natural order, as *abcdfg*, etc., *the number of inversions in the subscript figures of any term will determine the sign of that term*, i.e., *if the number of inversions is odd, the sign is minus, but if the number is even the sign is plus.*

Since n is the greatest subscript in any term, and since $(\lambda_n)_n$ occupies the most advanced place in the series of letters, no inversions are introduced in multiplying D_{n-1} by $(\lambda_n)_n$. Hence, *all terms of* $D_{n-1}(\lambda_n)_n$ *will have the same signs as the corresponding terms of* D_{n-1}.

Similarly, since n is greater than $(n - 1)$, the product $(\lambda_n)_{n-1}(\lambda_{n-1})_n$ involves one inversion. Hence, *all terms of*

$$D_{n-2}(\lambda_n)_{n-1}(\lambda_{n-1})_n$$

will have signs contrary to those of the corresponding terms of D_{n-2}.

The number of terms in D_n will evidently be the sum of the numbers of terms in D_{n-1} and D_{n-2}.

An examination of the notation will at once show that:

$$(\lambda_n)_n = 2(l_n + l_{n+1})\ ; \quad (\lambda_n)_{n-1} = l_n\ ; \quad \text{and} \quad (\lambda_{n-1})_n = l_n.$$

Hence there will result:

$$D_n = 2D_{n-1}(l_n + l_{n+1}) - D_{n-2}l_n^2 \quad . \quad . \quad . \quad . \quad (11)$$

The minus sign before the last term of the second member is on account of the inversion introduced, as already explained.

The general value of the determinant D_i (shown in Eq. (4) when $i = 3$) can be most easily expanded by considering it the sum of two determinants; and in order to illustrate this method let it be supposed that M_3 is desired. It will then be necessary to expand the determinant D_3, given in Eq. (4). As is known from the theory of determinants, D_3 may be written as follows:

$$D_3 = \begin{vmatrix} a_1, b_1, 0, 0, 0, 0, \dots \\ a_2, b_2, u_2, 0, 0, 0, \dots \\ 0, b_3, u_3, d_3, 0, 0, \dots \\ 0, 0, u_4, d_4, f_4, 0, \dots \\ 0, 0, 0, d_5, f_5, g_5, \dots \\ \dots\dots\dots\dots \\ 0, 0, 0, 0, 0, 0, 0, \dots p_n, q_n \end{vmatrix} + \begin{vmatrix} a_1, b_1, u_1, 0, 0, 0, \dots \\ a_2, b_2, 0, 0, 0, 0, \dots \\ 0, b_3, 0, d_3, 0, 0, \dots \\ 0, 0, 0, d_4, f_4, 0, \dots \\ 0, 0, u_5, d_5, f_5, g_5, \dots \\ \dots\dots\dots\dots \\ 0, 0, u_n, 0, 0, 0, \dots p_n, q_n \end{vmatrix} \quad (12)$$

or :

$$D_3 = D_3' + D_3'' \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Eq. (12) shows at a glance what D_3' and D_3'' represent.

D_3' is precisely the same in form as D, and is given at once by the Eqs. (6) to (11) after writing u_2, u_3 and u_4 for c_2, c_3 and c_4.

In general, D_i' is found by simply writing u_{i-1}, u_i and u_{i+1} for $(\lambda_{i-1})_i$, $(\lambda_i)_i$ and $(\lambda_{i+1})_i$ in the determinant D.

As a general method, that of alternate numbers is probably as simple as any for the expansion of the determinant D_i''. For example :

$$D_3'' = (a_1e_1 + b_1e_2 + u_1e_3)\,(a_2e_1 + b_2e_2)\,(b_3e_2 + d_3e_4)\ldots$$

$$\ldots(u_ne_3 + p_ne_{n-1} + q_ne_n)\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

in which e_1, e_2, e_3, etc., are the units of the alternate numbers.

The circumstances of any particular case will frequently either furnish a more expeditious method than that of alternate numbers, or allow the expansion of D_i'' to be written at once from an inspection of the array given in Eq. (12).

In any case the method of alternate numbers may be used as a check.

Special Method for Ordinary Use.

If the number of spans is large, the expansion of the determinant D_i will, at best, be found somewhat tedious. Special methods may be employed which involve only the determinant D, given in Eqs. (6) to (11); and it has already been seen that that determinant admits of a very simple expansion.

Let any one span carry *any load whatever*, while all other spans carry no load. In such a case, P will be zero for every span but one, and, in consequence of the notation employed, all but two quantities in the series u_1, u_2, u_3, u_4, u_5, etc., will also become equal to zero.

If l_i (the i^{th} span) carries the load, there will result :

$$u_{i-1} = -\frac{1}{l_i}\overset{i}{\Sigma}P(l_i^2 - z^2)z \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$u_i = -\frac{1}{l_i}\overset{i}{\Sigma}P(l_i^2 - z^2)z \quad . \quad . \quad . \quad . \quad . \quad (16)$$

All other us reduce to zero. Although Eqs. (15) and (16) have the same form, they are not identical except in special cases, since z is not measured from the same end of the span in both expressions.

Now let u_{i-1} and u_i take the place of those letters in that column of D formed with the i^{th} letter of the series a, b, c, d, etc., which have the subscripts i and $i - 1$; u_{i+1} is equal to zero. Or in the notation already employed, let u_{i-1} and u_i take the place of $(\lambda_{i-1})_i$ and $(\lambda_i)_i$, while zero takes the place of $(\lambda_{i+1})_i$. The resulting determinant, D_i, will then be precisely the same as D in general form. The expansion of D_i can then be at once made by simply putting in D the substitutions above indicated. There will then result :

$$M_i = \frac{D_i}{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

In order to find M_{i-1}, with the same loading on the same span, u_{i-1} and u_i must take the place of $(\lambda_{i-1})_{i-1}$ and $(\lambda_i)_{i-1}$, respectively, while $(\lambda_{i-2})_{i-1}$ becomes equal to zero. Making these substitutions in the determinant D, there will result the determinant D_{i-1}. Then :

$$M_{i-1} = \frac{D_{i-1}}{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The values of M_i and M_{i-1}, thus obtained, placed in the i^{th} and $(i-1)^{th}$ of the Eqs. (2) will at once give :

$$M_{i-2} \quad \text{and} \quad M_{i+1}.$$

Similar substitutions in the other equations will give all the moments. Thus the solution is complete, for the span and loading taken, with the use of the expanded determinant D only.

Each span may be treated in the same manner and the same expansion of D will be the only one necessary.

This method is equivalent to splitting the elements u_1, u_2, u_3, u_4, etc., of the general determinant D_i.

In order to determine the bending moment at any point of support, for loading which covers more than one span, or portions of more than one span, it is only necessary to take the algebraic sum of the separate moments (as above determined), at the point of support in question, found for the loading in each single span. The result will be the moment due to the combined action of all the loading.

It is thus seen that the solution of the most general case is made to depend on the one expansion of the determinant D.

Example.

Let there be a continuous beam of six spans, and let any loading rest upon the fourth ; it is required to find the expansions of the determinants D_i and D_{i-1}.

The expansion of D is given in Eq. (10), and need not be repeated here.

Using the preceding notation :

$$i \quad = 4. \qquad (\lambda_{i-1})_i = d_3.$$

$$i - 1 = 3. \qquad (\lambda_i)_i \quad = d_4.$$

$$u_i \quad = u_4. \qquad (\lambda_{i+1})_i = d_5.$$

$$u_{i-1} \quad = u_3.$$

In Eq. (10) then, d_4 and d_3 are to be displaced by u_4 and u_3, while zero is to take the place of d_5. Hence :

$$D_i = a_1b_2c_3u_4f_5 - a_1b_3c_2u_4f_5 - a_2b_1c_3u_4f_5 - a_1b_2c_4u_3f_5 + a_2b_1c_4u_3f_5 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Again :

$$(\lambda_{i-1})_{i-1} = c_3 \qquad (\lambda_i)_{i-1} = c_4.$$

Then, in Eq. (10), placing u_3 and u_4 for c_3 and c_4, and placing

$$(\lambda_{i-2})_{i-1} = c_2 = 0,$$

there will result :

$$D_{i-1} = a_1b_2u_3d_4f_5 - a_2b_1u_3d_4f_5 - a_1b_2u_4d_3f_5 + a_2b_1u_4d_3f_5 - a_1b_2u_3d_5f_4 + a_2b_1u_3d_5f_4 \quad . \quad . \quad . \quad . \quad . \quad (20)$$

These values placed in Eqs. (17) and (18) will give M_4 and M_3.

The lengths of span may be any whatever ; if they are equal, the results will be simplified.

Special Case of Equal Spans.

If all the spans are of equal length, each may be represented by l. There will then result :

$$\left.\begin{array}{l} a_2 = b_3 = c_4 = d_5 = \ldots = p_i = b_1 = c_2 = d_3 = f_4 = \ldots s_i = l \\ a_1 = b_2 = c_3 = d_4 = f_5 = \ldots = q_i = 4l. \end{array}\right\} \quad (21)$$

These values of a, b, c, etc., placed in Eqs. (6) to (10) give :

For two spans :

$$D = 4l.$$

For three spans :

$$D = 15l^2.$$

For four spans :

$$D = 56l^3.$$

For five spans :

$$D = 209l^4.$$

For six spans :

$$D = 780l^5.$$

Others may be easily and rapidly written by the aid of Eq. (11), which now becomes :

$$D_n = 4lD_{n-1} - l^2D_{n-2} \quad . \quad . \quad . \quad . \quad . \quad (22)$$

If the determinant for seven (*i.e.*, $n + 1$) spans is desired :

$$D_{n-1} = 780l^5 \quad \text{and} \quad D_{n-2} = 209l^4.$$

Hence :

$$D_n = D_6 = 3120l^6 - 209l^6 = 2911l^6.$$

Similarly for eight spans :

$$D = 4 \times 2911l^7 - 780l^7 = 10864l^7.$$

For nine spans :

$$D = 4 \times 10864l^8 - 2911l^8 = 40545l^8.$$

For ten spans :

$$D = 4 \times 40545l^9 - 10864l^9 = 151316l^9.$$

The values given in Eq. (21) will correspondingly simplify the expansion of the determinant D_i, either in its general form as exemplified in Eq. (4) or as given in the special method. As an illustration, Eqs. (19) and (20) become, respectively :

$$D_i \quad = 224l^4u_4 - 60l^4u_3.$$

$$D_{i-1} = 225l^4u_3 - 60l^4u_4.$$

These values then give :

$$M_4 = \frac{D_i}{D} = \frac{56u_4 - 15u_3}{195l}.$$

$$M_3 = \frac{D_{i-1}}{D} = \frac{15u_3 - 4u_4}{52l}.$$

Then by Eqs. (2) :

$$M_2 = \frac{u_3}{l} - (4M_3 + M_4).$$

$$M_5 = \frac{u_4}{l} - (M_3 + 4M_4) = -\frac{M_4}{2}.$$

$$M_1 = -\frac{M_2}{2}.$$

Thus all the moments are known for this example, *i.e*, with six spans and loading on the fourth span only.

Reactions.

After the moments are found either by the general or special method, for any condition of loading, the reactions will at once result from the substitution of the values thus found in the Eqs. (15) to (21) of the preceding Art., which it is not necessary to reproduce here.

Art. 24.—The Neutral Curve for Special Cases.

The curved intersection of the neutral surface with a vertical plane passing through the axis of a loaded, and originally straight, beam may be called the " neutral curve." The neutral curve is the locus of the extremities of the ordinates w of Art. 19; it therefore gives the deflection at any point of the beam.

The method of finding the neutral curve for any particular case of beam or loading can be well illustrated by the operations in the following three cases.

Case I.

This case is shown in the accompanying figure, which represents a cantilever carrying a uniform load with a single weight W at its free end. As usual, the intensity of the uniform loading will be represented by p.

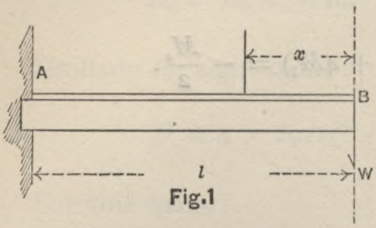


Fig.1

Measuring x and w from B, as shown, the general value of the bending moment is :

$$M = EI\frac{d^2w}{dx^2} = Wx + \frac{px^2}{2} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Integrating between x and l, remembering that :

$$\frac{dw}{dx} = 0$$

for $x = l$:

$$EI\frac{dw}{dx} = \frac{W}{2}(x^2 - l^2) + \frac{p}{6}(x^3 - l^3) \quad . \quad . \quad . \quad (2)$$

Hence :

$$w = \frac{1}{EI}\left\{\frac{W}{2}\left(\frac{x^3}{3} - xl^2\right) + \frac{p}{6}\left(\frac{x^4}{4} - l^3x\right)\right\} \quad . \quad . \quad (3)$$

The greatest deflection, w_1, occurs for $x = l$. Hence :

$$w_1 = -\frac{1}{EI}\left(\frac{Wl^3}{3} + \frac{pl^4}{8}\right) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The greatest moment, M_1, exists at A, and its value is :

$$M_1 = Wl + \frac{pl^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

These equations are made applicable to a cantilever with a uniform load by simply making $W = 0$. They then become :

$$M = EI\frac{d^2w}{dx^2} = \frac{px^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$EI\frac{dw}{dx} = \frac{p}{6}(x^3 - l^3) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$w = \frac{p}{6EI}\left(\frac{x^4}{4} - l^3x\right) \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$w_1 = -\frac{pl^4}{8EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$M_1 = \frac{pl^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Again, for a cantilever with a single weight only at its free end, p is to be made equal to zero in the first set of equations. Those equations then become :

$$M = EI\frac{d^2w}{dx^2} = Wx \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$EI\frac{dw}{dx} = \frac{W}{2}(x^2 - l^2) \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$w = \frac{W}{2EI}\left(\frac{x^3}{3} - xl^2\right) \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$w_1 = -\frac{Wl^3}{3EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$M_1 = Wl \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The general expressions for the shear and the intensity of loading are :

$$S = EI\frac{d^3w}{dx^3} = W + px \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$EI\frac{d^4w}{dx^4} = p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Case II.

This case, shown in the figure, is that of a non-continuous beam, supported at each end, and carrying both a uniform load

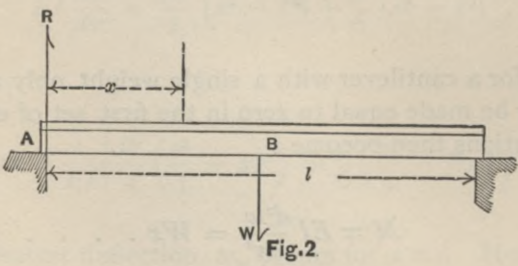


Fig.2

(whose intensity is p) and a single weight W at its middle point. The reaction R, at either end, will then be :

$$R = \frac{pl + W}{2}.$$

The general value of the moment will then be :

$$M = EI\frac{d^2w}{dx^2} = Rx - \frac{px^2}{2} \quad . \quad . \quad . \quad . \quad (18)$$

The origin of x and w is taken at A.
Remembering that :

$$\frac{dw}{dx} = 0 \quad \text{for} \quad x = \frac{l}{2},$$

and integrating between the limits x and $\frac{l}{2}$:

$$EI\frac{dw}{dx} = \frac{R}{2}\left(x^2 - \frac{l^2}{4}\right) - \frac{p}{6}\left(x^3 - \frac{l^3}{8}\right) \quad . \quad . \quad (19)$$

Again integrating :

$$w = \frac{1}{EI}\left\{\frac{R}{2}\left(\frac{x^3}{3} - \frac{xl^2}{4}\right) - \frac{p}{6}\left(\frac{x^4}{4} - \frac{xl^3}{8}\right)\right\} \quad . \quad . \quad (20)$$

The greatest deflection w_1 occurs at the centre of the span, for which :

$$x = \frac{l}{2}.$$

Hence :

$$w_1 = -\frac{l^3}{48EI}\left\{W + \frac{5}{8}pl\right\} \quad . \quad . \quad . \quad . \quad (21)$$

The greatest moment, also, is found by putting :

$$x = \frac{l}{2}.$$

It has the value :

$$M_1 = \frac{l}{4}\left(W + \frac{pl}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

These formulæ are made applicable to a non-continuous beam carrying a uniform load only, by putting $W = 0$. They then become :

$$R = \frac{pl}{2}.$$

$$M = EI\frac{d^2w}{dx^2} = \frac{px}{2}(l - x) \quad . \quad . \quad . \quad . \quad . \quad (23)$$

$$EI\frac{dw}{dx} = \frac{p}{2}\left(\frac{x^2 l}{2} - \frac{x^3}{3} - \frac{l^3}{12}\right) \quad . \quad . \quad . \quad . \quad (24)$$

$$w = \frac{p}{24EI}(2x^3 l - x^4 - l^3 x) \quad . \quad . \quad . \quad . \quad (25)$$

$$w_1 = -\frac{5pl^4}{384EI} = -\frac{5}{8} \cdot \frac{pl^4}{48EI} \quad . \quad . \quad . \quad (26)$$

$$M_1 = \frac{pl^2}{8} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The formulæ for a beam of the same kind carrying a single weight at the centre, are obtained by putting $p = 0$ in the first set of equations. Those for the greatest deflection and greatest moment, only, however, will be given. They are :

$$w_1 = -\frac{Wl^3}{48EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$M_1 = \frac{Wl}{4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

The general values of the shear and intensity of loading are :

$$S = \frac{dM}{dx} = R - px \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

$$\frac{d^2M}{dx^2} = -p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Case III.

The general treatment of continuous beams requires the use of the theorem of three moments. The particular case to

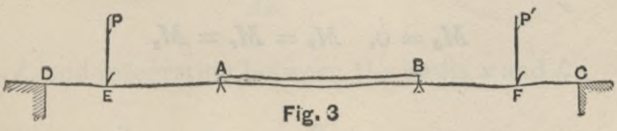


Fig. 3

be treated is shown in Fig. 3. The beam covers the three spans, *DA*, *AB* and *BC*, and is continuous over the two points of support *A* and *B*.

$$\left.\begin{array}{l}\text{Let } DA = l_1 \\ \text{"} \quad AB = l_2 \\ \text{"} \quad BC = l_3\end{array}\right\} \text{Let } l_2 = nl_1 = n'l_3.$$

Let the intensity of the uniform load on AB be represented by p and let the two single forces P and P' *only*, act in the spans DA and BC respectively. Also let the two distances :

$$DE = z_1 = al_1 \quad \text{and} \quad CF = a'l_3$$

be given. *It is required to find the magnitudes of the forces P and P', if the beam is horizontal at A and B.*

Since the beam is horizontal at A and B, the bending moments over those two points of support will be equal to each other, for the load on AB is both uniform and symmetrical. Let this bending moment, common to A and B, be represented by M_2. As the ends of the beam simply *rest* at D and C, the moments at those two points reduce to zero.

Because the four points D, A, B and C are in the same level, the first member of Eq. (14), of Art. 23, becomes equal to zero.

If that equation be applied to the three points D, A and B, the conditions of the present problem produce the following results :

$$M_a = 0, \quad M_b = M_c = M_2$$

and

$$\frac{1}{l_c} \sum^{c} P(l_c^2 - z^2)z = p\frac{l_2^3}{4}.$$

Hence the equation itself will become :

$$M_2(2l_1 + 3l_2) + \frac{P}{l_1}(l_1^2 - z_1^2)z_1 + p\frac{l_2^3}{4} = 0 \quad . \quad (32)$$

$$\therefore \quad M_2 = -\frac{4P(l_1^2 - z_1^2)z_1 + pl_1l_2^3}{4l_1(2l_1 + 3l_2)};$$

$$\therefore \quad M_2 = -l_1\frac{4P(1 - a^2)a + pn^3l_1}{4(2 + 3n)} \quad . \quad . \quad . \quad . \quad (33)$$

$$\therefore \quad \text{Reaction at} \quad D = R_1 = P\frac{l_1 - z_1}{l_1} + \frac{M_2}{l_1} \quad . \quad . \quad . \quad . \quad (34)$$

As the origin of z_1 is at D, x will be measured from the same point.

Separate expressions for moments must be obtained for the two portions, DE and EA, of l_1, because the law of loading in that span is not continuous.

Taking moments about any point of EA :

$$EI\frac{d^2w}{dx^2} = R_1x - P(x - z_1) \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Remembering that :

$$\frac{dw}{dx} = 0$$

for $x = l_1$, and integrating between the limits x and l_1 :

$$EI\frac{dw}{dx} = \frac{R_1}{2}(x^2 - l_1^2) - \frac{P}{2}(x^2 - l_1^2) + Pz_1(x - l_1) \quad (36)$$

Again, remembering that $w = 0$ for $x = l_1$, and integrating between the limits x and l_1 :

$$EIw = \frac{R_1}{2}\left(\frac{x^3}{3} - l_1^2 x + \frac{2l_1^3}{3}\right) - \frac{P}{2}\left(\frac{x^3}{3} - l_1^2 x + \frac{2l_1^3}{3}\right)$$

$$+ Pz_1\left(\frac{x^2}{2} - l_1 x + \frac{l_1^2}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad (37)$$

Taking moments about any point in DE:

$$EI\frac{d^2w}{dx^2} = R_1 x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

$$\therefore \quad EI\frac{dw}{dx} = R_1\frac{x^2}{2} + C \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Making $x = z_1$ in Eqs. (36) and (39), then subtracting:

$$C = -\frac{R_1}{2}l_1^2 - \frac{P}{2}(z_1^2 - l_1^2) + Pz_1(z_1 - l_1).$$

$$\therefore \quad EI\frac{dw}{dx} = \frac{R_1}{2}(x^2 - l_1^2) - \frac{P}{2}(z_1^2 - l_1^2) + Pz_1(z_1 - l_1) \quad (40)$$

Remembering that $w = 0$ for $x = 0$, and integrating between the limits x and 0:

$$EIw = \frac{R_1}{2}\left(\frac{x^3}{3} - l_1^2 x\right) - \frac{P}{2}(z_1^2 - l_1^2)x + Pz_1(z_1 - l_1)x. \quad (41)$$

Making $x = z_1$ in Eqs. (37) and (41), then subtracting:

$$\frac{R_1 l_1^3}{3} - \frac{P}{3}(l_1^3 - z_1^3) + \frac{Pz_1}{2}(l_1^2 - z_1^2) = 0 \quad . \quad . \quad (42)$$

Putting the value of M_2 from Eq. (33) in Eq. (34), then inserting the value of R_1, thus obtained, in Eq. (42), after making $z_1 = al_1$:

$$P\left[(1-a) - \frac{a(1-a^2)}{2+3n} - (1-a^3) + \frac{3}{2}a(1-a^2)\right]$$

$$= \frac{pn^3l_1}{4(2+3n)}.$$

$$\therefore \quad P = \frac{pn^2l_1}{6a(1-a^2)} = \frac{pnl_2}{6a(1-a^2)} \quad . \quad . \quad . \quad (43)$$

This is the desired value of P, which will cause the beam to be horizontal over the two points of support A and B when the span AB carries a uniform load of the intensity p.

By the aid of Eq. (43), Eq. (33) now gives:

$$M_2 = -pl_1^2\frac{(2n^2+3n^3)}{12(2+3n)} = -\frac{pn^2l_1^2}{12} = -\frac{pl_2^2}{12} \quad . \quad (44)$$

It is to be noticed that M_2 is entirely independent of l_1 or l_3. Eq. (43) also gives:

$$p = P\,\frac{6a(1-a^2)}{n^2l_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Hence:

$$M_2 = -\frac{Pl_1}{2}(1-a^2)a \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Thus any of the preceding equations may be expressed in terms of p or P.

R_1, also, becomes :

$$R_1 = \frac{pnl_2}{6a(1+a)} - \frac{pnl_2}{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

or :

$$R_1 = P(1-a)\left[1 - \frac{1}{2}a(1+a)\right] \quad . \quad . \quad . \quad (48)$$

It is clear that there cannot be a point of no bending in DE. Hence, the point of contra-flexure must lie between E and A, Fig. 3. In order to locate this point, according to the principles already established, the second member of Eq. (35) must be put equal to zero. Doing so and solving for x :

$$x = \frac{P}{P - R_1} z_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Since P is always greater than R_1, there will always be a point of contra-flexure.

All these equations will be made applicable to the span BC, by simply writing a' for a, l_3 for l_1, and n' for n.

As an example, let :

$$a = \frac{1}{2} \quad \text{and} \quad n = 1.$$

Eqs. (43), (44) and (47) then give :

$$P = \frac{4}{9} pl ;$$

$$M_2 = -\frac{pl^2}{12} = -\frac{3Pl}{16} ;$$

$$R_1 = pl_1\left(\frac{2}{9} - \frac{1}{12}\right) = \frac{5}{36}pl = \frac{5}{16}P;$$

after writing :

$$l_1 = l_2 = l_3 = l.$$

In general, the span l_1 is called "a beam fixed at one end, simply supported at the other and loaded at any point with the single weight, P."

Let it, again, be required *to find an intensity, "p'," of a uniform load, resting on the span l_1, which will cause the beam to be horizontal at the points A and B.*

Since the load is continuous, only one set of equations will be required for the span. The equation of moments will be :

$$EI\frac{d^2w}{dx^2} = R_1x - \frac{p'x^2}{2} \quad . \quad . \quad . \quad . \quad . \quad (50)$$

Integrating between the limits x and l_1 :

$$EI\frac{dw}{dx} = \frac{R_1}{2}(x^2 - l_1^2) - \frac{p'}{6}(x^3 - l_1^3) \quad . \quad . \quad . \quad (51)$$

Integrating between the limits x and 0 :

$$EIw = \frac{R_1}{2}\left(\frac{x^3}{3} - l_1^2x\right) - \frac{p'}{6}\left(\frac{x^4}{4} - l_1^3x\right) \quad . \quad . \quad (52)$$

But, also, $w = 0$ when $x = l_1$. Hence :

$$R_1\frac{l_1^3}{3} = \frac{p'l_1^4}{8} \quad \therefore \quad R_1 = \frac{3}{8}p'l_1. \quad . \quad . \quad . \quad (53)$$

This equation gives the value R_1 when p' is known. Making $x = l_1$ in Eq. (50) and using the value of R_1 from Eq. (53):

$$M_2 = p'l_1^2\left(\frac{3}{8} - \frac{1}{2}\right) = -\frac{p'l_1^2}{8} \quad . \quad . \quad . \quad . \quad (54)$$

Adapting Eq. (32) to the present case:

$$M_2(2l_1 + 3l_2) + \frac{1}{4}(p'l_1^3 + pl_2^3) = 0.$$

$$\therefore \quad M_2 = -\frac{(p' + pn^3)l_1^2}{4(2 + 3n)} \quad . \quad . \quad . \quad . \quad . \quad (55)$$

Equating these two values of M_2:

$$p' = \frac{2}{3}pn^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

Thus is found the desired value of p'. In this case the span l_1 is called "a beam fixed at one end, simply supported at the other and uniformly loaded."

The points of contra-flexure are found by putting the second member of Eq. (50) equal to zero and solving for x, after introducing the value of R_1 from Eq. (53). Hence:

$$\frac{3}{4}l_1x - x^2 = 0.$$

or:

$$x = 0 \quad \text{and} \quad x = \frac{3}{4}l_1.$$

Between the simply supported end and point of contra-flexure the beam is evidently convex *downward*, and convex upward in the other portion of the spans l_1 and l_3, whether the load is single or continuous. Moments of different signs will, then, be found in these two portions, and there will be a maximum for each sign. The location of the sections in which these greatest moments act may be made in the ordinary manner by the use of the differential calculus; but the *negative* maximum is evidently M_2, given by Eqs. (44) and (55). On the other hand the *positive* maximum is clearly found at the point of application of P in the case of a single load, and at the point

$$x = \frac{3}{8} l_1,$$

in the case of a continuous load. These conclusions will at once be evident if it be remembered that the portion of the beam between the supported end and point of contra-flexure is, in reality, *a beam simply supported at each end.* These moments will have the values:

$$M_1 = Pl_1(1 - a)a - l_1 \frac{4P(1 - a^2)a^2 + pan^3l_1}{4(2 + 3n)}. \quad (57)$$

$$M_1' = \frac{9}{128} p'l_1^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

In case of a single load if P is given, and not p, Eq. (45) shows:

$$M_1 = Pl_1(1 - a)a \left[1 - \frac{1}{2} a(1 + a) \right].$$

The points of greatest deflection are found by putting the second members of Eqs. (36), (40) and (51) each equal to zero, and then solving for x. They are not points of great importance, and the solutions will not be made.

The following are the general values of the shears for a single load on l_1:

$$\text{In } AE; \quad S = EI\frac{d^3w}{dx^3} = R_1 - P; \text{ [from Eq. (35)].}$$

$$\text{In } ED; \quad S_1 = EI\frac{d^3w}{dx^3} = R_1; \qquad \text{[from Eq. (38)].}$$

The shear in l_1 for the uniform load p' is:

$$S' = EI\frac{d^3w}{dx^3} = R_1 - p'x; \qquad \text{[from Eq. (50)].}$$

Also:

$$\textit{Intensity of load} = EI\frac{d^4w}{dx^4} = -p'.$$

As has already been observed, all the equations relating to the span l_1 may be made applicable to the span l_3 by changing a to a' and n to n'.

The span l_2 remains to be considered.

Since the bending moments at A and B are equal to each other, and since the loading is uniformly continuous, half of it (the load pl_2) will be supported at A and the other half at B. In other words, the vertical shear at an indefinitely short distance to the right of A, also to the left of B, will be equal to $\frac{pl_2}{2}$. Let x be measured to the right and from A. The bending moment at any section x will be:

$$EI\frac{d^2w}{dx^2} = M_2 + \frac{pl_2}{2}x - \frac{px^2}{2}.$$

or :

$$EI\frac{d^2w}{dx^2} = M_2 + \frac{p}{2}(l_2x - x^2) \quad . \quad . \quad . \quad . \quad (59)$$

Integrating between the limits x and o :

$$EI\frac{dw}{dx} = M_2x + \frac{p}{2}\left(\frac{l_2x^2}{2} - \frac{x^3}{3}\right) \quad . \quad . \quad . \quad (60)$$

Again integrating between the same limits :

$$EIw = \frac{M_2x^2}{2} + \frac{p}{12}\left(l_2x^3 - \frac{x^4}{2}\right) \quad . \quad . \quad . \quad . \quad (61)$$

Since :

$$\frac{dw}{dx} = 0$$

for l_2, Eq. (60) will give M_2 independently of preceding equations. Following this method, therefore :

$$M_2 = -\frac{pl_2^2}{12}.$$

This is the same value which has already been obtained. Introducing the value of M_2 :

$$EI\frac{d^2w}{dx^2} = \frac{p}{2}\left(l_2x - x^2 - \frac{l_2^2}{6}\right) \quad . \quad . \quad . \quad . \quad (62)$$

$$EI\,\frac{dw}{dx} = \frac{p}{2}\left(\frac{l_2x^2}{2} - \frac{x^3}{3} - \frac{l_2^2}{6}\,x\right). \quad . \quad . \quad (63)$$

$$EIw \quad = \frac{px^2}{12}\left(l_2x - \frac{x^2}{2} - \frac{l_2^2}{2}\right). \quad . \quad . \quad . \quad (64)$$

The points of contra-flexure are found by putting the second member of Eq. (62) equal to zero. Hence :

$$x^2 - l_2x = -\frac{l_2^2}{6}$$

$$\therefore \quad x = l_2\left(\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{6}}\right) = \begin{cases} 0.789l_2. \\ 0.211l_2. \end{cases}$$

The moment at the centre of the span is found by putting,

$$x = \frac{l_2}{2}$$

in Eq. (62) :

$$M_1 = \frac{pl_2^2}{24}.$$

This is the greatest *positive* moment.

The general value of the shear is :

$$S = EI\,\frac{d^3w}{dx^3} = p\left(\frac{l_2}{2} - x\right).$$

And the intensity of load :

$$EI \frac{d^4w}{dx^4} = -p.$$

The span l_2 is generally called "a beam fixed at both ends and uniformly loaded."

It is sometimes convenient to consider a single load at the centre of the span l_2 while the beam remains horizontal at A and B; in other words, to consider "a beam fixed at each end and supporting a weight at the centre."

Let W represent this weight: then a half of it will be the shear at an indefinitely short distance to the right of A and left of B. As before, let x be measured from A, and positive to the right. The moment at any point will be:

$$EI \frac{d^2w}{dx^2} = M_2 - \frac{Wx}{2} \quad . \quad . \quad . \quad . \quad . \quad (65)$$

Integrating between x and o:

$$EI \frac{dw}{dx} = M_2 x - \frac{Wx^2}{4} \quad . \quad . \quad . \quad . \quad . \quad (66)$$

If $x = \frac{l_2}{2}$, then will

$$\frac{dw}{dx} = 0;$$

hence:

$$M_2 = \frac{Wl_2}{8}.$$

The general value of the moment then becomes:

$$M = EI \frac{d^2w}{dx^2} = \frac{Wl_2}{8} - \frac{Wx}{2} \quad . \quad . \quad . \quad . \quad (67)$$

If $x = \frac{l_2}{2}$ in this equation, the bending moment at the centre (where W is applied) has the value :

$$Centre\ moment = -\frac{Wl_2}{8}.$$

Hence, *the bending moments at the centre and ends are each equal to the product of the load by one eighth the span, but have opposite signs.*

A second integration between x and 0 gives :

$$w = \frac{1}{EI}\left(\frac{Wl_2x^2}{16} - \frac{Wx^3}{12}\right) \quad . \quad . \quad . \quad . \quad (68)$$

Hence, the deflection at the centre has the value :

$$Centre\ deflection = \frac{Wl_2^3}{192EI}.$$

By placing $M = 0$, the points of contra-flexure are found at the distance from each end :

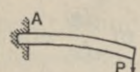
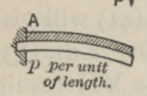
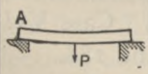
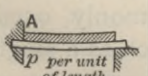
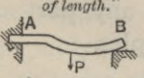
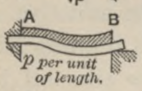
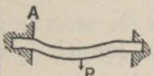
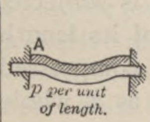
$$x_1 = \frac{l_2}{4}.$$

Addendum to Art. 24.

The formulæ of this Art. furnish the solutions of many practical questions of maxima deflections and moments. The latter for several ordinary cases are given in the following tabulation.

P is the weight in pounds at *end of beam* or *centre of span.*

p is the load in pounds *per lin. ft.* of beam.

	BEAM.	MAX. MOMENT.	MAX. DEFLECTION.	POINT OF CONTRAFLEXURE.
I		Pl at A.	$576\,\frac{Pl^3}{EI}$ at A.	
II		$\frac{1}{2}pl^2$ at A.	$216\,\frac{pl^4}{EI}$ at A.	
III		$\frac{1}{4}Pl$ at centre.	$36\,\frac{Pl^3}{EI}$ at centre.	
IV		$\frac{1}{8}pl^2$ at centre.	$22.5\,\frac{pl^4}{EI}$ at centre.	
V		$-\frac{3}{16}Pl$ at A. $\frac{5}{32}Pl$ at centre.	$16.16\,\frac{Pl^3}{EI}$ at $0.447l$ from B.	$\frac{8}{11}l$ from B. Reaction at $B = \frac{5}{16}P$.
VI		$-\frac{1}{8}pl^2$ at A. $+\frac{9}{128}pl^2$ at $\frac{3}{8}l$ from B.	$9.35\,\frac{pl^4}{EI}$ at $0.4215l$ from B.	$\frac{3}{4}l$ from B. Reaction at $B = \frac{3}{8}pl$.
VII		$-\frac{1}{8}Pl$ at A. $\frac{1}{8}Pl$ at centre.	$9\,\frac{Pl^3}{EI}$ at centre.	$\frac{1}{4}l$ from each end.
VIII		$-\frac{1}{12}pl^2$ at A. $\frac{1}{24}pl^2$ at centre.	$4.5\,\frac{pl^4}{EI}$ at centre.	$0.211l$ from each end.

l is the length of beam or of span in *feet.*

E is the coefficient of elasticity in pounds per sq. inch.

I is the moment of inertia of the normal section of the beam with all dimensions of section in *inches.*

The "Max. Moments" will be in *foot pounds;* and the "Max. Deflections" will be in *inches.*

In the use of Eq. (2), Art. 62, in its many practical applications, it is best to have the moment M in inch pounds, which will result from simply multiplying the "Max. Moments" of the preceding Table by 12.

Case I results from Eqs. (14) and (15); Case II, from Eqs. (9) and (10); Case III, from Eqs. (28) and (29); Case IV, from Eqs. (26) and (27). In Case V the reaction is found by putting $a = \frac{1}{2}$ in Eq. (48); the point of "Max. Deflection" is found by placing $z_1 = \frac{1}{2}l$ in Eq. (40), and the resulting value of $\frac{dw}{dx}$ equal to zero and solving for x, which latter value in Eq. (41) will give "Max Deflection." Case VI results from treating Eqs. (53), (51) and (52) in precisely the same manner. Case VII results directly from the formulæ on pages 189 and 190. Case VIII results directly from the equations on pages 187 and 188.

The preceding cases are those which commonly occur with constant values of E and I. Other cases, such as a single load at any point, or partial uniform load over any part of span, are to be treated by the same general principles.

Art. 25.—The Flexure of Long Columns.

A "long column" is a piece of material whose length is a number of times its breadth or width, and which is subjected to a compressive force exerted in the direction of its length. Such a piece of material will not be strained, or compressed, directly back into itself, but will yield laterally as a whole, thus causing flexure. If the length of a long column is many times the width or breadth, the failure in consequence of flexure will take place while the pure compression is very small.

As with beams, so with columns, the ends may be "fixed," so that the end surfaces do not change their position however great the compression or flexure. Such a column is frequently, perhaps usually, said to have "flat" ends. If the ends of the column are free to turn in any direction, being simply supported, as flexure takes place, the column is said to have "round" ends. It is clear that if the column has freedom in one or several directions, only, it will be a "round" end column in that one direction, or those several directions, only. It is also evident that a column may have one end "round" and one end "flat" or "fixed."

In Fig. 1 let there be represented a column with flat ends, vertical and originally straight. After external pressure is imposed at A, the column will take a shape similar to that represented. Consequently the load P, at A, will act with a lever arm at any section equal to the deflection of that section from its original position. Let y be the general value of that deflection, and at B let $y = y_1$. Let x be measured from A, as an origin, along the original axis of the column. In accordance with principles already established, the condition of fixedness at each of the ends A and C is secured by the application of a *negative* moment $-M$. Now it is known from the general condition of the column that the curve of its axis will be convex toward the axis of x at and near A, while it will be concave at and near B (the middle point of the column). Hence, since y is positive toward the left, and since the ordinate and its second derivative must have the same sign when the curve is convex toward the axis of the abscissas, the general equation of moments must be written as follows :

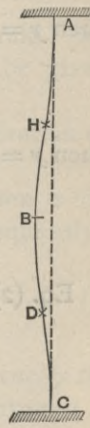


Fig. 1

$$-EI\frac{d^2y}{dx^2} = -M + Py \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Multiplying by $-2dy$:

$$EI\frac{2dy\ d^2y}{dx^2} = 2M\ dy - P\ 2y\ dy.$$

$$\therefore\ EI\left(\frac{dy}{dx}\right)^2 = 2My - Py^2 + (c = 0) \quad . \quad . \quad . \quad (2)$$

$c = 0$ because the column has flat ends, and,

$$\frac{dy}{dx} = 0$$

when $y = 0$. Also:

$$\frac{dy}{dx} = 0$$

when $y = y_1$.

$$\therefore\ M = \frac{Py_1}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Eq. (2) now becomes:

$$\sqrt{\frac{EI}{P}}\ \frac{dy}{\sqrt{y_1 y - y^2}} = dx.$$

$$\therefore\ x = \sqrt{\frac{IE}{P}}\ ver\ sin^{-1}\frac{2y}{y_1} \quad . \quad . \quad . \quad . \quad (4)$$

If $y = y_1$:

$$x = \frac{l}{2} = \pi\sqrt{\frac{EI}{P}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In this equation l is the length of the column. From Eq. (5) there may be deduced :

$$P = \frac{4\pi^2 EI}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

It is to be observed that P is *wholly independent of the deflection*, *i. e.*, it remains the same, whatever may be the amount of deflection, after the column begins to bend. Consequently, if the elasticity of the material were perfect, the weight P would hold the column in any position in which it might be placed, after bending begins.

Eq. (6) forms the basis of "Hodgkinson's Formula" for the resistance of long columns, of which more will be given hereafter. It was first established by Euler.

Some very important results flow from the consideration of Fig. 1 in connection with the preceding equations.

The bending moment at the centre, B, of the column is obtained by placing $y = y_1$ in Eq. (1) ; its value is, consequently :

$$M' = -M + Py_1 = M \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence *the bending at the centre of the column is exactly the same (but of opposite sign) as that at either end.* Between A and B, then, there must be a point of contra-flexure.

Putting the second member of Eq. (1) equal to zero, and introducing the value of M from Eq. (3) :

$$y = \frac{y_1}{2}.$$

Introducing this value of y in Eq. (4), and bearing in mind Eq. (5) :

$$x = \frac{\pi}{2}\sqrt{\frac{EI}{P}} = \frac{l}{4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The points of contra-flexure, then, are at H and D, $\frac{1}{4}\,l$ and $\frac{3}{4}\,l$ from A.

Hence, *the middle half of the column* (HD) *is actually a column with round ends*, and it is equal in resistance to a fixed-end column of double its length.

Hence writing l' for $\frac{l}{2}$ and putting $2l'$ for l in Eq. (6) :

$$P = \frac{\pi^2 EI}{l'^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Eq. (9) gives the value of P for a round-end column.

Again, either the upper three quarters (AD) or the lower three quarters (CH) of the column is very nearly equivalent to a column with one end flat and one end round, and its resistance is equal to that of a fixed-end column whose length is $\frac{4}{3}$ its own. Putting, therefore :

$$l_1 = \frac{3}{4}\,l$$

and introducing :

$$l = \frac{4}{3}\,l_1$$

in Eq. (6) :

$$P = 2.25 \frac{\pi^2 EI}{l_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The last case is not quite accurate, because the ends of the columns *HC* and *AD* are not exactly in a vertical line.

In reality, the column under compression may be composed of any number of such parts as *HD*, with the portions *HA* and *CD* at the ends, thus taking a serpentine shape, so far as pure equilibrium is concerned. In such a condition the column would be subjected to considerably less bending than in that shown in the figure. In ordinary experience, however, the serpentine shape is impossible, because the slightest jar or tremor would cause the column to take the shape shown in Fig. 1. Hence, the latter case only has been considered.

If r is the radius of gyration and S the area of normal section of the column, Eqs. (6) and (9) will take the forms :

$$\frac{P}{S} = \frac{4\pi^2 Er^2}{l^2} \quad \text{and} \quad \frac{P}{S} = \frac{\pi^2 Er^2}{l^2}.$$

Eq. (10) will, of course, take a corresponding form.

These equations evidently become inapplicable when $\frac{P}{S}$ approaches C, the ultimate compressive resistance of the material in short blocks. The corresponding values of $\left(\frac{l}{r}\right)$ at the limit, are :

$$\frac{l}{r} = 2\pi\sqrt{\frac{E}{C}}\,; \quad \text{and} \quad \frac{l}{r} = \pi\sqrt{\frac{E}{C}} \quad . \quad . \quad . \quad (11)$$

for fixed and round ends respectively ; other conditions of ends will be included between those two.

If, for wrought iron :

$$E = 28,000,000 \quad \text{and} \quad C = 60,000,$$

the above values become 136 and 68, nearly.

Euler's formula, therefore, is strictly applicable only to wrought-iron columns, with ends fixed or rounded, for which $l \div r$ exceeds 136 and 68, respectively.

If, for cast iron :

$$E = 14,000,000 \quad \text{and} \quad C = 100,000,$$

Eqs. (11) give :

$$\frac{l}{r} = 74, \quad \text{and} \quad \frac{l}{r} = 37, \text{ nearly.}$$

Euler's formula evidently becomes inapplicable considerably above the limits indicated, since columns in which $\frac{l}{r}$ has those values will not nearly sustain the intensity C.

The analytical basis of "Gordon's Formula" for the resistance of long columns is so closely associated with the empirical, that both will be treated together, hereafter.

Art. 26.—Graphical Determination of the Resistance of a Beam.

The graphical method is well adapted to the treatment of beams whose normal sections are limited either wholly or in part by irregular curves. In Fig. 1 is represented the normal section of such a beam, the centre of gravity of the section being situated at C. The lines HL, AB and DF are parallel. As is known by the common theory of flexure, the neutral axis will pass through C.

Let *aa* be any line on either side of *AB*, then draw the lines *aa'* normal to *AB*, having made *MN* and *HL* equidistant from *AB*. From the points *a'*, thus determined, draw straight lines to *C*. These last lines will include intercepts, *bb*, on the original lines *aa*. Let every linear element parallel to *AB*, on each

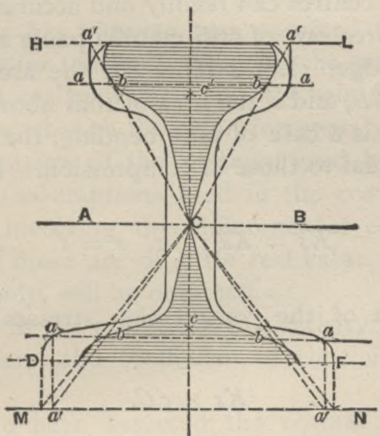


Fig. 1

side of *C*, be similarly treated. All the intercepts found in this manner will compose the shaded figure.

This operation, in reality, and only, determines an amount of stress with a uniform intensity identical with that developed in the layer of fibres farthest from the neutral axis, and equal to the total bending stress existing in the section; this latter stress, of course, having a variable intensity. *HL* represents the layer of fibres farthest from the neutral surface, consequently *MN* was taken at the same distance from *AB*. Any other distance might have been taken, but the intensity of the uniform stress would then have had a value equal to that which exists at that distance from the neutral axis. Again, a different intensity might have been chosen for the stress on each side of AB. It is most convenient, however, to use the greatest intensity in the section for the stress on both sides of the neutral axis; this intensity, which is the modulus of rupture by bending, will be represented, as heretofore, by K.

Let c and c' be the centres of gravity of the two shaded figures. These centres can readily and accurately be found by cutting the figures out of stiff manilla paper and then balancing on a knife edge. Let s represent the area of the shaded surface below AB, and s' the area of that above AB.

Because this is a case of pure bending, the stresses of tension must be equal to those of compression. Hence:

$$Ks = Ks' \text{; or, } s = s' \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The moment of the compression stresses about AB will be:

$$Ks \times c'C.$$

The moment of the tensile stresses about the same line will be:

$$Ks \times cC.$$

Consequently the resisting moment of the whole section will be:

$$M = Ks(c'C + cC) = Ks \times cc' \quad . \quad . \quad . \quad . \quad (2)$$

Thus, the total resisting moment is completely determined. In some cases of irregular section the method becomes absolutely necessary.

It is to be observed that the centre of gravity, c or c', is at the same normal distance from AB as the centre of the actual stress on the same side of AB with c or c'.

Art. 27.—The Common Theory of Flexure with Unequal Values of Coefficients of Elasticity.

In all cases it has hitherto been assumed that the coefficient of elasticity for tension is equal to the same quantity for compression. In reality, this is exactly true for probably no material whatever, though the error, fortunately, is not serious for the greater portion of the material used by the engineer. By the aid of the assumptions used in the common theory of flexure, formulæ involving this difference of coefficients may be deduced. As these are of little real value, however, a few general results, only, will be obtained.

Let E represent the coefficient of elasticity for tension.

Let E' represent the coefficient of elasticity for compression.

As has before been assumed, the normal sections of the beam, which are plane before flexure, will be taken as plane and normal to the neutral surface after flexure. Also, as before (Art. 18), let u represent the rate of strain (strain for unit of length of fibre) at unit's distance from the neutral surface; let the variable width of the section be represented by b, while y represents the variable normal distance of the element bdy from the neutral axis of the section. The element of the tensile stress in the section will be :

$$Euy \,.\, b\,dy.$$

The elementary moment of the same will be :

$$Euy^2\, b\,dy.$$

In precisely the same manner, the elementary compressive moment will be :

$$E'uy^2b\ dy.$$

Consequently, the total resisting moment will have the value :

$$M = u\left[E\int_0^{y_1} y^2b\ dy + E'\int_{-y'}^{0} y^2b\ dy\right].$$

$$= \frac{K}{y_1}\int_0^{y_1} y^2b\ dy + \frac{K'}{y'}\int_{-y'}^{0} y^2b\ dy\ .\ \ .\ \ .\ \ .\ \ . \quad (1)$$

The ordinates y and y_1 are those belonging to the extreme fibres of the section, while K and K' represent stress intensities in those fibres. The general value of y is also affected with the negative sign on the compression side of the beam.

It has been shown in Art. 18 that :

$$u = \frac{1}{\rho}\ ;$$

also, in the case of straight beams, that :

$$\frac{1}{\rho} = \frac{d^2w}{dx^2}\ ;$$

w being the deflection and x the abscissa measured along the axis of the beam. For the sake of brevity, let the quantity in the brackets in the second member of Eq. (1) be represented by EJ, in which, consequently, E' will be displaced by nE, n being the ratio between E and E'. Eq. (1) may then take the form :

$$M = EJ\frac{d^2w}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or :

$$\frac{M}{EJ} = \frac{d^2w}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If M and J are expressed in terms of x, w may at once be found. If, as is usual, the section is uniform, then will J be constant and M, only, will be a function of x.

If the section is rectangular, b will be constant and J will take the following value :

$$J = \frac{by_1^3}{3} + n\frac{by'^3}{3}.$$

$$\therefore \quad J = \frac{b}{3}(y_1^3 + ny'^3) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Because the internal tensile stress in any section must equal the internal compressive stress in the same section :

$$Eu\int_0^{y_1} by\,dy = E'u\int_0^{y'} by\,dy \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Eq. (5) will enable the neutral axis of any section to be located. If the section is symmetrical, the neutral axis will evidently be situated on that side of the centre of gravity of the section on which is found the greatest coefficient of elasticity.

Art. 28.—Greatest Stresses at any Point in a Beam.

If the approximate conditions on which are based the formulæ found in the latter part of Art. 17 are assumed, some interesting and important results may be very easily obtained.

The Eqs. (13), (14) and (15) of Art. 6 are those which lead to the ellipsoid of stress, and hence to all of its special cases and consequences. The equation representing the ellipsoid of stress might first be found, and then the special form relating to the case considered. It will be more simple and direct, however, to use those equations immediately.

If, as in Art. 17, a rectangular beam carrying a load at its end be assumed, in which :

$$T_1 = T_3 = N_2 = N_3 = 0,$$

Eqs. (13), (14) and (15) of Art. 6 reduce to :

$$N_1 \cos p + T_2 \cos r = P \cos \pi ;$$

$$T_2 \cos p \qquad = P \cos \rho.$$

But since all stress is assumed to be found in planes parallel to ZX :

$$\cos r = \sin p, \quad \text{and} \quad \cos \rho = \sin \pi.$$

Hence :

$$N_1 \cos p + T_2 \sin p = P \cos \pi \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$T_2 \cos p \qquad = P \sin \pi \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which P is the intensity of the *resultant stress* on *any* plane at *any* point; p the angle which the normal to that plane makes with the axis of X (the axis of the beam); and π the angle which the direction of P makes with the same axis.

Let it first be required *to find the plane, at any point, on which the normal or direct stress is the greatest.*

It is known from the theory of internal stress that this greatest normal stress will be the resultant and, hence, a principal stress. Hence the relation : $\pi = p$; or:

$$N_1 + T_2 \tan p = P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$T_2 \qquad = P \tan p \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If F is the weight carried by the beam at its end; I the moment of inertia of the beam's cross section; and d its half depth, or greatest value of z, it has been shown in Arts. 17 and 18 that :

$$N_1 = \frac{Fx}{I} z, \quad \text{and} \quad T_2 = \frac{F}{2I}(d^2 - z^2) \quad . \quad . \quad . \quad (5)$$

Inserting the value of P from Eq. (4) in Eq. (3) :

$$T_2 - T_2 \tan^2 p = N_1 \tan p.$$

$$\therefore \quad \tan^2 p + \frac{N_1}{T_2} \tan p = 1.$$

Solving this quadratic equation and then inserting the values of T_2 and N_1 from Eq. (5) :

$$\tan p = -\frac{xz}{d^2 - z^2} \pm \frac{\sqrt{(d^2 - z^2)^2 + x^2 z^2}}{d^2 - z^2} \quad . \quad . \quad (6)$$

This value of $\tan p$ put in Eq. (3), or Eq. (4), will give the greatest value of the direct or normal stress (also resultant) at any point in the beam.

At the exterior surface, $d = z$; hence:

$$\tan p = 0 \quad \text{or} \quad -\infty.$$

Since for this point $T_2 = 0$, the first value gives, by Eq. (3), $P = N_1$. The second value, by Eq. (4), gives, $P = 0$. These results might have been anticipated.

At the neutral surface, $z = 0$; hence:

$$\tan p = \pm 1 = \tan \pm 45°.$$

Hence, *at the neutral surface there are two planes on which the stress is wholly normal, and these planes make angles of* 45° *with the neutral surface, or* 90° *with each other* (i.e., *they are principal planes*).

Since $N_1 = 0$ at the neutral surface, either of the Eqs. (3) or (4) gives:

$$P = \pm T_2 = \pm \frac{Fd^2}{2I} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence each of these normal or principal stresses equals in intensity that of the transverse or longitudinal shear at the neutral surface; also, one of these principal stresses is a tension and the other a compression.

$$P = T_2 \cot p = -\frac{F}{2I}\left\{\frac{(d^2 - z^2)^2}{xz \mp \sqrt{(d^2 - z^2)^2 + x^2z^2}}\right\}$$

is the equation of the locus of the point of constant greatest normal intensity of stress, if P be taken constant and equal to any possible value.

Let it next be required to find the plane of greatest shear at any point in the beam, and the value of that shear.

The shear on any plane will be :

$$P \sin(\pi - p) = T \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Multiplying Eq. (1) by $(-\sin p)$ and Eq. (2) by $\cos p$, then adding :

$$-N_1 \cos p \sin p + T_2 (\cos^2 p - \sin^2 p) = P(\sin \pi \cos p$$

$$- \cos \pi \sin p) = P \sin(\pi - p) = T.$$

$$\therefore \quad T = -\frac{N_1}{2} \sin 2p + T_2 \cos 2p \quad . \quad . \quad . \quad . \quad (9)$$

It is now required to find what value of p will make the general value of T [given by Eq. (9)] a maximum. Hence :

$$\frac{dT}{dp} = -N_1 \cos 2p - 2T_2 \sin 2p = 0.$$

$$\left. \begin{aligned} \therefore \quad \tan 2p &= -\frac{N_1}{2T_2} = -\frac{xz}{d^2 - z^2} \\ \therefore \quad \cos 2p &= \pm \frac{d^2 - z^2}{\sqrt{x^2z^2 + (d^2 - z^2)^2}} \end{aligned} \right\} \quad . \quad . \quad . \quad (10)$$

Eqs. (10) give the value of p which is to be placed in Eq. (9), in order to obtain the greatest value of T at any point of the beam.

From Eq. (9):

$$T = T_2 \cos 2p \left\{ - \frac{N_1}{2T_2} \tan 2p + 1 \right\}$$

$$\therefore \quad T = \pm \frac{F}{2I} \sqrt{x^2 z^2 + (d^2 - z^2)^2} \quad . \quad . \quad . \quad . \quad (11)$$

At the exterior surfaces of the beam:

$$z = \pm d.$$

Hence:

$$T = \pm \frac{Fxd}{2I} = \pm \frac{N_1}{2}.$$

For this case, also:

$$\cos 2p = 0, \quad \text{or} \quad p = 45°.$$

Hence, *at the exterior surfaces of the beam the planes of greatest shear make angles of* 45° *with the axis of the beam, and the intensity of the shear is half that of the direct stress at the same place.*

At the neutral surface: $z = 0$. Hence:

$$T = \pm \frac{Fd^2}{2I} = T_2; \text{ and } \cos 2p = \pm 1.$$

Hence, $2p = 0$ or $180°$; or $p = 0$ or $90°$; *i.e.*, the planes of greatest shear are the transverse and longitudinal planes, and the greatest shear itself is, consequently, the transverse or longitudinal shear.

If T is given any possible value and considered constant, Eq. (11) will give the locus of the point of constant greatest shear.

The result expressed in Eq. (7) is of great value in determining the thickness of the web of flanged beams, as will be seen hereafter.

PART II.—TECHNICAL.

CHAPTER V.

TENSION.

Art. 29.—General Observations.—Limit of Elasticity.

HITHERTO, certain conditions affecting the nature of elastic bodies and the mode of applying external forces to them, have been assumed as the basis of mathematical operations, and from these last have been deduced the formulæ to be *adapted* to the use of the engineer. These conditions, in nature, are never realized, but they are approached so closely, that, by the introduction of empirical quantities, the formulæ give results of sufficient accuracy for all engineering purposes ; at any rate, they are the only ones available in the study of the resistance of materials.

In determining the quantity called the "coefficient of elasticity," it is supposed that the body is perfectly elastic, *i.e.*, that it will return to its original form and volume when relieved of the action of external forces, also, that this "coefficient" is constant. There is reason to believe that no body known to the engineer is either perfectly elastic or, possesses a perfectly constant coefficient of elasticity. Yet, within certain not well defined limits the deviations from these assumptions are not sufficiently great to vitiate their great *practical* usefulness.

The " not well defined " limit for any one given material is called its " limit of elasticity," or " elastic limit." The " limit of elasticity," then, may be defined as *that degree of stress within which the coefficient of elasticity is essentially constant and equal to the stress divided by the strain.*

In some materials, like many grades of wrought iron and steel, the limit of elasticity approximates, to a greater or less degree, to the condition of a well defined point. If a piece of such a material is subjected to stress in a testing machine, at the elastic limit, the amount of strain caused by a given increment of stress will be observed, comparatively speaking, to rapidly increase. This increase may be uniform for a considerable range of stress, but it finally becomes irregular, after which failure takes place.

In other materials, there seems to be no simple relation between stress and strain for any condition of stress whatever. For such a material it obviously is impossible to assign either any definite elastic limit or coefficient of elasticity.

Between these limits, of course, all grades of material are found.

It should be stated that some authorities have given arbitrary definitions of the elastic limit, and that these definitions have been very much used. Wertheim and others have considered the elastic limit to be that force which produces a permanent elongation of 0.00005 of the length of a bar. Again, Styffe defines, as the limit of elasticity, a much more complicated quantity. He considers the external load to be gradually increased by increments, which may be constant, and that each load, thus attained, is allowed to act during a number of minutes given by taking 100 times the quotient of the increment divided by the load. Then the " limit of elasticity " is " that load by which, when it has been operating by successive small increments as above described, there is produced an increase in the permanent elongation which bears a ratio to the length of the bar equal to 0.01 (or approximates most nearly to 0.01) of the ratio which the increment of weight bears to the total load." (Iron and Steel, p. 30.)

The most natural value, however, seems to be that stress which exists at the point where the ratio between stress and strain ceases to be essentially constant, though the assignment of the precise point be difficult in many cases and impossible in some; and in that sense it is here used, though seldom in ordinary testing.

Again, in the common theory of flexure, modes of application of external forces and a constitution of material are assumed, which are never realized; yet the resulting formulæ are of inestimable value to the engineer.

Finally, it will be shown in the first section of Art. 32 that it is in general impossible to produce a uniform intensity of stress in a normal cross section of a body subjected to pure tension, and, consequently, that the ultimate resistance, as experimentally determined, is a mean intensity which may be, and usually is, considerably less than the maximum sustained by the test piece.

These general observations are to be carefully borne in mind in connection with all that follows.

Art. 30.—Ultimate Resistance.

After a piece of material, subjected to stress, has passed its elastic limit, the strains increase until failure takes place. If the piece is subjected to tensile stress, there will be some degree of strain, either at the instant of rupture or somewhat before, accompanied by an intensity of stress greater than that existing in the piece in any other condition. This greatest intensity of internal resistance is called the "Ultimate Resistance."

In very ductile materials this point of greatest resistance is found considerably before rupture; the strains beyond it increasing very rapidly while the resistance decreases until separation takes place.

The ultimate resistances of different materials used in engineering constructions can only be determined by actual tests, and have been the objects of many experiments.

It has been observed in these experiments that many influences affect the ultimate resistance of any given material, such as mode of manufacture, condition (annealed or unannealed, etc.), size of normal cross section, form of normal cross section, relative dimensions of test piece, shape of test piece, etc. In making new experiments or drawing deductions from those already made, these and similar circumstances should all be carefully considered.

Art. 31.—Ductility.—Permanent Set.

One of the most important and valuable characteristics of any solid material is its "ductility," or that property by which it is enabled to change its form, beyond the limit of elasticity, before failure takes place. It is measured by the permanent "set," or stretch, in the case of a tensile stress, which the test piece possesses after fracture; also, by the decrease of cross section which the piece suffers at the place of fracture.

In general terms, *i.e.*, for any degree of strain at which it occurs, "permanent set" is the strain which remains in the piece when the external forces cease their action. It will be seen hereafter that in many cases, and perhaps all, permanent set decreases during a period of time immediately subsequent to the removal of stress. Indeed, in some cases of small strains it is observed to disappear entirely.

Some experimenters, with the aid of very delicate measuring apparatus, have observed permanent set even within what is ordinarily termed the limit of elasticity, and have been led to believe that a very small permanent set exists with any degree of stress whatever. In such cases, however, it is probable that the greater part or all of the permanent set disappears after the lapse of a few hours.

Art. 32.—Wrought Iron.—Coefficient of Elasticity.

Before considering the experimental results which are to follow, it will be interesting as well as important to examine some of the circumstances which attend the experimental determination of the coefficient of elasticity.

If tensile stress is uniformly distributed over each end of a test piece, it will not be so distributed over any other normal section. For since lateral contraction takes place, the exterior molecules of the piece must move towards the centre. But if this motion takes place, the molecules in the vicinity of the centre must be drawn farther apart, or suffer greater strains, than those near the surface.

Hence the stress will no longer be uniformly distributed, but the greatest intensity will exist at the centre and the least at the surface of the piece. These effects will evidently increase, for a given kind of cross section, with its area. But the stretch, or strain, from which the coefficient of elasticity is computed, is measured on the surface of the piece, and corresponds, as has just been shown, to an intensity of stress less than the mean, while the latter is actually used in the computation. In the notation of Eq. (1), Art. 2, p is too great and l too small; hence E will be too large.

As these effects increase with the area of the cross section, while other things are the same, *larger bars should give greater coefficients of elasticity than smaller ones.*

These effects will evidently be intensified, also, if the external force is applied with its greatest intensity near, or at, the centre of the bar, as is the case in testing eye-bars.

Again, on the other hand, if the ends of the test piece are gripped on the surface, or skin, as is usually the case with small pieces, these effects will be very much modified, and possibly entirely counteracted, so that the greatest intensity will exist at the surface. In the latter case, the resulting coefficient would be too small.

Between these extreme cases, all grades will be found.

From these considerations, it is clear that the manner of gripping the test piece, length, character and area of cross section all affect the value of the coefficient of elasticity, and should be given in connection with the latter.

These conclusions apply to any other material, as well as to wrought iron.

Table I. gives the results of some experiments made by the Phœnix Iron Co., of Phœnixville, Penn., on some flats and rounds of the dimensions shown in the column headed " Size."

TABLE I.

NO. OF BARS	SIZE.	LENGTH.	STRETCH.	p.	E.
	Inches.	Ft. In.	Inches.	Pounds.	Pounds.
12	$4 \times 1\frac{3}{8}$	35 0	0.2692	20,000.00	31,203,000.00
9	$4 \times 1\frac{3}{16}$	27 6	0.2033	" "	32,464,700.00
24	$3\frac{1}{2} \times 1\frac{1}{4}$	35 0	0.2500	" "	33,600,000.00
24	$3\frac{1}{2} \times 1\frac{1}{8}$	35 0	0.2617	" "	32,098,000.00
23	$3 \times \frac{7}{8}$	35 0	0.2587	" "	32,470,000.00
24	$3 \times \frac{3}{4}$	35 0	0.2633	" "	31,902,000.00
24	2×1	24 $9\frac{1}{2}$	0.1948	" "	30,544,000.00
36	$2\frac{7}{8}$ O	11 9	0.0953	" "	29,380,000.00
68	$2\frac{3}{4}$ O	11 11	0.0998	" "	28,056,000.00
120	$2\frac{5}{8}$ O	11 9	0.0947	" "	29,567,000.00
48	$2\frac{1}{2}$ O	11 9	0.0955	" "	29,319,000.00
72	$2\frac{3}{8}$ O	11 9	0.0940	" "	29,787,000.00
48	$2\frac{1}{8}$ O	11 9	0.1008	" "	27,777,777.00

The column "p" is the intensity per square inch which caused the stretches shown in the column headed " Stretch."

From Eq. (1) of Art. 2 :

$$E = \frac{p}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In this case, for any individual bar :

$$l = \frac{Stretch}{Length};$$

remembering that the stretch and length must be reduced to the same unit.

Let the above formulæ be applied to the twenty-four bars $3 \times \frac{3}{4}$ inches $\times$ (35 ft. = 420 ins.) long.

$$E = \frac{20000 \times 420}{0.2633} = 31{,}902{,}000.00 \text{ pounds.}$$

The other values are found in precisely the same way. The quantities in the column E are the averages of the number of experiments given in the extreme left hand column. The fact that the results are the averages of a great number of experiments gives the table peculiar value. This table is taken from "Useful Information for Architects and Engineers," published by the Phœnix Iron Co. The following reference to the table is taken from the same source: "The annexed table gives the results attained in testing with the proof load of 20,000 pounds per square inch, a number of bars for the International Bridge over the Niagara River, near Buffalo, N. Y. The recovery of each bar, after the removal of the load, was perfect, no permanent set occurring at less than 25,000 pounds. It will be observed that the stretch per foot of the flat bars is less than that of the rounds, giving them higher moduli of elasticity." It is interesting and important to observe this last point.

It is to be observed, finally, that these coefficients of elasticity are determined for one intensity of stress, only, *i.e.*, 20,000.00 pounds per square inch. It is probable that values a little different might be given by other intensities.

Table II. contains coefficients of elasticity for tension in

TABLE II.

NUMBER.	SIZE OF BAR. INCHES.	GAUGED LENGTHS. INCHES.	STRETCH IN GAUGED LENGTH. INCHES.	E IN POUNDS PER SQ. IN.
S 1	3.03 × 1.01	80	.029	27,586,240
S 2	3.03 × 1.01	80	.0279	28,673,760
S 3	3.03 × 1.01	80	.0272	29,411,760
S 4	3.03 × 1.01	80	.0238	33,613,440
D 5	3.03 × 1.01	80	.0281	28,469,760
D 6	3.03 × 1.01	80	.0215	37,209,300
D 7	3.03 × 1.01	80	.0273	29,304,000
D 8	3.03 × 1.01	80	.0260	30,769,200
S 9	5.05 × 1.28	80	.0275	29,090,880
S 10	5.04 × 1.27	80	.029	27,586,240
S 11	5.03 × 1.27	80	.0278	28,776,960
D 12	5.02 × 1.26	80	.026	30,769,200
D 13	5.03 × 1.26	80	.0256	31,250,000
D 14	5.03 × 1.26	80	.0277	28,880,860
S 15	3.05 × 1.01	80	.028	28,571,430
S 16	3.05 × 1.01	80	.0284	28,168,960
S 17	3.05 × 1.00	80	.0279	28,673,760
D 18	3.05 × 1.00	80	.0285	28,070,160
D 19	3.05 × 1.00	80	.032	25,000,000
D 20	3.05 × 1.02	80	.0315	25,396,800
S 21	5.08 × 1.26	80	.025	32,000,000
S 22	5.08 × 1.26	80	.028	28,571,430
S 23	5.09 × 1.26	80	.0256	31,250,000
D 24	5.05 × 1.25	80	.026	30,769,200
D 25	5.06 × 1.26	80	.0272	29,411,760
D 26	5.06 × 1.25	80	.027	29,629,600

pounds per square inch, computed from data given in "Report of Tests of Metals" for 1881, made on the government testing machine at Watertown, Mass. These results, like those in the preceding table derive additional interest and importance from the fact that they belong to full size bars and such as are ordinarily used in engineering practice.

Those bars whose numbers are preceded by an " S " are of single rolled material, while those preceded by a " D " are double rolled. A portion of the bars given in Table II. are those on which the subsequent tests shown in Table XIII*a*. were made.

The values of "*E*," the coefficient of elasticity, were computed by Eq. (1), l representing the stretch in 80 inches divided by 80. The stretch in every case was measured at a stress of 10,000 pounds per square inch, which is the limit quite generally specified as a maximum in railway bridges of ordinary length. It is seen that the values thus determined are not on the whole very different from those shown in table I. for double this intensity of stress.

Bar No. 9 was slightly warped and No. 25 not originally straight, but the coefficients do not seem to be appreciably affected.

A comparison between the results for single and double rolled iron shows that there is no appreciable difference between them either in uniformity or magnitude. In the aggregate, the values run from 25,000,000 to 37,209,300 pounds; giving a variation of fifty per cent. of the lowest amount. This fact has a most important bearing on those theories of continuous girders which assume E to be constant.

Double rolling, which materially increases the cost of the metal, is thus seen to give it no elastic advantage.

Prof. Woodward, in " The Saint Louis Bridge," gives the results of 67 experiments on specimens varying from 6 to 18 inches long and from 0.45 inch to 1.13 inches in diameter, from 17 different producers. In these results the range of variation was very great; in fact the coefficient of tensile elasticity varied from 9,500,000 lbs. per sq. in. to 65,500,000, and some of the widest variations were in specimens of the same brand.

Table III. gives the results of the experiments of Mr. Eaton

TABLE III.

Tensile Experiments on two Annealed "Best" Wrought Iron Bars ten feet long and one inch square.

BAR NO. 1.

p.	*Ll.*	*Sets.*	*E.*
	Inches.	Inches.	
2,668	.00986	——	32,457,000
5,335	.02227	——	28,198,000
8,003	.03407	.000305	28,180,000
10,670	.04556	.000407	28,101,000
13,338	.05705	.000509	28,056,000
16,005	.06854	.000610	28,020,000
18,673	.07993	.000813	28,033,000
21,340	.09193	.001525	27,855,000
24,008	.10485	.003966	27,475,000
26,676	.12163	.009966	26,308,000
29,343	.15458	.031424	22,782,000
32,011	.26744	——	14,361,000
——	.28271 in 5 minutes	.13566	——
34,678	.5148	.36864	8,083,000
37,346	1.095	1.01695	4,077,000
Repeated	1.1949	1.02966	——
40,013	1.220 in 5 minutes	1.093	3,924,000
Repeated and left on	1.411 after 1 hours	——	——
"	1.424 after 2 hours	——	——
"	1.433 after 3 hours	——	——
"	1.434 after 4 hours	——	——
"	1.436 after 5 hours	——	——
"	1.437 after 6 hours	——	——
"	1.443 after 7 hours	——	——
"	1.443 after 8 hours	——	——
"	1.443 after 9 hours	——	——
"	1.443 after 10 hours	——	——
42,681	2.148 in 5 minutes	1.983	2,384,000
Repeated	2.339 in 5 minutes	——	——

BAR NO. 2.

p.	*Ll.*	*Sets.*	*E.*
	Inches.	Inches.	
1,262	.00520	——	29,125,000
2,524	.01150	——	26,347,000
3,786	.01690	.00050	26,851,000
5,047	.02240	.00060	26,990,000
6,309	.02772	.00050	27,312,000
7,571	.03298	.00045	27,551,000
8,833	.03790	.00050	27,953,000
10,095	.04300	.00050	28,198,000
11,357	.04854	——	28,077,000
12,619	.05370	.00070	28,199,000
13,880	.05950	——	27,984,000
15,142	.06480	——	28,041,000
16,404	.06980	——	28,186,000
17,666	.07530	.00130	28,153,000
18,928	.08170	——	27,794,000
20,190	.08740	.00270	27,734,000
21,452	.09310	——	27,644,000
22,713	.09920	.00410	27,474,000
23,795	.10570	——	27,213,000
25,237	.11250	.00680	26,919,000
26,499	.12040	——	26,420,000
27,761	.12880	.0120	25,872,000
29,023	.14500	——	23,986,000
30,285	.1991	——	18,244,000
30,285	.2007 after 5 min.	——	18,030,000
30,285	.2018 after 10 min.	0736	——
30,285	.2054 after 15 min.	.0774	——
Repeated	.2080 after 20 min.	.0796	——
"	.2096 after 1 hour	.0814	——
"	.2366 after 17 hours	.1082	——
31,546	.242 after 5 min.	.1083	15,617,000
Repeated	.2449 after 5 min.	.1111	——
32,808	.5506	.4141	7,132,000
Repeated	.7024 after 5 min.	.5635	——
"	.70966 after 10 min.	.6558	——

TABLE III.—*Continued.*

BAR NO. 1.				BAR NO. 2.			
p.	*Ll.*	*Sets.*	*E.*	*p.*	*Ll.*	*Sets.*	*E.*
	Inches.	Inches.			Inches.	Inches.	
Repeated	2.383 in 10 minutes	2.212	——	Repeated	1.014 after 15 min.	.866	——
"	2.428 after 46 hours	2.237	——	34,070	1.346 after 1 min.	——	2,839,000
45,348	2.580 after 5 min.	2.377	2,109,000	34,070	1.400 after 2 min.	——	——
Repeated	2.605 after 1 hour	——	——	34,070	1.600	1.44	——
"	2.606 after 2 hours	——	——	Repeated	1.65 after 1 min.	——	——
"	2.606 after 19 hours	2.403	——	"	1.786 after 1 hour	1.628	——
48,016	2.975 after 5 min.	2.733	1,936,000	35,332	2.04 after 5 min.	1.874	2,078,000
Repeated	3.019 after 1 hour	——	——	Repeated	2.18 after 5 min.	2.01	——
"	3.029 after 11 hours	——	——	"	2.254	2.08	——
50,684	4.195 in 10 minutes	3.941	1,448,000	36,594	2.54 after 6 min.	——	1,743,000
Repeated	4.226	——	——	37,856	2.894	——	1,571,000
"	4.227 in 7 hours	——	——				
"	4.227 in 12 hours	——	——				
53,351	Broke	——	——				

Hodgkinson on the tensile elasticity and permanent set of two wrought iron bars. The coefficients of elasticity E have been computed from the data contained in the first three columns as given by Mr. B. B. Stoney in his "Theory of Strains in Girders and similar Structures." The following is the notation used :

p = pounds per square inch ;
Ll = total elongation, or strain, for the bar ;
"*Sets*" = permanent set ;
E = coefficient of tensile elasticity $= p \times L \div Ll$.

These experiments show some very interesting results.

In the first place permanent sets were observed with the low intensities of stress of 8,003 and 3,786 pounds, and it becomes a question whether permanent sets would not have been observed with lower intensities and more delicate apparatus, at least for a short time after the material is subjected to stress.

In both bars the largest value of E is found for the smallest intensity of stress. In bar No. 1, the values of E decrease, with one exception, regularly from the greatest. In bar No. 2, however, greater irregularity is observed; there are two maxima, one for the intensity 1,262 pounds, and the other for about 12,000 with nearly regular gradations from these values.

Considering the whole range in both bars, E may be considered nearly constant until an intensity of about 24,000 pounds per square inch is reached in each case; it then begins to fall off very rapidly. 24,000 pounds per square inch, then, may be considered about the limit of elasticity for both bars.

It is very important to observe the increase of strain with the lapse of time after the limit of elasticity has been considerably passed.

Values of the coefficient of elasticity, therefore, mean little after that limit is exceeded.

The results of the experiments on bar No. 1 are shown graphically in Fig. 1. The values of "p" are laid off vertically through O to a scale of 20,000 pounds to the inch; the tensile strains are the horizontal co-ordinates of the curve laid down at full size. The essentially straight portion of the curve between O and a is within what is ordinarily known as the "elastic limit."

The equation for this portion of the line is:

$$p = El;$$

E being assumed constant if Oa is considered straight.

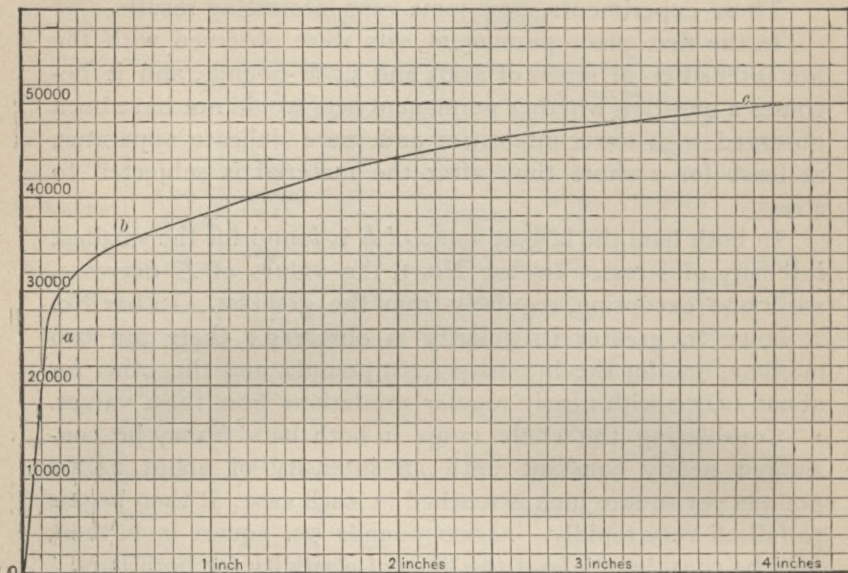


Fig. 1

The point a is at a vertical distance above O indicating about 24,000 pounds per square inch, *i.e.*, about the elastic limit. Above this point the curvature of the line is very sharp, indicating a rapid fall in the value of E and a rapid rise in the values of the strains l or Ll. For "p" = 27,000 (nearly) the table shows $E = 23,000,000$ (nearly) and $Ll = 0.12$ inch; while for "p" = 37,000, $E = 4,100,000$ and $Ll = 1.095$ inches (nearly). These phenomena are always characteristic of the limit of elasticity.

Above the point b the curvature is slight, indicating (what the table shows) a comparatively slow change in the values of E.

The table shows that bar No. 2 would exhibit a curve of precisely the same character but with a more rapid decrease to E above the elastic limit. The tests of this bar were not carried to failure on account of the breaking of one of the holding details.

Within the elastic limit, the mean values of E may be taken about as follows :*

For bar No. 1 :

$$E = 28,000,000 \text{ pounds.}$$

For bar No. 2 :

$$E = 27,500,000 \text{ pounds.}$$

The next, Table IV., contains values of the coefficient of tensile elasticity (E) determined by Knut Styffe (" The Elasticity, Extensibility and Tensile Strength of Iron and Steel," translated from the Swedish by Christer P. Sandberg).

TABLE IV.

KIND OF IRON.	PER CENT OF CARBON.	AREA OF SECTION.	SET.	E.
		Sq. inch.	Inch.	Pounds.
Hammered Bessemer Iron (square)	0.1	0.1003	0.002	32,320,020
" " " "	0.15	0.1107	0.001	34,241,380
Puddled, from Low Moor (round)	0.20	0.1961	0.006	31,976,920
" " Dudley "	0.09	0.1844	0.008	28,408,680
" " " "	0.09	0.2006	0.077	27,448,000
" " Motala, Sweden "	0.05	0.1942	0.008	30,261,420
" " " " (square)	0.20	0.1229	——	29,575,220
From Surahammar "	0.14	0.2176	0.018	31,084,860
" " "	0.20	0.1269	0.002	30,467,280
Swedish Rolled Iron from Åryd "	0.07 to 0.18	0.2087	0.037	26,761,800
" " " " " "	0.18	0.2279	0.003	27,791,000
" " " " Hallstahammer (square)	0.07	0.1891	0.013	28,957,640
" " " " " "	0.07	0.1965	0.001	30,810,380

The " Set " is the permanent elongation which " the bar had just before the modulus (E) was taken." In the per cent. of carbon no distinction is here made between " in the bar tested " and " in the bars of the same kind," the two quantities given by Styffe.

As a result of his experiments in regard to the effect of a change of temperature on the coefficient of tensile elasticity, he states (page 112 of the work above cited):

"That the modulus (coefficient) of elasticity in both iron and steel is increased on reduction of temperature and diminished on elevation of temperature; but that these variations never exceed .05 per cent. for a change of temperature of 1.8° Fahr., and therefore such variations, at least for ordinary purposes, are of no special importance."

In his "Physique Mécanique," page 58 of the "Premier Mémoire," M. G. Wertheim gives three coefficients of tensile elasticity for wrought iron, each having about the value of 29,680,000 pounds per square inch, and one for iron wire of about 26,474,000 pounds per square inch.

Redtenbacher (Resultate für den Maschinenbau, Zweite Auflage, page 36) gives as the limits of the values of the coefficient of elasticity, expressed in pounds per sq. in., about 21,330,000 and 35,550,000.

Reviewing the preceding values, therefore, it would appear that the coefficient of tensile elasticity for good wrought iron may be ordinarily taken to lie between 25,000,000 to 30,000,000 pounds per square inch, with extreme values arising from variation of mode of manufacture, chemical constitution, size of bar, etc., lying some distance either side of those limits.

Since $E = \frac{p}{l}$, if $p = 1$, $l = \frac{1}{E}$ will be the elongation or tensile strain for each unit of stress; hence, *the coefficient of elasticity is the reciprocal of the strain for a unit of stress.* For an intensity of stress of 20,000 pounds, for example, then:

$$l = \frac{20,000}{25,000,000} \text{ to } \frac{20,000}{30,000,000}$$

$$= \frac{1}{1250} \text{ to } \frac{1}{1500};$$

or a bar of wrought iron will be stretched :

$$\frac{1}{1250}\text{th} \quad \text{to} \quad \frac{1}{1500}\text{th}$$

of its length.

The coefficient of elasticity is thus seen to be a measure of the stiffness of the material.

Ultimate Resistance and Elastic Limit.

It has been found by experiment that bars of wrought iron which are apparently precisely alike, in every respect, except in area of normal section, *do not give the same ultimate tensile resistance per square inch.* Other things being the same, *bars of the smallest cross section give the greatest intensity of ultimate tensile resistance.*

Aside from the absence of uniform distribution of stress in the interior of the bar, as was shown in the section "*coefficient of elasticity,*" and the intensified effects of the processes of production on pieces with comparatively small cross sections, this result is to be expected from the circumstances which attend fracture. When a piece of material is subjected to tension to the point of rupture, not only a tensile strain of essentially uniform character, from end to end, takes place, but also a very considerable local transverse strain, or contraction, at the place of fracture. This latter manifests itself only shortly before rupture as a short " neck " in the piece. Now a given *percentage* of " local " contraction in the case of a large section involves a much larger absolute lateral movement of the molecules than in the case of a small section. But it is evident that this absolute lateral movement will exert a much more potent influence toward severing the molecules sufficiently for rupture, than the percentage of contraction. Hence the degree of local and lateral movement, required by rupture, will be reached with a less mean intensity of stress in the cases of large section than in those of small ones. But this is equivalent to a greater intensity of ultimate resistance for the small sections, and, as has been indicated, this conclusion is verified by experiment.

The same considerations result in the additional conclusion that, other things being equal, the smaller sections will give the *greater final contraction*. But a greater intensity of ultimate resistance with greater final contraction involves a *greater final stretch*, for the same length of piece.

These last two conclusions will also be found to be hereafter verified by experiment.

Again, it is found independently of the effects of the processes of production, as might be anticipated, that the length in terms of the lateral dimensions of the test piece, within certain limits, affects very perceptibly the ultimate resistance.

If a specimen of the shape shown in Fig. 2 be broken by a tensile stress, it will, of course, fail in the reduced section *MN*. But before failure takes place, the reduced portion will be considerably elongated and the normal section correspondingly reduced, in consequence of the shearing strains in the oblique planes shown by the dotted lines.

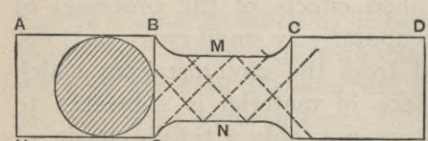


Fig. 2

(See Arts. 3 and 4.) When the reduced portion in the vicinity of *MN* is very short in comparison with its lateral dimensions, it includes the *whole* of very few of these oblique planes, if any at all, consequently very little movement of these oblique layers over each other can take place; in other words little or no reduction of section can take place before rupture. In this latter case, then, a greater area of metal section will offer its resistance to the external tensile force, at the instant of failure, than in the former, and a correspondingly greater intensity of ultimate resistance will be found. Thus the shape and dimensions of the test piece will considerably influence the ultimate resistance and strains, as will soon be shown by experimental results.

All the preceding conclusions, though given in connection with wrought iron, are independent of the nature of the material, and apply equally to steel and cast iron.

Since the reduction of area of the fractured section and the elongation of the bar are true measures of the ductility of the iron, these are or should be always measured with care.

Table V. exhibits in a very plain manner the decrease of ultimate tensile resistance with the increase of sectional area of round bars; it is taken from the "Report of the Committees of the U. S. Board appointed to test Iron, Steel and other Metals, etc.," by Commander L. A. Beardslee, U.S.N.

This decrease is probably partly due to the effect produced upon the iron by the rolls as it passes through them; the bars of smaller sections being more "drawn," and at a lower temperature in consequence of the lesser mass cooling more quickly.

The notation of the table is the following:

"*Dia.*" = diameter of the round bar in inches;
"*T.*" = ultimate tensile resistance;
"*E. L.*" = elastic-limit.

It will be observed that the ultimate resistance per square inch varies between widely separated limits, in some cases, for the same diameter of bar. This is due to the fact that the different bars, even of the same diameter, were from a number of different mills, and consequently involved different treatment in manufacture, chemical constitution, etc. A general view of the table, however, shows in a marked and satisfactory manner the decrease of T with the increase of the diameter or area of normal section. The last fourteen bars of the table are of the same manufacture, and show a decrease in T as nearly uniform as could be expected.

TABLE V.

Ultimate Resistance and Elastic Limit in Pounds per square inch of Original Normal Section.

DIA.	*T.*	*E. L.*	DIA.	*T.*	*E. L.*	DIA.	*T.*	*E. L.*
¼	59,085	——	1⅜	53,016	35,379	1 13/16	50,969	30,814
⅜	54,090	40,980	"	51,296	31,992	"	50,307	29,767
½	62,700	——	"	50,594	34,940	"	48,953	——
"	59,000	——	1½	57,052	38,417	1⅞	55,803	31,031
"	57,700	——	"	56,505	32,496	"	53,100	32,074
"	55,400	——	"	55,131	33,771	"	52,875	35,641
"	52,275	39,126	"	54,540	——	"	52,505	32,312
⅝	55,450	——	"	55,415	32,869	"	51,459	27,816
"	52,050	——	"	54,354	34,617	"	50,363	——
"	57,660	——	"	54,544	33,027	"	51,039	33,067
¾	51,546	35,933	"	53,512	——	"	49,744	35,615
⅞	50,630	33,931	"	52,819	34,840	"	48,670	23,250
1	61,727	——	"	52,736	34,901	2.0	60,213	31,441
"	57,363	37,415	"	52,700	35,880	"	52,914	31,198
"	57,807	39,230	"	52,155	27,708	"	49,164	——
"	56,790	36,885	"	51,994	32,054	"	51,684	33,104
"	51,921	31,300	"	51,456	34,591	"	52,127	32,461
"	52,819	32,267	"	51,047	——	"	52,011	34,702
"	51,400	34,600	1⅝	56,344	35,889	"	51,146	28,567
1⅛	60,458	37,344	"	57,402	35,701	"	50,000	36,184
"	57,470	31,900	"	56,227	33,207	"	50,171	28,983
"	57,498	41,311	"	54,334	32,163	"	47,812	35,864
"	55,927	37,250	"	53,339	33,540	"	48,249	31,413
"	54,644	34,695	"	53,614	30,664	"	46,151	36,050
"	53,900	26,787	"	52,675	33,745	2 1/16	51,559	——
"	53,035	34,410	"	52,314	29,364	"	49,422	——
"	52,267	32,019	"	52,401	34,012	2⅛	50,481	——
1¼	59,461	36,501	"	51,205	33,318	"	51,225	——
"	57,897	32,469	"	50,970	33,625	"	48,382	30,459
"	55,782	35,596	1 11/16	56,595	38,310	2 3/16	51,666	——
"	56,334	33,921	"	54,114	——	2¼	51,530	——
"	55,253	34,784	"	57,789	34,160	"	49,290	32,163
"	53,893	32,712	1¾	57,874	——	"	48,898	——
"	53,247	32,520	"	54,410	31,354	"	46,866	28,241
"	53,752	——	"	53,846	36,573	2½	48,475	28,932
"	52,970	32,075	"	55,018	34,283	"	47,428	29,941
"	53,022	——	"	53,264	——	"	47,344	29,758
"	50,040	30,730	"	53,154	35,323	2¾	46,446	26,333
1⅜	58,926	37,548	"	51,509	29,404	3.0	47,761	26,400
"	58,021	32,152	"	50,395	36,254	3¼	47,014	24,591
"	54,949	31,030	"	50,547	35,954	3½	47,000	24,961
"	54,277	33,622	"	49,816	31,214	3¾	46,667	23,636
"	52,733	34,606	"	50,129	32,271	4.0	46,322	23,430
"	52,557	33,650	1 13/16	56,577	——			
"	52,537	34,469						

In the words of the Report, as given by Wm. Kent, C.E., in the abridgment, "The elastic limit as given is not from perfectly accurate data; it is simply the amount of stress which produced the first perceptible change of form, divided by the bar's area."

TABLE V*a*.

Rectangular Bars.

NO.	KIND OF IRON.	SIZE OF BAR.	STRESS IN LBS. PER SQ. IN.		PER CENT. OF	
			Elastic Limit.	Ultimate.	Final elongat'n in 80 inches.	Final contraction.
		Inches.				
1	Single Refined	3 × 1	29,000	52,470	18.0	31.0
2	Double "	3 × 1	31,000	53,550	16.0	27.7
3	Single "	5 × 1¼	27,330	50,410	16.6	24.1
4	Double "	5 × 1¼	27,170	50,920	19.0	25.7
5	Single "	3 × 1	28,330	48,700	13.1	27.1
6	Double "	3 × 1	29,170	51,370	22.2	35.6
7	Single "	5 × 1¼	24,830	49,240	16.0	18.1
8	Double "	5 × 1¼	27,170	51,010	19.7	29.5

Table V*a*. shows the results of some tests in the U. S. Govt. machine during 1881, at Watertown, Mass. Nos. 1 and 2 are means of four tests; the others are means of three. Nos. 1, 2, 3 and 4 are for bars from the Elmira Iron and Steel Rolling Mill Co.; Nos. 5, 6, 7 and 8 are from the Passaic Rolling Mill Co. As a rule the large bars give the least elastic limit and ultimate resistance.

It is also important to observe that the double refined iron, with two exceptions, gives the highest results of all kinds.

It appears from an examination of the tables that the elastic limit varies, approximately, from a half to two-thirds the ultimate resistance.

The ultimate resistance, it is to be particularly observed, is given in pounds per square inch of *original* sectional area. On account of the reduction of the fractured section, the ultimate resistance should be specifically referred either to its own section (to be noticed hereafter) or to the original section. The customary reference is to the latter, though it is frequently interesting and important to make an accompanying reference to the former.

The influence of the reduction of the piles between the rolls was next examined by the same committee. It was found that the additional working involved in the increased reduction of the pile, as it passes through the successive rolls, in the process of manufacture, considerably increases both the ultimate resistance and elastic limit. Tables VI. and VII., condensed from those containing the results of the committee's experiments, show this effect in a very satisfactory manner. The notation is as follows :

D = diameter of bar in inches ;
A = area of normal section of original pile in square inches ;
Per cents. = area of bar in per cent. of area of pile ;
T = ultimate tensile resistance in pounds per square inch of entire bar ;
T' = ultimate tensile resistance in pounds per square inch of core of bar ;
$E. L.$ = elastic limit in pounds per square inch of entire bar ;
$E'. L'.$ = elastic limit in pounds per square inch of core of bar.

As is to be anticipated in such cases, some irregularities are exhibited in the tables, but they are very few, while the general result is unmistakable. On the whole, a considerable *increase* in the values of T is observed in connection with a decrease in the values of "*Per cents.*" Values of the elastic limit show greater irregularities.

TABLE VI.

Comparison of the Reductions by the Rolls, with the Effects upon Tenacity, and Elastic Limit of Round Bars.

D.	*A.*	PER CENTS.	*T.*	*T'.*	*E. L.*	*E'. L'.*
4	80	15.70	——	46,322	——	23,430
3½	80	12.03	——	47,000	——	24,961
3	80	8.83	——	47,761	——	26,400
2½	80	6.13	47,344	47,428	29,758	29,941
2	72	4.36	47,872	48,280	35,864	31,892
1¾	36	6.68	50,547	48,792	35,954	38,992
1½	36	4.90	50,820	51,838	35,087	36,467
1¼	36	3.41	52,729	49,801	39,608	40,534
1	25	3.14	51,921	51,128	39,066	38,596
¾	12¼	3.60	50,673	50,276	33,933	35,933
½	9	2.17	52,275	52,775	38,445	39,126
¼	3	1.60	57,000	59,585	Lost	Lost

TABLE VII.

Another Table showing Similar Results, with T' and E' L', for Core, omitted.

D.	*A.*	PER CENTS.	*T.*	*E. L.*
2	27	11.63	51,848	32,461
1½	15	11.78	53,550	34,690
1⅞	27	10.22	54,034	33,610
1⅜	15	9.90	54,277	33,622
1¾	27	8.90	55,018	34,283
1¼	15	8.18	56,478	33,251
1⅝	27	7.68	56,344	35,889
1⅛	15	6.62	56,143	32,267

The opinion of the committee on the effect of *underheating* or *overheating* is thus given in the abridgment of their report by Wm. Kent, M.E.: "The indications are that if a bar is *underheated* it will have an unduly high tenacity and elastic limit, and that if *overheated* the reverse will be the case."

In the words of the report: "The evidence submitted is of sufficient value to justify us in asserting that variations in the amount of reduction by the rolls of different bars from the same material produce fully as much difference in their physical characteristics as is produced by differences in their chemical constitution."

The committee also made some valuable experimental investigations with the object of ascertaining the influence of the

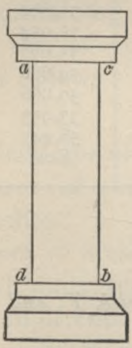


Fig. 3

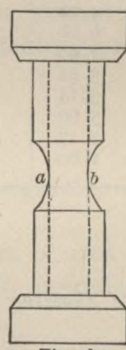


Fig. 4

relative dimensions of the test piece, already remarked upon in connection with Fig. 2. Eighteen specimens were prepared, of which Figs. 3 and 4 represent types.

Fig. 3 represents a specimen whose middle portion is turned down to a uniform diameter. Seventeen of the specimens were of this kind, with lengths of cylindrical portions varying from ½ inch to 10 inches. Fig. 4 represents the eighteenth specimen with simply a groove in the centre, in which, at *ab*, the fracture took place. In this latter specimen, the reduction of area at the section of failure must necessarily be much less than in those like Fig. 3; hence, the ultimate resistance will be correspondingly greater.

Table VIII. is taken from the report already cited, and contains the results of the experiments on the eighteen specimens prepared in the manner indicated above.

$L =$ original length in inches;
$l =$ *per cent.* of elongation;
$a =$ *per cent.* of contraction of fractured area;
$t =$ stress in pounds per square inch at first stretch;
$T =$ ultimate tensile resistance in pounds per square inch of original section.

TABLE VIII.

NO.	*L.*	*l.*	*a.*	*t.*	*T.*	REMARKS.
1	10	23.1	38.2	29,678	54,888	Slight seam.
2	9½	24.3	36.5	28,011	55,288	
3	9	21.5	31.1	29,345	55,355	
4	8½	22.0	31.2	29,345	55,622	
5	7½	25.0	39.9	30,840	54,890	Slight seam.
6	7	25.8	38.6	30,412	55,488	
7	6½	22.1	40.0	28,562	51,800	Bad seam.
8	6	22.3	34.7	30,600	55,418	
9	5½	25.4	39.3	29,475	55,333	
10	5	21.2	32.2	29,278	55,887	Slight seam.
11	4	25.7	37.4	29,705	55,532	
12	3½	26.7	36.6	31,817	55,482	
13	3	27.0	38.3	31,123	56,190	
14	2	27.0	36.2	33,428	56,428	Seamy.
15	1½	26.0	34.0	42,249	57,096	"
16	1	37.0	34.3	34,288	58,933	
17	½	30.0	37.9	57,565	59,388	Seamy.
18	Groove	——	20.6	45,442	71,300	

The diameters at the section of failure were nearly uniform and originally about 0.97 inch.

The values of l, a and T are as nearly uniform as could be expected until the length decreases to about 4 diameters (2 inches).

For the grooved specimen t and T are very large, and a very small.

Other experiments on a still softer iron were made with the same general results.

" In conclusion," states the committee, " our results lead us to the decision that, in testing iron, no test piece should be less than one half inch in diameter, as inaccuracy is more probable with a small than with a large piece, and the errors are more increased by reduction to the square inch; that the length should not be less than four times the diameter in any case; and that, with soft, ductile metal, five or six diameters would be preferable."

In Vol. II. of the "Transactions of the American Society of Civil Engineers," Mr. C. B. Richards has given a paper in which are recorded the results of some experiments exhibiting the influence of the relative dimensions of the specimens. The average of eight tests of Burden's "best" iron, with "long" specimens (similar to Fig. 3) varying from 5 to 5½ inches in length and 0.62 to 1.00 inch in diameter, gave:

$$T = 49{,}588 \text{ pounds}; \qquad a = 46.7 \text{ per cent.};$$

$$l = 30.4 \text{ per cent.}$$

With "short" specimens (like Fig. 4) of the same iron, the average of six tests gave:

$$T = 62{,}089 \text{ pounds}; \qquad a = 29.5 \text{ per cent.}$$

The large value of T and small value of a, for the "short" specimens, are thus seen to be very marked in contrast with the same quantities for the "long" specimens.

Other experiments of Mr. Richards, showing the same results, will be given in connection with the resistance of boiler plates.

It has long been the impression that there exists a considerable difference between the ultimate tensile resistance of the "skin" of a bar of iron and that of the portion of the bar underneath the skin. The U. S. committees, therefore, broke a number of bars first with the skin on, or "in the rough," and then with the skin turned off. In a large majority of the cases, the rough bars gave the highest ultimate resistance per square inch, by a small amount, while in a few cases the results were of the opposite character. On the whole, however, "the accumulated evidence indicates that the strength of the skin of the bar is greater in proportion to its area than that of the rest of the bar."

All the tests, of which the results have hitherto been given, were made on round bars, or on specimens turned from them. Results of tests on other iron will now be detailed, and it will be convenient to use the following and customary symbols for the various kinds of "shape" irons :

L, for angle irons :
⊥, for tee irons ;
⊏, for channel bars ;
I, for eye beams ;
□, for rectangular bars or "flats" ;
O, for rounds ;
+, for star sections ;

in short, any shape iron, or steel, is represented by a skeleton of its section.

Table IX. contains the results of tests of a wide range of full sized eye bars as ordinarily manufactured for bridge building purposes. Some were made and tested in 1887.

TABLE IX.

SECTION. Inches.	LENGTH. Inches.	*E. L.*	*T.*	CONT.	STRAIN.	REMARKS.
4 × 1	237	27,500	51,000	0.25	0.143	"Strain" for 17 ft. (*a*)
4 × 1	66	25,318	48,070	0.24	0.117	" " 4 ft. (*a*)
3 × $\frac{15}{16}$	230	32,590	51,870	0.365	0.216	" " 16 ft. (*b*)
$3\frac{1}{2}$ × $1\frac{1}{4}$	173	30,360	49,040	0.326	0.208	" " 13 ft. (*b*)
4 × $1\frac{1}{8}$	251	33,000	50,350	0.336	0.208	" " 18 ft. (*b*)
5 × $1\frac{3}{8}$	251	28,010	47,440	0.17	0.175	" " 18 ft. (*b*)
5 × $1\frac{1}{4}$	370	26,500	51,200	0.16	0.11	" " 29 ft. (*a*)
6 × $2\frac{1}{16}$	304	26,750	49,000	0.211	0.129	" " 23 ft. (*c*)
6 × $2\frac{1}{16}$	304	26,000	48,000	.....	0.124	" " 23 ft. (*d*)
8 × $1\frac{1}{2}$	300	24,200	41,280	0.097	0.061	" " 23.3 ft. (*e*)
8 × $1\frac{1}{2}$	300	24,030	41,860	0.099	0.056	" " 23.3 ft. (*f*)

E. L. = elastic limit in pounds per sq. in.

T. = ultimate resistance in pounds per sq. in. of original section.

Cont. = ultimate contraction or reduction of original area.

Strain = ultimate stretch in length given under "Remarks."

All bars except *e* and *f* were rolled and manufactured by the Phœnix Iron Co. The bars *a* were of single rolled iron, while all the others were of double rolled material. The bars *e* and *f* were made by the Central Bridge Co. of Buffalo, N.Y., and were tested at the U. S. Arsenal, Watertown, Mass., as sample bars selected from those used in the Niagara Cantilever bridge; and the results are taken from the paper on that structure by Mr. C. C. Schneider, Ch. Engr., presented to the Am. Soc. of C. E. March 4th, 1885.

Bars *a* were tested at the works of the Phœnix Iron Co.; bars *b* at the works of the Keystone Bridge Co., and bars *c* and *d* at the works of the Union Bridge Co., at Athens, Pa.

The bar *d* broke through the eye, for the reason that about $\frac{3}{8}$ inch in thickness of material was planed off the face of the head in order to get it into the testing machine.

Considered as a comparison between single and double rolled bars, the table possesses both interest and importance. The first three of the bars *b* show unusually high elastic limits, which may have been the result of the approximate methods used in the observation. On the whole, however, the elastic limits of the single rolled bars are not essentially different from those determined for the double rolled material.

With the exception of the last two ultimate resistances, which are very low, even for large bars, that column shows much more nearly uniform results for the two grades of bars than found among the elastic limits, and the values for the single rolled metal are fully equal to the best of the double rolled. As was to be expected, the smaller bars gave results appreciably in excess of those belonging to the larger ones.

The percentages of contraction for the single rolled bars are seen to be on the whole somewhat smaller than those belonging to the others, although the advantage is not maintained by the latter throughout the entire Table. The last preceding observation holds, but less markedly, in the column of ultimate stretch or "strains." While the decision of such questions should be made only on a far greater number of tests than given in the table, it is proper to say that the latter shows precisely what is found in extended experience, *i.e.*, that double rolled iron, as produced by the most reputable iron companies, possesses only a possible small and unimportant advantage in ductility and uniformity, but with less welding properties. Its cost is from twenty to twenty-five per cent. over that of single rolled iron; which is out of all proportion to the very small advantage gained.

Two kinds of tests are usually required by engineers in determining the suitability for a given purpose, of the finished member and the material of which it is fabricated. Those tests which determine the character of the finished tension member (*i.e.*, eye bar) have been exemplified by Table IX.; such tests fix the character of the finished member by showing the effect of the mode of manufacture. The other class of tests is made on specimens cut from the bar as it comes from the rolling mill and before it is manufactured into the bridge member; such tests simply fix the quality of the material.

Table X. shows the results of specimen tests from the full size bars given in the first column on the left. The material of these specimens was double rolled iron produced by The

TABLE X.

Double Rolled Iron.

ORIGINAL BAR. INCHES.	TEST SPECIMEN.		LBS. PER SQ. IN. ORIG. SECTION.		PER CENTS OF FINAL	
	Section. Inches.	Length. Inches.	Elastic Limit.	Ultimate Resistance.	Contraction.	Stretch in 8 Ins.
$5 \times 1\frac{11}{16}$	1×1.7	8	29,150	50,090	34.3	32.3
$5 \times 1\frac{5}{8}$	1×1.63	8	25,050	49,350	36.1	32.5
$5 \times 1\frac{5}{8}$	1×1.63	8	25,720	48,760	33.8	31.8
$5 \times 1\frac{7}{16}$	1×1.5	8	27,220	50,550	39.1	34.3
$5 \times 1\frac{1}{2}$	1×1.53	8	26,920	50,620	35.3	28.3
$5 \times 1\frac{3}{8}$	1×1.4	8	26,990	50,470	37.3	30.6
$5 \times 1\frac{3}{8}$	1×1.4	8	27,030	51,200	37.8	30.0
$5 \times 1\frac{9}{16}$	1×1.6	8	26,100	50,420	36.5	30.3
$5 \times 1\frac{1}{8}$	1×1.14	8	26,850	50,000	36.2	29.3
$5 \times 1\frac{1}{8}$	1×1.14	8	26,860	50,220	34.8	31.3
$3\frac{1}{2} \times 1\frac{3}{16}$	1×1.2	8	28,140	50,630	37.7	30.6
$3\frac{1}{2} \times 1\frac{3}{16}$	1×1.2	8	27,180	49,620	26.8	23.8

Phœnix Iron Company, and the tests were made at the works of that company in the latter part of 1886. The sizes of the specimens and the lengths for which the stretches were measured are seen at a glance at the Table. In preparing such test pieces care is always taken to have opposite sides of the test piece in exactly the same condition. It will be observed, for instance, that one dimension of each of the pieces is just equal to the original thickness of the bar from which the specimen was cut. Hence two opposite sides of each specimen had the rough surfaces of the original bar, while the other two sides were the machine-finished surfaces along which the cutting was done. The two machine-finished surfaces, moreover, should be smoothly cut. If all these precautions are not taken, the specimens may resist unequally on opposite sides, and fail in detail by pulling apart gradually from one surface; or by rough cutting, the material in the vicinity of the machine surfaces may be so injured as to lose a large portion of its resistance.

The variety in the sizes of the bars shown in Table X. is not nearly so great as that given in Table IX., but for the same sizes the elastic limits as a whole run a little lower for the specimens than for the full size bars. While the ultimate resistances in the two tables are not very different, the general

TABLE XI.

Double Rolled Iron.

DIAMETER ORIGINAL BAR. INCHES.	TEST SPECIMEN.		LBS. PER SQ. IN. ORIG. SECTION.		PER CENT. OF FINAL	
	Diameter. Inches.	Length. Inches.	Elastic Limit.	Ultimate Resistance.	Contraction.	Stretch in 8 Inches.
$1\frac{7}{8}$	1.25	8	26,820	50,480	45.7	31.5
$1\frac{7}{8}$	1.25	8	26,740	50,580	46.1	35.3
$1\frac{5}{8}$	1.65	8	30,300	51,340	45.5	33.3
$1\frac{1}{2}$	1.5	8	28,858	51,040	32.3	21.0
$1\frac{1}{2}$	1.51	8	30,119	52,429	39.5	30.2
$1\frac{3}{8}$	1.41	8	29,740	50,546	43.8	31.7
$1\frac{3}{8}$	1.41	8	26,800	48,527	32.8	25.5
$1\frac{1}{4}$	1.26	8	28,760	50,130	32.7	26.7
$1\frac{1}{4}$	1.26	8	26,390	51,340	44.3	30.0
$1\frac{1}{8}$	1.13	8	31,880	51,400	43.4	28.7
$1\frac{1}{8}$	1.13	8	27,200	49,900	48.7	31.2

result is a little higher for the specimens than for the bars, and more nearly uniform. The contractions and stretches for the specimens in Table X. are far larger as well as more nearly uniform in character than those given for the full size bars in Table IX.

No class of materials used by engineers possesses more widely varying characteristics of a physical nature than plates used in bridge construction. The very wide plates forming the webs of large plate girders give a high elastic limit, comparatively low ultimate resistance, final stretch and contraction ; these are always sheared plates. Narrow plates, either rolled in grooves or universal mill, approximate more nearly in character to bars in all respects of elastic and ultimate resistances and final stretch and elongation.

TABLE XII.

Bridge Plate Specimens.

ORIGINAL PLATE. INCHES.	TEST SPECIMEN.		LBS. PER SQ. IN. ORIG. SECTION.		PER CENT. OF FINAL	
	Section. Inches.	Length. Inches.	Elastic Limit.	Ultimate Resistance.	Contraction.	Stretch in 8 in.
8 × 4	1 × 0.30	8	31,900	47,700	22.4	19.8
8 × $\frac{1}{4}$	1 × 0.3	8	36,000	52,800	23.4	17.5
10 × $\frac{1}{4}$	1 × 0.26	8	35,110	52,670	20.6	16.3
10 × $\frac{1}{4}$	1 × 0.26	8	34,600	51,540	21.9	16.8
10 × $\frac{5}{16}$	1 × 0 32	8	37,000	53,700	27.2	19.1
10 × $\frac{7}{16}$	1 × 0.45	8	32,810	51,670	30.8	26.3
10 × $\frac{7}{16}$	1 × 0 45	8	32,960	52,250	28.3	21.8
10 × $\frac{1}{2}$	1 × 0.50	8	27,160	49,500	25.96	21.0
10 × $\frac{9}{16}$	1 × 0.57	8	27,400	49,220	29 67	23.0
12 × $\frac{3}{8}$	1 × 0.40	8	30,230	50,380	20.91	15.0
12 × $\frac{5}{8}$	1 × 0.63	8	34,700	51,200	22.5	18 2
14 × $\frac{3}{8}$	1 × 0.39	8	33,850	50,390	28.9	19.8
14 × $\frac{3}{8}$	1 × 0.39	8	34,870	54,100	44.1	23.0
20 × $\frac{7}{16}$	1 × 0.45	8	36,860	53,930	18.9	13.3
20 × $\frac{7}{16}$	1 × 0.45	8	36,580	53,720	25.6	20.8
24 × $\frac{5}{16}$	1 × 0.31	8	30,290	46,900	18.6	11.0
24 × $\frac{5}{16}$	1 × 0.31	8	28,100	46,920	18.1	15.0
27 × $\frac{3}{8}$	1 × 0.39	8	39,470	52,370	28.2	18.0
30 × $\frac{5}{16}$	1 × 0.32	8	33,330	49,210	15.2	12.5
30 × $\frac{5}{16}$	1 × 0.31	8	34,290	49,680	15.1	11.5
69 × $\frac{3}{8}$	1 × 0.40	8	35,800	49,870	25.6	11.9
72 × $\frac{3}{8}$	1 × 0.39	8	44,380	53,060	14.5	7.0
72 × $\frac{3}{8}$	1 × 0.35	8	44,030	55,960	23.0	9.8

Table XII. gives results for bridge plates throughout a great range of width. All the specimens were of the same thickness as the original plates; hence two sides were as they came from the rolls and two were machine finished. Although these re-

TABLE XIII.

Angle Iron Specimens.

SIZE OF ORIGINAL ANGLE. INCHES.	TEST SPECIMEN.		LBS. PER SQ. IN. ORIG. SEC.		PER CENT. OF FINAL	
	Section. Inches.	Length. Inches.	Elastic Limit.	Ultimate Resistance.	Contraction.	Stretch in 8 Inches.
Pounds.						
6 × 4 —71	1 × 0.7	8	26,800	49,420	25.6	24.5
6 × 4 —46	1 × 0.46	8	28,570	49,460	24.6	20.0
6 × 4 —46	1 × 0.46	8	28,850	49,340	27.3	21.3
6 × 3½—36	1 × 0.38	8	29,890	47,090	19.5	11.8
6 × 3½—36	1 × 0.38	8	30,520	48,420	18.2	13.0
5 × 3 —25	1 × 0.34	8	29,200	46,900	22.7	14.4
5 × 3 —25	1 × 0.34	8	29,370	47,770	20.7	15.6
4 × 3 —34	1 × 0.49	8	28,250	48,860	26.8	22.8
4 × 3 —34	1 × 0.47	8	29,780	49,360	25.1	17.8
4 × 3 —34	1 × 0.49	8	34,440	52,900	18.5	18.8
4 × 3 —34	1 × 0.49	8	31,680	50,620	31.9	21.3
3½ × 3 —23	1 × 0.38	8	26,960	45,790	26.8	13.75
3½ × 3 —23	1 × 0.38	8	28,600	49,080	28.1	23.00
3 × 3 —17	1 × 0.30	8	30,660	51,000	34.3	22.3
3 × 3 —21	1 × 0.38	8	30,260	51,050	30.8	23.1
3 × 3 —21	1 × 0.37	8	29,030	48,920	29.0	26.0
3 × 3 —28	1 × 0.46	8	31,080	51,300	35.4	27.3
3 × 3 —18	1 × 0.34	8	32,220	50,300	30.1	20.8
3 × 3 —18	1 × 0.33	8	33,230	52,430	36.3	22.8
3 × 2½—17	1 × 0.33	8	33,030	51,350	27.6	23.0
3 × 2½—17	1 × 0.33	8	32,320	51,360	34.7	23.3
3 × 2½—18	1 × 0.35	8	29,760	50,280	32.1	21.8
3 × 2½—18	1 × 0.35	8	30,430	50,140	29.5	19.3

sults are somewhat irregular, the lowest ultimate resistances and relatively highest elastic limits are found with the greatest widths, with the exception of the 72 inch plates. It is altogether probable that both the high elastic limit and ultimate resistance for those were produced by the addition of steel to the piles from which the plates were rolled. This practice has of late obtained some footing in order to meet the extreme requirements of some very exacting specifications based upon insufficient knowledge regarding the actual capacities of plate iron in great widths.

Table XIII. gives the results of tests of specimens cut from all sizes of angles used in ordinary bridge work. These angles were all produced by the Phœnix Iron Company, and the tests were made at the works of that company in 1887. The results are most excellent, as well as being typical for the shapes tested. Although the results at one or two points are a little irregular, on the whole the elastic and ultimate resistances, as well as the final contraction and elongation, increase with considerable uniformity from the heaviest sections to the lightest, showing clearly the improved qualities in the smaller angle bars. The Table demonstrates in a very marked manner the varying characteristics which always accompany varying dimensions of bars of the same kind, even when produced of absolutely uniform material in the original piles.

Table XIV. exhibits some very interesting results obtained by testing full size bars to destruction, then allowing the portions to rest during the periods given, and finally retesting those portions. Some of the latter were heated previous to testing and allowed to cool in the air.

Those bars whose numbers are preceded by D were of double rolled iron and the others of single rolled material.

The tests were made in the government machine at Watertown, Mass., and are reported at page 205 of " Ex. Doc. No. 1, 47th Congress, 2d Session."

This subject of the effect of repeated stress separated by intervals of rest will again receive attention in a later section of this article ; it is sufficient here to observe the influence on the elastic limit and ultimate resistance of these full size bars. The intervals of rest after the first test varied from four to ten months, and in all these instances the elastic limit was raised to about nine-tenths the ultimate resistance found subsequently by the same test, and in every instance it had a value higher than the ultimate resistance found in the original test. The original ultimate resistance is, also, but about five-sixths of that found after a period of rest. It is a singular fact that the general effect of the different periods of rest is to *reduce* the final contraction about one-third of its original value, but to *increase* the final stretch from about fifteen to about forty per

TABLE XIV.

NO.	CHARACTER OF TEST.	SIZE OF BAR WHEN TESTED. INCHES.	GAUGED L'GTH. INCHES.	POUNDS PER SQ. IN. ORIGINAL SECTION.		PER CENT. OF FINAL	
				Elastic Limit.	Ultimate Resist.	Cont.	Stretch in L'gth.
D 1	Original	3.05 × 1.0	80	29,500	51,150	31.5	21.00
	Rested 4 months				60,000		24.00
D 2	Original	3.05 × 1.0	80	28,500	51,110	36.1	21.00
	Rested 4⅚ months			53,440	61,960	33.8	25.00
S 3	Original	3.05 × 1.0	80	28,000	50,390	35.1	14.00
	Rested 5 months				60,390	21.8	20.00
S 4	Original	5.05 × 1.28	80	27,500	50,500	24.2	13 00
	Rested 8 months		50	56,500	58,980	17.0	18.00
	Heated cherry red and cooled in air	4.75 × 1.17		21,670	{ 36,220 / 41,640 }		33.00
S 5	Original	3.03 × 1.01	80	29,500	53,630	36 0	16.0
	Rested 10 months	2.84 × 0.94	50	53,920	63,130	21.9	23 0
	Heated dull red and cooled in air	2.80 × 0.92	20	22,800	41,240	34.0	32.0
D 6	Original	3.03 × 1.01	80	29,000	53,560	37.9	15.0
	Rested 10 months	2.87 × 0.95	50	53,920	63,070	19.6	19.0
	Heated to 370° Fahr. and cooled in air	2.83 × 0.92	20	57,520	57,520	34.6	19.0
	Rested 15 days after 3d test				62,420		
D 7	Original	3.03 × 1.01	80	32,500	53,500	27.5	16.0
7	Rested 9 months	2.84 × 0.94	50	55,880	63,360	19.9	21.0
8	Rested 9 months	2.84 × 0.92			66,010	19.9	21.0

cent. of the original. The qualitative effects are the same for both single and double rolled iron, but the amounts of change in the final contractions and elongations are markedly greater, as a rule, for the double rolled material. It must not be supposed that these increased resistances indicate improved material, for other investigations show that it has been fatigued, and has less capacity to resist a repetition of large stresses than before.

In all the preceding tables the length for which the "*Strain*" is given should be carefully borne in mind. A considerable "local" strain takes place at the section of fracture, which causes the per cent. of elongation, or strain, to be much greater for a very short length than for a longer one.

Wrought-Iron Boiler Plate.

A committee of the Franklin Institute made a very extensive series of tests of boiler plate, and reported the results of their investigations to that body in 1837. The Report of that committee can be found complete in the "Franklin Institute Journal" for that year. The sectional area of the test specimens varied from 0.10 to 0.20 square inch. This fact, coupled with the hardness of much of the iron which they tested, rendered many of their ultimate resistances very high and extremely irregular.

These considerations deprive the results which they give in the many "Tables" of their Report of the greater part of their value for present practical purposes. They are therefore not given.

This committee made numerous experiments to determine the resistance of boiler plate in different directions in reference to the fibre of the iron. The results were by no means of a uniform character. In one set of forty strips cut in each direction (along the fibre and across it), the length strips showed an excess of resistance varying from one per cent. to twenty. This comparison was made principally on the minimum resistance of each bar, but the committee state that the result would not have been much different if the mean had been taken.

On reviewing all their experiments, the committee concluded *that lengthwise of the fibre, the boiler iron which they tested was about six per cent. stronger than across the fibre.*

TABLE XIV*a*.

NO.	SPEC.	SECTION.	*T'*.	T_1.	*T*.	CONT.	STRAIN.	BRAND.
		Inches.						
6	LL	1.25 × 0.29	47,785	46,484	47,017	0.27	0.19	B. S.
3	LC	0.75 × 0.29	49,113	46,815	47,884	0.14	0.12	B. S.
6	SL	1.25 × 0.29	52,993	50,770	51,943	0.15	——	B. S.
3	SC	0.75 × 0.29	53,161	51,597	52,262	0.10	——	B. S.
14	LL	1.25 × 0.30	51,378	44,036	48,098	0.25	0.16	B. S.
12	LC	1.25 × 0.30	49,023	39,898	46,277	0.11	0.11	B. S.
4	LL	1.25 × 0.30	48,819	46,086	47,725	0.15	0.12	B. S.
4	LC	1.25 × 0.30	45,240	42,961	44,301	0.09	0.07	B. S.
4	LL	1.25 × 0.30	71,139	70,100	70,672	0.52	0.20	B. S. H
2	SL	0.87 × 0.27	47,245	46,410	46,827	——	——	T.
2	SC	0.87 × 0.27	44,355	43,165	43,760	——	——	T.
3	SL	0.87 × 0.16	54,699	44,581	49,227	——	——	Penn.
3	SC	0.87 × 0.16	54,031	43,436	47,641	——	——	Penn.
3	SL	0.87 × 0.28	56,429	53,870	52,986	——	——	Penn.
2	SC	0.87 × 0.28	55,218	53,395	54,306	——	——	Penn.
1	SL	0.87 × 0.28	——	——	54,466	——	——	Penn.
2	SC	0.87 × 0.28	54,819	51,184	53,001	——	——	Penn.
10	SL	0.87 × 0.28	58,450	48,650	53,129	——	——	B. S.
4	SC	0.87 × 0.28	53,145	50,449	51,928	——	——	B. S.
4	SL	0.87 × 0.28	57,934	54,377	57,529	——	——	B. S.
2	SC	0.87 × 0.28	53,998	53,395	53,696	——	——	B. S.
2	SL	0.87 × 0.28	53,791	52,546	53,168	——	——	S. F.
2	SC	0.87 × 0.28	54,394	50,272	52,333	——	——	S. F.
1	SL	1.25 × 0.33	——	——	60,911	——	——	S. F.

They also determined that the weakest direction of all was *diagonally* across the fibres, but their experiments did not enable them to determine quantitative results.

Table XIV*a*. is taken from the "Transactions of the American Society of Civil Engineers," Vol. II. It contains the results of some experiments on several different kinds of plate iron by C. B. Richards, M.E., and among other things it reveals the difference between "long" and "short" specimens.

> Column "*No.*" shows the number of tests, of which T is the average ultimate tensile resistance in pounds per square inch, T' the highest and T_1 the lowest, all being referred to the original section.
>
> Column "*Cont.*" shows per cent. (of original section) of contraction at section of failure.
>
> Column "*Strain*" shows per cent. (of original length) of elongation.
>
> Column "*Spec.*" shows kind of specimen, *i.e.*, "long" or "short," also direction of stress in reference to fibre; "*LL*" signifies "long and along fibre;" "*LC*" "long and across fibre;" while "*SL*" and "*SC*" signify "short and along" or "across fibre," respectively.
>
> In column "*Brand*," "*B. S.*" signifies "Bay State;" "*B. S. H.*" "Bay State Homogeneous Metal;" "*T.*" "Thorneycroft," English; "*Penn.*" "Pennsylvania;" "*S. F.*" "Sligo Fire Box.'

Different brands of the same make, though given by Mr. Richards, have been neglected.

The lengths for which the "*Strains*" existed are not given, although they should be. The long specimens were three or four inches between the shoulders.

In his "Treatise on the Resistance of Materials," Prof. De

Volson Wood gives the following results of some boiler-plate tests at the shops of the Camden and Amboy R. R. by Mr. F. B. Stevens.

"Av. breaking weight in	*pounds*	*per*	*square*	*inch*			....54,123.00
Highest "	"	"	"	"	"	"	....57,012.00
Lowest "	"	"	"	"	"	"	....51,813.00"

The experiments of Sir Wm. Fairbairn on English boiler plate (" Useful Information for Engineers, First Series," p. 259) along and across the fibres, gave irregular results, but other English experiments of Easton and Anderson would seem to make the resistance across the fibres from 5 to 15 per cent. less than that along the fibres.

Effect of Annealing.

The Franklin Institute Committee determined the effect of annealing, at different temperatures, on about 56 specimens of boiler plate and wire iron. Table XV. is condensed from that giving their results on boiler plate.

The mean value of T for five specimens of iron wire 0.19 inch in diameter, before annealing, was :

$$T = 73,880.$$

After annealing by heating to redness and cooling in dry ashes, the mean of five specimens was :

$$T' = 58,101.$$

After annealing at red heat and quenching in water, the mean of another five specimens was :

$$T' = 53,578.$$

TABLE XV.

NO.	T.	ANNEALING TEMP. FAHR.	T'.	DECREASE BY ANNEALING.
1	57,133	1,037°	56,678	.025
3	53,774	1,111°	52,186	.029
6	53,185	1,159°	46,212	.131
9	52,040	1,237°	44,165	.151
12	48,407	Bright welding heat.	39,333	.187
15	48,407	" " "	38,676	.201
18	76,986	" " "	50,074	.349

T is the ultimate tensile resistance in pounds per square inch at ordinary temperatures, before annealing.
T' is the same after annealing and cooling.

The means of sets of five, three and four specimens of wire 0.156 inch in diameter, exactly similarly treated, were, respectively :

$$T = 89,162. \qquad T' = 48,144.$$

$$T_1' = 50,889.$$

The process of annealing is thus seen to decrease the ultimate tensile resistance a very considerable amount. In many cases, however, this may make the iron very much more valuable, since annealing renders it much more ductile. If a structure or machine is subject to shocks or sudden applications of loading, a very stiff, hard iron, originally utterly unfit for the purpose, after being annealed might be used in its construction with safety.

Effect of Hardening on the Tensile Resistance of Iron and Steel.

It has been seen that annealing reduces the ultimate resistance of wrought iron. Experiments have shown that hardening, on the other hand, increases the resistance of both iron and steel, provided the hardening is done in a proper manner. If the hardening is accomplished by heating and sudden cooling in water, without subsequent tempering, the resistance of hard steel is very much diminished. This is probably due to the internal stresses induced by the sudden cooling.

Knut Styffe (" Iron and Steel ") concluded from his experiments that " by heating and sudden cooling (hardening), the limit of elasticity is raised while the extensibility is diminished, not only in steel but also in iron." This results of the experiments by David Kirkaldy will be given hereafter.

Variation of Tensile Resistance with Increase of Temperature.

Table XVI. has again been condensed from a similar one given in the Report of the Franklin Institute Committee.

The third column gives the temperatures at which the ultimate tensile resistances in the fourth column were observed.

The committee observed that the resistance of many irons *increased* with the temperature, to nearly the boiling point of mercury in some cases, while others remained unchanged until a temperature of 572° was reached. Above this point, however, as a rule, they found the decrease of resistance, below the greatest, to vary about as the 2.6 power of (*Temp.* – 80°).

TABLE XVI.

	ULTIMATE TENSILE RESISTANCE AT ORDINARY TEMPERATURE.	TEMP. FAHR.	ULTIMATE TENSILE RESISTANCE AT OBSERVED TEMPERATURE.
1	56,736	212	67,939
5	62,646	394	67,765
9	49,782	440	59,085
13	52,542	552	55,939
17	53,385	562	59,623
21	66,724	572	66,620
25	76,071	574	65,387
29	59,234	576	66,065
33	45,757	578	53,465
37	59,530	630	60,010
41	52,542	732	53,378
45	59,219	819	55,892
49	59,219	1,022	37,410
53	54,768	1,142	18,672
56	53,426	1,187	21,910
59	54,758	1,317	18,913

In the London "Engineering" of 30th July, 1880, is given a synopsis of some German experiments by Herr Kollmann, which is reproduced in Table XVII. The resistance of the materials at 0° *Cent.*, or 32° *Fahr.*, is taken as 100, and that at other temperatures as the proper proportional part of that number.

It will be noticed that these German experiments show a much earlier decrease of resistance than those of the Franklin Institute.

The results of some tests of a grade of charcoal boiler plate at three different temperatures are given in "*Effect of low and high temperatures on steel.*"

Some French experiments by M. Baudrimont are given in the "Journal of the Franklin Institute" for 1850, by which he found that at the temperatures 32°, 212° and 392° *Fahr.*, iron

TABLE XVII.

TEMPERATURE.		FIBROUS IRON.	FINE GRAINED IRON.	BESSEMER STEEL.
Cent.	Fahr.			
0°	32°	100	100	100
100	212	100	100	100
200	392	95	100	100
300	572	90	97	94
500	932	38	44	34
700	1,292	16	23	18
900	1,652	6	12	9
1,000	1,832	4	7	7

wire gave the following tensile resistances, in pounds per square inch, respectively :

291,510.00 ; 271,602.00 ; 298,620.00 ;

These resistances are most extraordinarily high, but, so far as the influence of variation of temperature is concerned, show nothing discordant with the preceding results.

The same experimenter found the tensile resistances of gold, platinum, copper, silver and palladium to decrease, in every instance, as the temperature increased from 32° to 392° *Fahr.*

In his " Useful Information for Engineers," Second Series, Sir Wm. Fairbairn gives the results of numerous experiments made on " short " specimens of plate and rivet iron at different temperatures.

TABLE XVIII.

TEMP., FAHR.	BREAKING WEIGHT IN POUNDS PER SQUARE INCH.	STRESS IN REFERENCE TO FIBRE.
0°	49,009	*With.*
60	40,357	*Across.*
50	43,406	*Across.*
60	50,219	*With.*
110	44,160	*Across.*
112	42,088	*With.*
120	40,625	*With.*
212	39,935	*With.*
212	45,680	*Across.*
212	49,500	*With.*
270	44,020	*With.*
340	49,968	*With.*
340	42,088	*Across.*
395	46,086	*With.*
Scarcely red.	38,032	*Across.*
Dull red.	30,513	*Across.*

In Table XVIII. will be found the results of his experiments on plate iron. On the whole, the table would seem to show a point of greatest resistance at about 270° to 300°, though so many irregularities exist that little or no law can be observed. In other words little or no decrease takes place at 395° or below. Much diminution, however, is seen at "scarcely red" and more at "dull red."

Table XIX. shows the results of Fairbairn's experiments on rivet iron at different temperatures. The irregularities are less than those seen in Table XVIII., and a maximum would seem to exist at about 325°.

The areas of the normal sections of the plate specimens varied from 0.6 to 0.8 square inch, while the sectional areas of the rivet-iron specimens were about 0.2 or 0.25 square inch.

Other results for wrought iron will be found in Table IX. of Art. 35.

TABLE XIX

TEMPERATURE, FAHR.	BREAKING WEIGHT IN POUNDS PER SQUARE INCH.	TEMPERATURE, FAHR.	BREAKING WEIGHT IN POUNDS PER SQUARE INCH.
− 30°	63,239	250	82,174
+ 60	61,971	270	83,098
66	63,661	310	80,570
114	70,845	325	87,522
212	82,676	415	81,830
212	74,153	435	86,056
212	80,985	*Red heat.*	36,076

All the preceding results, while irregular to some extent, show conclusively that no essential decrease in the tensile resistance of wrought iron takes place below about 500° *Fahr.*, while a possible increase at that temperature may exist over that at any below, but that at about 1,000° it may lose more than a half of its resistance. These conclusions are of the greatest importance in the construction of boilers.

Effect of Low Temperatures on Wrought Iron.

It is a matter of common observation that many articles, large and small, are much more easily broken in very cold weather than at higher temperatures. These breakages are undoubtedly frequently due to the circumstances in which the piece broken is found at the time of failure, either partly or wholly.

The frozen, and consequently less yielding, condition of the ground in the winter is unquestionably a very potent factor in failures or tires and axles of railway rolling stock, but it is at least an open question whether it is the sole cause.

A number of investigators have made numerous experiments with the object of determining the effect of low temperatures on the resistance of wrought iron in different forms.

From the results of these experiments, however, they have drawn the most discordant conclusions. In some cases this arises from the fact that the tests have not been made under the same circumstances, or *have not been of the same kind.*

Knut Styffe ("Iron and Steel") made the following "Résumé of Results of Experiments on Tension at different Temperatures:"

1. "That the absolute strength of iron and steel is not diminished by cold, but that even at the lowest temperature which ever occurs in Sweden it is at least as great as at the ordinary temperature (about 60° *Fahr.*). . . .
3. "That neither in steel nor in iron is the extensibility less in severe cold than at the ordinary temperature; . . .
4. "That the limit of elasticity in both steel and iron lies higher in severe cold; . . ."

He concluded from his experiments that the common impression of increased weakness and brittleness with a low degree of temperature is entirely erroneous. His tests, however, were wholly with tension gradually applied, and could support no conclusion in regard to other conditions.

The translator of Styffe's work, Christer P. Sandberg, made some experiments in order to determine the effect of *shocks* at different temperatures, *i.e.*, ordinary and low. These were also made in Sweden, and by dropping heavy weights, from different heights, on rails supported at each extremity. The records of these tests may be found in the translator's Appendix to Styffe's work.

The following are Sandberg's conclusions, and they will be observed to be directly opposed to those of Styffe:

1. "That for such iron as is usually employed for rails in the three principal rail-making countries (Wales, France and Belgium), the breaking strain, as tested by sudden blows or shocks, is considerably influenced by cold; such iron exhibiting at 10° *Fahr.* only one-third to one-fourth of the strength which it possess at 84° *Fahr.*
2. "That the ductility and flexibility of such iron is also much affected by cold; rails broken at 10° *Fahr.* showing on an average a permanent deflection of less than one inch, whilst the other halves of the same rails, broken at 84° *Fahr.*, showed a set of more than four inches before fracture.
3. "That at summer heat the strength of the Aberdare rails was 20 per cent. greater than that of the Creusot rails; but that in winter the latter were 30 per cent. stronger than the former."

All these experiments were made previous to 1869, and with iron rails.

Prof. Thurston, from his own experiments and those of others, concludes (Trans. Am. Soc. of Civ. Engrs., Vol. III., p. 30), "That with good materials, cold does not produce injury, but actually improves their power of resisting stress and increases their resilience.

"That the influence of impurities, of various methods of manufacture, of changes of density with temperature, and of the causes which produce a concentration of the action of rapidly produced distortion and of quick blows, are subjects which still require careful investigation."

He considers it probable that the cold-shortening effect of phosphorus is intensified at low temperatures.

After observing the failures on the railroads coming under their observation, the Railroad Commissioners of Massachusetts reported in 1874 that, in their opinion, neither iron nor steel attained any greater degree of brittleness, or became any more "unreliable for mechanical purposes" at low temperatures than at ordinary. They did not observe as a "rule that the most breakages" occurred "on the coldest days."

They further stated that "the introduction of steel in place of iron rails, has caused an almost complete cessation of the breakage of rails."

Thus it is seen that the subject is most thoroughly involved in confusion. It seems, however, to be established that the resistance of iron, at a low temperature, to a steady strain, is not diminished, while it may, perhaps, be increased.

Its resistance to shocks, at low temperatures, is probably very much affected by its quality, mode of manufacture or chemical composition, and these should always be taken into consideration when experiments are made.

The Report of the Mass. Railroad Commissioners would indicate that steel rails resist shocks at low temperatures better than iron ones.

Iron Wire.

Mr. John A. Roebling found by his tests that the English wire used in the Niagara Falls Suspension Bridge gave an ultimate tensile resistance of about 98,500.00 pounds per square inch ("Papers and Practice Illustrative of Public Works." John Weale, London, 1856). This wire was about 0.145 inch in diameter.

The Committee of the Franklin Institute made thirteen tests of some iron wire one-third of an inch in diameter, of which the highest, lowest and mean ultimate resistances in pounds per square inch of original section were as follows:

Highest........................ 88,354.00 pounds.
Mean........................... 84,186.00 pounds.
Lowest......................... 72,325.00 pounds.

The results of other tests by the same committee have already been given under "*Effect of annealing.*"

TABLE XX.

ORIGINAL DIAMETER IN INCHES.	ULTIMATE TENSILE RESISTANCE IN POUNDS PER SQUARE INCH OF		CONTRACTION OF ORIGINAL AREA OF SECTION.
	Original Area.	Fractured Area.	
0.122	94,871	179,032	0.47
0.123	87,395	162,500	0.462
0.124	89,256	145,946	0.388
0.125	88,618	137,974	0.358
0.122	92,308	168,750	0.453
0.124	91,735	156,338	0.413
0.124	90,082	170,313	0.471
0.122	92,308	168,750	0.453
0.124	91,735	173,437	0.471
0.124	86,776	164,063	0.471
0.125	87,805	156,522	0.439
0.124	86,776	152,174	0.43

Table XX. is a condensed form of one given in the "Transactions of the Am. Soc. Civ. Engrs.," Vol. III., p. 212, and contains an account of the tests made by Prof. R. H. Thurston on some wires that had been in use 32 years in the cables of the Fairmount Suspension Bridge at Philadelphia. It is both interesting and important to observe that the long service cannot have appreciably injured either the ducility or ultimate resistance of the wire.

Table XXI. contains the records of tests on other wire, at the same time (1875), by Prof. Thurston. The small reduction of diameter at fracture shows the iron to have been not very ductile. It will also be noticed that the smaller diameters give much the highest resistances.

TABLE XXI.

ORIGINAL DIAMETER.	DIAMETER AFTER FRACTURE.	ULTIMATE RESISTANCE IN POUNDS PER SQUARE INCH OF ORIGINAL AREA.
0.134	0.133	92,890
0.1205	0.1185	84,442
0.08	0.0795	94,299
0.071	0.068	90,384
0.0535	0.0532	105,871
0.029	0.029	113,546

According to Weisbach (" Mechanics of Engineering, etc.," Vol. I., 4th Edit.), Lagerhjelm and Brix found the mean value of the ultimate resistance, for a large number of tests of wrought-iron wire with diameters varying from 0.0833 to 0.125 inch, to be 98,000.00 pounds per square inch for unannealed wire, and 64,500.00 pounds for annealed.

Morin, in his " Mécanique Practique," gives the following for unannealed iron wire, after changing his results to pounds per square inch :

Mean for diameters of 0.039 to 0.118 inch....	85,000.00	(nearly).
Highest for diameters of 0.02 to 0.039 inch...	114,000.00	(nearly).
Lowest for large diameter....................	71,000.00	(nearly).
For a special grade ("l'Aigle")..............	128,000.00	(nearly).

Sir Wm. Fairbairn (" Useful information for Engineers, 3d Series," p. 282) gives the following as the results of experiments on various kinds of English iron wire. These experiments resulted from investigations relating to the fabrication of a submarine Atlantic cable.

KIND OF WIRE.	DIAMETER.	ULT. RESIST.	STRETCH.
	Inch.	Pounds.	Inch.
Hæmatite	0.087	109,300	0.280
Homogeneous	0.095	134,000	0.366
Special Homogeneous	0.097	115,000	0.267
Charcoal	0.093	110,400	0.173
Galvanized	0.098	86,200	0.198
Homogeneous	0.089	104,500	0.190
Homogeneous	0.091	192,200	0.712
Charcoal	0.091	92,200	0.198
Homogeneous	0.088	106,900	0.218
Charcoal	0.093	80,960	0.320
Hæmatite, S. 3	0.089	88,400	0.171
Hæmatite, S. 4	0.095	105,800	0.366
Homogeneous	0.180	45,200	0.480
Homogeneous	0.148	61,050	0.550
Homogeneous	0.095	134,000	0.346
Homogeneous	0.095	77,600	0.116
Special Charcoal	0.095	105,800	0.170

The ultimate resistance is in pounds per square inch, and the stretch is the total amount for 50 inches of length.

Reviewing the values given, it appears :

1. *That wire is the strongest form in which iron can be used to resist tensile stress ;*
2. *That, as a rule, the ultimate tensile resistance increases as the diameter of the wire decreases.*

Tensile Resistance of Shape Iron.

The phenomena exhibited in the fracture of shape iron depend, to a great extent, on the character of the piles from which it is rolled. The webs of [s and Is are sometimes rolled from old rails in connection with double refined iron in the flanges. In such cases, specimens cut from the web will frequently, if not usually, show a high intensity of ultimate resistance, but very little ductility, while those cut from the flanges will give good records of both kinds.

In general, shapes will offer less tensile resistance than either bars or rods, yet small specimens cut from good shape iron will give values ranging from 52,000 to 58,000 pounds per square inch, with ductility little less than that of ■s and ●s.

English Wrought Iron.

A great number of experiments on English wrought iron have been made by Sir Wm. Fairbairn, David Kirkaldy, and others. A record of Fairbairn's experiments may be found in his "Useful Information for Engineers," while an account of those of the latter is given in "Experiments on Wrought Iron and Steel," by David Kirkaldy, Glasgow, 1863.

B. B. Stoney, in his "Theory of Strains in Girders and Similar Structures," summarizes Kirkaldy's results, in pounds per square inch, as follows:

Mean of 188 rolled bars	57,555.00
Mean of 72 angle irons and straps	54,729.00
Mean of 167 plates, lengthwise	50,737.00
Mean of 160 plates, crosswise	46,171.00

It should be stated that these means include some Russian and Swedish irons, also that the bars were small ones.

These results do not differ much from quantities for corresponding grades of American iron.

Fracture of Wrought Iron.

The characteristic fracture of wrought iron broken in tension, either directly or transversely, is rather coarsely fibrous, not unfrequently exhibiting a few bright granular spots which, in rare cases, may possibly be crystalline. This characteristic (fibrous) fracture is always produced by the *steady* application of an external force, under the influence of which the piece is drawn out in jagged points at the place of failure.

The best of fibrous wrought iron, however, will exhibit a granular fracture if broken suddenly. In making tests, therefore, it is of the greatest importance to observe and direct the mode of application of the external forces producing fracture.

When some grades of iron in bars are broken transversely by shocks (such as are produced by falling weights), a phenomenon known as " barking " is produced. A skin of metal from a sixteenth to an eighth of an inch in thickness, on the tension side of the bent piece, tears apart and separates from the core of the bar. At the place of fracture and on each side of it, this skin or " bark " remains essentially straight. This kind of fracture shows remarkably well the fibrous character of wrought iron ; it is simply the separation of the fibres near the outside of the bar from those within.

Crystallization of Wrought Iron.

The subject of crystallization of wrought iron is one about which there is much dispute. In " Strength of Wrought Iron and Chain Cables," by Beardslee, as abridged by Kent, p. 36, the following is given as the opinion or view of the United States Testing Commission : " The question as to whether crystallization can be produced in iron by stress, or by repetition of stress with alternations of rest, or by vibration, has been much discussed, and very opposite views are entertained by experts.

" We have met with but one unmistakable instance of crystallization which was probably produced by alternations of severe stress, sudden strains, recoils and rest."

The committee then state the case of a connecting-rod, carefully made of the best quality of wrought-iron scrap, which had been used in a testing machine for forty years, in the Navy Yard at Washington. It was five inches in diameter, but one day, while in use it suddenly broke under a stress (total) of less than 200,000 pounds. "The surface of the fractured ends showed well-defined crystallization, the facets being large and bright as mica."

The data at hand, at present, are not sufficient for a decision of the question, but it may be confidently stated that in many cases granulation has been mistaken for crystallization.

Elevation of Ultimate Resistance and Elastic Limit.

It was first observed by Prof. R. H. Thurston and Commander L. A. Beardslee, U. S. N., independently, in this country, that if wrought iron be subjected to a stress beyond its elastic limit, but not beyond its ultimate resistance, and then allowed to "rest" for a definite interval of time, *a considerable increase of elastic limit and ultimate resistance may be experienced.* In other words, the application of stress and subsequent "rest" increases the resistance of wrought iron.

This "rest" may be an entire release from stress or a simple holding the test piece at a given intensity.

Prof. Thurston's investigations were on torsion, while those of the United States Commission were on tension, and will be given here.

The Commission prepared twelve specimens and subjected them to an intensity of stress equal to the ultimate resistance of the material, without breaking the specimens. These were then allowed to rest. entirely free from stress, from twenty-four to thirty hours, after which period they were again stressed until broken.

The gain in ultimate resistance by the rest was found to vary from 4.4 to 17 per cent.

These tests, remark the committee, seem to indicate that the tough fibrous irons gained the most, while those which broke with a steel-like fracture gained the least.

Before the rest, the stress which produced the first permanent elongation was about 65 per cent. of the ultimate resistance, but after the rest the two were nearly identical.

The committee then took forty-two other specimens and subjected them to precisely the same operations, except that the rest periods varied from one minute to six months.

The gains were as follows:

In less than 1 hour	1.1 per cent., mean of 5 tests.
In less than 8 and over 1 hour	3.8 per cent., mean of 8 tests.
In 3 days	16.2 per cent., mean of 10 tests.
In 8 days	17.8 per cent., mean of 2 tests.
Between 8 and 43 days	15.3 per cent., mean of 5 tests.
In 6 months	17.9 per cent., mean of 12 tests.

After seven other experiments involving a rest of 24 hours, with an average gain of 15.4 per cent., the committee concluded "that at the end of one day the result is, with very ductile irons, practically accomplished."

The manifestation of this phenomenon in different grades of iron was then investigated.

"Thirteen pieces were prepared, five of which were of soft charcoal bloom boiler iron, five of coarse contract chain iron, and three of a fine-grained bar of . . . very pure iron with high tenacity."

After testing these specimens subsequent to an eighteen hours' rest, the committee state (Kent's abridgment):

"These experiments confirmed the opinion already formed, and indicate that a bridge, cable, or other structure, composed of iron of either of the latter two varieties, will receive comparatively slight benefit from the operation of this law; while ductile fibrous metal . . . gains . . . to a great extent by the effect of strains already withstood." The gain in these specimens varied from about 3 per cent. (for the coarse iron) to about 18 per cent. (for the soft iron).

Again, two sets of specimens were prepared: one from the two portions of fractured bars after having been pulled asunder, the other from the bars in their normal condition. After a rest of several days the first set showed a gain over the second in ultimate resistance, varying from about 8 to 39 per cent., the higher values belonging to the more ductile irons.

Bauschinger's Experiments on the Change of Elastic Limit and Coefficient of Elasticity.

In "Der Civilingenieur," Heft 5, for 1881, are contained the results of the experiments of Prof. Bauschinger, of Münich. The observations in these experiments were made by the aid of a piece of apparatus which gave the elongations (all experiments were tensile) in ten-millionths of a metre, or approximately in $\frac{1}{250000}$ of an inch. An extraordinarily high degree of accuracy was therefore attained.

Prof. Bauschinger's elastic limit was strictly a proportionality limit between stresses and strains. He also observed what may be called the "stretch-limit" (Ger., Streckgrenze), at which point the stretching or elongation suddenly increases and continues to increase for more than a minute after the application of the stress. In ordinary experimenting this point has probably frequently been considered the elastic limit.

The test pieces were subjected to loads which gradually increased from zero by an increment a little less than 3,000 pounds per square inch, each load having been allowed to act one minute before adding the succeeding increment. At intervals of the loading separated by about 11,500 or 12,000 pounds per square inch, each piece was entirely unloaded and allowed to remain so for 15 or 20 minutes. After the "stretch-limit" was found the piece was subjected to a final load somewhat greater than the "stretch-limit," and then entirely unloaded.

In some cases the piece was immediately put through the same process of testing either once or a number of times, and the results of such tests will be found in the columns of the following tables, indicated by the contraction "*Im'y.*"

In the remaining cases intervals of time, shown at the tops of the columns, were allowed to elapse between any one test and the succeeding one.

The tables, Nos. 1 to 7 inclusive, give the results of the experiments on seven specimens of a grade of iron called "Schweisseisen" (weld iron). These specimens were a very little less than 1 inch (25 millimetres) in diameter. Nos. 1 and 2 were about 32 inches long, and the others about 16 inches long.

Tables No. 8 to 13, inclusive, give the results obtained with Krupps "Flusseisen." These specimens were about one inch in diameter and sixteen inches long.

The tables have been condensed from those given by Bauschinger and reduced to English measures.

The following is the notation :

E. L. = elastic limit in pounds per square inch.
S.-L. = stretch limit in pounds per square inch.
F. L. = final load in pounds per square inch.
E. = coefficient of elasticity in pounds per sq. in.

Weld Iron.

NO. 1.	IN ORIGINAL CONDITION.	IM'Y.	IM'Y.	IM'Y.
E. L.	20,110	14,370	14,900	15,500
S.-L.	27,300	31,600	41,700	49,500
F. L.	31,600	40,200	47,700	——
E.	39,293,000	27,928,000	27,672,000	27,544,000

Weld Iron.

NO. 2.	IN ORIGINAL CONDITION.	AFTER 19 HRS.	AFTER 27 HRS.	AFTER 24 HRS.
E. L.	20,110	28,970	35,500	39,100
S.-L.	28,700	34,750	44,300	45,100
F. L.	31,600	40,540	47,300	——
E.	29,037,000	28,923,000	28,198,000	28,241,000

Weld Iron.

NO. 3.	IN ORIGINAL CONDITION.	AFTER 51 HRS.	AFTER 41 HRS.	AFTER 45 HRS.
E. L.	23,164	28,440	39,080	45,580
S.-L.	29,000	34,750	45,090	——
F. L.	31,900	40,540	48,100	——
E.	29,208,000	28,397,000	28,483,000	28,170,000

Weld Iron.

NO. 4.	IN ORIGINAL CONDITION.	AFTER 80 HRS.	AFTER 68 HRS.	AFTER 64 HRS.
E. L.	22,890	31,900	35,340	43,800
S.-L.	30,050	34,750	44,170	——
F. L.	31,470	40,540	47,110	——
E.	29,293,000	28,810,000	28,227,000	28,696,000

Weld Iron.

NO. 5.	IN ORIGINAL CONDITION.	IM'Y.	AFTER 63 HRS.	IM'Y.
E. L	21,070	14,720	42,090	15,260
S.-L.	30,670	35,320	48,110	51,860
F. L.	34,611	42,700	51,110	——
E.	29,293,000	28,312,000	28,056,000	26,705,000

Weld Iron.

NO. 6.	IN ORIGINAL CONDITION.	AFTER 48.5 HRS.	AFTER 44.5 HRS.	AFTER 49 HRS.
E. L.	27,730	26,720	33,350	24,940
S.-L.	32,120	37,110	45,480	——
F. L.	35,040	43,040	51,550	——
E.	29,720,000	28,639,000	28,483,000	28,881,000

Weld Iron.

NO. 7.	IN ORIGINAL CONDITION.	AFTER 47 HRS.	AFTER 50.5 HRS.	AFTER 42.5 HRS.
E. L.	20,110	26,720	27,170	18,540
S.-L.	30,160	38,590	45,290	——
F. L.	34,470	43,040	51,320	——
E.	28,668,000	28,611,000	28,568,000	29,592,000

Melted Wrought Iron.

NO. 8.	IN ORIGINAL CONDITION.	IM'Y.	IM'Y.	IM'Y.
E. L.	35,340	——	8,990	9,230
S.-L.	36,750	46,910	53,890	58,490
F. L.	45,230	52,770	59,880	——
E.	31,256,000	——	31,483,000	30,488,000

Melted Wrought Iron.

NO. 9.	IN ORIGINAL CONDITION.	IM'Y.	IM'Y.	IM'Y.
E. L.	37,870	5,770	14,720	15,160
S.-L.	42,080	46,140	53,000	60,630
F. L.	44,880	51,920	58,880	——
E.	32,379,000	31,796,000	29,947,000	28,213,000

Melted Wrought Iron.

NO. 10.	IN ORIGINAL CONDITION.	AFTER 3 HRS.	AFTER 15 HRS.	AFTER 7 HRS.
E. L.	33,790	11,490	17,730	15,260
S.-L.	36,600	43,090	53,210	61,020
F. L.	42,230	51,704	59,130	——
E.	31,881,000	31,953,000	31,895,000	32,393,000

Melted Wrought Iron.

NO. 11.	IN ORIGINAL CONDITION.	AFTER 2.5 HRS.	AFTER 15.5 HRS.	AFTER 5.5 HRS.
E. L.	33,930	11,590	11,774	12,070
S.-L.	39,570	43,442	53,000	60,380
F. L.	42,404	52,130	58,880	——
E.	32,536,000	32,237,000	32,222,000	31,085,000

Melted Wrought Iron.

NO. 12.	IN ORIGINAL CONDITION.	AFTER 51 HRS.	AFTER 47 HRS.	AFTER 46 HRS.
E. L.	36,900	43,800	39,820	40,950
S.-L.	39,731	49,640	55,940	60,630
F. L.	42,630	52,560	58,880	——
E.	32,479,000	31,754,000	31,696,000	31,568,000

Melted Wrought Iron.

NO. 13.	IN ORIGINAL CONDITION.	AFTER 43.5 HRS.	AFTER 54 HRS.	AFTER 44.5 HRS.	AFTER 45.5 HRS.	AFTER 10 DAYS.
E. L.	——	35,340	38,930	41,740	42,720	61,560
S. L.	——	36,745	47,600	56,640	——	——
F. L.	——	42,400	51,920	59,630	——	——
E.	31,853,000	32,165,000	31,298,000	31,454,000	31,440,000	32,364,000

During the progress of the various tests, the bars Nos. 6, 7, 9, 11 and 12 were subjected to shocks in addition to the static tests. These shocks were produced by striking the test piece on its end by a hammer. It does not appear that these blows of the hammer perceptibly influenced the results.

The ultimate resistance of the weld iron was found to vary from 55,300 to 58,870 pounds per square inch. That of the melted wrought iron was about 65,000 pounds per square inch.

Although there are some irregularities, the following general conclusions may be drawn from the tables:

By "immediate" testing the elastic limit of weld iron is very much decreased.

With a rest (entirely free from load) between the tests, the elastic limit of weld iron is very much increased.

The greatest proportional gain, except in the case of previous immediate testing, seems to be acquired after a rest no greater than twenty hours.

Bar No. 6 is seen to give anomalous results.

In all cases of the weld iron the stretch-limit is considerably raised by repeated testing.

In no case is the coefficient of elasticity, after once testing, equal to its original value; as a rule, a steady decrease is seen to take place by repeated testing, but there are some exceptions.

The elastic limit of "Flusseisen," after repeated testing, is found to be much less than its original value until the length of rest becomes about fifty hours.

The stretch-limit of the same metal is invariably raised by repeated testing, either with or without "rests."

In nearly all the cases of Nos. 8 to 13, the coefficient of elasticity is found to be slightly decreased by repeated testing.

For a very clear and detailed account of these experiments reference must be made to the "Civilingenieur.'

Resistance of Bar Iron to Suddenly Applied Stress.

If tensile stress is suddenly applied to a bar of wrought iron, both its ultimate resistance and elongation will be very materially decreased.

As a mean of a number of tests, Mr. David Kirkaldy (" Experiments on Wrought Iron and Steel ") found with suddenly applied stress an ultimate resistance of 46,500 pounds per square inch, while with stress gradually applied it rose to 57,200 pounds.

In the former case the elongation was about 20 per cent., and as high as 24.6 per cent. in the latter.

It is thus seen that the mode of application of external force not only affects the character of the fracture of the iron, but also its ultimate resistance and elongation.

It will hereafter be seen that similar observations apply to other metals than wrought iron.

Reduction of Resistance Between the Ultimate and Breaking Point.

It has already been observed that the ultimate tensile resistance of wrought iron is the *greatest* tensile resistance which it offers to being pulled asunder, and that a test specimen finally parts at much less than the ultimate resistance. This is due to the ductility of the iron, which allows it to " pull out " or stretch, thus decreasing the cross section as well as the actual resisting capacity of the metal.

The ultimate resistance, therefore, is not exerted on the final section of fracture, but on a section somewhat greater; referring it (the ultimate resistance) to the section of fracture, then, may mean little or nothing.

The United States Commission made six tests, for the purpose of determining this reduction, on some specimens which had previously been stressed with a subsequent rest. The highest, lowest, and mean losses were as follows:

Highest	14.5 per cent.
Mean	13.8 per cent.
Lowest	12.9 per cent.

It was observed from a number of specimens, by the same commission, that the reduction of area at the instant of ultimate resistance (or greatest resistance) was about one-half, and the elongation or strain a little over three-quarters, of the corresponding quantities at the instant of fracture, supposing failure to be produced by a steady strain.

Some further observations seemed to show that if failure were produced by shock, the final contraction would be nearly the same as at the instant of greatest resistance in the case of a steady failure.

Effects of Chemical Constitution.

While it is well known that the resistance of wrought iron to tension varies greatly with the chemical composition, it is yet uncertain just what influence most of the foreign elements, found in iron exert, either individually or collectively. This will be apparent on examining Table XXII., taken from the report of the three committees of the United States Commission, to which allusion has here been so frequently made before.

The first part of the table represents the relative values of sixteen different irons in reference to their physical characteristics, one being the highest. The second part shows the amount of the various elements named in the left-hand lower column, found in the corresponding irons, *i. e.*, each vertical column belongs to one iron.

An inspection of the table will make very evident the diffi-

TABLE XXII.

NUMBER—																
In tenacity	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
In reduction of area	15	12	9	11	10	6	14	8	3	5	7	13	2	16	4	1
In elongation	16	13	9	12	8	2	10	7	1	11	5	14	4	15	6	3
In welding value	—	13	6	8	12	10	9	11	4	—	7	1	2	—	5	3
In power of resisting shock.	14	12	6	7	11	9	13	3	7	6	2	9	1	14	5	4
PERCENTAGES OF PHOSPHORUS, SILICON, ETC.																
Phosphorus	0.072	0.152	0.203	0.169	0.225	0.250	0.191	0.187	0.095	0.154	0.193	0.231	0.178	0.202	0.201	0.072
Silicon	0.095	0.149	0.147	0.154	0.184	0.182	0.169	0.163	0.028	0.160	0.163	0.156	0.139	0.271	0.160	0.072
Carbon	0.350	0.068	0.029	0.042	0.044	0.033	0.055	0.032	0.066	0.033	0.032	0.015	0.021	0.036	0.026	0.043
Copper	0.009	0.058	0.011	0.046	0.353	0.081	0.032	0.010	0.008	0.014	0.006	0.038	0.172	0.009	0.002	0.046
Manganese	0.014	0.022	0.030	0.021	0.020	0.033	0.038	0.031	0.009	0.084	0.039	0.017	0.031	0.023	0.048	0.006
Cobalt	0.008	0.019	0.026	0.029	0.070	0.037	0.029	0.026	0.020	0.023	0.042	0.047	0.068	0.006	0.018	0.032
Nickel	0.014	0.040	0.027	0.031	0.132	0.057	0.023	0.013	0.023	0.028	0.042	0.037	0.078	0.010	0.028	0.031
Slag	0.331	0.455	0.722	.....	1.044	0.848	1.760	.....	1.214	0.546	.....	.....	1.210	1.156	0.650	1.071

culty of drawing definite conclusions in regard to any one element.

For a detailed discussion of these results reference must be made to the report.

Kirkaldy's Conclusions.

The following conclusions were deduced by Mr. Kirkaldy from the results of his experiments. As will be seen, they belong to both wrought iron and steel in tension, and are taken from his "Experiments on Wrought Iron and Steel," 1861 :

1. The breaking strain does *not* indicate the quality, as hitherto assumed.

2. A *high* breaking strain may be due to the iron being of superior quality, dense, fine, and moderately soft, or simply to its being very hard and unyielding.

3. A *low* breaking strain may be due to looseness and coarseness in the texture, or to extreme softness, although very close and fine in quality.

4. The contraction of area at fracture, previously overlooked, forms an essential element in estimating the quality of specimens.

5. The respective merits of various specimens can be correctly ascertained by comparing the breaking strain *jointly* with the contraction of area.

6. Inferior qualities show a much greater variation in the breaking strain than superior.

7. Greater differences exist between small and large bars in coarse than in fine varieties.

8. The prevailing opinion of a rough bar being stronger than a turned one is erroneous.

9. Rolled bars are slightly hardened by being forged down.

10. The breaking strain and contraction of area of iron plates are greater in the direction in which they are rolled than in a transverse direction.

11. A very slight difference exists between specimens from the centre and specimens from the outside of crank shafts.

12. The breaking strain and contraction of area are greater in those specimens cut lengthways out of crank shafts than in those cut crossways.

13. The breaking strain of steel, when taken alone, gives no clue to the real qualities of various kinds of that metal.

14. The contraction of area at fracture of specimens of steel must be ascertained as well as in those of iron.

15. The breaking strain, *jointly* with the contraction of area, affords the means of comparing the peculiarities in various lots of specimens.

16. Some descriptions of steel are found to be very hard, and, consequently, suitable for some purposes ; whilst others are extremely soft, and equally suitable for other uses.

17. The breaking strain and contraction of area of *puddled steel* plates, as in iron plates, are greater in the direction in which they are rolled ; whereas in *cast* steel they are less.

18. Iron, when fractured suddenly, presents invariably a crystalline appearance ; when fractured slowly, its appearance is invariably fibrous.

19. The appearance may be changed from fibrous to crystalline by merely altering the shape of specimen, so as to render it more liable to snap.

20. The appearance may be changed by varying the treatment, so as to render the iron harder and more liable to snap.

21. The appearance may be changed by applying the strain so suddenly as to render the specimen more liable to snap, from having less time to stretch.

22. Iron is less liable to snap the more it is worked and rolled.

23. The "skin" or outer part of the iron is somewhat harder than the inner part, as shown by appearance of fracture in rough and turned bars.

24. The mixed character of the scrap iron used in large forgings is proved by the singularly varied appearance of the fractures of specimens cut out of crank shafts.

25. The texture of various kinds of wrought iron is beautifully developed by immersion in dilute hydrochloric acid, which, acting on the surrounding impurities, exposes the metallic portion alone for examination.

26. In the fibrous fractures the threads are drawn out, and are viewed externally, whilst in the crystalline fractures the threads are snapped across in clusters, and are viewed internally or sectionally. In the latter cases the fracture of the specimen is always at right angles to the length ; in the former it is more or less irregular.

27. Steel invariably presents, when fractured slowly, a silky fibrous appearance ; when fractured suddenly, the appearance is invariably granular, in which case also the fracture is always at right angles to the length ; when the fracture is fibrous, the angle diverges always more or less from 90°.

28. The granular appearance presented by steel suddenly fractured is nearly free of lustre, and unlike the brilliant crystalline appearance of iron suddenly fractured ; the two combined in the same specimen are shown in iron bolts partly converted into steel.

29. Steel which previously broke with a silky fibrous appearance, is changed into granular by being hardened.

30. The little additional time required in testing those specimens, whose rate of elongation was noted, had no injurious effect in lessening the amount of breaking strain, as imagined by some.

31. The rate of elongation varies not only extremely in different qualities, but also to a considerable extent in specimens of the same brand.

32. The specimens were generally found to stretch equally throughout their

length until close upon rupture, when they more or less suddenly drew out, usually at one part only, sometimes at two, and, in a few exceptional cases, at three different places.

33. The ratio of ultimate elongation may be greater in short than in long bars in some descriptions of iron, whilst in others the ratio is not affected by difference in the length.

34. The lateral dimensions of specimens forms an important element in comparing either the rate of, or the ultimate, elongation—a circumstance which has been hitherto overlooked.

35. Steel is reduced in strength by being hardened in water, while the strength is vastly increased by being hardened in oil.

36. The higher steel is heated (without, of course, running the risk of being burned) the greater is the increase of strength by being plunged into oil.

37. In a highly converted or hard steel the increase in strength and in hardness is greater than in a less converted or soft steel.

38. Heated steel, by being plunged into oil instead of water is not only considerably *hardened*, but *toughened* by the treatment.

39. Steel plates hardened in oil, and joined together with rivets, are fully equal in strength to an unjointed soft plate, or the loss of strength by riveting is more than counterbalanced by the increase in strength by hardening in oil.

40. Steel rivets, fully larger in diameter than those used in riveting iron plates of the same thickness, being found to be greatly too small for riveting steel plates, the probability is suggested that the proper proportion for iron rivets is not, as generally assumed, a diameter equal to the thickness of the two plates to be joined.

41. The shearing strain of steel rivets is found to be about a fourth less than the tensile strain.

42. Iron bolts, case-hardened, bore a less breaking strain than when wholly iron, owing to the superior tenacity of the small proportion of steel being more than counterbalanced by the greater ductility of the remaining portion of iron.

43. Iron highly heated and suddenly cooled in water is hardened, and the breaking strain, when gradually applied, increased, but at the same time it is rendered more liable to snap.

44. Iron, like steel, is softened, and the breaking strain reduced, by being heated and allowed to cool slowly.

45. Iron subject to the cold-rolling process has its breaking strain greatly increased by being made extremely hard, and not by being "consolidated," as previously supposed.

46. Specimens cut out of crank-shafts are improved by additional hammering.

47. The galvanizing or tinning of iron plates produces no sensible effects on plates of the thickness experimented on. The result, however, may be different, should the plates be extremely thin.

48. The breaking strain is materially affected by the shape of the specimen. Thus the amount borne was much less when the diameter was uniform for some

inches of the length than when confined to a small portion—a peculiarity previously unascertained, and not even suspected.

49. It is necessary to know correctly the exact conditions under which any tests are made before we can equitably compare results obtained from different quarters.

50. The startling discrepancy between experiments made at the Royal Arsenal, and by the writer, is due to the difference in the shape of the respective specimens, and not to the difference in the two testing machines.

51. In screwed bolts the breaking strain is found to be greater when old dies are used in their formation than when the dies are new, owing to the iron becoming harder by the greater pressure required in forming the screw thread when the dies are old and blunt than when new and sharp.

52. The strength of screw-bolts is found to be in proportion to their relative areas, there being only a slight difference in favor of the smaller compared with the larger sizes, instead of the very material difference previously imagined.

53. Screwed bolts are not necessarily injured, although strained nearly to their breaking point.

54. A great variation exists in the strength of iron bars which have been cut and welded; whilst some bear almost as much as the uncut bar, the strength of others is reduced fully a third.

55. The welding of steel bars, owing to their being so easily burned by slightly overheating, is a difficult and uncertain operation.

56. Iron is injured by being brought to a white or welding heat, if not at the same time hammered or rolled.

57. The breaking strain is considerably less when the strain is applied suddenly instead of gradually, though some have imagined that the reverse is the case.

58. The contraction of area is also less when the strain is suddenly applied.

59. The breaking strain is reduced when the iron is frozen; with the strain gradually applied, the difference between a frozen and unfrozen bolt is lessened, as the iron is warmed by the drawing out of the specimen.

60. The amount of heat developed is considerable when the specimen is suddenly stretched, as shown in the formation of vapor from the melting of the layer of ice on one of the specimens, and also by the surface of others assuming tints of various shades of blue and orange, not only in steel, but also, although in a less marked degree, in iron.

61. The specific gravity is found generally to indicate pretty correctly the quality of specimens.

62. The density of iron is *decreased* by the process of wire-drawing, and by the similar process of cold rolling, instead of *increased*, as previously imagined.

63. The density in some descriptions of iron is also decreased by additional hot-rolling in the ordinary way; in others the density is very slightly increased.

64. The density of iron is decreased by being drawn out under a tensile strain, instead of increased, as believed by some.

65. The most highly converted steel does not, as some may suppose, possess the greatest density.

66. In cast steel the density is much greater than in puddled steel, which is even less than in some of the superior descriptions of wrought iron.

Art. 33.—Cast Iron.

Coefficient of Elasticity and Elastic Limit.

Cast iron is a material of much less value to the engineer than wrought iron, and consequently has been the subject of much less experimental investigation.

The following table (Table I.) contains values of the coefficient of tensile elasticity for three (Nos. 1, 2 and 3) different irons used in the fabrication of cast-iron cannon. They are computed by the aid of Eq. (1), Art. 2, from data contained in "Reports of Experiments on the Properties of Metals for Cannon," etc., by the late Captain T. J. Rodman,

TABLE I.

	NO. 1.	NO. 2.	NO. 3.	NO. 4.
W.	*E.*	*E.*	*E.*	*E.*
1,000	28,011,000	50,000,000	33,333,000	25.000.000
2,000	28,011,000	28,571,000	28,571,000	16,667,000
3,000	25,000,000	23,810,000	27,273,000	15,000,000
4,000	22,962,000	22,727,000	25,000,000	15.385,000
5,000	23,031,000	20,833,000	23,810,000	13,889,000
10,000	20,960,000	17,000,000	20,000,000	12,195,000
15,000	16,773,000	13,204,000	17,241,000	10,000,000
20,000	13,384,000	7.370,000	14,085,000	8,000,000
24,000	10,150,000	3,454,000	11,060,000	——

W and *E* are expressed in pounds per square inch.

U. S. A. The iron was an excellent charcoal gun iron, and the specimens were from 30 to 35 inches long turned to a diameter of 1.382 inches. The data were selected at random (pages 158, 212 and 228 of the work cited) from the large amount accumulated by Captain Rodman.

Column No. 4 contains values of E given by Wm. Kent, M. E. (Van Nostrand's Magazine, Vol. 20); they belong to a piece of cast iron $1\frac{1}{8}$ inches in diameter and 5 inches long.

The left-hand column, headed "W," gives the stress per square inch, while the three columns "E" give the corresponding ratios between stress and strain for the three different irons. Such ratios are the "coefficients of elasticity," properly speaking, below the elastic limit only. It will be observed, however, that none of these specimens can really be considered to possess an elastic limit, unless possibly No. 1, whose elastic limit may be taken at, or a very little above 2,000 pounds per square inch.

In No. 1 first permanent set was observed at 4,000 pounds per square inch.
In No. 2 first permanent set was observed at 4,000 pounds per square inch.
In No. 3 first permanent set was observed at 8,000 pounds per square inch.

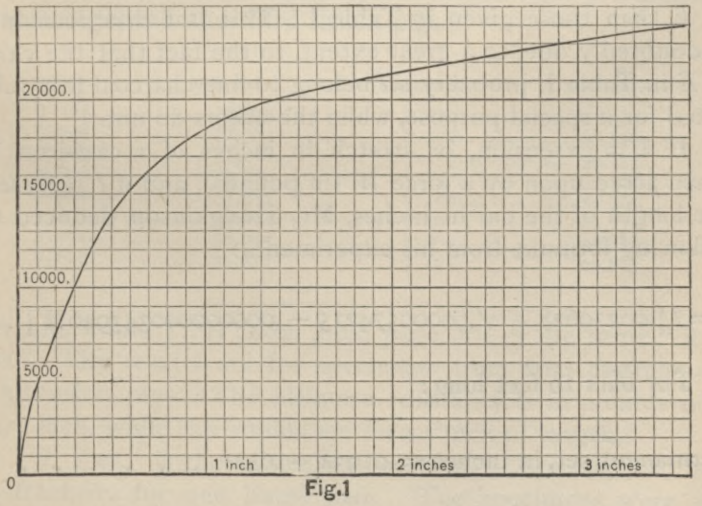


Fig.1

Fig. 1 represents graphically the results of the experiments on specimen No. 2. The constantly varying value of the ratio between stress and strain is shown in a very evident manner by the continually varying inclination of the curve. The strains (stretches) are laid down as if belonging to a bar 1,000 inches long.

The following results are deduced by B. B. Stoney (Theory of Strains in Girders and similar Structures, p. 369) from experiments by Eaton Hodgkinson on a bar of English cast iron 10 feet long.

$W =$ 2,240 pounds per square inch	.....	$E =$ 13,603,520 pounds per square inch.
$W =$ 4,480 " " " "	.....	$E =$ 13,260,800 " " " "
$W =$ 6,720 " " " "	.....	$E =$ 12,382,720 " " " "
$W =$ 8,960 " " " "	.....	$E =$ 11,596,480 " " " "
$W =$ 11,200 " " " "	.....	$E =$ 10,843,840 " " " "
$W =$ 13,440 " " " "	.....	$E =$ 9,856,000 " " " "
$W =$ 14,560 " " " "	.....	$E =$ 9,549,120 " " " "

These results show a limit of elasticity at about 6,000 pounds per square inch; they also show much smaller values of E than those given in Table I. This last disagreement is undoubtedly due, to a great extent, to the fact that the values of E in Table I. probably *all* belong to fine charcoal iron fabricated for a special purpose, while the others do not.

If $\lambda =$ extension, or stretch in inches of a cast-iron bar when acted upon by a force W (in pounds), and if l represents the length of the bar in inches, Mr. Hodgkinson deduced the following formulæ from his experiments:

$$\lambda = l\{.00239628 - \sqrt{.0000574215 - .0000000000343946W}\} \quad . \quad (1)$$

For bars 10 feet long:

$$\text{Permanent set, in inches} = .0193\lambda + .64\lambda^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Although the preceding results are only a few of a great many similar results that may be computed in the same manner, yet they give a fair representation of the general character of the elastic properties of cast iron. The metal is seen to be very irregular and unreliable in its elastic behavior. A large portion of the material can scarcely be said to have an elastic limit, although no apparent permanent set takes place under a considerable intensity of stress; in other words, although perhaps all tested specimens resume their original shape and dimensions for small intensities of stress, yet the ratio between stress and strain is seldom constant for essentially any range of stress.

Ultimate Resistance.

On page 5 of Captain Rodman's "Reports" are given the following densities and ultimate tensile resistances, expressed in pounds per square inch, of 16 specimens of warm-blast, charcoal Greenwood and Salisbury iron, taken from preliminary castings of second and third fusion pigs:

DENSITY.	ULT. RESIST.	DENSITY.	ULT. RESIST.
7.184	33,079	7.210	22,547
7.198	31,384	7.172	28,518
7.307	35,486	7.159	36,373
7.099	23,776	7.137	33,268
7.304	31,317	7.106	22,290
7.273	42,884	7.100	22,179
7.272	38,993	7.109	22,888
7.219	25,372	7.191	23,873

Again, Table II. is taken from page 261 of the same "Reports." The results are for specimens from trial castings of second-fusion pigs. The ultimate resistance is in pounds per square inch, while the strains are for an inch of length.

"*Ult. Ext.*" is the ultimate extension, or stretch, just before fracture, for one lineal inch. The specimens were 30 inches long and 1.382 inches in diameter.

TABLE II.

SPECIMEN.	DENSITY.	ULT. EXT.	ULT. RESIST.
Ao	7.267	.00303	30,117
A1	7.274	.00334	31,681
Bo	7.178	.00291	23,617
B1	7.202	.00161	24,260
Co	7.255	.00287	28,220
C1	7.280	.00382	27,147
Do	7.221	.00424	25,627
D1	7.230	.00223	24,767

On page 42 of "Reports of Experiments on Metals for Cannon," Major Wade gives the following for 15 proof bars cast with 8- and 10-inch guns and 6-pounder trial guns, at South Boston, 1844:

Greatest resistance per square inch	31,027	pounds.
Mean " " " "	27,232	"
Least " " " "	22,402	"

He states that these specimens show the general quality of the iron used at that time.

Again, on page 179 of the same "Reports," Major Wade gives for 25 specimens from 32-pounder cannon made at West Point foundry in 1850:

Greatest resistance per square inch	36,728	pounds.
Mean " " " "	32,023	"
Least " " " "	28,990	"

He states that the character of this iron was "that of good *foundry* iron, of the different grades of Numbers 1, 2, and 3;" it was composed of first, or first and second fusion pigs.

The preceding results give correct representations of the character of the best quality of American cast iron, produced for use in cases requiring such a metal.

Three specimens, turned down to a diameter of about 0.625 inch, taken from the iron used in the Boston water mains, and broken at the Warren Foundry, Phillipsburg, N. J., gave the following ultimate resistances in pounds per square inch:

13,070............ 15,470............ 18,300

As with all material, the character of cast iron affects, to a great extent, its resistance; *i. e.*, whether it is fine or coarse grained, gray or white, etc. It (the resistance) also depends upon the character of the ore from which it is produced.

Major Wade ("Reports," pages 378 and 388) shows that the cold-blast iron which he tested gave much higher resistance than the hot-blast metal.

It is to be remembered that all the specimens from which the preceding results were deduced were what may be called "small specimens." Specimens with several square inches in area of normal section would probably give somewhat different results.

It is interesting to observe that, in experimenting upon cast-iron cannon, Major Wade ("Reports," pages 77 and 78) found that water was forced through the "pores" of the metal of one cannon at a pressure of 7,000 pounds per square inch, and through those of another with thicker metal (thickness equal to radius of bore) at a pressure of 9,000 pounds per square inch.

Capt. Rodman ("Reports," page 262) forced water through the pores of the metal of cylinders 5 inches long, 1 inch thick, and 1 inch bore, at pressures ranging from 15,276 to 25,464 pounds per square inch.

The experiments of Eaton Hodgkinson ("Experimental Researches on the Strength and Other Properties of Cast Iron"), on English metal gave the following resistances in pounds per square inch:

Cannon iron No. 2, hot blast	13,505	pounds.
Cannon iron No. 2, cold blast	16,683	"
Cannon iron No. 3, hot blast	17,755	"
Cannon iron No. 3, cold blast	14,200	"
Devon (Scotland) iron No. 3, hot blast	21,907	"
Buffery iron No 1, hot blast	13,434	"
Buffery iron No. 1, cold blast	17,466	"
Coed-Talon iron No. 2, hot blast	16,676	"
Coed-Talon iron No. 2, cold blast	18,855	"
Low Moor iron No. 3	14,535	"
Mixture	16,542	"

Several of these results are the means of those of a number of tests. The areas of the normal sections of the test specimens varied from 1.54 inches to 4.27 inches, being considerably larger than those of the specimens tested by Major Wade and Captain Rodman.

The characteristic fracture of cast iron is granular and crystalline, with very little (scarcely perceptible by the unaided eye) reduction of area or elongation. Fracture takes place suddenly and without warning, and its ultimate resistance is influenced by many causes whose action may not be observed by any ordinary means; for these reasons, it is a treacherous and unreliable material in tension, as indeed any brittle material must be.

Effect of Remelting.

Crude pigs are said to be "*first-fusion*" metal.
Once remelted pigs produce "*second-fusion*" iron.
Twice remelted pigs produce "*third-fusion*" iron.

etc., etc., etc.

On page 237 of Major Wade's "Reports," the following values are given for Greenwood *first-fusion iron* (iron in original pigs):

	ULT. RESIST. IN POUNDS PER SQ. IN.	
No. 1 iron	15,129	(mean of 3 tests).
No. 2 iron	27,153	(mean of 2 tests).
No. 3 iron	34,923	(mean of 4 tests).

"No. 1 is the softest gray iron,
"No. 2 is intermediate,
"No. 3 is the hardest gray iron."

Again on page 240 :

	ULT. RESIST.
Greenwood, No. 1, 1st fusion	20,900 pounds per sq. in.
Greenwood, No. 1, 2d fusion	30,229 pounds per sq. in.
Greenwood, No. 1, 3d fusion	35,786 pounds per sq. in.
Guns cast from 3d fusion	33,815 pounds per sq. in.

The last result is a mean of four tests.
Finally on page 242 :

Nos. 1 and 2 mixed	2d fusion	27,588 pounds per sq. in.
	3d fusion	40,987 pounds per sq. in.
Nos. 1, 2, and 3 mixed.	2d fusion	37,789 pounds per sq. in.
	3d fusion	32,485 pounds per sq. in.

It is seen that "the softest kinds of iron will endure a greater number of meltings with advantage, than the higher grades." The greatest ultimate resistance, in pounds per square inch, is obtained with :

No. 1 iron at the	4th fusion,
Nos. 1 and 2 mixed at the	3d fusion,
Nos. 1, 2 and 3 mixed at the	2d fusion.

These results probably indicate about the limits to which the remelting of this iron could be advantageously carried.

On page 279 of the same "Reports," is given the result of the test of a specimen of third-fusion iron, of a mixture of Nos. 1, 2 and 3, taken from a gun. The ultimate resistance found was 45,970 pounds per square inch; a most remarkable specimen of cast iron.

Effect of Continued Fusion.

Major Wade ("Reports," pp. 38–41) tested the effects of continued fusion on different grades of iron, both in relation to transverse and tensile resistance.

The general result was an increase of tensile resistance up to 3¾ hours in fusion, which was the longest period tried.

The following results are taken from pp. 40 and 41 of the "Reports."

IRON.	TIME IN FUSION.	ULT. RESIST.
Stockbridge	0.5 hour	17,843 pounds per square inch.
	1.0 "	20,127 " " " "
	1.5 "	24,387 " " " "
	2.0 "	34,496 " " " "
Proof bars.. No. 5	0.5 "	25,969 " " " "
No. 6	1.5 "	29,143 " " " "
No. 7	3.0 "	27,755 " " " "
No. 8	3.75 "	30,039 " " " "
10-inch Howitzer, 2d fusion from pigs.	0.00 "	15,861 " " " "
	1.00 "	20,420 " " " "
	2.00 "	24,383 " " " "
	3.00 "	25,773 " " " "

These tests show well the effect of continued fusion for a period not exceeding 3.75 hours.

Effect of Repetition of Stress.

Capt. Rodman ("Reports," p. 262) experimented on the effect of repeated stresses with the following results:

SPECIMEN.					
Ao	broke at	2301st	repetition	of 22,000	pounds per square inch.
Ao	" "	282d	"	" 26,000	" " " "
Bo	" "	252d	"	" 20,000	" " " "
Bo	" "	150th	"	" 20,000	" " " "
Co	" "	651st	"	" 22,500	" " " "
Co	" "	457th	"	" 23,500	" " " "
Do	" "	172d	"	" 21,600	" " " "

The repetition of the letters representing the specimen indicates that duplicates were tested.

A reference to Table II. will show what single loads per square inch broke the same irons, and a comparison of the two will exhibit the " fatigue " of the metal.

On pages 166 and 167 he also gives some very interesting results of intermittent repetitions of stresses. He subjected a cylinder of cast iron, 1.382 inches in diameter and 35 inches long to intermittent repetitions of 15,000 pounds per square inch (about three-quarters of its ultimate resistance) as follows: 250 repetitions, then a rest of 40 hours; next, 375 additional repetitions, then a rest of 30 days; next, 155 additional repetitions, then a rest of 29 days; next, 1,020 additional repetitions, then a rest of 26 days; finally, 156 additional repetitions followed by breakage at the 1,956th repetition. In every case " rest " signifies entire freedom from load. Capt. Rodman's table gives a detailed account of these experiments. He remarks upon them as follows: " The most interesting point . . . is the fact that at every interval of rest, of any considerable time, the permanent set, and the extension due to the last previous application of the force, diminished. And in some instances it required some fifty repetitions to bring up the extension and set to the same points where they had been at the beginning of the period of rest; thus indicating clearly that the specimen was partially restored, by the interval of rest, from the injury which it had received; and that it endured a greater number of repetitions, owing to the intervals of rest, than it would have done had the repetitions succeeded each other continuously, and at short intervals of time."

These experiments show the " fatigue " of cast iron and the *increase* of the ratio of stress over strain produced by " rest "—so far as tensile stress is concerned.

An examination of the tables also shows that in any series of repetitions, between any two consecutive rests, both the extension and set *were constantly increasing*, consequently, that the ratio of stress over strain was constantly *decreasing*.

Effect of High Temperatures.

A few experimental results bearing on this point will be found in Table IX. of Article 35.

Art. 34.—Steel.

Coefficient of Elasticity.

The great number of the varieties and grades of "steel" renders possible the existence of a correspondingly great number of the mechanical quantities and coefficients used in its consideration in connection with the "Resistance of Materials." In every case, therefore, the kind and character of the steel on which experiments are made, should be stated. In some cases, however, this is impossible.

In Table I. are contained the coefficients of elasticity of the hardened and tempered steel wire (see Table XXII.), supplied by the different makers named, in response to the call for bids for the steel cable wire for the New York and Brooklyn suspension bridge. (Washington A. Roebling's "Report," 1st Jan., 1877).

In the same "Report," page 72, the specifications state: "The elastic limit must be no less than $\frac{47}{100}$ of the breaking strength. . . . Within this limit of elasticity, it must stretch at a uniform rate corresponding to a modulus of elasticity of not less than 27,000,000 nor exceeding 29,000,000."

TABLE I.

PRODUCER.	COEFFICIENTS OF ELASTICITY.		NUMBER OF TESTS.
	Greatest E.	Least E.	
J. Lloyd Haigh	29,817,067	28,815,797	12
Cleveland Rolling Mills	30,142,026	28,917,715	6
Washburne & Moen	29,757,300	28,887,006	6
Sulzbacher, Hymen, Wolff & Co	30,389,946	29,103,238	6
Jno. A. Roebling's Sons Co	30,231,929	28,788,619	13
Carey & Moen	31,261,041	29,418.025	12

Table I. gives the greatest and least results of these tests in pounds per square inch, in the columns headed "E," together with the number of tests of the product of each maker. All the wire was No. 8 Birmingham gauge; *i. e.*, 0.165 inch in diameter.

It is not evident from the "Report" whether these values were obtained for some particular intensity of stress, or whether they are mean values for the entire range below the elastic limit.

Table II. gives a very condensed statement of the results of a very elaborate investigation in the "constants" of steel by Prof. P. C. Ricketts at the mechanical laboratory of the Rensselaer Polytechnic Institute during the year 1886. The paper containing the detailed account of this series of very important tests was given to the Am. Soc. of C. E. in Feb. 1887. Although this table contains other values than those immediately desired, the opportunity of directly comparing different physical constants from the same quality of steel is a sufficient reason for inserting the entire table at this place. All the test

TABLE II.

	MARK.	PER CENT. CARBON.	TENSION. SPECIMEN. Diam. Inches.	Per cen. Reduc. of Area.	Per cen. Elong. in 8 Ins.	POUNDS PER SQUARE INCH. Elastic Limit.	Ultimate Resist.	Coefficient of Elas.
Rivet steel*........	1_1	15	0.756	61.7	30.5	39,600	63,600	30,939,000
"	1_2		0.758	61.7	30.5	38,800	63,300	30,010,000
"	1_3		0.757	60.8	28.9	37,800	63,000	31,160,000
"	4_1	15	0.757	65.3	29.6	37,800	62,000	31,063,000
"	4_2		0.758	65.1	29.4	38,600	63,200	30,471,000
"	4_3		0.758	62.3	29.9	39,400	62,800	29,965,000
"	6_1	15	0.760	61.6	30.1	37,400	60,600	30,456,000
"	6_2		0.760	60.6	29.6	36,900	61,300	30,885,000
"	6_3		0.760	61.8	32.2	39,100	61,900	27,335,000
"	8_1	15	0.760	57.9	29.2	38,100	62,500	30,618,000
"	8_2		0.759	62.4	28.4	37,100	62,300	30,172,000
"	8_3		0.758	61.0	28.2	36,600	61,400	30,424,000
"	10_1	15	0.756	65.7	28.6	35,600	61,700	29,696,000
"	10_2		0.755	64.7	29.0	36,800	61,600	30,075,000
"	10_3		0.754	64.3	29.1	36,900	62,100	30,371,000
"	3_1	16	0.757	63.4	27.9	36,700	61,200	30,918,000
"	3_2		0.758	64.0	30.4	37,700	61,900	30,801,000
"	3_3		0.758	64.3	29.2	37,100	61,800	31,091,000
"	5_1	16	0.757	51.7	30.1	37,800	62,900	30,032,000
"	5_2		0.755	49.4	29.2	38,500	63,600	31,646,000
"	5_3		0.757	51.2	28.1	37,800	61,300	30,031,000
"	7_1	16	0.750	62.1	30.9	36,200	61,200	30,166,000
"	7_2		0.749	60.5	29.6	36,800	62,400	30,415,000
"	7_3		0.751	61.3	31.7	37,800	62,000	30,232,000
"	11_1	16	0.752	64.3	29.4	36,400	62,400	30,030,000
"	11_2		0.754	63.0	29.4	36,400	61,700	30,556,000
"	11_3		0.749	62.3	29.2	36,700	62,200	30,011,000
"	2_1	17	0.752	55.1	29.9	37,200	61,600	30,210,000
"	2_2		0.757	53.7	31.0	36,700	60,100	32,965,000
"	2_3		0.753	53.2	32.0	39,300	61,000	30,097,000
Bessemer †	N_1	11	0.748	60.3	28.4	41,500	66,600	28,950,000
"	N_2		0.754	58.3	28.2	41,400	65,200	29,391,000
"	N_3		0.750	57.0	28.2	43,400	67,000	29,890,000
"	O_1	12	0.751	59.7	27.4	41,500	65,300	29,186,000
"	O_2		0.750	59.2	28.5	41,100	65,100	29,252,000
"	O_3		0.750	57.4	27.0	41,400	65,700	29,464,000
"	T_1	12	0.747	57.3	30.6	42,000	66,100	29,907,000
"	T_2		0.750		30.1	41,900	65,400	29,899,000
"	T_3		0.751	57.1	28.7	41,300	65,400	29,270,000
"	S_1	13	0.763	58.1	26.8	48,100	69,400	29,706,000
"	S_2		0.760	59.5	27.0	47,400	69,300	29,500,000
"	S_3		0.760	56.4	27.1	47,100	70,100	29,238,000
"	U_1	13	0.763	59.1	28.2	42,200	65,300	29,439,000
"	U_2		0.760	56.6	27.6	42,300	65,600	29,678,000
"	U_3		0.756	58.3	27.0	42,300	66,400	29,390,000
"	R_1	16	0.747	54.8	28.9	42,000	68,300	30,083,000
"	R_2		0.745	55.7	27.6	41,700	68,500	30,266,000
"	R_3		0.745	55.0	27.4	41,000	68,600	29,442,000
"	P_1	17	0.746	56.3	27.1	42,100	70,400	29,375,000
"	P_2		0.744	57.2	27.4	42,700	70,500	30,158,000
"	P_3		0.749	55.8	27.1	41,500	79,600	30,784,000
"	V_1	36	0.761	40.7	20.5	60,900	97,500	29,045,000
"	V_2		0.755	38.5	19.1	60,400	99,600	30,236,000
"	V_3		0.759	39.5	19.4	69,700	99,100	29,989,000
"	W_1	39	0.763	39.0	20.0	69,500	95,800	30,025,000
"	W_2		0.762	36.8	19.2	69,600	96,200	30,944,000
"	W_3		0.765	36.7	19.0	69,100	95,200	29,291,000

* Open hearth from Steelton, Pa. † From Troy, N. Y.

TABLE II.—*Continued.*

COMPRESSION. POUNDS PER SQUARE INCH. Elastic Limit.	Coefficient of Elasticity.	SHEAR. POUNDS PER SQUARE INCH. SINGLE SHEAR. Elastic Limit.	Ultimate Resist.	DOUBLE SHEAR. Elastic Limit.	Ultimate Resist.	DOUBLE SHEAR ULTIMATE OVER SINGLE SHEAR ULTIMATE.
39,000	29,897,000					
39,500	27,113,000	39,600	45,440	43,600	46,460	1.022
39,000	28,444,000					
41,100	29,110,000					
41,100	29,025,000	34,600	45,260	38,200	47,450	1.048
41,000	29,045,000					
40,200	30,045,000					
40,200	28,853,000	31,500	46,020	33,800	47,590	1.034
40,400	29,411,000					
41,600	30,192,000					
41,600	29,302,000	31,700	46,910	33,500	48,390	1.032
41,600	29,216,000					
38,600	29,013,000					
38,600	29,963,000	31,100	44,780	34,000	46,590	1.040
38,600	29,478,000					
38,300	29,090,000					
38,300	29,807,000	35,900	44,600	38,500	47,350	1.062
38,300	28,961,000					
41,700	29,630,000					
41,700	28,941,000	33,800	46,440	39,400	48,890	1.053
41,700	29,696,000					
39,900	29,437,000					
40,000	30,009,000	33,700	45,190	35,700	47,210	1.045
40,000	28,730,000					
39,500	29,005,000					
39,700	29,740,000					
39,900	29,963,000					
40,000	31,433,000					
40,000	29,782,000	35,800	46,100	40,700	47,210	1.024
39,700	29,391,000					
41,800	28,567,000					
41,700	29,144,000	30,500	49,210	38,600	51,000	1.036
41,700	28,747,000					
41,100	28,503,000					
41,400	29,531,000	34,400	51,470	39,500	51,470	1.000
41,200	28,730,000					
42,600	29,162,000					
42,400	29,210,000	37,000	49,740	40,300	50,940	1.024
41,900	28,635,000					
44,400	28,070,000					
44,800	28,729,000					
45,000	29,025,000					
44,100	29,281,000					
44,300	29,830,000	36,600	51,000	40,800	51,510	1.010
44,200	29,324,000					
41,100	28,812,000					
41,400	29,342,000	36,700	51,280	43,800	52,550	1.025
41,000	28,666,000					
41,400	28,860,000					
41,600	29,241,000	41,500	53,260	46,000	53,390	1.002
41,800	29,802,000					
55,200	29,162,000					
54,400	29,454,000	52,500	70,190			
54,400	29,281,000					
59,500	28,602,000					
59,200	28,981,000	51,900	67,760			
59,500	29,281,000					

pieces were uniformly about three-quarters of an inch in diameter, and the stretch was in all cases measured on eight inches. The elongations given are per cents of the original length of eight inches.

The reductions of area are the per cents of original sections of the test pieces which indicate the differences between the original and fractured areas.

As indicated, the first half of the Table belongs to specimens of open hearth rivet steel from Steelton, Pa., while the second half contains results drawn from tests on a comparatively wide range of metal from the Bessemer process of the Troy Steel and Iron Co. of Troy, N. Y. The open hearth steel is seen to vary only from 0.15 to 0.17 per cent. of carbon, while the Bessemer metal had carbon varying from 0.11 per cent. to 0.39 per cent. with a wide gap between 0.17 and 0.36 per cent.

The specimens 1_1, 1_2 and 1_3 were cut from the two ends and centre of bar 1, and those subjected to tension were located adjacent to specimens of the same name subjected to compression. Similar observations apply to other sets of specimens affected by the same figure or same letter. Hence there is shown in this Table, the relation of different physical quantities belonging to as nearly identically the same material as the possibilities of the case admit.

The coefficients of tensile elasticity exhibit unusual uniformity. Those for the open hearth steel show no variation with the small variation in carbon. Although the tensile coefficients for the Bessemer steel are slightly lower for the lowest per cents of carbon than for the highest, yet some of the lowest coefficients are found for the highest carbons, and it is difficult to determine any essential variation with varying proportions of that element.

While the average of the tensile coefficients is a very little more for the open hearth than for the Bessemer steel, there is really no sensible difference between them. The average tensile coefficient may be taken at 30,000,000 pounds per square inch.

Too much importance should not be attached to the percentage of carbon alone in these specimens, as the presence of other elements not given, such as manganese, phosphorus, etc., exert marked influences on the physical characteristics of steel.

The "Report of the Naval Advisory Board" prepared by Asst. Naval Constructor R. Gatewood, U. S. N., contains on pages 71 to 75, a large number of tensile tests.

The least coefficient of tensile elasticity given by Lieut. Gatewood is 24,360,000 pounds per square inch, while the greatest value is 30,890,000 pounds per square inch, and the mean 27,720,000 pounds. These values belong to 42 tests of accepted material, and were distributed about continuously over the range covered by the limiting values. The carbon varied from 0.11 per cent. to 0.24 per cent., and these extreme values belonged to about average values of the coefficient of elasticity.

Table III. has been computed from data obtained by David Kirkaldy during his experiments on Fagersta steel plates ("Experimental Inquiry into the Properties of Fagersta Steel," series D, Part 1). The test specimens were, in the clear, 2¼ inches wide and 100 inches long. The thickness is given in the horizontal row, as shown. The values of the coefficient of elasticity (E) are the greatest and least, in pounds per square inch, for the various intensities "p," for five unannealed ⅛, ¼, ⅜, ½ and ⅝-inch (nominally) plates and five similar annealed ones.

These show very irregular elastic behavior. The ⅝ inch annealed specimen is the only one which can properly be considered as possessing a true "coefficient of elasticity" (about 29,000,000 pounds per square inch) above the stress intensity of 10,000 pounds, the ratios of stress to strain are so very variable. Prof. Bauschinger's "stretch-limit" is clearly shown, for the different specimens, at that point of stress where the

TABLE III.

p.	UNANNEALED.		ANNEALED.	
	Greatest E.	Least E.	Greatest E.	Least E.
10,000	45,455,000	33,333,000	37,037,000	29,412,000
14,000	38,889,000	30,435,000	34,146,000	29,167,000
18,000	36,000,000	29,032,000	32,727,000	29,032,000
22,000	34,375,000	28,205,000	31,429,000	28,947,000
26,000	33,333,000	25,000,000	30,952,000	20,968,000
30,000	31,915,000	1,714,000	29,412,000	1,765,000
34,000	30,631,000	1,107,500	4,151,000	1,066,000
38,000	29,008,000	821,000	1,214,000	805,000
42,000	13,084,000	——	——	——
46,000	4,670,000	——	——	——
50,000	2,294,000	——	——	——
Thickness.	0.125 inch.	0.380 inch.	0.255 inch.	0.625 inch.

values of E are almost annihilated. In these four specimens the first permanent sets were noted at 40,000, 20,000, 30,000 and 20,000 pounds per square inch respectively.

Table III*a*. contains a most valuable set of results obtained from the tests of full size steel bars for the Blair Crossing bridge by Geo. S. Morison, chief engineer, and it is taken from his report on that structure.

"E. L." is the elastic limit.

"Ult." is the ultimate tensile resistance.

"E." is the coefficient of elasticity in pounds per square inch.

"C." is carbon and "Mn." manganese.

The bars were broken in the government testing machine at Watertown, Mass., but the chemical analyses were made at the Pittsburgh testing laboratory.

The coefficients of elasticity are seen to vary from 28,089,-000 to 30,627,000 and thus sensibly coincide with the coefficients for test specimens found by Prof. Ricketts and Lieut.

Gatewood, with much lower percentages of carbon. The phosphorus was determined for these bars, although it is not given in Table III*a.*; it varied from about 0.06 to 0.2 of one per cent. with an average of about 0.09. It is a significant fact that two of the bars which broke in the head contained the the highest phosphorus.

Table VII. contains other coefficients of tensile elasticity for full size bars.

TABLE III*a.*

Full Size Steel Bars.

BAR. INCHES.	GAUGED L'NGTH. INCHES.	LBS. PER SQ. IN.		PER CENT. FINAL		E.	PER CENT. OF	
		E. L.	Ult.	Stretch.	Cont'n.		C.	Mn.
6.5 × 1.02	120	34,890	69,250	12.2	44.2	30,456,000	0.27	0.63
6.5 × 1.38	120	33,670	66,700	22.0	45.0	28,846,000	0.25	0.885
6.5 × 1.54	120	34,470	69,920	16.6	42.1	29,702,000	0.295	0.68
6.5 × 1.14	200	32,570	64,740	20.0	46.6	29,498,000	0.24	0.78
6.0 × 1.02		24,550	52,390		18.7	28,089,000	broke	in head.
7.0 × 1.49	200	24,280	45,270	7.9	17.5	28,776,000		
6.5 × 1.03		36,270	broke	in	head	29,940,000	0.30	0.745
6.5 × 1.26		37,250	broke	in	head	30,120,000	0.25	0.82
6.5 × 1.26	200	34,680	64,750	16.5	45.9	30,627,000	0.295	0.75
6.5 × 1.40	200	35,230	64,020	20.7	43.9	30,030,000	0.23	0.76
6.5 × 1.39	200	37,440	70,470	13.1	37.9	29,585,000		
6.5 × 1.52		32,690				30,030,000	bar	not broken.
*6.5 × 1.02	200	37,250	67,630	15.0	43.1	28,901,000		
6.5 × 0.98	160	37,480	67,800	15.8	33.7	29,880,000	0.30	0.665
*6.5 × 0.98	160	36,650	72,050	13.8	38.4	30,270,000	0.28	0.655
6.5 × 0.98	200	37,600	68,720	12.3	34.1	29,630,000	0.28	0.66
6.5 × 0.98	200	35,810	65,850	12.0	39.2	29,960,000	0.30	0.65
6.5 × 0.97	200	33,230	64,410	16.4	49.5	29,670,000	0.29	0.645
6.5 × 0.97	200	37,640	68,290	13.9	42.4	29,960,000	0.28	0.68

Prof. Alex. B. W. Kennedy (London "Engineering," Vol. XXXI., 1881) determined the coefficients of tensile elasticity of specimens of mild steel plates containing about 0.18 per cent. of carbon, and of some specimens of still milder rivet steel.

Twelve specimens of plates (1⅜ × ¼ ; 4 × ¼ ; 2 × ⅜ ; 3½ × ⅜ ; *and* 2¾ × ½, *all in inches*) *gave :*

GREATEST.	MEAN.	LEAST.
33,670,000	29,882,000	25,440,000

all in pounds per square inch.

* Broke in head and retested.

Eight other specimens of the same plates gave:

GREATEST.	MEAN.	LEAST.
31,940,000	29,001,000	26,110,000

all in pounds per square inch.

As a rule, *the thinner plates gave the higher values of E.* There were, however, some marked exceptions.

Eleven specimens of $\frac{15}{16}$ inch round rivet steel, turned to about $\frac{5}{8}$ inch diameter; two each of $\frac{11}{16}$ and $1\frac{1}{16}$ inch round, turned to $\frac{1}{2}$ and $\frac{7}{8}$ inch diameter, respectively, gave:

GREATEST.	MEAN.	LEAST.
31,750,000	30,670,000	29,790,000

all in pounds per square inch.

Hay Steel.

Some experiments on three different bars of the Hay steel used in the bridge at Glasgow, Missouri, by Gen. Wm. Sooy Smith, gave the following results ("Annales des Ponts et Chaussées," Feb., 1881):

Experiment No. 1.

A bar of rectangular section 2.09 × 1.1 inches, reduced by hammering from a bar 2.6 inches square, was subjected to different intensities of stress varying from about 20,500 to 54,000 pounds per square inch, at which the following values of the coefficients of elasticity (in pounds per square inch) were found:

GREATEST.	MEAN.	LEAST.
32,900,000	28,824,000	26,094,000

At 54,000 pounds per square inch there was a "trace" only of permanent elongation or set. The length of this bar, between the observation marks, was about 38.5 inches.

Experiment No. 2.

A round bar 1.04 inches in diameter was subjected to a stress of about 51,200 pounds per square inch, with a stretch of 1.66 millimetres per metre, at which a "trace" only of permanent set was observed. The resulting coefficient of elasticity was:

$$E = 30,857,000 \text{ pounds per square inch.}$$

The distance between observation marks was about 18.7 inches.

Experiment No. 3.

A bar 5.2 × 1.34 inches was subjected to a stress of about 49,200 pounds per square inch, with a trace only of permanent set and a strain of 0.00171 metre per metre. Consequently the resulting coefficient of elasticity was:

$$E = 28,772,000 \text{ pounds per square inch.}$$

These experiments show that the coefficient of elasticity of Hay steel is not essentially different from that of other material of the same class.

In his "Report on the Renewal of Niagara Suspension Bridge," Mr. Leffert L. Buck, C. E., gives the following values for the Hay steel used in that work:

GREATEST.	MEAN.	LEAST.
30,830,000	28,000,000	26,400,000

all in pounds per square inch. These results are for eighteen experiments on small specimens.

Ultimate Resistance and Elastic Limit.

In this section, it is to be observed, the "elastic limit" is seldom that point at which the coefficient of elasticity (stress over strain) ceases to be essentially constant, but more nearly Prof. Bauschinger's "stretch-limit," at which the increment of strain, due to a constant increment of stress, very suddenly increases, involving a correspondingly great permanent set.

TABLE III*b*.

MAKERS.	MEAN OF.	ULTIMATE RESISTANCE, POUNDS PER SQUARE INCH.			REMARKS.
		Greatest.	Mean.	Least.	
Coleman, Rahm & Co., Pittsburg	3	118,400	91,200	74,000	"Very poor steel."
Am. Tool Steel Co., Brooklyn.	1	——	106,500	——	For lathe tools.
Butcher & Co., Philadelphia	7	144,300	112,100	93,500	
Park Bros., Pittsburg	2	118,000	118,000	118,000	
Steel Works, New York	1	——	85,400	——	
Jessup, Sheffield, Eng	4	86,000	78,500	74,000	{ 2 Annealed. 2 Unannealed.
Anderson & Woods, Pittsburg	2	100,000	100,000	100,000	
Coleman, Rahm & Co., Pittsburg	1	——	68,000	——	
Miller, Barr & Parkin, Pittsburg	1	——	90,000	——	Annealed.
Miller, Barr & Parkin, Pittsburg	2	103,200	102,200	101,200	Unannealed.
Hussey, Wells & Co., Pittsburg	3	128,000	126,500	125,000	Steel plate.
Brown & Co., Pittsburg	7	81,300	75,450	45,000	Cast "machinery steel."
Thos. Firth, Sheffield, Eng	3	113,900	112,600	112,000	"Gun metal."
Butcher & Co., Philadelphia	4	110,100	103,500	99,200	Chrome steel stave.
" " "	4	107,500	106,000	103,500	Chrome steel stave.
" " "	3	151,000	148,700	147,000	Chrome steel ingot.
" " "	5	129,000	99,900	69,800	Carbon rivet steel.
" " "	6	128,300	98,300	65,300	Carbon steel.
" " "	5	142,000	120,100	100,000	Carbon rivet steel.
" " "	5	143,600	139,300	136,300	Chrome steel stave.
" " "	4	135,400	119,200	111,700	Chrome steel stave.
N. Y. Chrome Steel Co	16	193,260	146,400	115,800	Chrome steel bar.
Parks Bros., Pittsburg } Black Dia.	3	181,864	119,500	109,500	Steel normal untemp.
" " "	3	227,500	217,400	201,341	Temp. in oil at 82° *F.*
" " "	3	176,100	165,500	152,500	Temp. in water at 79° *F.*
" " "	3	150,500	141,900	132,700	Temp. in water at 79° *F.*

Table III*b* is condensed from Prof. Woodbury's history of the St. Louis arch. The last four results are from the experiments of Chief Engineer Shock, U.S.N., while the "N. Y. Chrome Steel Co." result is from Kirkaldy's tests.

The diameters of the (circular) specimens varied from 0.357 inch to 1.00 inch, and their lengths from 3 to 12 inches. The elastic limit varied from 45 to 55 (nearly) per cent. of the ultimate resistance.

TABLE IV.

Rail Steel.

NO.	UNANNEALED.			ANNEALED.			CARBON, PERCENTS.
	T.	*E. L.*	STR.	*T.*	*E. L.*	STR.	
8	79,625	37,625	19.6	78,250	35,750	20.5	0.324
8	81,250	36,625	15.6	77,375	37,125	14.7	0.379
4	72,750	32,250	22.5	70,750	29,250	22.0	0.308
4	76,750	38,250	11.5	74,000	30,500	9.5	0.438
4	75,750	36,500	12.8	73,750	34,000	15.2	0.405
4	83,500	37,000	14.5	80,750	40,000	15.2	0.384
8	72,375	36,625	17.5	70,500	33,875	19.2	0.282
8	79,875	38,000	14.5	78,000	36,875	14.4	0.381
4	70,500	34,250	17.0	69,500	30,500	17.5	0.367
4	82,000	40,250	12.7	76,000	38,500	13.2	0.394
4	78,000	36,750	10.7	74,250	33,250	13.7	0.378
4	77,000	40,250	15.0	76,000	35,750	14.5	0.388
24	74,542	35,833	18.9	72,958	33,167	19.8	0.314
24	80,167	37,958	14.1	76,792	36,167	13.5	0.392
8	76,875	36,625	11.7	74,000	33,625	14.4	0.391
8	80,250	38,625	14.7	78,375	37,875	14.8	0.386
32	75,125	36,031	17.1	73,219	33,281	18.5	0.334
32	80,188	38,125	14.2	77,188	36,594	13.8	0.390

Table IV. contains the results of one hundred and ninety-two tests of specimens from steel rails which had been in use on the Penn. R. R. during periods of time of different lengths, These results were given by Charles B. Dudley, Ph.D., Chemist to Penn. R. R. Co., in his "Report to the Supt. of Motive

Power," and published in the Journal of the Franklin Institute, March, 1881.

The specimens were circular and turned to a diameter of 0.75 inch between shoulders five inches apart.

The following is the notation :

"*No.*" is the number of specimens for which the other quantities are the average.

"*T.*" is the ultimate tensile resistance in pounds per square inch.

"*E. L.*" is the elastic limit in pounds per square inch.

"*Str.*" is the per cent. of original length of ultimate elongation or stretch (*i. e.*, at instant of rupture).

TABLE V.

Bars Unannealed.

CARBON.	STRESS IN POUNDS PER SQUARE INCH.				PER CENTS.			
Per cent.	Ultimate.		Elastic Limit.		Str.		Cont.	
0.30	94,760	Av. = 95,390	55,712	Av. = 55,635	15.1	Av. = 14.3	30.	Av. = 28.6
0.30	95,380		56,009		12.9		26.	
0.30	95,830		55,120		15.3		31.	
0.30	96,020		55,830		14.5		27.	
0.30	94,970		55,512		13.8		29.	
0.50	112,340	Av. = 112,630	65,790	Av. = 65,904	10.8	Av. = 10.1	19.	Av. = 19.6
0.50	112,470		66,040		8.9		16.	
0.50	111,980		66,160		10.5		22.	
0.50	113,320		65,550		10.9		21.	
0.50	113,040		65,980		9.4		20.	

"*Str.*" is the elongation of original length.

"*Cont.*" is the contraction of original section.

TABLE VI.

Annealed Eye-Bars.

CARBON PERCENTAGE.	DIMENSIONS OF: Stem.	DIMENSIONS OF: Head.	DIMENSIONS OF: Pin Hole.	MODE OF MANUFACTURE.	TENSILE STRESS IN POUNDS PER SQUARE INCH AT: Elastic Limit.		TENSILE STRESS IN POUNDS PER SQUARE INCH AT: Ultimate.		STRETCH PERCENTAGE OF ORIGINAL LENGTH.		REDUCTION OF AREA—PER CENT. OF ORIGINAL AREA.		DISTANCE FROM PIN-HOLE AT WHICH THE BAR BROKE.
0.30	3 inches × $\frac{7}{8}$ inch × 10 feet.	$1\frac{1}{4}$ inches thick, $7\frac{1}{2}$ inches across eye.	$3\frac{1}{8}$ inches in diameter.	Upset.	54,026	Av. = 54,083.	87,400	Av. = 90,400.	8.2	Av. = 8.1	44.	Av. = 44.	1 foot 3 inches.
0.30					54,113		94,500		9.2		46.		4 feet $3\frac{3}{4}$ inches.
0.30					54,113		89,300		7.0		42.		1 foot $8\frac{1}{2}$ inches.
0.30				Rolled.	51,762	Av. = 52,781.	92,672	Av. = 93,007.	8.2	Av. = 9.2	0.0	Av. = 34.5	Head into 3 pieces.
0.30					54,065		91,570		9.3		29.		5 feet $2\frac{1}{4}$ inches.
0.30					52,518		94,780		11.8		40.		2 feet $1\frac{3}{4}$ inches.
0.30				Welded.	58,473	Av. = 56,614.	69,140	Av. = 67,180.	2.0	Av. = 2.3			0 feet $5\frac{1}{4}$ inches.
0.30					56,059		63,000		2.6				0 feet $5\frac{1}{2}$ inches.
0.30					55,310		69,400		2.2				0 feet $5\frac{3}{8}$ inches.

It will be observed that the process of annealing decreased both the ultimate resistance and elastic limit. The results are irregular, however, so far as the strains or elongations are concerned.

Tables V. and VI. are taken from "Steel in Construction," a paper read before the Engineers' Society of Western Pennsylvania, by Albert F. Hill, C. E., 20th April, 1880. Table V. contains the results of tests of specimen bars 3″ × ⅞″ × 30″. These specimens were cut from rolled bars which were subsequently manufactured into eye-bars.

Nine eye-bars containing 0.30 per cent. of carbon were made, besides nine others containing 0.50 per cent. of carbon. Each group of nine was divided into three classes, with welded, rolled and upset heads, respectively. These eighteen eye-bars were then put into the testing machine, and the results belonging to the nine containing 0.30 per cent. carbon are given in Table VI.

The results of the tests of the bars containing 0.50 per cent. carbon, were not given, but it was stated by Mr. Hill that they verified the conclusions drawn from Table VI.

Mr. Hill considered that these experiments "clearly establish the fact that welding of high grade steel for purposes of construction is out of the question in general practice."

The metal in these bars was "open hearth" steel made by Messrs. Anderson & Co., of Pittsburg, Penn.

Table III*a*. contains the results of Mr. Morison's tests, to which reference has already been made on pages 290 and 291. These bars are seen to be of high steel for tension members, and as they were annealed it is important to observe that their elastic limits average about half their ultimates, *i. e.*, relatively much lower than the unannealed test specimens. Lower steel would be affected less.

Table VII. gives the results of tests of some full size eye bars of Pernot open hearth steel. The latter was made by the Cambria Iron Co., but the bars were formed by the Keystone

Bridge Co. and tested in the gov't machine at Watertown, Mass. The tests are reported on pages 194 to 207 of Ex. Doc.

TABLE VII.

$6\frac{1}{2}$ *Inch* × 1 *Inch Full Size Bars.*

LENGTH OF BAR. INCHES.	POUNDS PER SQUARE INCH.				PER CENT. OF	
	Elastic Limit.	Ultimate Resistance.	Coefficient of Elasticity.	Bearing on Pin Hole.	Final Stretch	Final Contraction.
222	37,480	67,800	29,880,000	86,100	15.9	33.7
222	36,650	64,000	30,270,000	79,430		
Re-	test.	71,560				36.5
Re-	test.	72,050				38.4
263	37,600	68,720	29,630,000	83,650	12.6	34.1
262	35,810	65,850	29,960,000	80,060	11.5	39.2
262	33.230	64,410	29,670,000	79,060	15.1	49.5
262	37,640	68,290	29,960,000	83,960	13.5	42.4

No. 5, 48th Congress, 1st Session. The carbon in these bars varied from 0.27 to 0.3 per cent., while the manganese ran from 0.58 to 0.71 per cent., and the phosphorus from 0.074 to 0.098 per cent.

No bar broke in the head. Tests 3 and 4 were the retests of the two parts of the second bar. The parts of the bar rested five days before being retested. The ultimate resistance is markedly increased in these two experiments.

Some of these bars showed considerable dishing of the head as failure took place, although no head failed. This shows that a bar whose width is $6\frac{1}{2}$ times its thickness has reached about the *limit of thinness*, *i. e.*, that a thinner bar of the same width would be liable to tear in detail through the eye with developing the full resistance of the bar.

Mr. Leffert L. Buck, C.E., in his "Report on the Renewal of the Niagara Suspension Bridge," 1880, gives the results of tests of plate specimens of Hay steel used in that work, seven to ten inches long between clamps. These specimens were subjected to various treatments, such as punching, annealing, blows while under stress, nicking on edges, etc., etc., and gave the following results under tension:

Hay Steel.	Elastic limit........	Greatest......	59,431	pounds per square inch.
		Least	43,300	" " " "
	Ultimate resistance.	Greatest......	97,600	" " " "
		Least	59,370	" " " "
	Final stretch........	Greatest......	19.4	per cent.
		Least	7.0	" "
	Final contraction of ruptured section..	Greatest......	42.0	" "
		Least	13.0	" "

The "stretch" and "contraction" are per cents of ten inches and original section, respectively.

Table VIII. contains the results of the experiments of Sir Wm. Fairbairn on the different varieties of English steel given in the left-hand column. The specimens were one inch square, and had previously been subjected to a transverse load. The per cents of strain or elongation are for a length of eight inches, which, it is presumed, included the section of fracture.

Table IX. contains the results of tensile experiments on Bessemer and crucible steel specimen bars by Mr. Kirkaldy for the "Steel Committee" (English).

The first part of the table gives the results of experiments on bars turned accurately to a diameter of 1.382 inches with a length "in the clear" of 50 inches. It is presumed that the per cents of elongation apply to that length.

The second part of the table gives the results of experiments on bars "in their natural skins," with a diameter of 1.5 inches and length of 120 inches; to which length the per cents of elongation apply.

TABLE VIII.

Bar Specimens.—1867.

PRODUCERS.	ULTIMATE RESISTANCE PER SQ. INCH.	FINAL STRAIN OR ELONGATION.
Messrs. Brown & Co.	Pounds.	Per cent.
Best cast steel, for turning tools	68,404	0.25
Best cast steel, milder	91,250	1.5
Cast steel from Swedish iron for tools ...	106,714	1.0
Cast steel, milder, for chisels...........	116,183	3.62
Cast steel, mild, for welding............	110,055	3.31
Bessemer steel........................	91,972	19.62
Double shear steel from Swedish iron....	92,555	5.43
Foreign bar, tilted direct...............	76,474	13.56
English tilted steel.....................	59,538	21.06
C. Cammel & Co.		
Cast steel, termed "Diamond Steel"....	110,055	1.53
Cast steel, termed "Tool Steel"........	109,072	1.50
Cast steel, termed "Chisel Steel".......	120,398	2.50
Cast steel, termed "Double Shear Steel".	96,665	2.37
Hard Bessemer steel....................	89,121	20.87
Soft Bessemer steel.....................	81,483	20.43
Messrs. Naylor, Vickers & Co.		
Cast steel, called "Axle Steel".........	88,665	16.25
Cast steel, called "Tire Steel".........	91,520	9.00
"Vickers Cast Steel, Special"..........	134,145	1.00
"Naylor & Vickers' Cast Steel"........	118,066	1.75
S. Osborne.		
Best tool cast steel.....................	98,942	0.93
Best chisel cast steel...................	123,686	3.18
Sates-cup, shear blades, etc.............	115,849	2.12
Best cast steel for taps and dies.........	98,790	1.68
Toughened cast steel for shafts, etc.....	103,116	5.25
Best double shear steel..................	87,931	2.43
Extra best tool cast steel	85,724	0.43
Boiler plate, cast steel..................	111,676	13.50

TABLE VIII.—*Continued.*

PRODUCERS.	ULTIMATE RESISTANCE PER SQ. INCH.	FINAL STRAIN OR ELONGATION.
H. Bessemer.	Pounds.	Per cent.
Hard Bessemer steel	103,085	1.87
Milder Bessemer steel	88,175	20.00
Soft Bessemer steel	78,606	19.12
Sanderson Brothers.		
Cast steel from Russia iron for welding	83,484	2.25
Double shear steel	107,940	3.31
Single shear steel	107,182	2.81
Fagot steel, welded	75,199	1.25
Drawn bar, not welded	103,960	3.43
Messrs. Turton & Sons.		
Steel for cups	100,155	2.75
" " drills	87,552	1.06
" " cutters	95,372	1.37
" " turning tools	80,273	0.12
" " machinery	102,915	1.43
" " punches	102,567	1.62
" " mint dies	106,237	2.87
" " dies	87,471	0.87
" " taps	97,994	1.87
Double shear steel	73,266	0.81

The "*Area of Fracture Section*" (Table IX.) is the *per cent. of original sectional area*, which, multiplied by that original area, will give the area of the fractured section. The per cent. of contraction will then be given by taking the difference between 100 and the number expressing the "*Area of Fracture Section.*"

The various grades of steel in the bar specimens of Table IX. exhibit, in the results, the great variations arising with different qualities of that metal.

TABLE IX.

Bar Specimens.—1868 *and* 1870.

NUMBER AND KIND OF SAMPLES.		LIM. OF ELAS. PER SQ. INCH.	ULT. RESIST. PER SQ. INCH.	PER CENT. FINAL ELONGATION.	AREA OF FRACTURE SECTION.
		Pounds.	Pounds.		
5, Hammered, tires.....	Bessemer.	52,200	78,500	11.1	55.5
5, Hammered, axles.....	Bessemer.	49,000	75,000	12.1	51.4
4, Hammered, rails,.....	Bessemer.	48,000	74,500	12.8	52.3
4, Rol'd tires, axles, rails..	Bessemer.	43,200	71,500	17 5	65.0
5, Hammered, tires	Crucible.	46,200	79,500	9.17	62.5
4, Hammered, axles.....	Crucible.	57,300	91,700	8.72	72.1
1, Hammered, rails.....	Crucible.	44,000	85,400	2.96	96.9
1, Rolled, axles....	Crucible.	42,000	68,600	10.56	89.9
3, Chisel...............	Crucible.	58,240	118,200	5.3	94.3
3, Samples.............	Crucible.	57,120	114,300	7.3	80.0
Tires.................	Crucible.	58,240	97,400	4.7	94.8
Rods	Crucible.	60,500	93 700	1.1	100 0
2, Samples.............	Crucible.	45,900	90,800	4.1	95.9
3, Gun-barrels..........	Crucible.	37,600	86,300	8.0	95.7
Hammered..........	Crucible.	56,000	83,000	8.0	98.5
Hammered..........	Crucible.	44,800	——	——	——
2, Rods................	Crucible.	59,900	75,400	0.9	98.7
2, Rolled..............	Crucible.	45,900	67,100	2.0	97.1
3, Faggoted............	Bessemer.	43,700	79,300	11.1	55.8
3, Samples.............	Bessemer.	44,800	76,600	11.9	54.4
2, Samples.............	Bessemer.	39,200	75,300	11.5	80.8
3, Tires and axles	Bessemer.	37,000	75,400	13.6	58.8

Table II. contains a synopsis of the valuable series of tests of specimens by Prof. P. C. Ricketts. This Table has already been explained on page 287. The tension tests show remarkably uniform results in elastic limit and ultimate resistance, and characterize a most excellent material. With the exception of the two Bessemer specimens containing 0.36 and 0.39 per cent. carbon, all specimens were of very mild steel.

Table X. contains the results of twelve experiments by Mr. Kirkaldy ("Experimental Enquiry, etc., of Fagersta Steel," 1873) on 2-inch square hammered bars of Fagersta steel, turned

TABLE X.

Fagersta Steel Bars.

MARK.	POUNDS OF STRESS PER SQ. INCH, AT		PER CENT. OF FINAL		FRACTURE.
	Elastic Limit.	Ultimate Resist.	Contraction.	Elongation.	
1.2 1.2 1.2	62,400 60,200 63,500 Av. = 62,033	81,952 81,424 92,224 Av. = 85,200	3.23 1.48 3.23 Av. = 2.65	1.7 1.4 2.2 Av. = 1.8	Granular.
0.9 0.9 0.9	63,600 62,400 63,200 Av. = 63,066	112,976 109,952 96,912 Av. = 106,613	4.97 8.39 4.97 Av. = 6.11	3.7 7.9 3.6 Av. = 5.1	Granular.
0.6 0.6 0.6	62,500 53,200 58,600 Av. = 58,100	101,232 97,968 108,696 Av. = 102,632	21.46 10.08 11.75 Av. = 14.43	5.5 5.7 8.7 Av. = 6.6	Granular.
0.3 0.3 0.3	44,200 41,500 40,600 Av. = 43,100	61,288 63,120 59,528 Av. = 61,312	61.52 60.42 62.61 Av. = 61.52	22.1 16.2 11.2 Av. = 16.5	Silky.

to 1.128 inches diameter, with a length between shoulders of nine diameters or 10.15 inches.

"*Mark*" indicates the relative hardness, 1.2 being the hardest and 0.3 the softest.

The per cents are of the original sectional area (*i. e.*, of one square inch) and of the original length, which, of course, includes the section of failure.

Sir Joseph Whitworth manufactures his compressed steel by subjecting the molten metal to an intensity of pressure of 13,000 to 14,000 pounds per square inch, immediately after it is taken from the furnace.

Table XI. contains the result of some tensile experiments on some specimens of this steel. Each specimen is turned to a diameter of 0.798 inch (0.5 square inch in normal section) for a length of two inches, for which the per cent. of final elongation is expressed (see "Proc. Inst. of Mech. Engrs.," 1875). The specimens are thus seen to be so formed as to give very high results, both for ultimate resistance and elongation.

TABLE XI.

Whitworth's Compressed Steel.

DISTINGUISHING COLORS FOR GROUPS.	ULT. RESIST. LBS. PER SQUARE INCH.	PER CENT. FINAL ELONGATION.	REMARKS.
Red, Nos. 1, 2, 3........	89,600	32.0	Axles, boilers, cranks, propellor shafts, rivets, etc.
Blue, Nos. 1, 2, 3.......	107,500	24.0	Shafting, drill spindles, hammers, etc.
Brown, Nos. 1, 2, 3.......	129,900	17.0	Large planing tools, Large shears, drills, etc.
Yellow, Nos. 1, 2, 3.....	152,300	10.0	Boring tools, finishing tools for planing, etc.
Special alloy with Tungsten....................	161,300	14.0	

TABLE XII.

Plates.—Unannealed.

Carbon Percentage.	Dimensions.	Tensile Stress in Pounds per Square Inch at Elastic Limit.		Tensile Stress in Pounds per Square Inch at Ultimate.		Av. per Cent. of Elongation.	Character of Fracture.
0.30	$\frac{3}{8}$ inch by 12 inches by 6 feet long, as they came from rolls.—50 inches between jaws of machine.	43,260	Av. = 44,380	79.120	Av. = 78,320	19.3	Fine and silky.
0.30		44.820		77.840			
0.30		45,110		78,390			
0.30		43,990		77,970			
0.30		44,720		78,280			
0.40		51,620	Av. = 51,170	81,990	Av. = 82,480	13.9	Very fine.
0.40		50.980		81,720			
0.40		51.260		83,730			
0.40		51,100		81,830			
0.40		50,890		83,130			
0.50		58,950	Av. = 58,960	85,790	Av. = 85,980	10.5	Good; slightly granular on edges.
0.50		59.200		86,220			
0.50		58,540		85,560			
0.50		58,880		86,000			
0.50		59,330		86,330			

Boiler Plate.

Table XII. was also taken from Mr. Hill's paper, and contains results obtained from tests of large pieces of boiler plate of mild steel of the same character as that used in the bars the results of the tests of which are given in Table VII. *The stress was in the direction of rolling.*

TABLE XIII.

Plates.—Unannealed.

MEAN OF	DIMENSIONS OF SPECIMENS IN INCHES.	STRESS IN POUNDS PER SQ. INCH. Elastic Limit.	STRESS IN POUNDS PER SQ. INCH. Ultimate.	PER CENT. OF STRETCH OF ORIG'L LENGTH.	
3	1⅜ × ¼	47,680	66,440	19.6	For 10 inches.
3	4 × ¼	53,400	68,900	19.9	
3	2 × ⅜	39,670	62,360	26.3	
3	3½ × ⅜	38,940	62,920	27.7	
4	2¾ × ½	38,390	64,860	24.8	
3	1⅜ × ¼	53,790	69,560	19.0	
3	4 × ¼	44,050	65,830	26.3	
3	2 × ⅜	40,160	65,690	23.9	
3	3½ × ⅜	41,690	65,200	21.5	
2	1 × ¼	49,485	77,270	23.6	For 4 inches.
2	1 × ¼	50,745	76,925	24.8	
2	1 × ⅜	42,985	70,375	29.8	
2	1 × ⅜	43,160	70,530	22.5	
9	1 × ¼	45,280	64,970	29.1	
7	1 × ⅜	41,810	64,640	28.6	
5	2 × ⅜	41,368	67,130	——	
4	2 × ⅜	38,390	66,990	——	
13	1¾ × ⅜	39,430	65,700	For 10 ins. 23.2	

Greatest ult. resist. in all these experiments = 78,550
Least " " " " " = 60,150

Table XIII. contains the mean results of Prof. Kennedy's experiments on mild boiler plate containing about 0.18 per cent. of carbon (London "Engineering," Vol. XXXI., 1881). The left column shows the number of tests in each group, for which the other columns contain the mean results. The dimensions of specimens are not exact, but closely approximate. The length for which the per cent. of elongation is given, in all cases, contained the fracture. Consequently those "*Per cents.*" include the "local" extension which exists at the section of fracture. This accounts for the larger values, as a rule, which are found for the 4-inch lengths; 2½-inch lengths, containing the section of fracture, gave much higher values.

The first five groups were pulled from pins, and the next four from wedge grips. The manner of holding the test pieces, however, was not observed to have any influence on the results.

Within the limits of these experiments, also, the ratio of width to thickness of the specimen seemed to have no influence. It will be observed that Prof. Kennedy's specimens were all (what may be called) "long" specimens.

His experiments on some annealed specimens of this steel showed that the process of annealing reduced the ultimate resistance only 3 or 4 per cent.

TABLE XIV.

Fagersta Plate.

THICKNESS, INCH.		ELASTIC LIMIT IN LBS. PER SQUARE INCH.				ULT. RESIST. IN LBS. PER SQUARE INCH.			
		Large.	Small.	Long.	Mean.	Large.	Small.	Long.	Mean.
Unannealed.	$\frac{1}{8}$	53,300	50,500	38,900	47,567	74,915	71,940	55,135	67,330
	$\frac{1}{4}$	37,900	35,400	35,600	36,300	60,480	56,740	54,140	57,120
	$\frac{3}{8}$	29,500	29,300	25,400	28,067	51,456	50,345	48,925	50,243
	$\frac{1}{2}$	31,100	30,800	27,500	29,800	55,803	54,425	50,160	53,463
	$\frac{5}{8}$	28,000	28,300	26,100	27,467	52,924	52,475	49,280	51,560
Mean.		35,960	34,860	30,700	33,840	59,116	57,185	51,528	55,943
Annealed.	$\frac{1}{8}$	35,500	33,200	26,700	31,800	57,485	55,459	45,460	52,801
	$\frac{1}{4}$	33,800	30,500	29,800	31,367	54,543	52,715	49,605	52,288
	$\frac{3}{8}$	28,900	28,100	25,900	27,633	51,076	50,350	46,740	49,389
	$\frac{1}{2}$	27,800	27,900	27,300	27,667	51,338	50,842	49,490	50,557
	$\frac{5}{8}$	25,500	25,700	25,200	25,467	50,432	50,025	47,455	49,304
Mean.		30,300	29,080	26,980	28,787	52,975	51,878	47,750	50,868

Fractures all "silky."

TABLE XV.

Fagersta Plate.

THICKNESS, INCH.	PER CENT. CONTRACTION OF AREA.				PER CENT. FINAL STRAIN OR STRETCH.			
	Large.	Small.	Long.	Mean.	Large.	Small.	Long.	Mean.
Unannealed. 1/8	43.1	47.1	37.9	42.7	10.8	13.5	5.21	9.45
1/4	48.5	54.2	59.7	54.1	28.2	35.5	10.17	24.41
3/8	59.3	62.5	71.0	64.3	36.1	41.5	20.64	32.57
1/2	50.0	58.6	61.2	56.6	36.4	40.0	16.30	30.78
5/8	55.1	61.7	60.7	59.2	37.2	44.7	17.95	33.01
Mean.	51.2	56.8	58.1	55.4	29.7	35.0	14.05	26.04
Annealed. 1/8	57.1	60.8	64.6	60.8	22.9	28.4	10.98	20.37
1/4	60.9	63.5	67.5	64.0	33.8	40.1	16.88	29.99
3/8	63.4	63.6	69.6	65.5	35.8	42.0	18.19	31.76
1/2	61.0	65.1	64.3	63.5	38.5	42.5	19.15	33.08
5/8	62.0	64.3	63.1	63.1	34.4	43.5	17.45	31.51
Mean.	60.9	63.5	65.8	63.4	33.1	39.3	16.50	29.52

In his paper Prof. Kennedy explains in detail his "elastic limit." It is the point at which the material "breaks down," and considerably above the elastic limit as analytically defined in this work.

Tables XIV. and XV. exhibit the results of Mr. Kirkaldy's experiments (in the direction of rolling) on some Fagersta steel plate specimens. The plate from which these specimens were taken was marked 0.15, and the material was a mild steel. The "*Large*" and "*Small*" specimens were shaped as shown in

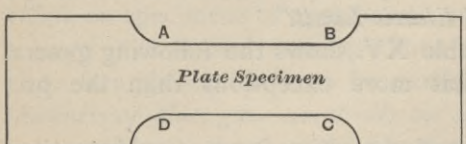


Fig.1

Fig. 1. The width *BC* or *AD* of the reduced portion was ten inches for the "Large" pieces, and one and one half inches for the "*Small*" ones. For the "*Large*" specimens, the length of the reduced portion (*AB* or *CD*) was ten inches (= width), and four and one half inches (= 3 widths) for the "*Small*." The "*Long*" specimens were 100 inches by 2¼ inches "in the clear."

The results embodied in these two tables are of greater interest and value in consequence of the variety in the *relative* dimensions of the specimens. They show the important part played by "lateral strains" both in the ultimate resistance and final strains, or elongations, of test specimens.

With very few exceptions the following general principle may be deduced from Table XIV.:

Both the elastic limit and ultimate resistance increase with the ratio of the width over the thickness of the plates.

Nearly all the exceptions are in the results which belong to the ⅜ unannealed, and the "*Long*" annealed, specimens. It may be observed in connection with Table III., that the character of the former specimen (possessing a low and irregular value of E) is decidedly abnormal, to which, undoubtedly, *its* exceptions are due. Annealing the long specimens seems to cause the disappearance of essentially all influence of the relative dimensions of the cross section, where the ratio of width over thickness is, comparatively speaking, small.

One origin of the results above stated is plainly to be found in the lack of lateral contraction in the plane of the plate, in accordance with the principles shown in Article 32, "*Ultimate Resistance and Elastic Limit.*"

An examination of Table XV. shows the following general result, which, however, has more exceptions than the preceding:

The final contraction and elongation increase as the ratio of width over thickness decreases.

With the long specimens, this does not seem to hold for less values of the ratio than $2\frac{1}{4} \div \frac{3}{8} = 6$. Whether these principles may hold true, as general ones, or whether they may hold within certain limits (a possibility indicated in the "*Long*" specimens), the number and character of these experiments does not permit to be decided. They show, however, that the partial prevention of lateral strains in one direction, whatever may be the cause, will affect, to a considerable extent, experimental results; also, that in testing plates the shape and relative dimensions of the test piece should be carefully noted.

TABLE XVI.

Open-Hearth Steel Plates—1880.

SPECIMEN, INCHES.	PER CENT. OF CARBON.	LENGTHWISE.			CROSSWISE.		
		Stress in pounds per square inch.		Per cent. final stretch.	Stress in pounds per square inch.		Per cent. final stretch.
		Elas. Lim.	Ult. Resist.		Elas. Lim.	Ult. Resist.	
$\frac{3}{4} \times 1\frac{1}{2} \times 18$	0.30	49,353	93,339	16	49,510	95,453	18
$\frac{3}{8} \times 1\frac{1}{4} \times 15$	0.40	63,227	86,410	14	63,723	87,780	16
$\frac{3}{16} \times 1 \times 12$	0.50	65,070	83,190	10	65,300	84 995	15

In Table XVI. are found the results of tests by Mr. Hill ("Engineers' Society of Western Pennsylvania," April 20th, 1880), on specimens of open-hearth steel plate. Each result is a *mean of three*, and each specimen was cut from unannealed plate in a planer. It is to be particularly observed that *each thickness of plate gave essentially the same elastic limit and ultimate resistance, whether the direction of the testing stress was along or across the direction of rolling.*

Although the elastic limit increases with the amount of carbon (consistently with the results in Table XII.), yet, *it is very remarkable to observe that the ultimate resistance decreases as the carbon increases,* which is *not* consistent with the results contained in Table XII.

TABLE XVII.

Siemens Steel Plate—1875.

		THICKNESS IN INCHES.	POUNDS OF STRESS PER SQ. IN. AT		PER CENT. FINAL STRETCH.	PER CENT. FINAL CONTRACTION.
			Elas. Limit.	Ult. Resist.		
LENGTHWISE.	Unannealed.	0.37	34,600	72,900	22.3	37.5
		0.71	30,400	66,900	24.5	44.7
	Annealed.	0.37	31,500	67,500	24.8	43.1
		0.40	31,200	66,400	21.1	44.7
		0.40	29,800	66,100	24.8	38.5
		0.50	29,400	65,800	26.4	44.5
		0.62	26,300	61,800	25.5	43.3
		0.70	24,500	60,100	25.0	45.5
CROSSWISE.	Unannealed.	0.37	34,300	72,700	22.4	37.5
		0.71	30,400	67,300	24.7	43.6
	Annealed.	0.37	31,200	66,900	26.4	46.6
		0.40	31,000	66,900	26.3	49.6
		0.42	30,000	65,800	20.4	39.0
		0.52	29,800	66,600	20.2	46.7
		0.62	26,300	60,600	22.7	35.3
		0.70	24,500	60,200	26.0	50.7

The ratio of width over thickness of specimen increases from 2 (for the $\frac{3}{4}$-inch, or, 0.30 per cent. carbon) to $5\frac{1}{3}$ (for the $\frac{3}{16}$-inch, or, 0.50 per cent. carbon), and Mr. Hill considers this an explanation of this disagreement in the two sets of results. The results of a large number of tests on Fagersta steel specimens of considerable variety in the ratio of width over thickness (Table XIV.) showed a regular increase, in both elastic limit and ultimate resistance, with an *increased* ratio of width over thickness. Agreeably to these results, therefore, the increase of carbon, in Mr. Hill's experiments, should have been accompanied by an increase in both elastic limit and *ultimate resistance*, since an increased ratio of width over thickness accompanied the increase of carbon. The disagreement seems inexplicable, but was probably due to the influence of some unnoticed peculiarity in the treatment of the material in the original plate, or of the specimens themselves.

Table XVII. contains the results of some specimen tests of Siemens steel plate, made by Mr. David Kirkaldy in 1875. The per cents of final stretch are for a length of eight inches, which contained the section of fracture.

Tables XIII., XIV., and XVII. show that, as a general rule, both *the elastic limit and ultimate resistance*, *in mild steel plates*, *increase as the thickness of the plate decreases.*

It is also seen that *the process of annealing decreases both those quantities.*

Although Table XVII. shows no very marked result in regard to final stretch and contraction, yet when it is taken in connection with Table XV., it is clear that *the process of annealing considerably increases both the final stretch and contraction;* in other words, increases the ductility of the material.

Again, Table XVII. shows that the ultimate resistance of steel plates is essentially the same, both in the direction of rolling and across it. This result is in agreement with that of Mr. Hill's experiments, as well as that of French experiments on Bessemer and Martin steel plates (Barba, on the "Use of Steel," translated by A. L. Holley, pages 26 and 29).

Effects of Hardening and Tempering Steel Plates.

In connection with the results given in Table XVII., Mr. Kirkaldy found the following quantities by testing the same sized specimens of the same plates:

Annealed.

THICKNESS.	ULTIMATE RESIST.	FINAL STRETCH.	FINAL CONTRACTION.
0.64 inch....	57,100 pounds....	24.1 per cent....	52.5 per cent.
0.62 inch....	60,500 pounds....	20.2 per cent....	48.7 per cent.

Hardened.

0.64 inch....	64,700 pounds....	22.4 per cent...	49.3 per cent.
0.62 inch....	65,050 pounds....	18.0 per cent .	45.5 per cent.

The hardening was done by heating to a cherry-red and cooling in water at a temperature of 82° Fahr.

Table XVIII. exhibits the effects of simply annealing, or of first oil tempering and subsequent annealing, on specimens of gun steel manufactured by the Midvale Steel Company of Phila., for the Ordnance Dep't U. S. Army, 1884. The results are taken from the Report of Chief of Ordnance for 1884. Oil tempering, or hardening in oil, may be said to almost universally increase both the elastic limit and ultimate resistance.

Annealing in all cases reduces the ultimate resistance and increases the final stretch and final elongation, *i. e.*, increases the ductility. Oil tempering with subsequent annealing, however, is seen in Table XVIII. to produce a very irregular effect, although on the whole it slightly reduces the final elongation, except in the case of high steel, for which the opposite effect is produced. In all cases the combined operations are seen to produce a very material increase in the final contraction. Tempering or hardening increases both the elastic limit and ultimate resistance, but decreases the ductility.

TABLE XVIII.

Specimen Tests.

TREATMENT.	RESULTS BEFORE TREATMENT.			RESULTS AFTER TREATMENT.		
	Ult. Resist. Pounds per square inch.	Per Ct. of Final Elong.	Per Ct. of Final Con.	Ult. Resist. Pounds per Square Inch.	Per Ct. of Final Elong.	Per Ct. of Final Con.
Annealed	100,500	10.5	15.6	87,700	16.0	20.0
"	80,400	9.0	8.0	71,500	25.5	39.7
Oil tempered and annealed	95,068	12.5	13.0	102,900	14.5	22.0
" " "	88,900	20.5	33.0	106,600	17.5	35.0
" " "	94,100	19.5	32.0	102,000	20.5	47.0
" " "	89,500	21.5	31.0	100,200	17.5	35.0
" " "	97,500	21.0	28.7	107,000	18.0	40.0
" " "	89,000	18.5	27.0	107,000	19.0	39.0
" " "	97,000	22.0	28.0	106,000	20.0	41.0
" " "	86,132	11.5	12.0	97,720	15.0	31.0
" " "	84,880	24.0	41.0	94,880	20.0	41.0
" " "	88,400	20.0	37.0	86,920	24.0	48.0
" " "	103,813	5.5	5.2	97,500	17.5	29.0
" " "	95,820	13.0	13.0	94,700	19.0	40.0
" " "	85,780	22.0	34.0	101,160	19.0	34.0

Rivet Steel.

In Table XIX. will be found the results of the experiments of Prof. Alex. B. W. Kennedy (" Engineering," 6th May, 1881). The steel was a very mild grade, for which coefficients of elasticity have already been given.

The specimens were turned down, as shown, from $\frac{11}{16}$, $\frac{15}{16}$ and $1\frac{1}{16}$ inch "rounds."

As in all other cases, the elastic limit and ultimate resistance are given per square inch of original section.

Effect of Reduction of Sectional Area, in connection with Hammering and Rolling.

Tables XX. and XXI. give the results of some of the experiments of Mr. Kirkaldy on Fagersta steel bars. The bars were originally three inches square, in normal cross section,

TABLE XIX.

Rivet Steel.

ORIGINAL DIA. OF BAR.	DIAMETER OF SPECIMEN.	POUNDS OF STRESS PER SQ. IN. AT Elas. Limit.	POUNDS OF STRESS PER SQ. IN. AT Ult. Resist.	PER CENT. OF FINAL STRETCH IN 10 INCHES.
Inch.	Inch.			
$\frac{11}{16}$	0.512	43,400	64,770	22.5
$\frac{11}{16}$	0.505	45,200	65,500	23.5
$\frac{11}{16}$	0.507	45,780	65,770	16.8
		Av. = 44,790	Av. = 65,350	Av. = 20.9
$\frac{15}{16}$	0.616	46,370	67,960	19.2
$\frac{15}{16}$	0.622	46,200	69,310	21.3
$\frac{15}{16}$	0.616	46,220	69,210	19.1
		Av. = 46,260	Av. = 68,830	Av. = 19.9
$1\frac{1}{16}$	0.804	48,600	60,280	21.6
$1\frac{1}{16}$	0.804	47,730	60,750	22.2
$1\frac{1}{16}$	0.786	46,750	63,500	26.0
		Av. = 47,690	Av. = 61,510	Av. = 23.3

and were hammered or rolled down to the dimensions shown in the second column in each table. Specimens were then turned down for testing to the diameters given in the third column, for a length of ten inches. The tables give results for duplicate specimens, one set having been unannealed and the other annealed. The fractures belonging to the 3×3 bars were all granular, and those belonging to the 0.5×0.5 bars

TABLE XX.

Fagersta Steel.—Unannealed.

	BARS IN INCHES.	DIA. OF SPECIMENS, INCHES.	POUNDS OF STRESS PER SQUARE INCH AT		PER CENT. FINAL STRETCH.	PER CENT. FINAL CONT.
			Elas. Limit.	Ult. Resist.		
Hammered..	0.5 × 0.5	0.357	78,300	95,960	6.9	47.0
Rolled......	0.5 × 0.5	0.357	46,800	90,730	16.0	43.0
Hammered..	1 × 1	0.619	49,800	83,720	16.0	44.7
Rolled......	1 × 1	0.619	43,100	87,760	16.2	29.3
Hammered..	1.5 × 1.5	1.009	46,700	77,720	12.6	38.8
Rolled......	1.5 × 1.5	1.009	40,500	79,280	10.2	15.8
Hammered..	2 × 2	1.382	44,800	80,920	19.2	35.5
Rolled......	2 × 2	1.382	38,300	84,073	15.9	20.8
Hammered..	2.5 × 2.5	1.694	34,700	78,840	21.4	26.2
Rolled......	2.5 × 2.5	1.694	36,600	72,585	8.2	10.3
Hammered..	3 × 3	1.994	38,800	70,080	2.3	4.4
Rolled......	3 × 3	1.994	30,400	62,393	2.5	4.4

all silky; the intermediate ones were partially silky and partially granular.

As a part of the hammering and rolling was done at such a temperature as to essentially amount to cold hammering or cold rolling, the annealed specimens show more truly the effects of the two kinds of treatment than the others.

The following results can be at once observed:

The elastic limit, ultimate resistance and final contraction at

TABLE XXI.

Fagersta Steel.—Annealed.

	BARS IN INCHES.	DIA. OF SPECIMENS, INCHES.	POUNDS OF STRESS PER SQUARE INCH AT Elas. Limit.	Ult. Resist.	PER CENT. FINAL STRETCH.	PER CENT. FINAL CONT.
Hammered..	0.5 × 0.5	0.357	47,800	82,120	7.7	55.0
Rolled......	0.5 × 0.5	0.357	41,200	80,210	9.8	51.0
Hammered..	1 × 1	0.619	40,800	78,650	15.2	54.0
Rolled......	1 × 1	0.619	40,100	83,720	11.3	39.7
Hammered..	1.5 × 1.5	1.009	42,300	77,810	13.7	47.7
Rolled......	1.5 × 1.5	1.009	37,800	82,780	15.2	38.7
Hammered..	2 × 2	1.382	41,300	78,893	17.7	41.2
Rolled......	2 × 2	1.382	36,100	80,330	16.8	38.9
Hammered..	2.5 × 2.5	1.694	31,300	66,140	14.7	45.4
Rolled......	2.5 × 2.5	1.694	32,700	71,630	13.8	35.5
Hammered..	3 × 3	1.994	29,800	69,640	7.7	8.4
Rolled......	3 × 3	1.994	27,600	60,193	3.8	5.4

section of fracture increase very much with the decrease of sectional area for either the hammered or rolled bars.

Other and similar experiments verified these conclusions for both higher and milder Fagersta steels.

The per cents. of final stretch are the greatest for the intermediate sectional areas, whether annealed or unannealed, while the relative effects of rolling and hammering are irregular.

The hammered specimens invariably give the greatest final contraction, whether unannealed or annealed.

If unannealed, the hammered specimens give the highest elastic limit and ultimate resistance; if annealed, while this holds true (essentially) for the elastic limit, the rolled specimens give the highest ultimate resistance in four out of the six tests.

Annealing decreases both the elastic limit and ultimate resistance; this was also found to be the case for both higher and milder Fagersta steel specimens, which were similarly tested.

In a set of 24 experiments (precisely the duplicates of those whose results are given in Tables XX. and XXI.) with a higher grade of steel, the greatest final stretch was found to belong to the smaller cross sections; while in a similar set with a milder grade of metal, the greatest final stretch was found with the larger bars, whether the specimens were unannealed or annealed.

Other relative effects of hammering and rolling were somewhat irregular, and seemed to depend on the grade of steel.

Effects of Annealing Steel.

It has not been convenient to separately classify the experimental results showing the effects of annealing, but it has been seen that the process, in general, decreases both the elastic limit and ultimate resistance, and increases the ductility; the lower grades of steel being the least influenced.

Steel Wire.

Table XXII. contains the results of testing, to ultimate resistance, the wire for which the coefficients of elasticity were given in Table I., together with some belonging to the Chrome Steel Co.'s wire, also tested by the engineers of the New York and Brooklyn bridge. The diameter of this wire was about 0.165 inch (No. 8 Birmingham gauge). As will presently be shown, some of the material was cast steel and other Bessemer steel, all having been hardened and tempered.

TABLE XXII.

Steel Wire.

PRODUCER.	NO. OF TESTS.	ULTIMATE RESISTANCE IN POUNDS PER SQ. IN.			PER CENT. FINAL STRETCH.	DIA. FRACTURE, INCH.
		Greatest.	Mean.	Least.		
J. Lloyd Haigh ....(1)	12	182,450	175,340	166,169	4.9 1.7	0.161 0.147
Cleveland Rolling Mills ....(2)	6	182,576	178,400	172,984	4.2 2.1	0.161 0.138
Washburn & Moen ....(3)	6	184,019	176,457	169,706	4.5 0.8	0.147 0.133
Sulzbacher, Hymen, Wolff & Co..(4)	6	179,833	175,291	167,807	4.4 1.8	0.162 0.139
John A. Roebling's Sons Co.......(5)	13	179,019	162,244	125,321	4.8 0.4	0.167 0.130
Johnson & Nephew................(6)	9	206,170	177,706	163,027	6.9 3.1	0.148 0.129
Carey & Moen......................(7)	12	194,227	167,880	126,814	4.2 0.5	0.160 0.125
Chrome Steel Co...................(8)	6	170,150	160,544	150,657	3.4 1.0	

The column "*Per cent. final stretch*" gives the highest values for the 5-feet lengths tested, and the lowest for the 100-feet lengths; these were the greatest and least found.

The column "*Dia. fracture*" gives the greatest and least values of the diameter of the fractured section in decimals of an inch. There seemed to be no definite relation between the ultimate resistance and contraction of section of rupture.

Col. W. A. Roebling states that the character of the above steel was believed to be as follows:

(1) English crucible cast steel.
(2) Open-hearth steel.
(3) English crucible cast steel.
(4) Krupp's Bessemer and cast steel.
(5) Crucible cast steel and American Bessemer steel.
(6) English crucible cast steel.
(7) English crucible cast steel.
(8) Crucible cast steel.

It is therefore seen that steel drawn into wire possesses an excess of resistance over that in larger masses, as bars; it thus exhibits the same general phenomenon as wrought iron under similar circumstances.

Shape Steel.

The results given in Table XXII*a*. belong to steel angles and deck beams rolled by the Phœnix Iron Co. for cruisers

TABLE XXII*a*.

Steel Angles and Deck Beams.

ORIGINAL ANGLE OR DECK BEAM. INCHES.	TEST SPECIMEN.		POUNDS PER SQARE INCH OF ORIGINAL SECTION.		PER CENT. OF FINAL	
	Section. Inches.	Length. Inches.	Elastic Limit.	Ultimate Resistance.	Contraction.	Stretch in 8 Inches.
2¾ × 2¾	1.23 × .40	8	39,500	64,154	52.7	28.5
3 × 2½	1.25 × .33	8		62,622	51.5	25.6
3 × 2½	1.25 × .33	8		64,595	49.4	25.4
3 × 2½	1.25 × .33	8		62,980	53.6	27.3
3 × 2½	1.25 × .32	8		62,733	54.1	25.6
3 × 3	1.25 × .38	8		57,990	61 1	26.9
3 × 3	1.25 × .41	8	39,209	61,991	51.8	25.31
3 × 3	1.25 × .43	8	38,791	62,943	51.1	27.9
3 × 3	1.25 × .46	8	39,414	66,538	51.3	26.3
4 × 3 } *a.*	1.26 × .38	8		67,356	53.8	26.6
4 × 3 } *a.*	1.27 × .38	8		66,307	51.7	25.5
4 × 3 } *a.*	1.27 × .38	8		66,307	52.3	25.7
4 × 3 } *a.*	1.26 × .38	8		67,878	50.8	23.1
4 × 3	1 24 × .41	8		62,390	52.2	26.4
4 × 3	1.25 × .38	8		64,842	49.0	24.4
4 × 3	1.25 × .40	8		63,256	51.8	24.6
4 × 3	1.25 × .40	8		63,485	47.7	28.6
5 × 3	1.24 × .40	8	37,820	61,407	53.1	25.6
5 × 3	1.25 × .42	8	38,920	62,809	50.9	22.8
5 × 3	1.26 × .40	8	37,685	62,675	51.4	25.7
5 × 3	1.26 × .40	8	37,303	63,078	47.0	25.5
6 × 3.25 } *b.*	1.24 × .38	8	37,000	63,803	47.5	23.3
6 × 3.25 } *b.*	1.24 × .34	8	41,321	66,777	48.2	27.0
6 × 3.25 } *b.*	1.25 × .33	8	41,206	63,671	50.3	28.2
6 × 3.25 } *b.*	1.24 × .38	8	39,178	64,632	51.3	29.0
6 × 3.25 } *b.*	1.25 × .39	8	38,777	62,513	51.6	30.3
2 × 2	1.24 × .26	8	42,342	65,237	47.0	27.5
2 × 2	1.24 × .26	8	41,315	63,984	48.5	27.8
2 × 2	1.25 × .26	8	43,600	66,873	45.0	26.0
2 × 2	1.24 × .26	8	42,012	60,310	43.7	25.0
2½ × 2½	1.50 × .28	8	40,596	63,720	48.6	27.8
2½ × 2½	1.50 × .29	8	40,900	62,770	44.6	27.8
2½ × 2½	1.50 × .29	8	41,616	62,933	48.1	26.5
2½ × 2½	1.50 × .26	8	42,460	63,300	47.6	23.8
3½ × 3	1.25 × .41	8	40,850	66,730	50.4	25.0
3½ × 3	1.25 × .42	8	40,380	65,190	57.9	25.8
3½ × 3	1.25 × .48	8	39,230	62,440	64.6	26.5
3½ × 3	1.25 × .42	8	39,570	63,130	56.2	28.3
8 × 5 } *b.*	1.50 × .47	8	42,709	65,177	44.8	....
8 × 5 } *b.*	1.50 × .47	8	39,534	63,375	43.9	26.5
8 × 5 } *b.*	1.50 × .55	8	40,272	66,677	44.3	26.8
8 × 5 } *b.*	1.50 × .50	8	37,007	61,924	50.2	31.3

The angles marked " a " are of Bessemer steel; all other members represented in the table are of open hearth steel. The members marked " b " are deck beams.

built by the U. S. Gov't in 1887. The tests were made at the works of the Phœnix Iron Co., and show excellent metal; the carbon varied from about 0.15 to 0.20 per cent.

Steel Gun Wire.

In 1875, W. E. Woodbridge, M.D., made a large number of tests on the mechanical properties of steel gun wires. The "wires" were about 0.3 inch square, having been drawn down from bars 0.375 inch square. The full, detailed account of these experiments is given in "Report on the Mechanical Properties of Steel, etc., by W. E. Woodbridge, M.D."

The results given in this section are abstracted from the "Report" mentioned.

TABLE XXIII.

Gun Wires—Annealed.

KIND AND MANUFACTURER.	POUNDS OF STRESS PER SQ. IN. AT		PER CENT. FINAL STRETCH.	PER CENT. FINAL CONTRACTION.
	Elas. Lim.	Ult. Resist.		
Crucible steel; Hussey, Welles & Co.....(10)	41,100	92,300	5.8	45.0
" " " " "	26,800	50,700	22.0	67.0
" " " " "	34,000	61,700	16.0	59.0
" " " " "	39,700	71,600	15.0	40.0
Martin steel; N. J. Steel & Iron Co.........	42,300	72,100	17.3	46.0
" " " " " " .....(10)	37,500	71,800	21.5	46.0
" " " " " "	39,200	71,500	18.0	46.0
" " " " " "	49,000	94,600	14.1	37.0
"Gun-screw wire" iron; Trenton Iron Co...	24,700	52,600	21.1	57.0
Chrome steel; Chrome Steel Co ....(10)	43,400	89,000	9.1	61.0
" " " " "	39,200	77,700	6.3	61.0
" " " " "	43,200	71,100	14.2	44.0
" " " " " .... (10)	39,100	89,100	8.7	41.0
Norway iron; Messrs. Naylor & Co	26,000	47,800	28.5	70.0
German steel; Messrs. Park Bros. & Co	22,100	51,700	14.2	42.0
Cemented cast steel; Messrs. Park Bros. & Co.	35,100	61,700	15.7	52.0
" " " " " " "	26,700	57,700	17.0	50.0
" " " " " " "	——	74,900	19.5	33.0

Table XXIII. gives results for wires which were annealed at bright red heat, without oxidation.

The per cents. of final stretch are for five inches of original length, except in the case of specimens marked "(10)," which indicates that the per cents. are for ten inches of original length.

Other tests of wires about 0.3 inch square and *unannealed*, gave the following ultimate resistances in pounds per square inch of original section. The wires were of different varieties of steel, including cast and Martin steel.

130,800.	84,400.
106,900.	58,700.
108,200.	59,200.
135,000.	

The elastic limit varied from 35 to 92 per cent. of the ultimate resistance; and the per cent. of final contraction varied from 11 to 43. The effect of annealing, both on resistance and ductility, is made very evident by comparing the two sets of results.

Effect of Low and High Temperatures on Steel.

The results of some German experiments and the experience of the Massachusetts Railroad Commissioners with steel rails for one year, have already been given in connection with wrought iron.

Table XXIV. contains the results of the experiments by Mr. Chas. Huston, as given in the "Journal of the Franklin Institute" for Feb., 1878.

"U. R." is the ultimate resistance in pounds per square inch, while "C." is the per cent. of contraction at the section of fracture.

Each result is a mean of three experiments.

TABLE XXIV.

KIND OF MATERIAL.	"COLD."		572° FAHR.		932° FAHR.	
	U. R.	C.	U. R.	C.	U. R.	C.
Charcoal boiler-plate, piled..........	55,400	26	63,100	23	65,300	21
Siemens-Martin (exceptionally soft)...	54,600	47	66,100	38	64,400	34
Crucible steel (ordinarily soft)	64,000	36	69,300	30	68,600	21
Crucible steel (not quite hard enough to temper)........................	78,400	27	82,800	16	77,300	20

The method of producing rupture at the desired place was such as to make the specimens partake, to some extent at least, of the nature of "short" ones, which, however, would not affect the *comparative* results.

It will be observed that the charcoal boiler-plate iron gave the highest resistance at the highest temperature, but that all the steels gave the highest "U. R." at the intermediate temperature 572° Fahr.

It is somewhat remarkable that in every case but the last (the hardest steel) the contraction of fractured section *decreased* with the rise in temperature.

Other results for steel will be found in Table IX. of Article 35, and it will be seen that they tend to confirm the conclusions just drawn.

In the "Annales des Ponts et Chaussées" for Feb., 1881, page 226, are given the number of breakages of steel rails which occurred in Russia in 1879. The following is the table showing the number of failures for each month of the year.

These results conflict somewhat with those given by the Massachusetts Railroad Commissioners, in Art. 32.

January	699
February	598
March	854
April	235
May	235
June	160
July	247
August	156
September	214
October	328
November	341
December	692

The greatest number is found in the coldest half of the year, but the greatest number for any one month belongs to March, which is not the coldest month. It is probable that this is due to the effect of long wear on the frozen ground of the entire winter in connection with the possible alternate freezing and thawing of the ground in the month of March.

Effect of Manipulations common to Constructive Processes; also Punched, Drilled, and Reamed Holes.

Table XXV. gives the results of the experiments of Mr. Hill (paper already cited in connection with Tables V. and VI.) on specimens of exactly the same size, and from the same steel plates, as those given in Table XVI.

The different methods of preparing and treating the specimens are shown in the column headed "*Treatment of specimen.*"

With the exception of those of the lowest two 0.30 per cent. specimens, the results are averages of a number of experiments.

From these results Mr. Hill concludes: 1st. "That both shearing and punching are injurious, *per se*, to all grades of steel, and cold punching far more so than shearing.

2d. "That both these operations affect the elastic limit . . . far more than they do the ultimate resistance.

TABLE XXV.

Open-Hearth Steel.

PER CENT. OF CARBON.	TREATMENT OF SPECIMEN.	POUNDS OF STRESS PER SQUARE INCH AT		PER CENT. OF FINAL STRETCH.
		Elas. Lim.	Ult. Resist.	
0.30	Cut in planer	49,431	94,396	17.00
0.30	Sheared	32,370	74,980	10.00
0.30	Punched	0	63,410	0.45
0.30	Punched and hammered cold	0	87,540	0.55
0.30	Punched, hammered and annealed	55,780	100,410	7.50
0.40	Cut in planer	63,475	87,095	15.00
0.40	Sheared	46,900	75,330	7.00
0.40	Punched	0	68,890	5.00
0.40	Sheared and annealed	59,350	86,160	16.00
0.40	Punched and tempered	52,780	103,560	7.00
0.50	Cut in planer	65,185	84,092	12.50
0.50	Sheared	51,666	79,900	5.00
0.50	Punched	0	78,400	4.00
0.50	Sheared and tempered	60,375	87,293	17.00
0.50	Punched and annealed	57,960	84,900	12.00

3d. "That apparently the lower grades of steel are proportionately more injuriously affected than the higher grades. . . .

4th. "That the injurious effects of shearing and punching can be almost entirely counteracted by subsequent annealing, or tempering in oil from a low heat.

5th. "That annealing restores the elastic limit to a greater extent than the ultimate, while tempering as above, on the contrary, largely increases the ultimate resistance and ductility, but does not so fully restore the elastic limit."

Table XXVI. shows the results of other experiments by

Mr. Hill, from the same paper, on the relative effects of drilling, punching and reaming, and punching (with and without annealing) rivet holes in steel plates. These plates were precisely the same as those from which the results given in Tables XVI. and XXV. were obtained. The dimensions of the different specimens are given in the second column from the left.

TABLE XXVI.

Open-Hearth Steel.

PER CENT. CARBON.	PLATE SPECIMEN.	HOLE.	ULT. RESIST., LBS. PER SQ. IN. OF EFFECTIVE SECTION.	PER CENT. ELONGATION OF HOLE.
0.30	¾ in. rolled plate, cut in planer on all edges. Strips 2½ ins. wide, 18 inches long.	Drilled, 1 in. diameter	98,966	22.0
0.30		Punched, 0.935 in. / Reamed to 1.1 in. } diameter	100,700	20.0
0.30		Punched and annealed, 0.935 in. diameter	78,970	21.0
0.30		Punched, 0.935 in. diameter	66,108	3.3
0.40	⅜ in. rolled plate, cut as above. Strips 1¼ ins. wide, 15 inches long.	Drilled, 0.6 in. diameter	99,747	15.6
0.40		Punched, 0.5 in. / Reamed to 0.62 in. } diameter	104,253	19.0
0.40		Punched and annealed, 0.62 in. diameter	87,910	18.9
0.40		Punched, 0.62 in. diameter	80,550	5.0
0.50	3/16 rolled plate, cut as above. Strips 1 inch wide 12 inches long.	Drilled, 0.4 in. diameter	86,963	29.0
0.50		Punched, 0.4 in. / Reamed to 0.5 in. } diameter	89,043	26.0
0.50		Punched and annealed, 0.45 in. diameter	84,951	31.0
0.50		Punched, 0.45 in. diameter	82,330	15.0

The relative influences of the different operations of drilling, punching, etc., will be emphasized by comparing the results in Table XXVI. with those given in Table XVI.

The operation of punching is seen to considerably injure the material in the vicinity of the punched hole. In every case, the punched specimen gives very much less resistance than any other. It is further to be observed that the injurious effect of the punch is only partially removed by annealing.

Mr. Hill draws the following conclusions:

1st. "That the 'reamed' hole is the strongest, and following in the order of strength come the 'drilled,' the 'punched and annealed,' and, lastly, the 'cold punched' hole. This graduation is well defined in all three groups. That the reamed hole should be stronger than the drilled hole, I am unable to account for.

2d. "That the injurious effect of punching is local, and can be entirely removed by enlarging the hole sufficiently with either drill or reamer. The amount of drilling or reaming required after punching varies with the thickness of the plate and grade of steel.

3d. "That although annealing is in a measure beneficial in partially restoring strength and ductility to the punched plate, it will hardly be found available for bridge work; for, if you attempt to anneal before riveting, the holes will not fit; if after riveting, you create internal strains of which no account can be taken, and which may subsequently produce failure. Moreover, with proper machinery, punching and reaming will be found much cheaper than 'punching and annealing.'"

In regard to the excess of resistance with the "punched and reamed" hole as compared with the drilled, it may be remarked that in every case the hole, as reamed, was greater in diameter than the drilled hole. There was, consequently, less material to be ruptured in the former case than the latter. This diminution of cross section makes the reamed specimen a "smaller" one than the other and intensifies the "shortening" effect of the rivet hole. Both these influences would increase the ultimate resistance, per square inch, of the reamed specimen beyond the drilled one; but whether they supply the explanation for the whole difference is an open question.

It will be observed that the ultimate resistances for the drilled and reamed specimens, in Table XXVI., run considerably higher than the corresponding quantities in Table XVI., or, indeed, those in Table XXV.; for which the explanation is simple and obvious. The specimens for Table XXVI. carried one rivet hole each; and at this rivet hole failure took place. The effect of the hole in any specimen was the restriction of the contraction to its immediate vicinity, and the partial prevention of the latter, which reduced the specimen, to a great extent, to a "short" one. An increase of ultimate resistance was, consequently, to be expected.

The *decrease* of ultimate resistance with the *increase* of carbon has already been remarked upon in connection with Table XVI.

The experiments of Prof. Alex. B. W. Kennedy, on the effect of punching and drilling holes, in mild-steel boiler plate, are well illustrated by Table XXVII., which is condensed from one given in London "Engineering," 6th May, 1881. None of these plates were annealed, but all were drilled or punched as received.

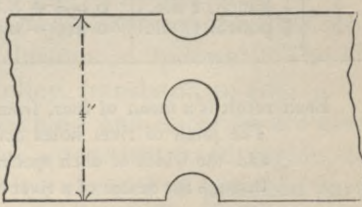


Fig. 2

Within the limits of these experiments, Prof. Kennedy observes, neither the width of the test piece nor the different diameters of die, had any essential influence on the results.

The injurious effect of punching is shown by the fact that the punched specimens gave only 92 to 98 per cent. of the resistance of the drilled ones.

It will be noticed that both the drilled and punched specimens gave higher resistances than the natural plate. This is due to the "shortening" and other influence (*i. e.*, the disturbance of the lateral strains) of the rivet holes, as before observed, and explained in Art. 32, "*Ultimate Resistance and Elastic Limit.*"

TABLE XXVII.

Punched and Drilled Holes.

THICKNESS IN INCHES.	HOLE, INCHES.	DIAMETER OF HOLE, INS.	TENACITY IN TONS, SQ. IN. NET SECTION.	TENACITY COMPARED WITH THAT OF	
				Nat. Plate.	Dril'd Plate.
$\frac{1}{4}$	Drilled.........	0.940	38.12	1.105	——
$\frac{1}{4}$	Drilled.........	0.940	38.22	1.108	——
$\frac{1}{4}$	$\frac{7}{8}$ punch, $\frac{31}{32}$ die.	0.912 — 0.876	35.04	1.000	0.918
$\frac{1}{4}$	$\frac{7}{8}$ punch, $\frac{29}{32}$ die.	0.892 — 0.871	34.44	1.025	0.902
$\frac{3}{8}$	Drilled.........	0.926	35.39	1.126	——
$\frac{3}{8}$	Drilled.........	0.934	34.90	1.111	——
$\frac{3}{8}$	$\frac{7}{8}$ punch, 1 die..	0.998 — 0.890	33.91	1.073	0.965
$\frac{3}{8}$	$\frac{7}{8}$ punch, $\frac{15}{16}$ die.	0 945 — 0.875	34.38	1.096	0.978

Each result is a mean of four, from plate specimens 2, 4, 6 and 8 inches wide. The pitch of rivet holes across the middle of specimens was 2 inches, and the width of each specimen was so chosen that each side passed through the centre of a rivet hole, as shown in Fig. 2. A "ton" is 2,240 pounds. The two diameters of punched holes are for the two sides of the plate.

A duplicate set of experiments on 32 specimens of a somewhat softer steel boiler plate, gave essentially the same results (see "Engineering," 6th May, 1881).

By experimenting on mild Fagersta steel plates with the thicknesses $\frac{1}{8}$, $\frac{1}{4}$, $\frac{3}{8}$, $\frac{1}{2}$ and $\frac{5}{8}$ inch, Mr. Kirkaldy found the ratio of the resistance of drilled specimens over that of punched ones to vary from about 1.1 (for $\frac{1}{8}$, $\frac{1}{4}$ and $\frac{3}{8}$-inch specimens) to 1.5 (for $\frac{5}{8}$ specimen) when unannealed, and to be about 1.1 for all the thicknesses when annealed. All the specimens were 12.5 inches wide, with three rows of 0.77 inch holes, pitched 2.5 inches apart, running across the specimens. The average resistance for square inch of net section was greater than that of the original plate for the drilled holes, but considerably less for the punched ones.

Mr. Kirkaldy states, " the loss from punching is not constant, but varies with the thickness, and also with the hardness of the material." He also concluded that punching hardens the material in the vicinity of the punch, and that the effect of punching is counteracted " to a considerable extent " by annealing.

The results of Mr. Hill's experiments, as given in Table XXVI., show, for the thickness of plates there used, that by enlarging the diameter of the punched hole from 0.1 inch to 0.165 inch, by reaming, the injurious effect of the punch is entirely removed.

Experiments on French steel plates, produced by the Bessemer and Martin processes (*métal fondu*), confirm this result and form a basis for other conclusions, as follows (" The Use of Steel," by J. Barba, A. L. Holley, translator, p. 40):

" 1st. That the effects of punching and shearing are essentially local and spread only over a very restricted region, less than 0.039 inch on the edges of the sheared or punched parts;

" 2d. That no cracks exist in this altered region;

" 3d. That tempering destroys the effects of shearing and punching by bringing the metal back to the state it would be in if drilling or planing had been substituted for punching or shearing;

" 4th. That annealing alone or after tempering destroys, as tempering alone does, the alterations caused by shearing and punching."

These conclusions relate to plates from 0.27 inch to 0.46 inch thick.

In first-class practice, holes in steel plates and shapes are frequently first punched and then reamed to a diameter 0.125 inch greater.

Experiments on some narrow specimens of steel plate seem to indicate that conical punching (the die 0.16 to 0.20 inch greater in diameter than the punch) injures the material less than cylindrical punching (with a clearance of perhaps $\frac{1}{16}$ inch).

In the working of steel plates and shapes, during ordinary constructive processes, all local pressure of great intensity, and hammering while cold or at a low temperature, tend to produce internal strains of great intensity or other changes in molecular condition which cause the finished plate or shape to be liable to great brittleness and unlooked-for failure of a local character.

For these reasons M. Barba gives the following directions in regard to the working of steel :

" 1st. Avoid any local pressure of whatever nature it may be ; 2d. If local pressures have been produced by blows of a hammer, the action of the punch, etc. (which may, as we have seen, cause ruptures), heat the piece to a cherry-red in a very regular manner and as much as possible in its entirety—the whole of it at once—and let it cool in the open air on a homogeneous surface, which has all over equal conducting power. This simple reheating, which may be considered as annealing for plates and bars, on account of their slight thickness, restores to the worked metal its original qualities, even if it was in a very unstable state of equilibrium."

If a large amount of working (such as bending or curving) of a single kind is to be done to a single piece, it is best, if possible, to heat to a cherry-red and do the work by stages, rather than all at once ; and then anneal after the working is completed. If the working is local and the heating irregular, it may be necessary to anneal once or more during the progress of the work.

Local heating in the production of the ordinary steel eye-bar head, for example, frequently gives much trouble, unless resort is had to subsequent annealing.

These difficulties in the working of steel are found more pronounced in the higher grades, and much experience is still needed before they can be entirely overcome.

On account of the homogeneous character of the metal, upsetting processes, as in riveting, etc., seem to injure the molecular condition of steel much less than that of iron.

Bauschinger's Experiments on the Change of Elastic Limit and Coefficient of Elasticity.

The details of these experiments are given in "Der Civilingenieur," Part 5, 1881. The manner of application of the tests, and remarks on the quantities, elastic limit, stretch limit, and final load, will be found by referring to page 262. The following is the notation :

E. L. = elastic limit in pounds per square inch.
S.-L. = stretch limit in pounds per square inch.
F. L. = final load in pounds per square inch.
E. = coefficient of elasticity in pounds per square inch.

Bessemer Steel.

	IN ORIGINAL CONDITION.	AFTER 69 HRS.	AFTER 0.5 HR.	AFTER 68 HRS.
E. L.	25,970	43,272	8,760	14,970
S.-L.	40,380	51,920	55,470	71,850
F. L.	46,140	57,690	70,080	——
E.	29,848,000	29,549,000	29,009,000	30,146,000

A small specimen of this Bessemer steel, about an inch in diameter, gave an ultimate resistance of 75,800 pounds per square inch.

The elastic limit rises twice after two long periods of rest, and falls in a very marked manner after the short rest of 0.5 hour.

The stretch limit rises steadily while the coefficient of elasticity falls twice and then rises above its original value.

Prof. Bauschinger was the first to determine, in regard to Bessemer steel, that by stretching the metal beyond its elastic limit its elasticity is elevated, not only during the time of action of the load, but also during a longer period of rest, without load, of one or more days; and that, in this manner, the elastic limit may exceed the load which caused the stretching. (Dingler's Journal, Band 224.)

Fracture of Steel.

The character of steel fractures has been incidentally noticed, in some cases, in the different tables.

If the steel is low, or if some of the higher grades are thoroughly annealed, the fracture is fine and silky, provided the breakage is produced gradually. In other cases the fracture is partly granular and partly silky, or wholly granular.

In any case a sudden breakage may produce a granular fracture.

Effect of Chemical Composition.

The ten sets of results given in Table XXVIII. are taken from a great number of similar ones established by the United States Test Board, "Ex. Doc. 23, House of Rep., 46th Congress, 2d Session." The physical phenomena developed in connection with a given chemical constitution may be observed at a glance.

The amount of final contraction of fractured section may be accurately estimated by comparing the ultimate resistances of the original and final sections.

TABLE XXVIII.

Percentages of									Ultimate resistance in pounds per sq. in. of		Per cent. of final stretch.	Coefficient of elasticity.	Elastic limit, lbs. per sq. in.
Sulphur.	Phosphorus.	Silicon.	Graphite.	Comb. Carbon.	Manganese.	Copper.	Cobalt.	Nickel.	Original Section.	Final Section.			
0.004	0.014	0.145	0.011	0.246	0.020	0.002	0.008	0.010	37,800	81,000	32.67	25,945,000	23,000
0.003	0.017	0.076	0.006	0.386	0.060	0.007	0.013	0.015	68,000	121,400	23.17	29,090,000	38,600
0.003	0.015	0.135	0.018	0.677	0.038	0.003	trace	0.020	95,666	132,000	15.92	24,581,000	51,174
trace	0.015	0.134	0.040	0.830	0.046	0.003	trace	0.023	113,106	133,100	8.17	26,076,000	50,553
trace	0.015	0.190	0.030	1.112	0.045	0.003	0.009	0.016	120,602	128,700	8.00	27,056,000	66,493
trace	0.015	0.106	0.033	1.285	0.273	0.005	0.013	0.018	120,602	129,500	5.83	25,398,000	71,709
—	—	—	—	—	—	—	—	—	93,711	154,300	10.75	33,054,000	50,522
none	0.017	0.246	0.082	1.244	0.262	none	none	none	135,269	144,800	7.33	27,542,000	75,294
0.029	0.045	0.011	0.014	0.116	0.192	0 002	trace	trace	52,000	146,100	34.33	28,085,000	28,500
0.011	0.109	0.168	none	0.042	0.051	0.028	0.028	0.044	48,245	101,300	29.00	26,282,000	28,350

The specimens were circular in section and either 0.625 inch or about 0.8 inch in diameter, while all possessed a length of 6 inches.

Art. 35.—Copper, Tin and Zinc, and their Alloys.—Phosphor Bronze.

Coefficients of Elasticity.

Table I. gives the coefficients of elasticity (E) of the various metals and their alloys, according to the various authorities. These coefficients were determined by experiments in tension, and E is given in pounds per square inch.

TABLE I.

METAL.	AUTHORITY.	E.	REMARKS.
Brass...........	Tredgold.	8,930,000	Cast metal.
Tin............	"	4,608,000	" "
Zinc............	"	13,680,000	" "
Gun Metal......	"	9,873,000	Copper, 8; Tin, 1.
Zinc............	Wertheim.	12,828,000	Ingot.
Zinc............	"	12,420,000	"*Étiré.*"
Copper.........	"	17,702,000	"
Copper.........	"	14,958,000	Annealed.
Brass...........	"	12,148,000	$ZnCu_2$.
"Berlin Brass."..	"	13,192,000	Zn_8Cu_{17}.
Gun Bronze.....	Thurston.	11,468,000	Copper, 0.90; Tin, 0.10 (nearly).
Alloy...........	"	13,514,000	Copper, 0.80; Zinc, 0.20.
Alloy...........	"	14,286,000	Copper, 0.625; Zinc, 0.375.
Tobin's Alloy ...	"	4,545,000	Composition, below table.
Copper.........	"	9,091,000	Cast metal.

Tobin's alloy is a composition of copper, tin, and zinc, in the proportions (very nearly) of 58.2, 2.3, and 39.5, respectively. The value of E for this metal, and those for the two preceding and one following it, are calculated for small stresses and strains given by Prof. Thurston in the "Trans. Am. Soc. Civ. Engrs.," for Sept., 1881.

TABLE II.

Cast Tin.

p.	E.	p.	E.
1,950	1,147,000	3,200	96,400
2,360	472,000	4,000	41,540
2,580	172,000	Broke at 4,200 lbs.	

TABLE III.

Cast Copper.

p.	E.	p.	E.
800	10,000,000	12,000	18,750,000
2,000	9,091,000	13,600	8,193,000
4,000	9,091,000	16,000	2,235,000
8,000	14,815,000	22,000	137,000

Broke at 29,200 lbs.

The values of E (stress over strain) for different intensities of stress (pounds per square inch) for cast tin, cast copper, and Tobin's alloy, are given in Tables II., III. and IV.

"p" is the intensity of stress in pounds per square inch, at which the ratio E exists.

Each of these metals is seen to give a very irregular elastic behavior.

Tables II., III. and IV. are computed from data given by Prof. Thurston in the United States Report (page 425) and "Trans. Am. Soc. Civ. Engrs.," already cited.

TABLE IV.

Tobin's Alloy.

p.	E.	p.	E.
2,000	4,545,000	18,000	5,455,000
4,000	4,545,000	24,000	5,941,000
6,000	4,688,000	30,000	6,250,000
8,000	4,938.000	40,000	6,390,000
10,000	5,263,000	50,000	4,744,000
14,000	5,110,000	60,000	3,436,000

Broke at 67,600 lbs.

Ultimate Resistance and Elastic Limit.

Table V. is abstracted from the results of the experiments of Prof. Thurston as given in the " Report of the U. S. Board Appointed to Test Iron, Steel and other Metals," and " Trans. Am. Soc. of Civ. Engrs.," Sept. 1881. The composition of the various alloys was as given in the table, which also contains results for pure copper, tin and zinc. All the specimens were of cast metal.

The mechanical properties of the copper-tin-zinc alloys have been very thoroughly investigated by Prof. Thurston (" Trans. Am. Soc. of Civ. Engrs.," Jan. and Sept., 1881). As results of his work he has found that the ultimate tensile resistance, in pounds per square inch, of " ordinary bronze, composed of copper and tin ...., as cast in the ordinary course of a brass founder's business," may be well represented by :

$$T_c = 30{,}000 + 1{,}000t\ ;$$

" where t isthe percentage of tin and not above 15 per cent."

TABLE V.

PERCENTAGE OF			POUNDS STRESS PER SQ. INCH AT		PER CENT. FINAL	
Copper.	Tin.	Zinc.	Elastic Limit.	Ult. Resist.	Stretch.	Contract'n.
100	00	00	11,620	19,872	0.05	10.0
100	00	00	11,000	12,760	0.005	8.0
100	00	00	14,400	27,800	0.065	15.0
90	10	00	15,740	26,860	0.037	13.5
80	20	00	——	32,980	0.004	00.0
70	30	00	5,585	5,585	——	00.0
62	38	00	688	688	——	00.0
52	48	00	2,555	2,555	——	00.0
39	61	00	2,820	2,820	——	00.0
29	71	00	——	1,648	——	00.0
21	79	00	——	4,337	——	00.0
10	90	00	3,500	6,450	0.07	15.0
00	100	00	1,670	3,500	0.36	75.0
	Queensl'd					
00	100	00	——	2,760	——	47.0
	Banca.					
00	100	00	2,000	3,500	0.36	86.0
Gun	Bronze.					
90	10	00	10,000	31,000	4.6	——
80	00	20	——	33,140	32.4	40.0
62.5	00	37.5	——	48,760	31.0	29.5
58.2	2.3	39.5	——	67,600	4.0	8.0
100	0.0	0.0	——	29,200	7.5	16.0
90.56	0.0	9.42	——	——	——	——
81.91	0.0	17.99	10,000	32,670	31.4	43.0
71.20	0.0	28.54	9,000	30,510	29.2	38.0
60.94	0.0	38.65	16,470	41,065	20.7	28.0
58.49	0.0	41.10	27,240	50,450	10.1	17 0
49.66	0.0	50.14	16,890	30,990	5.0	11.5
41.30	0.0	58.12	3,727	3,727	——	——
32.94	0.0	66.23	1,774	1,774	——	——
20.81	0.0	77.63	9,000	9,000	0.16	0.0
10.30	0.0	88.88	14,450	14,450	0.39	0.0
0.0	0.0	100.00	4,050	5,400	0.69	0.0
70.0	8.75	20.25	18,000 (?)	31,600	0.36	0.0
57.50	21.25	21.25	1,300	1,300	——	——
45.0	23.75	31.25	2,196	2,196	——	——
66.25	23.75	10 00	3,294	3,294	——	——
58.22	2.30	39 48	30,000 (?)	66,500	3.13	7.0
10.00	50.00	40 00	5,000 (?)	9,300	0.7	0.0
60.00	10.00	30.00	21,780 (?)	21,780	0.15	0.0
65.00	20.00	15.00	——	3,765	——	——

TABLE V.—*Continued.*

PERCENTAGE OF			POUNDS STRESS PER SQ. INCH AT		PER CENT. FINAL	
Copper.	Tin.	Zinc.	Elastic Limit.	Ult. Resist.	Stretch.	Contract'n.
70.00	10.00	20.00	24,000 (?)	33,140	0 31	——
75.00	5.00	20.00	12,000 (?)	34,960	3.2	5.4
80.00	10.00	10.00	12,000 (?)	32,830	1.6	4.0
55.00	0.50	44.50	22,000	68,900	9.4	25.0
60.00	2.50	37.50	22,000	57,400	4.9	6.6
72.50	7.50	2.00	11,000	32,700	3.7	11.0
77.50	12.5	10.00	20,000	36,000	0.7	0.0
85.00	12.5	2.50	12,000 (?)	34,500	1.3	3.0

The values of the elastic limit in the lower part of the table were not at all well defined.

"For brass (copper and zinc) the tenacity may be taken as :

$$T_z = 30{,}000 + 500z.$$

where z is the percentage of zinc and not above 50 per cent."

He found that a large portion of the copper-tin-zinc alloys is worthless to the engineer, while the other, or valuable portion, may be considered to possess a tenacity, in pounds per square inch, well represented by combining the above formulæ as follows :

$$T_{zt} = 30{,}000 + 1{,}000t + 500z.$$

These formulæ are not intended to be exact, but to give safe results for ordinary use within the limits of the circumstances on which they are based.

Prof. Thurston found the "strongest of the bronzes" to be composed of :

Copper	55.0
Tin	0.5
Zinc	44.5
	100.0

This alloy possessed an ultimate tensile resistance of 68,900 pounds per square inch of original section, an elongation of 47 to 51 per cent. and a final contraction of fractured section of 47 to 52 per cent.

The first and sixth alloys of copper, tin and zinc, in Table V., are called by Prof. Thurston "Tobin's alloy." "This alloy, like the maximum metal, was capable of being forged or rolled at a low red heat or worked cold. Rolled hot, its tenacity rose to 79,000 pounds, and when moderately and carefully rolled, to 104,000 pounds. It could be bent double either hot or cold, and was found to make excellent bolts and nuts."

As just indicated for the particular case of the Tobin alloy, the manner of treating and working these alloys exerts great influence on the tenacity and ductility.

Baudrimont found for a copper wire 0.0177 inch in diameter, an ultimate resistance of about 45,000 pounds per square inch, the wire being unannealed, while for a diameter of 0.064 inch, Kirkaldy found about 63,000 pounds per square inch.

Prof. Thurston states: "brass, containing copper 62 to 70, zinc 38 to 30, attains a strength in the wire mill of 90,000 pounds per square inch, and sometimes of 100,000 pounds."

All of Prof. Thurston's specimens were what may be called "long" ones, *i.e.*, they were turned down to a diameter of 0.798 inch for a length of five inches, giving an area of cross section of 0.5 square inch.

Gun Metal.

Major Wade ("Reports of Experiments on Metals for Cannon," 1856) made many experiments on a gun metal composed of copper 89 and tin 11 (very nearly), called gun bronze.

He found that different methods of manipulation of the molten metal and of treatment, as in cooling, affected to a great extent its resistance.

TABLE VI.

Gun Bronze.

MINUTES IN LADLE.		ULTIMATE TENSILE RESISTANCE, POUNDS PER SQUARE INCH.				
		Gun-heads.			Small bars.	
0	Highest....	17,698	17,825	17,761	50,973	31,132
15	Mean......	29,216	28,775	28,995	52,330	28,153
29	Lowest. ...	23,381	24,064	23.722	56,786	28,082

Density varied from 7.978 to 8.823.

Table VI. gives the average results of a large number of experiments made by Major Wade. It shows the great range in the tenacity of the different specimens.

General Results.

Table VII. gives general results of various European experimenters. T represents the ultimate tensile resistance in pounds per square inch.

Some of these results are from the experiments of early investigators, who attached little importance to the size and form of the test specimen. In all the cases the results would be more valuable if the circumstances of testing were given. Those belonging to the more unusual alloys, however, possess considerable general interest in spite of the uncertainty surrounding their experimental origin. The presence of a little phosphorus in copper is seen to increase its resistance in a marked manner.

TABLE VII.

METAL.	EXPERIMENTER.	*T.*
Copper, wrought	Anderson.	33,600
Copper, cast	"	19,000 to 26,100
Copper, bolts, with phosphorus 0.01	"	16,900
Copper, bolts, with phosphorus 0.015	"	38,400
Copper, bolts, with phosphorus 0.02	"	45,400
Copper, bolts, with phosphorus 0.03	"	47,900
Copper, bolts, with phosphorus 0.04	"	50,000
—Proportions.—		
Gun metal, copper 12, tin 1	"	29,000
Gun metal, copper 11, tin 1	"	30,700
Gun metal, copper 10, tin 1	"	33,000
Gun metal, copper 9, tin 1	"	38,100
—Weights in 100.—		
Alloy, copper 84.29, tin 15.71	Mallet.	36,100
Alloy, copper 82.81, tin 17.19	"	34,050
Alloy, copper 81.10, tin 18.90	"	39,650
Alloy, copper 78.97, tin 21.03, brasses	"	30,500
Alloy, copper 34.92, tin 65.08, small bells	"	3,140
Alloy, copper 15.17, tin 84.83, speculum metal	"	6,950
Tin	"	5,600
—Proportion.—		
Aluminium bronze, copper 90, Al. 1	Anderson.	73,000
Aluminium bronze, greatest	"	96,300
Tin, cast	Rennie.	4,740
Zinc, cast	Stoney.	3,000
Brass, yellow	Rennie.	18,000
Brass, yellow, copper 67, zinc 33	Anderson.	28,900
Brass, tube, copper 62, zinc 38	Everitt.	103,000
Brass, tube, copper 70, zinc 30	"	80,600
Brass, wire	Dufour.	91,300
Muntz metal, copper 60, zinc 40	Anderson.	49,300
Sterro-metal, copper 10, iron 10, zinc 80	"	7,100
Sterro-metal, copper 60, iron 3, zinc 39, tin 1.5	"	53,800
Sterro-metal, copper 60, iron 4, zinc 44, tin 2.0		
—cast in sand	"	43,100
—cast in iron, annealed	"	54,300
—cast in iron, forged red hot	"	69,400
Copper 60, iron 2, zinc 37, tin 1	"	76,200
Copper 60, iron 2, zinc 35, tin 2	"	85,100
Copper 55, iron 1.77, zinc 42.36, tin 0.83—		
cast	"	60,500
—forged red hot	"	76,200
—drawn cold	"	85,100

Phosphor Bronze, and Brass and Copper Wire.

Table VIII. contains the results of the experiments of Mr. Kirkaldy on phosphor bronze, with two results each for brass and copper wire.

TABLE VIII.

Phosphor Bronze.

METAL.	*E. L.*	ULT. RESIST., POUNDS PER SQUARE INCH.		FINAL STRETCH.
		Unannealed.	Annealed.	
Phosphor bronze	55,800	75,000	——	——
" "	55,200	74.000	——	——
" "	40.500	63.700	——	——
" "	26,300	54.100	——	——
" "	21,700	50,100	——	——
" " wire	——	102,750	49,400	37.5
" " "	——	121,000	47,800	34.1
" " "	——	121,000	53,400	42.4
" " "	——	139,100	54,200	44.9
" " "	——	159 500	58,900	46 6
" " "	——	151,100	64,600	42.8
Copper wire	——	63.100	37,000	34.1
Brass wire	——	81,200	51,500	36.5

The diameter of the phosphor bronze wire varied from about 0.06 inch to 0.11 inch ; that of the copper wire was 0.064 inch, and that of the brass wire 0.0605 inch.

The final stretch is the per cent. of the original length, and belongs to the annealed wire.

The contraction of fractured section for the phosphor bronze specimens varied from about four to thirty-two per cent. of original area.

The first five results belong to metal of the same composition but subjected to different treatment.

Some specimens tested by Mr. Kirkaldy gave as low as about 21,700 pounds per square inch.

Experiments on Rolled Copper by the "Franklin Institute Committee."

The results of these experiments are contained in the "Journal of the Franklin Institute," for 1837.

That committee found, as a mean of 66 experiments, the ultimate resistance of rolled copper to be 32,826 pounds per square inch. The temperature of the copper varied from 62° to 82° *Fahr.* "The irregularities of strength in the different specimens varied from 1.9 to 4.8 per cent. of the mean tenacity."

The resistance was found to be the greatest at ordinary temperatures, and to decrease with acceleration as the temperature increased.

Variation of Ultimate Resistance and Stretch at High Temperatures.

The results contained in Table IX. were obtained at Portsmouth (England) Dockyard, and were published in the *Engineer*, 5th Oct., 1877. "R" is the ultimate tensile resistance in pounds per square inch, and "*St.*" is the per cent. of stretch for a length of 10 inches in all except the last (steel) specimen.

At 250° to 350° the gun-metal specimens lose about half their ultimate resistance and nearly all their ductility. Phosphor bronze loses about one-third of its resistance and two-thirds of its ductility at 300° to 400°. Muntz metal and copper are not much affected, nor is cast iron. Wrought iron and steel gain in ultimate resistance but lose in ductility. These

TABLE IX.

Temperature, Fahr.	Gun metal rods 1 inch in diameter.											
	Copper...... 87.75 Tin.......... 9.75 Zinc......... 2.5		.... 87.75 .... 9.75 .... 2.5		.... 91.00 .... 7.00 .... 2.00		.... 85.00 .... 5.00 .... 10.00		.... 83.00 .... 2.00 .... 15.00		.... 94.5 .... 5.0 .... 2.5	
	R.	*St.*	*R.*	*St.*	*R.*	*St.*	*R.*	*St.*	*R.*	*St.*	*R.*	*St.*
Atmos.	34,240	12.5	36,800	8.8	33,600	16.0	33,600	21.0	31,040	26.0	35,840	20.0
100°	32,320	10.0	—	—	33,600	15.5	35,200	18.0	30,720	26.0	—	—
150°	33,600	11.0	—	—	33,600	14.0	33,920	19.5	28,800	25.5	—	—
200°	31,040	10.0	34,240	8.8	29,440	9.0	33,470	19.0	29,440	26.3	28,160	11.0
250°	32,320	10.0	24,640	5.0	16,320	3.0	32,960	16.0	28,160	26.0	23,040	6.0
300°	32,000	10.0	18,880	0.7	16,960	0.0	33,960	18.3	27,840	23.0	16,320	0.7
350°	28,800	8.3	18,880	0.0	—	—	31,680	17.0	27,840	25.0	—	—
400°	15,680	0.8	—	—	16,640	0.0	16,640	2.0	27,840	25.0	—	—
450°	16,960	0.0	—	—	—	—	16,000	2.0	9,728	1.2	—	—
500°	16,000	0.0	—	—	17,600	0.0	14,720	2.0	9,728	0.0	—	—

TABLE IX.—*Continued.*

TEMPERATURE, FAHR.	PHOSPHOR BRONZE.		MUNTZ METAL.		COPPER.		CAST IRON.		WROUGHT IRON.				LANDORE STEEL.	
	Dia. = 1 inch. Copper .... 92.5 Tin........ 7.0 Phosphor.. 0.5		Dia. = 0.74 inch. Copper...... 62 Zinc......... 38		Dia. = 0.72		Dia. = 1 inch. Welsh....... .25 Scotch....... .25 Scrap........ .50		Remanufactured Dia. = 0 74 inch.		Yorkshire. Dia. = 0.70 inch.		0.74 in. × 0.49 in.	
	R.	*St.*	*R.*	*St.*	*R.*	*St.*	*R.*	*St.*	*R.*	*St.*	*R.*	*St.*	*R.*	*St.*
Atmos.	38,980	17.5	81,900	2.5	56.810	2.5	30,720	0.0	58,500	22.0	60,030	25.0	76,920	26.0
100°	39,300	17.0	79,560	2.5	57,430	2.5	27,070	0.0	62,010	18.8	60,030	24.3	——	——
150°	39,040	18.0	84,240	3.9	54,960	4.0	28,220	0.0	62,010	15.0	——	——	——	——
200°	38,720	18.0	80.145	3.9	54.340	5.0	28,160	0.0	62,010	15.0	60,680	17 3	74,150	22.5
250°	37,120	15.0	78,390	5.0	53,110	7.0	30,080	0.0	63.180	15.0	——	——	——	——
300°	36,800	12.0	76,050	2.5	53,110	6.0	31,360	0.0	76,050	15.5	62,640	7.5	76,920	11.3
350°	30,080	7.0	76,050	2.3	53,110	6 0	30,080	0.0	77,220	12.5	——	——	——	——
400°	27,140	5.0	70,200	2.3	51,870	6 0	25,660	0.0	63,180	12.0	60,680	15.0	81,770	10.3
450°	24,320	4 0	71,955	3 8	51,260	6.0	25,600	0.0	79,560	15.0	——	——	——	——
500°	26,880	5.0	72,540	5.0	48,220	6.0	25,600	0.0	79,560	20.0	71,830	13.8	83,200	10.0

results would probably be somewhat varied by different processes of, and treatment in, manufacture and construction.

The Muntz metal and copper specimens were rolled.

Bauschinger's Experiments with Copper and Red Brass.

Prof. Bauschinger extended his experiments on the repeated application of stress so as to cover not only wrought iron and steel, the results of which have already been given, but also copper and red brass.

The notation is that already used :

E. L. = elastic limit in pounds per square inch.
S.-L. = stretch limit in pounds per square inch.
F. L. = final load in pounds per square inch.
E. = coefficient of elasticity in pounds per sq. inch.
Im'y = "immediately."

The copper specimens were of rolled material about 16 inches long with a cross section about 2.4 inches by 0.64 inch. These specimens gave an ultimate tensile resistance, per square inch, of 28,900 to 32,000 pounds and a final contraction of 27 to 46 per cent.

The red brass specimens were turned to about one inch in diameter and 16 inches long. They gave ultimate tensile resistances, in pounds per square inch, varying from 19,600 to 23,460.

With one exception, in the second case of red brass, the elastic limit and stretch limit were elevated by repeated application of stress, whether immediately or at the end of following periods of rest.

The effects on the coefficient of elasticity are seen to be somewhat irregular.

Copper.

	IN ORIGINAL CONDITION.	IM'Y.	IM'Y.	IM'Y.
E. L	5,475	8,030	8,790	11,450
S.-L.	——	11,670	14,650	22,880
F. L.	——	14,600	21,970	——
E.	16,651,000	17,249,000	16,154,000	15,770,000

Copper.

	IN ORIGINAL CONDITION.	AFTER 18 HRS.	AFTER 23 HRS.	AFTER 24 HRS.
E. L.	2,560	7,320	8,080	11,520
S.-L.	——	——	14.680	23,040
F. L.	——	14,650	22,010	——
E.	16,011,000	16,295,000	15,197,000	15,756,000

Copper.

	IN ORIGINAL CONDITION.	AFTER 43 HRS.	AFTER 44.5 HRS.	AFTER 51.5 HRS.
E. L.	5,840	8,030	10,340	15,390
S.-L.	——	11,670	14,760	23,080
F. L.	——	14,600	22,160	——
E.	16,097,000	16,780,000	16,069,000	15,472,000

Red Brass.

	IN ORIGINAL CONDITION.	IM'Y.	IM'Y.	
E. L.	7,680	9,090	9,260	——
S.-L.	13,960	16,070	19,240	——
F. L.	16,770	19,550	——	——
E.	12,030,000	12,485,000	12,727,000	——

Red Brass.

	IN ORIGINAL CONDITION.	AFTER 17.5 HRS.	AFTER 21 HRS.
E. L.	5,600	9,115	8,550
S.-L.	14,020	16,130	19,240
F. L.	16,820	19,640	——
E.	12,322,000	12,314,000	12,485,000

Red Brass.

	IN ORIGINAL CONDITION.	AFTER 53 HRS.	
E. L.	3,480	9,090	——
S.-L.	13,910	16,070	——
F. L.	16,690	——	——
E.	13,239,000	12,940,000	——

The explanation of the method of applying these repeated stresses will be found in connection with the results for wrought iron on page 262.

Art. 36.—Various Metals and Glass.

Coefficients of Elasticity.

The following values of the coefficients of elasticity, in pounds per square inch, contained in Table I. are taken from Wertheim's "*Physique Mécanique*," pages 57 and 58. The co-

TABLE I.

METAL.	EXPERIMENTER.	COEFFICIENT OF ELASTICITY.	
		Drawn.	Annealed.
Lead	Wertheim.	2,564,000	2,457,000
Cadmium	"	7,713,000	7,555,000
Gold	"	11,564,000	7,942,000
Silver	"	10,463,000	10,155,000
Palladium	"	16,721,000	13,920,000
Platinum	"	24,237,000	22,067,000

efficients are the means of a large number of tensile experiments, with the exception of that for cadmium, which was derived from experiments on transverse vibrations. This last method gave results which differed, in most cases, from the direct tensile ones not more than the latter did from each other.

Wertheim also gives for the tensile coefficients of elasticity of some different glasses:

Mirror glass	$E = 8,792,000$	pounds per square inch.
Goblet (common)	$E = 9,559,000$	" " " "
Goblet (fine)	$E = 8,589,000$	" " " "
Goblet (violet)	$E = 7,110,000$	" " " "
"Crystal"	$E = 5,830,000$	" " " "

Ultimate Resistance and Elastic Limit.

Wertheim determined the elastic limit of many of the more rare metals, such as those named in Table I., and they are here given in pounds per square inch :

	ANNEALED.		DRAWN.
Lead	284	to	355
Cadmium	142	to	171
Gold	4,266	to	19,200
Silver	4,266	to	16,350
Palladium	7,110	to	25,600
Platinum	20,600	to	37,000

His "limit of elasticity" is that force which will permanently elongate the metal 0.000,05 of its original length, and all his experiments were made on wires of very small diameters.

The following ultimate resistances were found for wires about $\frac{1}{60}$th inch in diameter by Baudrimont ("Annales de Chimie," 1850) :

Gold	17,100 to 26,200	pounds per square inch.
Silver	40,300 to 40,550	" " " "
Platinum	32,300 to 32,700	" " " "
Palladium	51,750 to 52,640	" " " "

The ultimate resistances of some other metals are :

METAL.	EXPERIMENTER.	ULT. RESIST.
Cast lead	Rennie	1,824 pounds per square inch.
Sheet lead	Navier	1,926 " " " "
Pipe lead	Jardine	2,240 " " " "
Soft solder ($\frac{2}{3}$ tin, $\frac{1}{3}$ lead)	Rankine	7,500 " " " "

Sir Wm. Fairbairn (" Useful Information for Engineers," second series, pages 226 and 267) found the following ultimate resistances in pounds per square inch by direct pull on straight tensile specimens :

Flint glass	2,413	pounds.
Green glass	2,896	"
Crown glass	2,546	"

The specimens were of circular section and about 0.53 inch in diameter.

By subjecting spherical glass shells to internal pressure he found the following ultimate resistances in pounds per square inch :

Flint glass	4,200	pounds.
Green glass	4,800	"
Crown glass	6,000	"

The thickness of these shells varied from about 0.02 (crown and green glass) to 0.08 (flint glass) inch.

Art. 37.—Cement, Cement Mortars, etc.—Brick.

The ultimate tensile resistance of these materials depends upon many circumstances, and only a few out of a great number of experimental results will be given. These results will be so chosen as to be representative, but a full and detailed knowledge of the action of cements and cement mortars, under different circumstances of testing and variety of composition, must be acquired by an examination of the original memoirs.

Mr. Bremermann, during the construction of the St. Louis bridge, found in 18 experiments with pure " Fall City " (Louisville) cement : Elastic limit, 16 to 104 pounds per square inch, with a mean of 72 ; ultimate resistance, 35 to 147 pounds per square inch, with a mean of 110 ; coefficient of elasticity, 800,000 to 6,930,000 pounds per square inch, with a mean of 2,239,000.

TABLE I.

KIND OF CEMENT.	NO. TESTS.	*C.*	NO. TESTS.	*T.*	
Toepffer, Grawitz & Co., Stettin, Germany	12	1,439	3	216	Portland cements.
Hollick & Co., London	10	1,330	3	216	
Wouldham Cement Co., London	12	1,140	3	199	
Saylor's Portland Cement, Coplay, Pennsylvania	8	1,078	2	184	
Wampun Cement & Lime Co., Newcastle, Pennsylvania	12	968	3	168	
Pavin de Lafarge, Teil, France	12	931	3	158	
A. H. Lavers, London	6	926	2	192	
Francis & Co	14	907	3	163	
Wm. McKay, Ottawa, Canada	10	882	3	141	
Borst & Roggenkamp, Delfzyl, Netherlands	12	826	3	132	
Louquéty & Co., Boulogne-sur-mer, France	12	764	3	108	
Riga Cement Co., Riga, Russia	5	693	2	134	
Scanian Cement Co., Lomma, Sweden	14	606	3	112	
Brunö Höfmark, Port-Kund, Esthland, Russia	6	580	2	154	
Coplay Hydraulic Cement, Coplay Cement Co., Coplay, Pa	8	292	2	38	Roman and other cements.
Charles Tremain, Manlius, N. Y	12	276	3	47	
Allen Cement Co., Siegfried's Bridge, Pennsylvania	12	276	3	43	
P. Gouvreau, Quebec, Canada	8	234	2	47	
Riga Cement Co., Riga, Russia	6	230	2	44	
Anchor Cement Co., Coplay, Pa	12	208	3	41	
Cumberland Hydraulic Cement Co., Cumberland., Md	12	196	3	41	
Societé Anonyme, France	7	184	2	29	
Howe's Cave Cement, No. 1, Howe's Cave, N. Y	6	183	2	28	
" " " No. 2, "	10	170	2	43	
" " " No. 3, "	8	170	2	31	
Società Anonima, Emilia, Italy, 1st quality	12	181	3	27	
" " " " 2d quality	12	154	3	22	
Thomas Gourdy, Limehouse, Ontario, Canada	8	126	2	23	
A. H. Lavers, London	6	122	2	24	
Scott's Selenitic Cement (Howe's Cave lime and plaster)	20	298	5	52	
Parian Cement, Francis & Co., London, 1st quality	8	1,175	3	181	
" " " " " 2d quality	12	696	3	169	
" " A. H. Lavers, "	6	205	2	51	

Table I. contains the results of the tests made by General Q. A. Gillmore at the Centennial Exposition, Philadelphia, in 1876 (Van Nostrand's Magazine, March, 1877). The specimens were prepared " by mixing them dry in each case with an equal measure of clean sand, tempering the mixture with water to the consistency of stiff mason's mortar, and then moulding it into briquettes of suitable form for obtaining the tensile strength in a sectional area 1½ inches square, equal to 2¼ square inches. The briquettes were left in the air one day to set, then immersed in water for six days, and tested when seven days old. After thus obtaining the tensile strength in each case, the ends of the broken specimens were ground down to 1½ inch cubes, which were used the same day for obtaining the compressive strength by crushing." The columns "*No. tests*" give the number of experiments from which were obtained the mean values contained in the columns C and T.

C = Ult. Compressive resistance in lbs. per sq. in.
T = " Tensile " " " " " "

The former is given here in order to avoid the repetition of the makers' names hereafter.

TABLE II.

AGE.	N. R. *T.*	N. N. J. *T.*	N. N. J. *T.*	N. Y. R. *T.*	L. H. *T.*	H. C. *T.*
16 hours	47	——	——	——	——	——
24 hours	55	63	58	45	44	60
36 hours	55	——	——	——	——	——
48 hours	60	64	——	79	50	49
60 hours	68	——	——	——	——	——
72 hours	——	——	——	——	——	77
4 days	——	——	——	——	73	——
7 days	97	——	71	85	——	——
10 days	——	——	——	98	105	——
14 days	120	——	——	——	——	——
15 days	——	134	——	88	——	——
19 days	140	——	——	——	——	——
21 days	——	——	——	——	97	——
25 days	——	——	——	——	——	117
1 month	——	——	92	——	108	——
64 days	——	——	——	297	——	——
69 days	——	——	——	325	——	——
2 months.....	——	——	157	——	202	——
3 months.....	——	——	——	——	——	175
3¼ months....	——	——	——	——	221	——
4 months.....	——	——	——	——	——	177
6 months.....	——	——	——	——	250	——
1 year	——	——	——	——	381	——
1¼ year	——	——	241	——	——	——
2 years	257	——	385	336	266	292

All specimens of neat cement mixed with fresh water at about 60° *Fahr.*

Table II. contains the results of a large number of tests of Rosendale cement (Trans. Am. Soc. of Civ. Engrs., Vol. VII., Feb. 1878—"Improvement of the South Boston Flats," by Edward S. Philbrick), also that from Howe's Cave.

"N. R." signifies "Newark & Rosendale Cement Co."

"N. N. J." signifies "Newark, N. J., Lime & Cement Co."

"N. Y. R." signifies "New York & Rosendale Lime & Cement Co."

"L. H." signifies "Lawrence Cement Co.—Hoffman Rosendale Cement."

"H. C." signifies "Howe's Cave Cement."

T = ultimate tensile resistance in pounds per square inch.

The values of T are averages of from 1 to 2,217 tests.

Table III. gives some results of the same neat cement specimens mixed with salt water at about 60° *Fahr.*

TABLE III.

AGE.	N. N. J. T.	L. H. T.	AGE.	N. N. J. T.	L. H. T.
24 hours....	39	19	2 months..	——	134
2 days.....	——	45	$3\frac{1}{4}$ months..	——	275
7 days.....	56	——	6 months..	——	237
10 days.....	——	50	1 year	——	367
20 days.....	——	63	$1\frac{1}{4}$ year	205	——
1 month ...	105	79	2 years....	311	234

The notation is the same as that of Table II.

Experiments and Conclusions of Wm. W. Maclay, C. E.

The following tables and conclusions are abstracted from "Notes and Experiments on the Use and Testing of Portland Cement," by Wm. W. Maclay, C. E. (Trans. Am. Soc. of Civ. Engrs., Vol. VI., Dec., 1877).

He made many valuable experiments in order to determine the effect of different temperatures at different periods in the life of the cement.

TABLE IV.

Portland Cement.

TEMP. (FAHR.) OF WATER IN WHICH BRIQUETTES WERE IMMERSED.	AVERAGE TENSILE STRENGTH PER SQ. IN.									
	7 days old. Temperature of cement paste when briquettes were moulded.					*21 days old.* Temperature of cement paste when briquettes were moulded.				
	32°	40°	50°	60°	70°	32°	40°	50°	60°	70°
	Lbs.	Lbs.	Lbs.	Lbs.	Lbs.	Lbs.	Lbs.	Lbs.	Lbs.	Lbs.
40°	156	147	131	133	113	265	307	236	244	212
50°	186	206	183	194	143	289	——	292	260	251
60°	259	275	245	240	191	348	——	318	309	282
70°	299	314	299	286	254	360	——	403	386	336
Temperature of air during last 24 hours.	40°	40°	45°	45°	45°	46°	46°	43°	43°	43°

Table IV. contains some of the results of Mr. Maclay's experiments which were made to determine the effect of the temperature of the water in which the specimens of neat cement were mixed. After the briquettes were moulded at temperatures shown in the upper horizontal column, and immersed in water either 7 or 21 days at the temperature shown in the left vertical column, they were taken out and dried in air at the temperature shown in the lower horizontal row. The resistances in pounds are averages of five or more results.

From these and many other similar results, Mr. Maclay concluded that the ultimate resistance follows "very closely the temperature of the water in which the sample briquettes were kept immersed, the warmest water giving the greatest tensile strength; that a change from 40° to 70° in the water, increases the tensile strength of the briquettes of neat cement, seven days old, from 63 to 168 pounds per square inch, or from 33 to 127 per cent.; of the briquettes of mortar, gauged 1 to 1, the same change in temperature increases them from 32 to 59 pounds per square inch, or from 87 to 133 per cent.; of the briquettes gauged 1 cement to 2 of sand, from 19 to 37 pounds per square inch, or from 95 to 176 per cent. After an interval of three weeks the changes in tensile strength, . . . become less marked, and, in some cases, an increase in the temperature of the water diminishes the tensile strength."

Other experiments seemed to "show quite conclusively that the tensile strength increases directly as the temperature of the air when the cement is being gauged, and inversely as the temperature of the air to which it is exposed for the last 24 hours before breaking."

"Exposing the briquettes after six days' immersion in water to a high drying temperature weakens them so invariably, that some interference with the setting seems clearly demonstrated. . . . ."

Some further experiments led him to conclude "that Portland cement gauged with either fresh or salt water, hardens more rapidly when immersed in salt or sewer water than in fresh water for the first seven days, and that this increase in the tensile strength probably continues for at least a year, more rapidly in the one than in the other."

Table V. contains the results of Mr. Maclay's tests on the relative influence of fine and coarse sand.

TABLE V.

Portland Cement.

Showing difference in mortars with fine and coarse sand.

AGE OF MORTAR.	1 VOL. CEMENT. 1 VOL. SAND.		1 VOL. CEMENT. 2 VOLS. SAND.		1 VOL. CEMENT. 3 VOLS. SAND.	
	Fine sand.	Coarse sand.	Fine sand.	Coarse sand.	Fine sand.	Coarse sand.
1 week....	85	95	33	63	19	41
1 month...	162	202	81	94	49	74

TABLE VI.

Portland Cement.

W.	BRAND.	*T.*	*P. c.*
101.5	Alsen & Son, Itzehoe, Germany.	326	93
108.0	" " " "	340	91
112.0	Burham.	289	87
113.0	"	317	87
114.0	"	285	88
115.0	Gibbs.	280	90
116.0	Burham.	316	88
117.0	"	301	85
118.0	"	276	85
119.0	"	305	85
120.0	"	252	84
121.0	Saylor's American Portland.	269	90
122.0	" " "	281	90
123.0	" " "	272	90
124.0	" " "	260	90
126.0	" " "	265	90
128.0	" " "	369	87
132.0	" " "	322	78

W = weight in lbs. per bushel.

T = ultimate tensile resistance in lbs. per sq. in.

P. c. = percentage that passed through sieve of 2,500 meshes per sq. in.

"The deduction from this table is, that by increasing the fineness of the sand of which the mortar is made the tensile strength is diminished, and that this reduction in tensile strength increases with the amount of sand used in the mortar." An English experimenter, Lieut. W. Innes, R. E., was led to the same conclusion.

Table VI. shows the results of experiments by Mr. Maclay, made to determine the connection between the weight per bushel and tensile resistance of specimens seven days old. Commenting on the results he says, "The close connection between the weight . . . and the tensile strength . . is now proved to be very uncertain, if not entirely fallacious." As will hereafter be seen, these experiments invalidate a contrary conclusion reached by the English experimenter, John Grant, M. Inst. C. E., in 1864.

TABLE VII.

AGE.	BURHAM.	FRANCIS.	TINGEY.	GILLINGHAM.	PORTLAND.		J. B. WHITE & BROS.	FRANCIS BROS.	SAYLOR'S
	Portland.	Portland.	Portland.	Portland.	Neat cement.	1 cement 1 sand.	Roman.	Medina.	American Portland.
1 week.	278	184	212	250	363	157	90	94	364
2 "	256	175	182	248	—	—	77	135	—
3 "	359	302	306	380	—	—	83	132	—
1 mo...	332	374	301	400	416	201	116	136	413
2 " ...	504	416	378	431	—	—	—	—	498
3 " ...	525	423	383	497	469	243	143	200	525
4 " ...	513	432	428	529	—	—	—	—	584
5 " ...	554	459	426	512	—	—	—	—	576
6 " ...	365	327	264	312	523	284	210	183	575
7 " ...	—	—	—	—	—	—	—	—	—
8 " ...	—	—	—	—	—	—	—	—	586
9 " ...	304	310	270	404	542	308	209	203	—
10 " ...	—	—	—	—	—	—	—	—	599
1 yr ..	443	414	367	569	546	318	286	212	—
1½ " ..	611	497	339	569	—	—	—	—	—
2 " ..	326	355	224	476	589	351	243	123	—
3 " ..	—	—	—	—	584	349	268	122	—
4 " ..	—	—	—	—	583	364	281	128	—
5 " ..	—	—	—	—	580	364	279	136	—
6 " ..	—	—	—	—	580	364	296	162	—
7 " .	—	—	—	—	590	384	315	168	—

Vertical columns give tensile resistance in pounds per square inch.

Table VII. shows the variation of ultimate tensile resistance, per square inch, of various neat cements (with one exception) with age. The left vertical column shows the time during which the specimens were kept under water, and the other vertical columns the ultimate tensile resistance in pounds per square inch.

The results for the Burham, Francis, Tingey, Gillingham and Saylor Portland cements are from Mr. Maclay's paper; the others are from "Experiments on the Strength of Cements," London, 1875, by John Grant, M. Inst. C. E.; all are means of great numbers of experiments.

The mean results of Mr. John Grant's experiments on Keene's and Parian cements are given in Table VIII.

TABLE VIII.

AGE AND TIME IMMERSED IN WATER.	KEENE'S CEMENT.		PARIAN CEMENT.	
	In water.	Out of water.	In water.	Out of water.
	T.	T.	T.	T.
1 week	242	243	264	285
2 "	216	260	267	298
3 "	224	258	242	310
1 month	218	260	242	332
2 "	202	288	222	322
3 "	226	320	232	380

$T =$ ultimate tensile resistance in pounds per square inch.

As the result of his experiments on Portland and Roman cements Mr. Grant was led to the following conclusions:

1. Portland cement, if it be preserved from moisture, does not, like Roman cement, lose its strength by being kept in casks, or sacks, but rather improves by age ; a great advantage in the case of cement which has to be exported.

2. The longer it is in setting, the more its strength increases.

3. Cement mixed with an equal quantity of sand is at the end of a year approximately three-fourths of the strength of neat cement.

4. Mixed with two parts of sand, it is half the strength of neat cement.

5. With three parts of sand, the strength is a third of neat cement.

6. With four parts of sand, the strength is a fourth of neat cement.

7. With five parts of sand, the strength is about a sixth of neat cement.

8. The cleaner and sharper the sand, the greater the strength.

9. Very strong Portland cement is heavy, of a blue-gray color, and sets slowly. Quick setting cement has, generally, too large a proportion of clay in its composition, is brownish in color, and turns out weak, if not useless.

10. The stiffer the cement is gauged, that is, the less the amount of water used in working it up, the better.

11. It is of the greatest importance that the bricks, or stone, with which Portland cement is used, should be thoroughly soaked with water. If under water, in a quiescent state, the cement will be stronger than out of water.

12. Blocks of brick-work, or concrete, made with Portland cement, if kept under water till required for use, would be much stronger than if kept dry.

13. Salt water is as good for mixing Portland cement as fresh water.

14. Bricks made with neat Portland cement are as strong at from six to nine months as the best quality of Staffordshire blue brick, or similar blocks of Bramley Fall stone, or Yorkshire landings.

15. Bricks made of four parts or five parts of sand to one part of Portland cement will bear a pressure equal to the best picked stocks.

16. Wherever concrete is used under water, care must be taken that the water is still. Otherwise, a current, whether natural or caused by pumping, will carry away the cement, and leave only the clean ballast.

17. Roman cement, though about two-thirds the cost of Portland, is only about one-third its strength, and is therefore double the cost, measured by strength.

18. Roman cement is very ill adapted for being mixed with sand.

Mr. Don J. Whittemore has proposed the following formula for the ultimate tensile resistance of cements :

$$T = A \sqrt[x]{N};$$

in which T is the ultimate tensile resistance in pounds per square inch ; A, an empirical coefficient, and N the age of the cement in days. For Portland cement (up to two years old) he gives $x = 10$, and $A =$ 267 to 356, by the aid of Mr. Grant's experiments. (See Trans. Amer. Soc. of Civ. Engrs., Vol. VII., Sept. 1878).

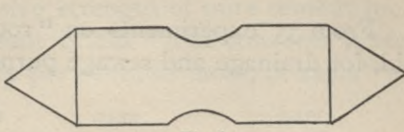


Fig.1

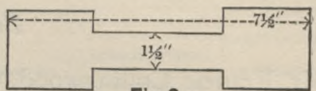


Fig.2

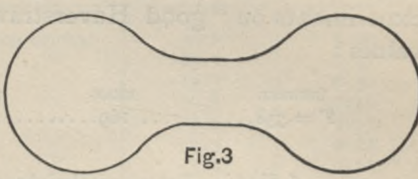


Fig.3

Fig. 1 shows the briquette used by Mr. Maclay; Fig. 2, that used by Mr. Grant, while that shown in Fig. 3 is the one generally used at the present time. Each briquette is 1½ inches thick, giving a breaking section of $1\frac{1}{2} \times 1\frac{1}{2} = 2.25$ square inches. In such testing it is very necessary that the pull should be central.

Artificial Stones.

The tensile resistances of many artificial stones and some natural British ones, can be found in " A Practical Treatise on Natural and Artificial Concrete," by Henry Reid, London, 1879.

On page 198 he gives the following results of Professor Ansted's experiments, T being the ultimate resistance per square inch :

Ransome stone (artificial)	$T = 360$	pounds.
Portland stone	$T = 201$	"
Bath stone	$T = 145$	"
Caen stone	$T = 140$	"

He also gives for " Victoria " (artificial) stone, three months old,

$T = 740$ pounds per square inch.

From 35 experiments on "rock concrete" pipe two years old, for drainage and sewage purposes, Mr. Reid found:

	HIGHEST.	MEAN.	LOWEST.
$T =$	700..........	444..........	213 pounds per square inch.

Bricks.

Mr. Francis Collingwood, C. E. (Trans. Amer. Soc. of Civ. Engrs., Vol. VII., Sept., 1878), found, as a result of twelve experiments on "good Haverstraw stock brick," the following values:

	HIGHEST.	MEAN.	LOWEST.
$T =$	358...........	169..........	90 pounds per square inch.

Adhesion between Bricks and Cement Mortar.

General Q. A. Gillmore ("On Limes, Hydraulic Cements and Mortars") cemented Croton bricks together crosswise and then separated them by a pull. He used pure cement paste and mortars of various proportions, by volume, of cement to sand, but never more sand than 1 volume of cement to 2 volumes of sand. Nearly all the cement was Rosendale, although some specimens were prepared with Hancock (Maryland) or James River cement. Bricks so cemented in pairs were kept 320 days and then separated. Reviewing the results, Gen. Gillmore says, "In tearing the bricks apart, at the expiration of the time specified, in a majority of cases the surface of contact of the brick and mortar remained intact, the adhesion to the brick overcoming the cohesive strength either of the bricks themselves, or of the mortar composing the joint between them. The results, therefore, although interesting for other reasons, furnish no entirely satisfactory measure of the power of adhesion."

Also, "At the age of 320 days (and perhaps considerably within that period) the cohesive strength of pure cement mortar exceeds that of Croton front bricks. The converse is true when the mortar contains fifty per cent., or more, of sand."

TABLE IX.

MORTAR OR PASTE.	MATERIALS CEMENTED.	ADHESION PER SQ. INCH IN LBS.	RATIO OF ADHESION TO RESISTANCE OF PURE CEMENT.
Pure cement	Croton bricks.	30.8	1.00
1 vol. cement 1 vol. sand...	" "	15.7	0.51
1 " " 2 " " ...	" "	12.3	0.40
1 " " 3 " " ...	" "	6.8	0.22
1 " " 4 " " ...	" "	5.2	0.17
1 " " 5 " " ...	" "	4.3	0.14
1 " " 6 " " ...	" "	3.3	0.11
Pure cement	Fine cut granite.	27.5	1.00
1 vol. cement 1 vol. sand...	" " "	20.8	0.76
1 " " 2 " " ...	" " "	12.6	0.46
1 " " 3 " " ...	" " "	9.2	0.33
1 " " 4 " " ...	" " "	7.9	0.29

Table IX. contains the results of another series of experiments by General Gillmore, made for the purpose of determining the adhesion to Croton front bricks and fine cut granite, of mortars containing different proportions of sand. "The bricks were used wet, and were well pressed together by hand. They were wetted with fresh water every alternate day for 29 days, the age of the mortar when tested. Each result is the average of five trials."

Art. 38.—Timber.

Table I. contains the results of experiments made by Chevandier and Wertheim ("Mémoire sur les Propriétés Mécanique du Bois;" by E. Chevandier and G. Wertheim, 1846). The

TABLE I.

KIND OF WOOD.	COEFFICIENTS OF ELAST.	ELASTIC LIMIT.	ULTIMATE TENSILE RESIST.
	Pounds.	Pounds.	Pounds.
Hornbeam......	1,335,000	3,060	4,250
Aspen..........	1,329,000	4,380	10,240 (1)
Alder..........	1,021,000	2,570	6,460 (1)
Sycamore.......	1,616,000	3,270	8,760 (1)
Maple.........	1,459,000	3,870	5,090 (1)
Oak...........	1,765,000 to 1,214,000	3,340	8,530
Birch..........	2,431,000 to 1,263,000	2,300	6,110
Beech..........	1,450,000	3,300	5,080
Ash............	1,798,000 to 1,364,000	2,890	9,640 (1)
Elm...........	1,436,000	2,620	9,940 (1)
Poplar.........	1,027,000 to 901,000	2,100	2,800 (1)
Acacia.........	2,206,000 to 2,018,000	4,540	11,280
Fir............	2,218,000 to 1,319,000	3,060	5,940
Pine...........	1,088,000	2,320	3,530

results are means, and were obtained from small, well seasoned rods, with cross-sectional areas varying from 0.30 square inch (some fir specimens) to 1.50 square inches (one oak specimen), and are given in pounds per square inch. The results indicated thus "(1)," belong to one tree only, others, to several.

The limit of elasticity is that force per square inch which will produce a permanent elongation of 0.00005 of the original length.

These experimenters found that the elongations produced by different weights were composed of two parts, one permanent and one elastic; the latter being essentially proportional

TABLE II.

KIND OF TIMBER.	EXPERIMENTER.	SP. GRAV.	ULT. RESIST. IN POUNDS PER SQ. INCH.	*B* IN POUNDS PER SQ. INCH.
Oak, English	Laslett.	0.858	3,837	Values for this column will be found computed in Art. 63.
Oak, English	"	0.893	7,571	
Oak, French	"	0.976	8,102	
Oak, Dantzic	"	0.838	4,217	
Oak, American White	"	0.969	7,021	
Oak, American, Baltimore	"	0.742	3,832	
Oak, African (or Teak)	"	0.971	7,052	
Teak, Moulmein	"	0.777	3,301	
Iron Wood, Burmah	"	1.176	9,656	
Chow, Borneo	"	1.134	7,199	
Greenheart, Guiana	"	1.141	8,820	
Sabicu, Cuba	"	0.917	5,558	
Mahogany, Spanish	"	0.765	3,791	
Mahogany, Honduras	"	0.659	2,998	
Mahogany, Mexican	"	0.655	3,427	
Eucalyptus, Australia :				
Tewart	"	1.169	10,284	
Mahogany	"	0.996	2,940	
Iron-Bark	"	1.150	8,377	
Blue Gum	"	1 049	6.048	
Ash, English	"	0.750	3,780	
Ash, Canadian	"	0.588	5,495	
Beech	"	0.705	4,853	
Elm, English	"	0.642	5,460	
Rock Elm, Canada	"	0.748	9,182	
Hornbeam, England	"	0.819	6,405	
Fir, Dantzic	"	0.603	3,231	
Fir, Riga	"	0.553	4,051	
Fir, spruce, Canada	"	0.484	3,934	
Larch, Russia	"	0.649	4,203	
Cedar, Cuba	"	0.469	2,870	
Red pine, Canada	"	0.553	2,705	
Yellow pine, Canada	"	0.551	2,759	
Yellow pine, Canada	"	0.552	2,259	
Pitch pine, American	"	0.659	4,666	
Kauri pine, New Zealand	"	0.544	4,040	
Georgia pine, American	Hatfield.	——	16,000	
Locust, American	"	——	24,800	
White oak, American	"	——	19,500	
Spruce, American	"	——	19,500	
White pine, American	"	——	12,000	
Hemlock	"	——	8,700	

to the load and the former measurable even for small loads and variable not only with the load but also with the time during which the load acted : that the coefficient and limit of elasticity augmented with the seasoning, but that the greatest elongation diminished under the same circumstances ; that if the coefficient of elasticity and ultimate resistance along the fibre be taken as units, the coefficients of elasticity along the radius and tangent to the tree, will be 0.165 and 0.091 respectively, while the ultimate resistances in the same directions will be respectively 0.163 and 0.159, these results being considered averages.

The ultimate tensile resistances of many woods, domestic and foreign, are given in Table II., as well as the specific gravities.

The column " B " will be explained hereafter, in the chapter on transverse resistance or bending.

TABLE III.

WOOD.	ULT. RESIST. IN POUNDS PER SQ. IN.	ELASTIC LIMIT IN POUNDS PER SQ. IN.	COEFFICIENT OF ELASTICITY IN POUNDS PER SQ. IN.	PER CENT. OF EXTENSION AT	
				Elas. Limit.	Fracture.
White Pine.....	6,880	3,900	183,490	0.40	0.73
Yellow Pine	20,700	13,200	240,240	0.63	1.65
Locust.........	28,930	19,200	373,830	1.10	1.85
Black Walnut ..	9,790	5,700	213,520	0.53	0.85
White Ash	15,490	9,700	206,540	0.78	1.48
White Oak	13,210	8,100	220,130	0.77	1.30
Live Oak	10,310	6,300	247,510	0.58	1.15

The table gives average results. Those determined from experiments of Mr. Laslett are of English origin ("Timber and Timber Trees, Native and Foreign," by Thomas Laslett, 1875); the others are from American experiments by the late R. G. Hatfield ("Transverse Strains," 1877). Mr. Laslett's specimens were 2 inches square in cross section, and genearily were 30 inches long, while those of Mr. Hatfield were about 0.35 inch round.

It will be observed that Mr. Hatfield reached far higher results than Mr. Laslett. This disagreement may be due to the larger cross-sectional area of the latter's specimens, which certainly brings his (Mr. Laslett's) results more nearly in accordance with what might be expected from such pieces as are ordinarily used by engineers. Mr. Hatfield's specimens were far too small for technical purposes.

Table III. is taken from a paper "On the Strength of American Timber," by Prof. R. H. Thurston (Jour. Frank. Inst., Oct., 1879). The specimens were turned down to about 0.5 inch diameter for a length of 4.00 inches.

The small values of the coefficient of elasticity, as compared with those given in Table I., are probably due to the fact that they were found at the elastic limit. Smaller intensities of stress would probably give much larger values.

Prof. Thurston also states that timber in tension takes a permanent set however small the intensity of stress.

The values given in Table IV. were found by Col. Laidley, U. S. Army, in the Government machine at Watertown, Mass. (Ex. Doc. No. 12; 47th Congress, 2d Session). Two of the specimens were about 0.63 inch in diameter, and one 1.25 inches. All the rest possessed diameters of about one inch each.

Such small specimens as those of Hatfield, Thurston, and Laidley, which were probably selected, give much larger results than would be found for large pieces of ordinary lumber; these considerations are highly prejudicial to the technical value of the results.

Far more importance attaches to the matter of size and character of timber specimens than to those of metallic ones. In the latter there is at least an approach to homogeneity of material, which the presence of knots, conditions of growth, seasoning, and other influences effectually prevent in timber specimens. Hence it is the more necessary to test timber in circumstances of condition and size as nearly identical as possible with those which attend its actual use.

TABLE IV.

Diameter of Test Specimens, 1 *inch.*

NO.	KIND OF WOOD.	ULTIMATE RSISTANCE PER SQUARE INCH IN POUNDS.			NO. OF TESTS.
		Greatest.	Mean.	Least.	
1	Yellow Pine.........	17,922	15,478	12,066	4
2	Oregon Pine	——	13,810	——	1
3	Oregon Spruce	——	16,160	——	1
4	White Pine	11,299	8,916	5,300	4
5	Spruce..............	17,044	14,283	11,600	4
6	White Wood	7,466	6,787	6,107	2
7	Gum Wood	19,400	14,313	4,586	4
8	White Maple........	15,714	11,164	7,312	3
9	Black Walnut	14,650	11,492	9,286	3
10	Red Birch	22,838	18,682	13,885	3
11	White Ash..........	27,532	24,120	18,961	3
12	Brown Ash..........	11,733	11,632	10,667	3
13	White Oak..........	22,703	17,410	12,670	4
14	Red Oak............	12,133	10,124	7,600	3
15	Yellow Oak.........	20,520	20,390	20,260	2
16	Hickory............	19,610	15,995	12,400	3

All specimens were of well seasoned wood.

CHAPTER VI.

COMPRESSION.

Art. 39.—Preliminary.

With the exception of material in the shape of long columns, but few experiments, comparatively speaking, have been made upon the compressive resistance of constructive materials.

Pieces of material subjected to compression are divided into two general classes—" short blocks " and " long columns ;" the first of these, only, afford phenomena of *pure compression.*

A " short block " is such a piece of material, that if it be subjected to compressive load it will fail by pure compression.

On the other hand, a long column (as has been indicated in Art. 25) fails by combined compression and bending.

Short blocks, only, will be considered in the articles immediately succeeding, while long columns will be separately considered further on.

The length of a short block is usually about three times its least lateral dimension.

It has already been shown in Art. 4 that the greatest shear in a short block subjected to compression, will be found in planes making an angle of 45° with the surfaces of the block on which the compressive force acts, *i. e.*, with its ends. If the material is not ductile, this shear will frequently cause wedge-shaped portions to separate from the block. But the friction at these end surfaces, and in the surfaces of failure will prevent those wedge portions shearing off at that angle. In fact the friction will cause the angle of separation to be considerably larger than 45° ; let it be called α. Then, in order that there may be perfect freedom in failure, the length of the block must not be less than its least width or breadth multiplied by 2 *tan* α. In some cases, α has been found to be about 55°, for which value

$$2 \tan \alpha = 2 \times 1.43 = 2.86.$$

It was shown in the first section of Art. 32, that the "ultimate resistance" to tension is in reality a mean, and not the greatest intensity which the material exerts. The same course of reasoning will show that it is, also, in general, impossible to subject a short block to a uniform intensity of compression throughout its mass, and that the "ultimate resistance to compression" is a mean, usually considerably less than the greatest intensity which exists at the centre of a normal section. As the inner portion will be supported laterally by that outside of it, large blocks of brittle material may give greater intensities of ultimate resistance than small ones.

Art. 40.—Wrought Iron.

It is difficult to fix the point of failure of a short block of wrought iron or other ductile material. An excessive compressive force causes the material to increase very considerably in lateral dimensions, or to "bulge" out, so that every increase of compressive force simply produces an increased area of resistance, while the material never truly fails by crumbling or shearing off in wedges.

A short block of wrought iron is usually considered to fail when its length is shortened by five to ten per cent.

If p_1 is any intensity of stress while l_1 is the compressive strain, or shortening per unit of length caused by p_1, then according to Eq. (2) of Art. 2, the coefficient of elasticity for compression at the intensity p_1, will be

$$E_1 = \frac{p_1}{l_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This ratio is not constant for all degrees of stress and strain, though for wrought iron, within the elastic limit, the divergences from a mean value are not great. Table I. contains coefficients of elasticity calculated by Prof. De Volson Wood, in the manner shown by Eq. (1), from the data determined by Mr. Eaton Hodgkinson and given in his work before cited. (See Prof. Wood's "Treatise on the Resistance of Materials").

TABLE I.

Ap_1.	Ll_1.	E_1.	Ll_1.	E_1.
Pounds.	Inch.	Pounds.	Inch.	Pounds.
5,098	0.028	20,796,500	0.027	21,864,000
9,578	0.052	21,049,000	0.047	23,595,000
14,058	0.073	21,979,000	0.067	24,273,000
16,298	0.085	21,343,000	——	——
18,538	0.096	22,156,000	0.089	24,108,000
20,778	0.107	22,160,000	0.100	24,038,000
23,018	0.119	23,587,000	0.113	23,587,000
25,258	0.130	22,095,000	0.128	23,679,000
27,498	0.142	22,111,000	0.143	22,259,000
29,738	0.152	21,938,000	0.163	21,139,000
31,978	0.174	20,979,000	0.190	19,478,000

The results belong to two square bars, and E_1 is in pounds per square inch. A is the area of cross section; it was 1.0506 square inches for the first bar and 1.0363 square inches for the other. Hence the bars were about one inch square. They were also ten feet long (L = 10.00 feet) and required lateral support to be kept in alignment so as to act like short blocks.

The table shows that the values of E_1 increase with p_1, when the latter is small; an opposite result was found for tension.

What may be called the elastic limit is found for p_1 =

30,000.00 pounds per square inch (nearly). Hence, it is seen that the greatest value of E_1 is found for p_1 equal to one-half to two-thirds the elastic limit.

The same general remarks in regard to the elastic limit, which were made in connection with tension, may be also applied to the compressive elastic limit.

The "Steel Committee" of British civil engineers, in 1870, made some experiments on twelve bars of Lowmoor wrought

TABLE II.

POUNDS PER SQUARE INCH FOR	
Elastic Limit.	Coefficient of Elasticity.
Pounds.	Pounds.
29,800	29,091,000
25,800	29,091,000
29,100	28,718,000
26,200	28,000,000

iron, 1.5 inches in diameter and 120 inches long. These twelve experiments were divided into four sets of three each, and the table gives the means of each of these sets or groups. The coefficients are computed at the elastic limit. Judging from the results in Table I., smaller values of p_1 would have given larger values of E_1.

As a mean value, the coefficient of elasticity for wrought iron in compression may be taken at 28,000,000 pounds per square inch. For every ton (2,000.00 pounds) of compression per square inch, therefore, a piece of wrought iron will be shortened by an amount equal to

$$\frac{2,000}{28,000,000} = \frac{1}{14,000} \text{ of its length.}$$

Table III. contains the results of some experiments made by Mr. Kirkaldy on some specimens of Swedish iron, in 1866. The last column gives the per cent. of compression of original length which the piece suffered at the point called the "ultimate compressive resistance." The results show well the great increase of resistance which a short block of ductile material offers with the increase of compression.

TABLE III.

SECTION OF SPECIMEN.	LENGTH.	POUNDS PER SQUARE INCH FOR		PER CENT. COMPRESSION.
		Elas. Lim.	Ult. Resist.	
In.	Ins.	Lbs.	Lbs.	
1.5 ○	1.5	24,050	148,800	45
1.5 ○	1.5	21,200	28,100	4
1.5 ○	3	23,300	84,900	33
1.0 □	1	——	184,100	53

Table IV. gives the results of experiments on some very short lengths of Phœnix and Keystone columns. The first six results are for Phœnix sections from experiments by the Phœnix Iron Co., in 1873; the two following are for the same section from experiments made at Watertown, Mass., in 1879; while the last result belongs to a Keystone section experimented upon by Mr. G. Bouscaren, in 1875. Unfortunately the amount of compression or shortening, in each instance, was not recorded.

Reviewing the results given in Tables II., III. and IV., it is seen that the "elastic limit" of wrought iron in compression, in

TABLE IV.

LENGTH, INCHES.	RATIO OF LENGTH TO DIAMETER OF SECTION.	AREA OF SECTION, SQ. IN.	ULT. RESIST. IN POUNDS PER SQ. IN.
8.00	1.46	6.97	60,570.00
8.00	1.46	6.97	60,390.00
4.00	0.92	5.62	65,870.00
4.00	0.92	5.62	65,870.00
4.00	1.01	2.92	56,890.00
4.00	1.01	2.92	55,560.00
8.00	1.00	11.90	57,130.00
8.00	1.00	11.90	57,300.00
9.00	1.12	14.25	51,500.00

short blocks, may be taken from 0.4 to 0.5 its ultimate compressive resistance, while the latter may be taken at about 60,000.00 pounds per square inch.

Art. 41.—Cast Iron.

The irregular elastic behavior of cast iron, as seen in tension, will also be discovered in compression. Table I. contains results computed from the data obtained by Captain Rodman by testing solid cylinders 10 inches long and 1.382 inches in diameter. The second column belongs to a specimen cylinder taken from a 10-inch columbiad, and the third or last to a trial cylinder of remelted Greenwood and Salisbury iron. Neither specimen can be considered to possess a true elastic limit, but what is ordinarily so termed may be taken at about 20,000 pounds per square inch.

In the first specimen the first permanent set took place at 3,000.00, and in the second at 5,000.00 pounds per square inch.

TABLE I.

INTENSITY OF STRESS.	COEFFICIENT OF ELASTICITY IN POUNDS PER SQUARE INCH.	
1,000	6,896,600	——
2,000	8,888,900	33,333,300
3,000	9,836,100	18,750,000
4,000	10,666,700	13,793,100
5,000	10,752,700	13,888,900
6,000	11,320,800	12,766,000
7,000	11,382,100	13,725,500
8,000	11,510,800	13,559,300
9,000	11,920,500	13,432,800
10,000	12,121,200	13,333,300
11,000	12,290,500	13,095,200
12,000	12,182,700	13,186,800
14,000	12,444,400	13,207,500
16,000	12,260,500	12,903,200
18,000	11,920,500	12,857,100
20,000	11,695,900	12,578,600
22,000	11,253,200	12,290,500
26,000	10,236,200	11,607,200
30,000	8,596,000	10,101,000
35,000	——	7,658,600
40,000	——	5,333,300

For a bar ten feet long and one inch square, Mr. Eaton Hodgkinson found the following values:

GREATEST.	MEAN.	LEAST.
13,216,000.00	12,134,100.00.......	10,837,100.00 ;

all in pounds per square inch. The greatest value was found at 2,240, and the least at 38,080 pounds per square inch.

Since the coefficient of elasticity measures the *stiffness* of a body, and since the coefficient of elasticity for wrought iron in compression has been seen to be at least twice as great as that of cast iron in the same condition, wrought iron is at least twice as stiff, compressively, as cast metal. A bar of the latter material will be compressed by 2,000 pounds per square inch, about

$$\frac{2,000}{12,000,000} = \frac{1}{6,000} \text{ of its length.}$$

If l is the length of a bar in inches, W the compressive stress in pounds per square inch, then Hodgkinson found the total decrement in inches for 10-feet Low Moor cast-iron bars to be

$$\lambda' = l(0.012363359 - \sqrt{0.000152853 - 0.00000000191212 W}). \quad (1)$$

and the permanent set, in inches:

$$0.543\lambda'^2 + 0.0013 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Major Wade tested a number of specimens of cast iron of different numbers of fusions, in order to determine the ultimate compressive resistance. His specimens were from 0.5 to 0.6 inch in diameter, and from 1.25 to 1.5 inches (nearly). The results were as follows:

FUSION.	NO. OF EXPS.	GREATEST.	MEAN.	LEAST.
2d	4	114,504	99,770	84,529
3d	2	140,415	139,540	138,666
2d and 3d	2	169,427	168,589	167.752
2d	2	140,415	136,868	133,321
3d	1	168,251	168,251	168,251
2d	5	163,528	154,576	144,141
3d	4	174,120	167,030	156,863

All results are in pounds per square inch.

As the specimens gave way, portions sheared off along planes making angles with the normal sections of specimens varying from 46° to 62.5°. This is the characteristic compressive fracture of cast iron.

The 3d fusion iron gave the highest resistance.

Mr. Hodgkinson ("Report of the Commissioners appointed to Inquire into the Application of Iron to Railway Purposes," 1849) took specimens of 16 different kinds of British irons, 0.75

inch in diameter and 0.75 and 1.5 inches long, with the following results :

GREATEST.	MEAN.	LEAST.
117,605	86,284	56,445 pounds per square inch.

As a rule, the short specimens gave from 5 to 10 per cent. greater resistance than the longer ones. From another set of experiments with 22 different kinds of iron (specimens 0.75 inch in diameter and 1.5 inches long) he found :

GREATEST.	MEAN.	LEAST.
115,995	84,200	54,761 pounds per square inch.

Mr. Hodgkinson found that the hardness and ultimate crushing resistance of thin castings were greatest near the surface, but that in thick castings the surface and heart gave essentially the same results. He also found that thin castings gave considerably greater ultimate resistance to crushing than thick ones.

Sir Wm. Fairbairn tested the effect of remelting on " Eglinton " No. 3, hot-blast iron with the following results :

REMELTINGS.	ULT. RESIST.	REMELTINGS.	ULT. RESIST.
1	98,560	10	129,250
2	97,660	11	156,350
3	92,060	12	163,740
4	91,170	13	147,840 defec.
5	92,060	14	214,820
6	92,060	15	171,810
7	91,620	16	157,920
8	92,060	17	——
9	123,420	18	197,190

All results are in pounds per square inch. It is observed that the 14th remelting gives the highest resistance.

From what precedes, it is seen that the ultimate compressive resistance of cast iron, in good ordinary castings, may safely be taken from 85,000 to 100,000 pounds per square inch.

Art. 42.—Steel.

Table II. of Art. 34 contains the results found by Prof. Ricketts in testing cylindrical specimens of mild steel in compression. These specimens were six inches long between carefully faced ends, and, as the table shows, their diameters were about 0.75 inch. The coefficients of elasticity for compression were found by measurements very carefully made with a micrometer on a length of four inches. The elastic limits, however, were determined by operating with a cylinder two inches long, and were taken at those points where the material of the specimens ceased to hold up the scale beam, and may have been somewhat above that point where the ratio between stress and strain ceases to be essentially constant.

The coefficients of elasticity are seen to be quite uniform, irrespective of the per cents of carbon, within the limits of the Table, and they are seen to be a very little less than the coefficients for tension. Yet the difference is so small that no essential error will arise if, for all engineering purposes, they are assumed the same.

A comparison of the elastic limits for tension and compression presents some irregularities; yet with the exception of the high percentages of carbon in the last two grades of Bessemer metal, the two sets of elastic limits as wholes are not very different from each other. In the Bessemer steel with the two high per cents of carbon, the tensile elastic limits are materially higher than those for compression. The following very important conclusion results from this comparison of the elastic limits for the mild structural steels: since these elastic limits are essentially equal, it is not only permissible but wholly rational to increase the working resistances of mild steel bridge columns over those for iron in at least the same proportion that the tensile working stress of the same steel is increased over that of iron in tension. Experiments on a sufficient number of full size steel columns are yet lacking to verify this conclusion.

Chief engineer Wm. H. Shock, U. S. N., 1868, gives the following results for Parker Bros. " Black Diamond " steel :

Normal untempered steel: *Ult. Resist.* from 100,100 to 112,400 pounds per square inch.

Heated to light cherry-red and plunged in oil at 82° *Fahr.*: *Ult. Resist.* from 173,200 to 199,200 pounds per square inch.

Heated as before, and plunged in water at 79° *Fahr.*, with final temper (plum-blue) drawn on heated plate: *Ult. Resist.* from 325,400 to 340,800 pounds per square inch.

Heated as before and plunged in water at 79° *Fahr.*, and tested at maximum hardness: *Ult. Resist.* from 275,640 to 400,000 pounds per square inch. In each of these cases there were three tests.

The following values (each is a mean of 8 tests) were found by the United States Test Board, " Ex. Doc. 23, House of Rep., 46th Congress, second Session," for small annealed specimens of tool steel, of about one inch in length and 0.715 inch in diameter :

Ult. comp. per sq. in. of original section. } 175,992 ; 174,586 ; 183,938 ; 193,413 ; 193,197 ; 174,586 ; 193,517 ; 174,895 pounds per square inch.

Final comp. per sq. in. of final section. } 134,717 ; 127,579 ; 149,881 ; 139,196 ; 145,751 ; 128,834 ; 125,126 ; 140,489 pounds per square inch.

The final lengths varied from 56 to 89 per cent. of the originals.

Kirkaldy's " Experimental Inquiry into the Mechanical Properties of Fagersta Steel," 1873, furnish data from which may be computed a series of values of the ratio (E_1) of stress over strain, or coefficient of elasticity, for different intensities of stress.

All the specimens were cut from plates of mild steel of the thickness shown in the table, and were 100 inches long and 2.25 inches wide. They were laterally supported in a trough arrangement designed by Mr. Kirkaldy.

TABLE I.

INTENSITY OF STRESS.	COEFFICIENTS OF ELASTICITY IN POUNDS PER SQUARE INCH.			
	Unannealed.		Annealed.	
	⅛ inch plate.	⅝-inch plate.	⅛-inch plate.	⅝-inch plate.
10,000	58,824,000	38,462.000	50,000,000	34,483,000
14,000	56,000,000	33,333,000	48,276,000	33,333,000
18,000	54,545,000	31,034,000	45,000,000	31,034,000
22,000	48,889,000	29,730,000	38,596,000	26,190,000
26,000	45.614,000	28,261,000	23,214,000	19,118,000
30,000	42,254,000	24,000,000	7,792,000	2,542,000
34,000	40,000,000	3,795,000	5,207,000	——
38,000	38,384,000	——	4,265,000	——
42,000	35,000,000	——	3,717,000	——
46,000	30,872,000	——	——	——
50,000	25,000,000	——	——	——

Below 18,000 pounds per square inch annealing does not much change the coefficients for the ⅝-inch specimens, but affects the thin ones more decidedly. In all the specimens the elastic behavior is very irregular, and none of them can be said to possess a true elastic limit. In the unannealed thin and thick plates, the first permanent sets took place at 40,000 and 20,000 pounds per square inch, respectively; in the corresponding annealed ones, at 30,000 and 20,000, respectively.

By referring to Table III. of Art. 34, it will be seen that the coefficients for compression are considerably larger than those for tension.

Table II. contains coefficients of elasticity computed from the results of experiments made under the supervision of the "Steel Committee of Civil Engineers" (English). All are computed for the "limit of elasticity." The upper portion of the table belongs to round specimens 1.382 inches in diameter and 50 inches (36 diameters) long, tested in 1868; the lower

TABLE II.

	QUALITY.	GREATEST E_1.	MEAN E_1.	LEAST E_1.
Hammered and rolled.	Bessemer.	36,130,000	35,000,000	34,461,000
	Crucible.	34,461,000	33,939,000	31,464,000
Chisel, tire, rod, rolled, etc.	Bessemer.	30,270,000	29,091,000	28,718,000
	Crucible.	31,550,000	29,474,000	28,000,000

E_1 = coefficient of compressive elasticity in pounds per square inch.

portion to 1.5 inch round specimens and 120 inches long, which were tested in 1870. Table I. shows that considerably different results might be expected with lower intensities of stress.

TABLE III.

	LIMITS OF ELASTICITY IN POUNDS PER SQUARE INCH.			LENGTH.
	Greatest.	Mean.	Least.	Inches.
Bessemer; tires, hammered, rolled, fagotted, etc. Diam. = 1.382 ins.	53,520	50,250	42,500	1.38
	52,240	48,540	40,990	2.76
	49,800	46,700	40,470	5.53
Crucible; hammered, rolled, chisel, tires, rods, etc. Diam. = 1.382 ins.	60,260	55,760	48,990	1.38
	59,000	53,780	43,990	2.76
	53,740	49,860	42,000	5.53

The upper Bessemer results are for a set of 18, and the lower for a set of 11 tests; the upper crucible results are for a set of 11, and the lower for a set of 20 (?) tests.

The "limits of elasticity" of specimens of the same steels to which the upper portion of Table II. belongs (and for the same number of experiments) are shown in Table III.

The following "limits of elasticity," in pounds per square inch, correspond to the lower portion of Table II.:

	GREATEST.	MEAN.	LEAST.
Bessemer	47,490	39,870	35,840 pounds.
Crucible	60,480	52,190	36,290 pounds.

TABLE IV.

LENGTH.	ELASTIC LIMIT IN POUNDS PER SQUARE INCH.			
	1.2.	0.9.	0.6.	0.3.
1 diam.	64,000	62,670	60,000	39,000
2 "	63,330	58,670	57,330	42,000
4 "	62,330	58,670	53,330	41,000
8 "	61,670	58,000	52,670	40,670
Means.	62,833	59,500	55,830	40,670
	ULTIMATE RESISTANCE IN POUNDS PER SQUARE INCH.			
1 diam.	——	——	——	——
2 "	169,910	173,290	156,000	121,330
4 "	133,330	117,560	105,330	81,760
8 "	102,170	95,210	84,830	47,510
Means.	135,140	128,690	115,387	83,540

The results of the experiments of Mr. Kirkaldy on specimens of different grades of Fagersta steel and of various lengths in terms of diameter, are given in Table IV. All the specimens were turned to 1.128 inches (1 square inch area) in diameter, and were of the lengths shown.

The numbers 1.2, 0.9, 0.6 and 0.3 were used to indicate the different grades of steel, the larger numbers belonging to the higher steels.

The specimens of one diameter in length shortened, under a load of 200,000 pounds per square inch, 21, 22, 26 and 48 per cent., respectively, for the marks 1.2, 0.9, 0.6 and 0.3. Three of the "2 diam." specimens failed by detrusion, or by portions shearing off obliquely; all the others either bulged or took a skew form, though one of the "8 diams." finally broke.

Table V. contains the results of Major Wade's experiments

TABLE V.

DESCRIPTION.	LENGTH OVER DIAMETER.	ULT. RESIST. IN POUNDS PER SQUARE INCH.
Not hardened	2.55	198,944
Hardened, low temper.....	2.47	354,544
Hardened, mean temper ...	2.52	391,985
Hardened, high temper....	2.48	372,598

All specimens about 1 inch long and 0.4 inch in diameter.

on specimens of cast steel, in 1851. The results are seen to be very high.

A piece of the Hay steel used by Gen. Smith in the Glasgow, Mo., bridge, about 1¼ inches square and 3⅛ inches long, gave an ultimate compressive resistance of 139,350 pounds per square inch ("Annales des Ponts et Chaussées," Feb., 1881).

Art. 43.—Copper, Tin, Zinc, Lead and Alloys.

Table I. shows some coefficients of elasticity (*i.e.*, ratios between stress and strain), computed from data determined by Prof. Thurston, and given by him in the " Trans. Amer. Soc. of Civ. Engrs.," Sept., 1881. The gun bronze contained copper, 89.97; tin, 10.00; flux, 0.03. The cast copper was cast very hot.

TABLE I.

STRESS IN POUNDS PER SQUARE INCH.	COEFFICIENTS OF ELASTICITY IN POUNDS PER SQUARE INCH.	
	Gun Bronze.	Cast Copper.
1,620	——	1,254,000
3,260	3,622,000	1,415,000
6,520	4,075,000	1,651,000
9,780	6,113,000	1,795,000
13,040	6,520,000	1,824,000
16,300	5,433,000	1,842,000
19,560	5,148,000	1,845,000
22,820	3,935,000	1,735,000
26,080	2,308,000	1,503,000
29,340	——	1,144,000
32,600	1,073,000	815,000
48,900	463,600	332,500

The ratios of stress over strain are far from being constant. Strictly speaking, therefore, there is no elastic limit in either case. In that of the gun bronze, however, it may be approximately taken at 20,000 pounds per square inch (Prof. Thurston takes it 22,820), and in that of the copper at 25,000 pounds. The test specimens were two inches long and turned to 0.625 inch in diameter.

At 38,000 pounds per square inch the gun bronze specimen was shortened about 41 per cent. of its original length, while its diameter had become 0.77 inch.

TABLE II.

COMPOSITION.		POUNDS PER SQUARE INCH CAUSING A SHORTENING OF			GREATEST LOAD IN LBS. PER SQ. IN.	PER CENT. OF SHORTENING CAUSED BY GREATEST LOAD.	ULT. CRUSHING RESISTANCE IN LBS. PER SQ. INCH.	MANNER OF FAILURE.
Copper.	Tin.	5%	10%	20%				
97.83	1.92	29,340	34,000	46,000	46,260	0.37	34,000	Flattened.
95.96	3.80	39,200	42,050	52,150	52,150	0.30	42,050	"
92.07	7.76	31,500	42,000	65,000	84,100	0.45	42,000	"
90.43	9.50	32,000	38,000	60,000	61,930	0.34	38,000	"
87.15	12.77	39,000	53,000	80,000	89,640	0.39	53,000	"
80.99	18.92	65,000	78,000	103,490	103,490	0.20	78,000	"
76.60	23.23	101,040	——	——	114,080	0.09	114,080	Crushed.
69.90	29.85	——	——	——	146,680	0.04	146,680	"
65.31	34.47	——	——	——	84,750	0.03	84,750	"
61.83	37.74	——	——	——	39,110	0.02	39,110	"
47.72	51.99	——	——	——	84,750	0.02	84,750	"
44.62	55.15	——	——	——	35,850	0.01	35,850	"
38.83	60.79	——	——	——	39,110	0.02	39,110	"
38.37	61.32	——	——	——	29,340	0.01	29,340	"
34.22	65.80	19,560	——	——	19,560	0.06	19,560	"
25.12	74.51	17,930	17,930	17,930	17,930	0.28	17,930	"
20.21	79.62	16,300	16,300	16,300	16,300	0.29	16,300	"
15.12	84.58	6,520	6,520	6,520	9,450	0.51	6,520	Flattened.
11.48	88.50	10,100	10,100	10,100	14,020	0.50	10,100	"
8.57	91.39	6,500	——	——	9,780	0.06	9,780	"
3.72	96.31	6,520	6,520	6,520	9,780	0.34	9,780	"
0.74	99.02	6,520	6,520	6,520	9,780	0.36	9,780	"
0.32	99.46	6,520	6,520	6,520	9,780	0.38	9,780	"
Cast copper.		26,000	39,000	51,000	74,970	0.45	39,000	"
"	"	33,000	45,500	58,670	78,230	0.43	45,500	"
"	"	34,000	42,000	58,000	71,710	0.32	42,000	"
"	"	30,000	36,000	50,000	104,300	0.52	36,000	"
"	"	30,000	37,000	50,000	91,270	0.48	37,000	"
"	"	35,000	48,000	65,000	97,790	0.41	48,000	"
Cast tin.		6,030	6,400	6,530	7,500	0.44	6,400	"

The copper specimen failed at 71,700 pounds per square inch, having been shortened about one-third of its length.

The results of a series of tests by Prof. Thurston, in connection with the United States testing commission, are given in Table II.; they were abstracted from "Mechanical and Physical Properties of the Copper-Tin Alloys," United States Report, edited by Prof. R. H. Thurston, 1879. All the specimens were 0.625 inch in diameter and 2 inches long. Scarcely one of them can be said to possess an elastic limit.

The series of alloys presents some interesting results. About the middle third of the series are seen to be brittle compounds giving (as a rule) high ultimate compressive resistances, while the other two-thirds are ductile, and give at the copper end high results, and low ones at the tin end.

It will be observed that Prof. Thurston took the load per square inch which gave a shortening of 10 per cent. of the original length as the ultimate resistance to crushing of the ductile alloys and metals, since such materials cannot be said to completely fail under any pressure, but spread laterally and offer increased resistance.

TABLE III.

PER CENT. OF		POUNDS PER SQUARE INCH FOR		PER CENT. OF SHORTENING.	MANNER OF FAILURE.
Copper.	Zinc.	E_1.	Ult. Resist.		
96.07	3.79	305,500	29,000	10.0	Flattened.
90.56	9.42	342,100	30,000	10.0	"
89.80	10.06	——	29,500	10.0	"
76.65	23.08	656,500	42,000	10.0	"
60 94	38.65	1,772,500	75,000	10.0	"
55.15	44.44	——	78,000	10.0	"
49.66	50.14	1,345,500	117,400	10.0	"
47.56	52.28	1,500,000	121,000	10.0	"
25.77	73.45	4,232,800	110,822	5.85	Crushed.
20.81	77.63	2,485,000	52,152	2.75	"
14.19	85.10	897,000	48,892	10.8	"
10.30	88.88	——	49,000	10.0	Flattened.
4.35	94.59	——	48,000	10.0	"
0.00	100.00	318,500	22,000	10.0	"

Table III. contains the results of Prof. Thurston's tests of the copper-zinc alloys made while he was a member of the United States Board. The data are taken from "Ex. Doc. 23, House of Representatives, 46th Congress, 2d Session." The specimens were two inches long and 0.625 inch in diameter of circular cross section.

The values of E_1 (ratios of stress over strain) are computed for about one-quarter the ultimate resistance. This ratio is so very variable for different intensities of stress that these alloys can scarcely be said to have a proper "elastic limit."

In the "Philosophical Transactions" for 1818, Rennie gives the following as the results of his experiments on 0.25 inch cubes :

Fine yellow brass (10 per cent. shortening)... 12,852 pounds per square inch.
Fine yellow brass (50 per cent. shortening)... 41,216 pounds per square inch.
Cast lead........(50 per cent. shortening)... 1,932 pounds per square inch.

Art. 44.—Glass.

The following results are taken from Sir Wm. Fairbairn's "Useful Information for Engineers," second series. The cylinders were about 0.75 inch in diameter and annealed.

TABLE I.

KIND OF GLASS.	SPECIMEN.	HEIGHT OF SPECIMEN.	CRUSHING RESISTANCE, LBS. PER SQ. INCH.
		Inch.	
Flint	Cylinder.	1.00	23,480
"	"	1.00	34,850
"	"	1.60	20,780
"	"	2.05	32,800
Green	"	1.00	22,580
"	"	1.50	35,030
"	"	2.00	38,020
Crown................	"	1.00	23,180
"	"	1.50	38,830
Flint	Cube.	1.15	14,240
"	"	1.16	13,200
"	"	1.10	13,260
"	"	1.10	11,820
Green	"	1.00	20,470
"	"	1.00	19,950
Crown................	"	0.90	21,870

It will be observed that the cubes give considerably less resistance than the cylinders.

All the glass was annealed, but Fairbairn remarks that the cubes may have been only imperfectly so, since they were cut out of the interior of larger masses, while the cylinders were cut from rods as they were drawn. The latter, also, thus retained their natural skins, which may have increased their resistances.

At the instant of failure the specimens were shattered into a great number of small pieces.

Art. 45.—Cement—Cement Mortar—Concrete—Artificial Stones.

Table I. of Article 37 contains the ultimate compressive resistances of a great number of pure cements, as tested by General Gillmore under the circumstances related in connection with the table. The results are given in pounds per square inch.

TABLE I.

CEMENT, VOL.	SAND, VOL.	LENGTH, INS.	DIAM., INS.	MEAN OF		THESE RESULTS ARE LBS. PER SQ. IN.			AGE, IN DAYS.
						Greatest.	Mean.	Least.	
1	0	12	2.5		E. L.	1,502	800	424	123 to 143.
1	0	"	"	114	R.	1,889	1,320	620	
1	0	"	"		E1.	1,500,000	800,000	500,000	
1	1	"	"		E. L.	587	365	191	117 to 141.
1	1	"	"	93	R.	783	494	261	
1	1	"	"		E1.	1,910,000	607,000	217,333	
1	2	"	"		E. L.	424	182	98	127 to 135.
1	2	"	"	45	R.	489	213	131	
1	2	"	"		E1.	6,633,330	1,285,000	220,450	

Table I. contains the results of tests of "Fall City" (Louisville) cement and cement mortar. The tests were made by Mr. Bremermann, by direction of Capt. Eads, during the construction of the St. Louis bridge. "E. L." is the elastic limit; "R." the ultimate resistance to compression; and "E_1" the coefficient of compressive elasticity, all in pounds per square inch.

The following results are from the same source:

Akron Cement.

4	$2\frac{1}{2}$-inch cubes;	1 vol. cement,	0 vol. sand; *ult. resist.*	= 2,140 lbs.
2	" " "	2 " "	1 " " " "	= 1,105 "
3	" " "	1 " "	1 " " " "	= 733 "
2	" " "	2 " "	3 " " " "	= 520 "
1	" " "	1 " "	2 " " " "	= 240 "
1	" " "	1 " "	4 " " " "	= 480 "

"*Fall City,*" *Louisville, Cement.*

3	$2\frac{1}{2}$-inch cubes;	1 vol. cement,	0 vol. sand; *ult. resist.*	= 1,587 lbs.
1	" " "	2 " "	1 " " " "	= 640 "
1	" " "	1 " "	1 " " " "	= 400 "
1	" " "	2 " "	3 " " " "	= 240 "

Louisville Cement from Beach & Co.

4	$2\frac{1}{2}$-inch cubes;	1 vol. cement,	0 vol. sand; *ult. resist.*	= 1,615 lbs.
1	" " "	2 " "	1 " " " "	= 1,280 "
1	" " "	1 " "	1 " " " "	= 560 "
1	" " "	2 " "	3 " " " "	= 400 "
2	" " "	1 " "	2 " " " "	= 280 "

Louisville Cement from Hulme & Co.

2	$2\frac{1}{2}$-inch cubes;	1 vol. cement.	0 vol. sand; *ult. resist.*	= 2,320 lbs.
2	" " "	1 " "	1 " " " "	= 740 "
1	" " "	2 " "	3 " " " "	= 600 "

These ultimate resistances are in pounds per square inch.

All specimens were "moulded under hand pressure only, left 12 days in water, and exposed six months to the air."

TABLE I*a*.

DESCRIPTION OF PORTLAND CEMENT AND MORTAR.		ULT. RESIST. IN POUNDS PER. SQ. IN.
Neat Portland cement	Made 3 months.	3,795
1 Portland cement to 1 pit sand		2,491
" 2 "		2,004
" 3 "		1,436
" 4 "		1,331
" 5 "		959
Neat Portland cement	Made 6 months.	5,388
1 Portland cement to 1 sand		3,478
" 2 "		2,752
" 3 "		2,156
" 4 "		1,797
" 5 "		1,540
Neat Portland cement	Made 9 months.	5,984
1 Portland cement to 1 pit sand		4,561
" 2 "		3,647
" 3 "		2,393
" 4 "		2,208
" 5 "		1,678

The results of a large number of experiments on the compressive resistance of Portland cement and mortar, at different ages, by Mr. John Grant, C. E. ("On the Strength of Cement," 1875), are given in Table I*a*. The specimens were made into bricks $9 \times 4.25 \times 2.75$ inches, and were compressed on their flat sides of $9 \times 4.25 = 38.25$ square inches area. The results are in pounds per square inch.

Table II. is taken from the same work, and shows the circumstances under which Mr. Grant made his experiments. Two sets of blocks were made in each case; one set was kept in air for one year, and the other in water for the same length of time. The cubes were then crushed with the results shown. It is to be observed that the results are pounds per square foot. Two series of the blocks were formed by compressing the material in layers one inch thick; the others were not compressed.

TABLE II.

Volumes of ballast of sand and gravel to one vol. of cement.	Ultimate compressive resistance in pounds per square foot.					
	Compressed.				*Not Compressed.*	
	Kept in air.	Kept in water.	Kept in air.	Kept in water.	Kept in air.	Kept in water.
1	(Exceptional.) 239,680	381,920	340,480	301,060	268,800	336,000
2	333,760	358,400	385,280	309,120	344,960	322,560
3	253,120	258,720	268,800	318,080	215,040	250,880
4	230,720	243,040	268,800	250,880	250,880	241,920
5	199,360	222,880	219,520	318,080	215,040	210,560
6	180,320	203,840	182,784	175,616	163,072	152,320
7	168,000	180,320	147,840	143,360	125,440	112,000
8	137,760	170,240	120,960	120,960	112,000	98,560
9	120,960	153,440	107,520	98,560	89,600	80,640
10	108,640	107,520	94,080	94,080	71,680	62,720
	12″ × 12″ × 12″ blocks.		6″ × 6″ × 6″ blocks.		6″ × 6″ × 6″ blocks.	

Table II*a.* is from the same source as Table I.

The concrete blocks were pressed evenly on 36 square inches until failure took place.

The following results for artificial stones are given by Mr. Henry Reid ("A Practical Treatise on Natural and Artificial Concrete," 1879):

A 4-inch cube of Ransome's "Siliceous Stone" gave 4,200 *pounds per square inch.*

By experimenting with 2-inch cubes of "Rock Concrete" pipes, Mr. Reid obtained the following results from two series:

GREATEST.	MEAN.	LEAST.
4,340	3,454	2,684 pounds per sq. in.
5,650	4,428	3,401 " " " "
5,650	4,763	3,107 " " " "

TABLE II*a*.

6″ × 6″ × 6″ *Concrete Blocks.*

All blocks were under water 12 days, and then exposed to the air for 6 months.

VOLS. SAND.	VOLS. AKRON CEMENT.	VOLS. LOUISVILLE CEMENT.	VOLS. BROKEN LIMESTONE.	ULT. RESIST., LBS. PER SQ. IN.
1	1	0	4	889
1	1	0	4	1,124
1	1	0	4	1,170
2	1	0	4	722
2	1	0	4	889
2	1	0	4	1,361
1	0	1	4	1,194
2	0	1	4	950
2	0	1	4	640
2	0	1	4	890
1	0.5	0.5	4	1,170
1	0.5	0.5	4	1,361
1	0.5	0.5	4	1,445
1	0.5	0.5	4	1,280
1	0.5	0.5	4	1,250
2	0.5	0.5	4	918
2	0.5	0.5	4	1,000
2	0.5	0.5	4	1,140
2	0.5	0.5	4	1,361
2	0.5	0.5	4	611
2	0.5	0.5	4	1,111

There were six experiments in each series.

Three-inch cubes of Victoria stone (six experiments in the first series, and ten in the second) gave:

GREATEST.	MEAN.	LEAST.
5,179	4,422	3,294 pounds per sq. in.
4,708	3,955	3,578 pounds per sq. in.

These cubes were made in February, 1879, and broken in May of the same year.

"Two-inch cubes of silicated stone made with 3 parts Thames ballast and 1 of Portland cement, gauged with water, and put in the silicate bath for 11 days, about 12 months old fractured as follows:

No. 1........................ 4,237 pounds per square inch.
No. 2........................ 5,650 pounds per square inch."

Four "granitic breccia" cubes, 3″ × 3″, about 25 years old, gave the following results:

GREATEST.	MEAN.	LEAST.
8,886	8,028	7,533 pounds per square inch.

Seven blocks of Sorel stone, varying from 1¼ × 1¼ × 1 inch to 2 × 2⅛ × 1⅞ inches gave:

AGE.	INERT MATERIAL.	ULT. RESIST.
1 year	Coral sand	6,240 lbs. per sq. in.
1 "	Pulverized quartz	7,270 " " "
2 years	Washed flour of emery	19,640 " " "
3 "	Fine marble	11,560 " " "
9 months	Mill sweepings	6,130 " " "
2 years	Marble and sand	4,920 " " "
Not known	Marble with colored veneer	7,680 " " "

The weight of the oxide of magnesium varied from 12 to 15 per cent. of the whole.

The results of a series of tests, by Gen. Gillmore, in 1870 and 1871, on coignet béton blocks, 3.5 × 5.5 × 3 inches, are given in Table III. Two blocks of each kind were tested. All the blocks were two months old. The results are in pounds per square inch.

With four 2-inch cubes of Frear stone, Gen. Gillmore obtained the following results:

Four weeks old	4,500 pounds per square inch.
Four weeks old	4,626 " " " "
Three weeks old	2,250 " " " "
Six months old	2,000 " " " "

These blocks were composed of one measure of hydraulic cement, two and a half of sand, moistened with an alkaline solution of gum shellac of sufficient strength to furnish one

TABLE III.

	PROPORTIONS BY VOLUME, LOOSELY MEASURED.	COMP. RESIST. IN POUNDS PER SQ. IN.
Boulogne Portland.	Cement, 1; common lime powder, 0.4; sand, 5.6..	935 831
	" 1; " " " 0.8; " 5.6..	805 987
	" 1; " " " 0.4; " 7.5..	416 519
	" 1; " " " 0.8; " 7.5..	551 571
	" 1; " " " 0.4; " 5.6 gravel and pebbles, 5	649 681
	" 1; " " " 0.4; sand, 5.6 gravel and pebbles, 13	675 831
	" 1; " " " 0.8; sand, 5.6 gravel and pebbles, 5	649 623
	" 1; " " " 0.8; sand, 5.6 gravel and pebbles, 13	649 753

ounce of the shellac to 1 cubic foot of the finished stone. Portland cement was used in the first three blocks and Louisville cement in the last.

Specimens of artificial stone made under the Van Derburgh system and used in the walls of the Howard University and Hospital buildings at Washington, D. C., in 1868 and 1869, varying in age from 3 to 16 months, gave resistances of 173 (4 months old) to 564 (10 months old) pounds per square inch. Another specimen of the same stone, ten years old, gave 1,455 pounds per square inch.

Addendum to Article 45.

The following Table exhibits results taken from "Ex. Doc. No. 35, 49th Congress, 1st Session." The compression tests of the cubes were made in the Gov't machine at Watertown, Mass. The compressed faces of the cubes were either made

Compression of Cubes of Cement, Cement Mortar and Concrete.

BRAND OF CEMENT.	NEAT CEMENT, MORTAR OR CONCRETE.	ULTIMATE COMP. RESIST. IN LBS. PER SQ. IN. FOR FOLLOWING CUBES.					WEIGHT PER CU. FT. LBS.
		4 Inch Cubes.	6 Inch. Cubes.	8 Inch Cubes.	12 Inch. Cubes.	16 Inch. Cubes.	
Dyckerhoff's Portland	neat cement.	4,860	4,272	4,865	5,436		125–132
National Portland	1—cement. 3—sand.	3,450	2,654	2,478	2,434	2,520	116–124
	1—cement. 3—sand. 6—broken stone.	4,014	2,627	3,027	2,690	2,979	137–145
Newark Co's Rosendale	1—cement. 3—sand.	762*	800*	707*	685*	614*	113–118
	1—cement. 3—sand. 2—gravel. 4—stone.	1,032*	1,025* / 1,230	846* / 1,194	831* / 1,113	674* / 1,039	127–136
Norton's Cement	1—cement. 1½-sand.	2,048	1,340	1,746	1,346	1,247	117.2–122.6
	1—cement. 1½-sand. 6—broken stone.	2,321	963	1,433	1,560	1,445	136–147.9
Norton's Cement	1—cement. 3—sand.	1,325	750	790	688	718	114–121
	1—cement. 3—sand. 6—broken stone.	1,635	1,000	862	766	843	128–140

* These results were with pine cushions or bearing surfaces. All other bearing surfaces were of steel. There is seen to be a loss of nearly 25 per cent. in passing from the latter to the former.

true in the original molds, or else were trued with plaster of paris. The ages of the cubes, and cushions between which they were crushed were as follows:

Dyckerhoff's Portland, age 22 months........cushion—steel.
National, " " 46 " .... " — "
Newark Co.'s Rosendale, " 22 " .... " — " and pine.
Norton's Cement, " 46 " .... " — "

Pyramidal or approximately pyramidal failure along planes of greatest shear took place very generally.

Each result for the Dyckerhoff Portland cement is a mean of six tests; each of all the others is a mean of two tests.

The results are somewhat irregular, although the smallest cubes generally give the greatest. Those for the 8, 12 and 16 inch cubes run comparatively uniform for any given mixture.

The following table shows the mean results of a great number of tensile tests of Portland and Rosendale cements by Eliot

Ultimate Tensile Resistance in Pounds per Square Inch.

Age.	Neat Cement.		Cement, 1. Sand, 1.		Cement, 1. Sand, 1.5.		Cement, 1. Sand, 2.		Cement, 1. Sand, 3.		Cement, 1. Sand, 5.	
	P.	*R.*	*P.*	*R.*	*P.*	*R.*	*P.*	*R.*	*P.*	*R.*	*P.*	*R.*
1 Day.........	102	71	......	......	......	......	......	......	......	......	......	......
1 Week........	303	92	160	56	......	41	126	24	95	14	55	5
1 Month.......	412	145	225	116	......	95	163	60	140	35	88	16
6 "	468	282	347	190	......	155	279	125	198	80	136	46
12 "	494	290	387	256	......	230	323	180	257	121	155	80

C. Clarke, as found in the "Trans. Am. Soc. C. E.," for 1885. "P" indicates Portland, and "R" Rosendale cement. The test briquette was quite similar to Fig. 3, page 363.

Art. 46.—Bricks and Brick Piers.

The first set of results given below is computed from data given by Gen. George S. Green, Jr., C.E., in Vol. II. of "Trans. Am. Soc. of Civ. Eng'rs.

Nos. 1, 2 and 4 cracked, but did not crush to pieces, as the others did.

NO.	SIZE OF BRICK.	SURFACE.	COMP. RESIST.
	Inches.	Sq. ins.	Lbs. per sq. in.
1	2.3 × 3.52 × 4.4	15.5	3,230
2	2.24 × 3.5 × 4.46	15.6	3,360
3	2.34 × 3.5 × 4.52	15.8	2,750
4	2.34 × 3.46 × 4.46	15.4	1,994
5	2.30 × 3.46 × 4.50	15.6	2,050
6	2.28 × 3.46 × 4.60	15.9	2,920

The pressure was applied on the two opposite largest faces of the bricks, giving blocks whose heights were only 0.7 their least widths.

In Vol. VII. of the "Trans. Am. Soc. of Civ. Engrs." Mr. Francis Collingwood, C. E., gives the following as the results of compressing ten whole bricks *on end:*

GREATEST.	MEAN.	LEAST.
3,060	2,065	1,524 pounds per square inch.

For ten half bricks on small side:

6,400	4,610	2,900 pounds per square inch.

For ten half bricks on flat side:

4,150	3,370	2,670 pounds per square inch.

In regard to these tests Mr. Collingwood says, "The bricks were selected to give a fair average of 'good Haverstraw stock brick,' not the hardest burned. No packing was inserted in the machine between the bricks and the compressing surfaces; so that the strength in compression represents the case of imperfect beds, etc., although it was found that it made but little difference." He attributes the higher values for the "ten half bricks on small sides," over those belonging to the half bricks on flat side, to the imperfect bearing surfaces of the latter.

Table I. exhibits the results of testing piers of brick masonry in the Gov't testing machine at Watertown, Mass.; it is taken from "Ex. Doc. No. 35, 49th Congress, 1st Session." The

TABLE I.

Crushing Strength of Brick Piers.

NO.	HEIGHT OF PIER FT.	IN.	SECTION OF PIER.	COMPOSITION OF MORTAR.	WEIGHT PER CU. FT. LBS.	ULTIMATE RESISTANCE. LBS. PER SQ. IN.	
1	1	4	8 × 8	1 lime, 3 sand.	137.4	2,520	Face bricks.
2	6	8	8 × 8	1 " 3 "	133.5	1,877	
3	1	4	8 × 8	1 Port. cement, 3 sand.	136.3	3,776	
4	6	8	8 × 8	1 " " 3 "	133.5	2,249	
5	2	0	12 × 12	1 lime, 3 sand.		1,940	
6	2	0	12 × 12	1 " 3 "		1,900	
7	10	0	12 × 12	1 " 3 "	131.7	1,511	
8	10	0	12 × 12	1 " 3 "	125.0	1,807	
9	2	0	12 × 12	1 Port. cement, 2 sand.		3,670	
10	10	0	12 × 12	1 " " 2 "	132.2	2,253	
11	1	4	8 × 8	1 lime, 3 sand.	135.6	2,440	Common bricks.
12	6	8	8 × 8	1 " 3 "	133.6	1,540	
13	2	0	12 × 12	1 " 3 "		2,150	
14	2	0	12 × 12	1 " 3 "		2,050	
15	9	9	12 × 12	1 " 3 "	131.5	1,118	
16	10	0	12 × 12	1 " 3 "	136.0	1,587	
17	10	0	12 × 12	1 Port. cement, 2 sand.	131.0	2,003	
18	2	8	16 × 16	1 " " 2 "		2,720	
19	10	0	16 × 16	1 " " 2 "		1,887	
20	2	0	12 × 12	1 lime, 3 sand.		1,370	Bay State b'ks.
21	6	0	12 × 12	1 " 3 "		1,133	
22	6	0	12 × 12	1 " 3 "	119.7	1,210	
23*	6	0	12 × 12	1 lime, 3 sand.	118.2	1,331	Face bricks.
24†	6	0	12 × 12	1 " 3 "	118.1	1,211	
25	7	10	12 × 12	1 " 3 "	120.3	1,174	
26	10	0	12 × 12	1 " 3 "	118.0	924	
27	10	0	8 × 12	1 " 3 "	107.0	940	
28	10	0	12 × 16	1 " 3 "	118.7	773	
29	6	0	12 × 12	1 " 3 ", 1 Rosendale cement.	120.6	1,646	
30	6	0	12 × 12	1 Rosendale cement, 2 sand.	123.0	1,972	
31	6	0	12 × 12	1 lime, 3 sand, 2 Port. cement.	120.3	1,411	
32	6	0	12 × 12	1 Port. cement, 2 sand.	119.7	1,792	
33	6	0	12 × 12	clear Port. cement.	126.6	2,375	

dimensions of piers are shown in the table, also the kinds of mortar used and the grades of brick. The "common" and "face" brick, both hard burnt, were from North Cambridge,

* Joints broken every 6 courses. † Bricks laid on edge.

Mass. The other bricks were from the Bay State Brick Co., of Boston and Cambridge, Mass., and were medium burnt.

The crushing strength of three bricks of each kind, between steel compression platforms were first determined as follows:

BRICKS.	CRUSHING RESISTANCE; LBS. PER SQ. IN.					SIZE; INCHES.
Face	11,056	13,984	16,734	13,925	Averages.	7.75 × 3.7 × 2.0
Common	19,785	22,351	12,995	18,337		8.0 × 3.6 × 2.1
Bay State	11,120	12,709	10,390	11,406		7.8 × 3.6 × 2.1

Care was taken to make the bed faces of all three bricks bear evenly against the compression platforms, and in order to accomplish this result thin sheets of brass were used for packing.

The brick piers were built of bricks "laid on bed, and joints broken every course, with the exception of two 12 by 12 piers, one of which had joints broken every sixth course, and one had bricks laid on edge.

"They were built in the month of May 1882," and "their ages when tested ranged from 14 to 24 months." They were all tested between cast-iron plates.

"Loads were gradually applied in regular increments, . . . returning at regular intervals to the initial load, . . ." "Cracks made their appearance at the surfaces of the piers and were gradually enlarged before the maximum loads were reached. Final failure occurred by the partial crushing of some of the bricks, and by the enlargement of these cracks, which took a longitudinal direction and occurred in the bricks of one course opposite the end joints of the bricks in the adjacent courses. This manner of failure was common to all the piers."

It is very important to notice that the resistance of the piers varies with the strength of the mortar used in the joints.

The remarkably high results for the single bricks given above are probably due to the excellent quality of the material tested and to the great care exercised to have even bearings on the compressed beds.

Art. 47.—Natural Building Stones.

The ultimate resistances and coefficients of elasticity given in Table I. were determined in connection with the construction of the St. Louis arch, and have been taken from Prof. Woodward's history. The following results for Missouri stones are from the same source:

TABLE I.

MATERIAL.	LENGTH IN INCHES.	DIAM. IN INCHES.	POUNDS PER SQ. IN. FOR Ultimate Resistance.	POUNDS PER SQ. IN. FOR E_1.
Grafton Magnesian limestone	6.46	1.14	7,200	10,500,000
" " "	5.87	1.06	8,500	8,400,000
" " "	5.96	1.06	2,000	8,500,000
" " "	5.99	1.07	6,000	6,000,000
" " "	3.00	3 × 3	15,400	——
" " "	8.00	2.38	10,100	12,000,000
" " "	13.00	1.13	10,800	5,000,000
Portland granite	5.88	2.36	16,000	5,500,000
" "	5.98	2.36	18,500	6,400,000
" "	5.97	2.38	17,000	5,000,000
Richmond "	6.00	2.30	16,400	13,500,000
Portland "	3.00	3 × 3	13,700	——
Missouri red granite	3.00	3 × 3	12,700	——
" " "	3.00	3 × 3	13,000	——
" " "	3.00	3 × 3	12,700	——
" " "	3.00	3 × 3	13,600	——

		ULT. RESIST.
3″ × 3″ × 3″	cube brown ochre marble	15,000 lbs. per sq. in.
3″ × 3″ × 3″	" sandstone from Ste. Genevieve	5,330 " " " "
$4\frac{7}{8} \times 4\frac{7}{8} \times 4\frac{7}{8}$	" " " " "	5,500 " " " "
$3\frac{1}{16} \times 3\frac{1}{16} \times 3\frac{1}{16}$	" " " " "	3,400 " " " "

TABLE II.

Two-inch Cubes.

KIND.	LOCALITY.		POSITION.	COMP. RESIST. LBS. PER SQ. INCH.	POUNDS PER CUBIC FOOT.	REMARKS.
Blue	Staten Island, N. Y.	Granite.	Bed.	22,250	178 8	Cracked before bursting.
	Dix Island, Me.			15,000	166.5	Burst suddenly.
Dark	Quincy, Mass			17,750	166.2	Cracked before bursting.
Light	Quincy, Mass			14,750	168.7	Cracked before bursting.
Flagging	North River			13,425	168.1	Broke suddenly.
Old Quarry	Westerly, R. I.			17,750	165.6	" "
Old Quarry	Westerly, R. I.			17,250	165.6	" "
Up River	Richmond, Va			21,250	——	" "
Up River	Richmond, Va			20,000	——	" "
Niantic River	New London, Conn			12,500	166.3	" "
Niantic River	New London, Conn		Edge.	14,175	166.3	" "
Porter's Rock	Mystic River, Conn		Bed.	18,125	164.4	" "
Porter's Rock	Mystic River, Conn		Edge.	22,250	164.4	" "
Gray	Westerly, R. I.		Bed.	14,687	166.9	" "
Gray	Westerly, R. I.		Edge.	14,937	166.9	" "
Gray	Richmond, Va		Bed.	14,100	164.4	" "
Gray	Richmond, Va.		Bed.	13,875	164.4	" "
Gray	New Haven, Conn.		Edge.	7,750	162.5	Waxy-looking.
Gray	New Haven, Conn.		Bed.	9,500	162.5	Broke suddenly.
Gneiss	Sachemshead Quarry, Conn		Edge.	15,937	163.7	" "
Gneiss	Sachemshead Quarry, Conn		Bed.	14,000	163.7	" "
Dark	Duluth, Minn.		Bed.	17,750	173.7	Syenitic.
Dark	Huron Island, Mich.		Bed.	18,125	164.4	
Bluish-gray	Keene, N. H.		Bed.	10,375	166.0	
Gray	Pompton, N. J.		Bed.	24,040	——	Average of 3.
Glen's Falls	Glen's Falls, N. Y.	Limestone.	Bed.	11,475	168.8	Burst without cracking.
Glen's Falls	Glen's Falls, N. Y.		Edge.	10,750	168.8	" " "
Lake	Lake Champlain, N. Y.		Bed.	25,000	171.9	" " "
Lake	Lake Champlain, N. Y.		Edge.	21,500	171.9	" " "
North River	Kingston, N. Y.		Bed.	13,900	168.2	" " "
North River	Kingston, N. Y.		Edge.	11,050	168.2	" " "
White	Joliet, Ill		Bed.	12,775	158.8	" " "
White	Joliet, Ill		Bed.	16,900	162.5	" " "
Drab	Lime Island, Mich.		Bed.	25,000	161.2	" " "
Drab	Lime Island, Mich.		Bed.	15,425	159.4	" " "
Drab	Marquette, Mich.		Bed.	7,825	146.3	" " "
Drab	Marquette, Mich.		Edge.	7,600	146.3	Rather a clay stone.
Dark	Bardstown, Ky		Bed.	16,250	166.9	
Dark	Bardstown, Ky		Edge.	15,000	166.9	
Drab	Canton, Mo		Bed.	9,250	146.0	
Drab	Canton, Mo		Bed.	5,650	146.0	
Caen	France		Bed.	3,650	118.8	
Caen	France		Bed.	3,450	118.8	
East Chester	Tuckahoe, N. Y.	Marble.	Bed.	12,950	179.7	
East Chester	Tuckahoe, N. Y.		Bed.	12,050	179.7	
Vermont	Dorset, Vt		Bed.	7,612	164.7	
Vermont	Dorset, Vt		Bed.	8,670	167.8	

TABLE II.—*Continued.*

KIND.	LOCALITY.		POSITION.	COMP. RESIST. LBS. PER SQ. INCH.	POUNDS PER CUBIC FOOT.	REMARKS.
Drab	Mill Creek Quarry, Ill.	Marble.	Bed.	9,687	160.6	
Drab	Mill Creek Quarry, Ill.		Edge.	9.787	156.9	
Drab	North Bay Quarry, Wis.		Bed.	20,025	175.0	
Drab	North Bay Quarry, Wis.		Edge.	13.700	175.0	
Common Ital.	Italy		Bed.	11,250	168.2	
Common Ital.	Italy		Bed.	13.062	168.2	
Brown	Little Falls, N. Y.	Sandstone.	Bed.	9.850	140.6	Broke sud'ly. Hardened by years of exposure.
Brown	Little Falls, N. Y.		Edge.	9,150	140.6	
Gray	Belleville, N. J.		Bed.	11.700	141.0	
Gray	Belleville, N. J.		Edge.	10,250	141.0	
Brown	Middletown, Conn		Bed.	6.950	148.5	
Brown	Middletown, Conn		Edge.	5,550	148.5	
Pink.	Medina, N. Y.		Bed	17,250	150.6	
Pink.	Medina, N. Y.		Edge.	14.812	149.3	
Drab	Berea, Ohio		Bed.	10,250	131.9	
Drab	Berea, Ohio		Bed.	8,300	133.1	
Drab	Berea, Ohio		Bed.	7.250	137.5	
Drab	Vermillion, Ohio		Bed	8,250	135.3	
Drab	Vermillion, Ohio		Bed.	6.000	135.3	
Purple	Fond du Lac, Wis.		Bed.	6,250	138.8	
Purple	Fond du Lac, Wis		Edge.	5,110	138.8	
Purple	Marquette, Mich		Bed.	7,450	135.0	
Purple	Marquette, Mich		Edge.	5,730	135.0	
Red-brown	Seneca Freestone, O.		Bed.	9.687	149.3	
Red-brown	Seneca Freestone, O.		Edge.	10,500	149.3	
Olive green	Cleveland, Ohio		Bed.	6,800	140.0	
Olive green	Cleveland, Ohio		Edge.	7,910	140.0	
Brown	Albion, N. Y		Bed.	13,500	151.2	
Brown	Albion, N. Y		Edge.	11,350	151.2	
Pink.	Kasota, Minn		Bed.	10,700	164.4	Calcareous.
Pink.	Kasota, Minn		Edge.	11,675	164.4	
Light buff	Fontenac, Minn		Bed.	6,250	145.3	
Light buff	Fontenac, Minn		Edge.	7,775	145.3	
Freestone	Dorch'ter, New Brunswick		Bed.	9,150	——	
Freestone	Dorch'ter, New Brunswick		Edge.	6,050	——	
Yellow drab.	Massillon, Ohio		Bed.	8.750	131.8	
Yellow drab.	Massillon, Ohio		Bed.	6,725	131.8	
Craigleith	Edinburgh, Scotland		Bed.	12.000	141.3	
Craigleith	Edinburgh, Scotland		Edge.	11.250	141.3	

Table II. and the other tables of this article contain the results of tests given in the "Report on the Compressive Strength, Specific Gravity and Ratio of Absorption of the Building Stones in the United States," by Gen. Q. A. Gillmore, 1876.

The specimens, whose tests are given in Table II., were 2-inch cubes. " Each cube was placed between two cushion blocks of soft pine wood, 2 inches by 2 inches square, and slightly more than 0.25 inch in thickness; one on the top and the other under the bottom; the grain of the wood being parallel in each to the other—though no difference was observed when this was changed, as regards amount of record." . . . " The cubes . . . . . were brought to a true, smooth and regular, but not a polished surface." The third column shows whether the specimen was crushed " on bed " or " on edge."

TABLE III.

Berea Sandstone Cubes.

EDGE OF CUBE.	COMP. RESIST., LBS. PER SQUARE INCH.	EDGE OF CUBE.	COMP. RESIST., LBS. PER SQUARE INCH.
Inch.	Pounds.	Inches.	Pounds.
0.25	4,992	2.00	8,955
0.50	6,080	2.25	9,130
0.75	6,347	2.50	8,856
1.00	6,990	2.75	9,838
1.25	7,342	3.00	10,125
1.50	8,226	4.00	11,720
1.75	9,310	——	——

General Gillmore showed that the size of the cube tested, affected very greatly the ultimate compressive resistance per unit of area of face of cube. Table III. shows the results of gradually increasing the size of cubes of Berea sandstone, crushed " on bed " between wooden cushion blocks increasing (with size of cube) from about 0.0625 inch to about 0.4 inch in thickness. The general result is very marked in spite of two or three irregularities.

These results are natural consequences of the character of stone and the cubical form of the specimens. A few of General Gillmore's experiments showed that such results would probably not appear if the length of the specimens had been two or three times the width or breadth.

The effect of different bearing surfaces on the ultimate compressive resistance of stone cubes is well shown by the results given in Table IV. All the results are in pounds per square inch, and belong to two-inch cubes, with the exception of the "Sandstone, drab" specimens, which were 1.5 inch cubes. Each result is a mean of two to five tests.

TABLE IV.

KIND OF STONE.	ULT. COMP. RESIST., POUNDS PER SQUARE INCH.			
	Steel	Wood.	Lead.	Leather.
Granite, Millstone Point, Conn....	23,190	22,880	15.730	——
Granite, Keene, N. H.	24,000	19,830	14.480	15,730
Marble, East Chester, N. Y.......	19,125	17,540	11.560	——
Sandstone, Berea, Ohio...........	11,260	10,290	7.380	6,730
Vermont marble, Vt...............	13,280	10,850	9,200	8,190
Limestone, Sebastopol............	1,075	1,075	1,075	1,075
Sandstone, drab..................	4,000	4,000	4.000	——
Sandstone, Massillon, Ohio........	8,500	8,750	7,250	——
Sandstone, Massillon, Ohio (softer).	5,660	6,730	5,500	3,640

The steel cushion gave the highest results by a little. A soft cushion seems to be driven into the small cavities and interstices of the specimen, and thus to produce a splitting action at the bearing surfaces. "The beds of the granite and marble cubes were rubbed to the border of polish; those of sandstone were rubbed smooth."

Again, polished and unpolished cubes give different resistances per square inch, as shown in Table V. The results there given are for two-inch cubes pressed upon by wooden cushions.

It is at once evident that the polished cubes gave considerably the highest resistances. This is probably due to the fact that the splitting action of the wooden cushions was reduced to a minimum on the polished surfaces.

TABLE V.

KIND OF STONE.	ULT. COMP. RESIST., PER SQUARE INCH.	
	Polished.	Unpolished.
	Pounds.	Pounds.
Granite, Quincy, Mass.	24,750	17,750
Granite, Staten Island, N. Y.	25,000	22,250
Granite, Garrison's, N. Y.	21,630	13,380
Granite, Tarrytown, N. Y.	23,750	18,250
Granite, Millstone Point, Conn.	22,880	18,750
Granite, Keene, N. H.	19,830	12,750
Granite, Westerly, R. I	23,500	17,750
Marble, East Chester, N. Y.	17,540	12,950
Marble, Vermont, Vt.	10,850	8,750

General Gillmore's experiments show, in a very conclusive manner, that variety in circumstances of testing will produce a variety of results for the same section of stone specimen. Attending circumstances and dimensions of specimens, therefore, should always be given.

Art. 48.—Timber.

Table I. is based upon results of experiments made at the Stevens Institute, which were given by Prof. Thurston in the Journal of the Franklin Institute for Oct., 1879. The specimens were well seasoned and turned to about 1.125 inches in

TABLE I.

WOOD.	POUNDS OF STRESS PER SQUARE INCH AT			PER CENT. OF FINAL SHORTENING.
	Ult. Resist.	Elas. Lim.	Coefficient of Elas.	
White pine	9,590	5,600	354,400	3.5
Yellow pine.	11,950	7,000	469,800	2.9
Locust	14,820	9,800	604,950	3.3
Black walnut....	7,000	5,700	1,079,500	1.25
White ash	8,150	5,180	713,300	2.3
White oak	7,140	5,600	361,300	3.3
Live oak	10,410	6,300	594,350	3.4

TABLE II.

WOOD.	NO. OF EXPERIMENTS.	ULT. RESIST. IN POUNDS PER SQUARE INCH.		
		Greatest.	Mean.	Least.
Georgia pine..........	9	11,500	9,520	8,170
White pine...........	9	7,500	6,640	5,880
Locust.........	9	12,580	11,720	11,010
White oak............	9	9,780	8,000	6,530
Spruce...............	9	8,410	7,860	7,170
Hemlock..............	9	6,280	5,690	5,210

diameter with a length of 2.25 inches; they were compressed in the direction of the fibre. The coefficients of elasticity were computed at the "elastic limit," *i.e.*, at the point at which permanent set began.

Table II. contains the results of experiments made by R. G. Hatfield ("Transverse Strains," 1877). The specimens were from one to two diameters high, and were compressed in the direction of the fibres.

The mean results of numerous English experiments by Thomas Laslett ("Timber and Timber Trees, Native and Foreign," 1875) are given in Table III. He found very little difference in the results for 1-inch, 2-inch, 3-inch and 4-inch cubes; those for the smaller cubes, as a rule, gave a slight excess over the others. The cubes were crushed in the direction of the fibre.

TABLE III.

TIMBER, 1, 2, 3 and 4-inch Cubes.	ULT. RESIST. IN LBS. PER SQ. IN.	TIMBER, 1, 2, 3 and 4-inch Cubes.	ULT. RESIST. IN LBS. PER SQ. IN.
Oak, English (unseasoned)	4,900	Mahogany, Mexican	5,600
Oak, English (seasoned)	7,480	Eucalyptus, Tewart	9,350
Oak, French	7,950	Eucalyptus, mahogany	7,170
Oak, Tuscan	5,470	Eucalyptus, iron-bark	10,300
Oak, Sardinian	5,835	Eucalyptus, blue-gum	6,900
Oak, Dantzic	7,480	Ash, English	6,970
Oak, American, white	6,070	Ash, Canadian	5,490
Oak, American, Baltimore	5,890	Elm, English	5,780
Teak, Moulmein	5,730	Elm, rock	8,580
Iron wood	11,670	Hornbeam	8,310
Chow	12,590	Fir, Dantzic	6,940
Greenheart	14,420	Fir, Riga	5,240
Sabicu	8,470	Fir, spruce	4,850
Mahogany, Spanish	6,400	Larch	5,820
Mahogany, Honduras	6,380	Cedar	4,480
Red pine	5,690	Pitch pine	6,470
Yellow pine	4,210	Kauri	6,430

The results of the compressive tests of short blocks of timber, during the construction of the St. Louis bridge, are given in Table IV. These are especially valuable, both in consequence of the large size of the blocks and the fact that the pressure was applied with and across the fibre.

The blocks are seen to be from two to eight times as strong with the fibre as across it.

TABLE IV.

KIND OF TIMBER.	WITH OR PERPENDICULAR TO FIBRE.	DIMENSIONS IN INCHES.	MEAN OF.	ULTIMATE RESISTANCE IN POUNDS PER SQ. IN.			REMARKS.
				Greatest.	Mean.	Least.	
White oak.......	Perp.	4 × 4 × 4	4	2,200	1,750	1,300	Not well seasoned.
White oak.......	With.	4 × 4 × 4	4	3,500	3,375	3,200	
Black oak	Perp.	3 × 3 × 3	2	2,000	1,800	1,600	Well seasoned.
Gum	Perp.	3 × 3 × 3	2	2,700	2,250	1,800	
Cypress	Perp.	3 × 3 × 3	2	440	385	330	
Ash	Perp.	3 × 3 × 3	2	3.100	2,550	2,000	
White pine.....	Perp.	6 × 6 × 6	3	722	610	555	
White pine......	With.	6 × 6 × 6	3	3,361	3,241	3,083	
Yellow pine	Perp.	6 × 6 × 6	3	1,222	1,092	1,000	
Yellow pine	With.	6 × 6 × 6	3	4,917	4,796	4,722	
Cypress	Perp.	6 × 6 × 6	3	444	426	417	
Cypress	With.	6 × 6 × 6	3	3.166	3,111	3,000	
White pine......	With.	6 × 6 × 6	2	3,694	3,291	2,889	
Yellow pine	With.	6 × 6 × 6	2	4,722	4,611	4,500	
White oak..... ..	With.	6 × 6 × 6	2	3,778	3,764	3,750	

Table V. contains the results of tests by Colonel Laidley, U.S.A., "Ex. Doc. No. 12, 47th Congress, 2d Session." A few other tests of short blocks from the same source will be found in the article on "Timber Columns." Unless otherwise stated, all the specimens were thoroughly seasoned.

In this table, the "length" of all those pieces which were compressed in a direction perpendicular to the grain might, with greater propriety, be called the thickness, since it is measured across the grain.

In the tests (24–60), the compressing force was distributed over only a portion of the face of the block on which it was applied; thus the compressed area was supported, on the face of application, by material about it carrying no pressure. In some cases, this rectangular compressed area extended across the block in one direction but not in the other. In all such instances the ultimate resistance was a little less than in those in which the area of compression was supported on all its sides.

TABLE V.

NO.	KIND OF WOOD.	LENGTH, INS.	COMPRESSED SECTION IN INCHES.	ULT. RESIST., LBS. PER SQ. INCH.	PERP. TO OR WITH GRAIN.	REMARKS.
1	Oregon pine	16.5	2.46 × 2.0	8,496	With.	
2	Oregon pine	19.9	1.21 × 1.21	9,041	"	
3	Oregon pine	19.9	1.21 × 1.21	8,253	"	
4	Oregon maple......	8.0	3.63 × 3.63	6,661	"	
5	Oregon spruce.....	24.02	3.92 × 5.75	5,772	"	Unseasoned.
6	California laurel ...	8.0	3.58 × 3.58	6,734	"	Worm-eaten.
7	Ava Mexicana.....	8.0	3.69 × 3.69	6,382	"	
8	Oregon ash	8.0	3.64 × 3.64	5,121	"	
9	Mexican white mahogany	8.0	3.77 × 3.77	6,155	"	
10	Mexican cedar	8.0	3.75 × 3.75	4,814	"	
11	Mexican mahogany	8.0	3.75 × 3.75	10,043	"	
12	White maple.......	12.0	4.00 × 4.00	7,140	"	
13	White maple.......	12.0	4.00 × 4.00	7,210	"	
14	Red birch	13.0	4.26 × 4.26	8,030	"	
15	Red birch..........	13.0	4.26 × 4.26	7,820	"	
16	Whitewood	12.0	4.00 × 4.00	4,440	"	
17	Whitewood	12.0	4.00 × 4.00	4,330	"	
18	White pine.........	12.0	4.00 × 4.00	5,475	"	
19	White pine	12.0	4.00 × 4.00	5,760	"	
20	White oak..........	12.0	4.00 × 4.00	7,375	"	
21	White oak..........	12.0	4.00 × 4.00	7,010	"	
22	Ash................	12.0	4.00 × 4.00	7,940	"	
23	Ash................	12.0	4.00 × 4.00	7,640	"	
24	Oregon pine	1.95	3.45 × 3.00	1,150	Perp.	
25	Oregon maple......	3.63	3.63 × 3.00	1,875	"	
26	Oregon spruce.....	3.92	5.75 × 4.75	710	"	Unseasoned.
27	Oregon spruce. ...	3.92	4.75 × 4.00	680	"	Unseasoned.
28	California laurel...	3.58	3.58 × 3.00	2,000	"	
29	Ava Mexicana.....	3.69	3.69 × 3.00	2,100	"	
30	Oregon ash	3.64	3.64 × 3.00	2,200	"	
31	Mexican white mahogany	3.77	3.77 × 3.00	2,150	"	
32	Mexican cedar	3.75	3.75 × 3.00	1,950	"	
33	Mexican mahogany	3.75	3.75 × 3.00	4,500	"	
34	White pine	3.06	6.20 × 4.75	875	"	
35	White pine	2.90	4.75 × 4.00	1,012	"	Mean of two.
36	Whitewood	3.15	4.75 × 6.20	900	"	Mean of two.
37	Whitewood	3.15	4.75 × 4.00	1,000	"	Mean of four.
38	Black walnut	0.875	4.75 × 4.00	2,450	"	Mean of two.
39	Black walnut	0.875	4.00 × 3.94	2,200	"	Mean of two.
40	Black walnut	0.875	4.00 × 2.50	2,525	"	Mean of two.
41	White oak..........	2.40	4.75 × 4.00	3,550	"	
42	Spruce.............	3.70	4.75 × 4.00	970	"	Mean of four.
43	Yellow pine........	3.90	4.00 × 4.00	1,900	"	
44	Black walnut	0.75	4.05 × 4.00	2,800	"	
45	" "	1.00	4.05 × 4.00	2,560	"	
46	" "	1.25	4.05 × 4.00	2,400	"	

TABLE V.—*Continued.*

NO.	KIND OF WOOD.	LENGTH, INS.	COMPRESSED SECTION IN INCHES.	ULT. RESIST., LBS. PER SQ. INCH.	PERP. TO OR WITH GRAIN.	REMARKS.
47	Black Walnut......	1.50	4.05 × 4.00	2,500	Perp.	
48	" "	1.75	4.05 × 4.00	2,400	"	
49	" "	2.06	4.05 × 4.00	2,360	"	
50	White pine	0.75	4.05 × 4.00	1,120	"	
51	" "	1.00	4.05 × 4.00	1,100	"	
52	" "	1.25	4.05 × 4.00	1,160	"	
53	" "	1.50	4.05 × 4.00	1,070	"	
54	" "	1.75	4.05 × 4.00	1,060	"	
55	" "	2.00	4.05 × 4.00	1,000	"	
56	Yellow birch.......	4.25	4.25 × 3.00	2,000	"	
57	Yellow birch	4.25	5.98 × 3.00	1,650	"	
58	White maple.......	4.00	3.95 × 3.00	1,700	"	
59	White maple	4.00	5.98 × 3.00	1,900	"	
60	White oak	3.95	3.96 × 3.00	2,500	"	Mean of two.

The "ultimate resistance" was taken to be that pressure which caused an indentation of 0.05 inch.

Nos. (44–55) show the effect of varying thickness of blocks. Within the limits of the experiments, the ultimate resistance is seen to decrease, somewhat, as the thickness increases.

The results of the experiments given in this article show conclusively that the ultimate compressive resistance of short blocks of timber depend upon a number of conditions, such as method of compression, quality of material, size of block, etc., etc. These reasons account for the different results obtained by different experimenters for the same kind of timber.

CHAPTER VII.

COMPRESSION.—LONG COLUMNS.

Art. 49.—Preliminary Matter.

THERE is a class of members in structures which is subjected to compressive stress, and yet those members do not fail entirely by compression. The axes of these pieces coincide, as nearly as possible, with the line of action of the resultant of the external forces, yet their lengths are so great, compared with their lateral dimensions, that they deflect laterally, and failure finally takes place by combined compression and bending. Such pieces are called "long columns," and the application to them, of the common theory of flexure, has been made in Art. 25.

Two different formulæ have been established for use in estimating the resistance of long columns; they are known as "Gordon's Formula" and "Hodgkinson's Formula." Neither Gordon nor Hodgkinson, however, gave the original demonstration of either formula.

The form known as Gordon's formula was originally demonstrated and established by Thomas Tredgold ("Strength of Cast Iron and other Metals," etc.), for rectangular and round columns, while that known as Hodgkinson's formula (demonstrated in Art. 25) was first given by Euler.

In 1840, however, Eaton Hodgkinson, F.R.S., published the results of some most valuable experiments made by himself, in cast and wrought iron columns (Experimental Researches on the Strength of Pillars of Cast Iron and other Materials; Phil.

Trans. of the Royal Society, Part II., 1840), and from these experiments he determined empirical coefficients applicable to Euler's formula, on which account it has since been called Hodgkinson's formula.

Mr. Lewis Gordon deduced from the same experiments some empirical coefficients for Tredgold's formula, since which time, Gordon's formula has been known.

The latter is now in almost, if not quite, universal use among engineers, and will be completely given in the next Article. Hodgkinson's coefficients and formula will be given farther on.

Before taking up either, however, it will be useful and convenient to determine the moments of inertia and squares of the radii of gyration of the various forms of cross sections of the columns now in common use.

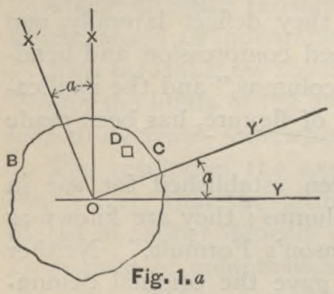


Fig. 1.*a*

It will also be both convenient and important to determine the conditions which exist with an isotropic character of section in respect to the moment of inertia.

In Fig. 1*a* let *BC* be any figure whose area is *A*, and and whose centre of gravity is at *O*. In the plane of that figure let any arbitrary system of rectangular co-ordinates X', Y' be chosen and let XY be any other system having the same origin ; also, let x', y' and x, y be the co-ordinates of the element *D* of the surface *A*, in the two systems. There will then result:

$$x = x' \cos \alpha + y' \sin \alpha.$$

$$y = y' \cos \alpha - x' \sin \alpha.$$

The moments of inertia of the surface about the axes y and x will then be:

$$\int x^2 dA = \cos^2\alpha \int x'^2 dA + 2\sin\alpha\cos\alpha \int x'y'dA + \sin^2\alpha \int y'^2 dA.$$

$$\int y^2 dA = \cos^2\alpha \int y'^2 dA - 2\sin\alpha\cos\alpha \int x'y'dA + \sin^2\alpha \int x'^2 dA.$$

If x and y are to be so chosen that they are *principal* axes, then must $\int xy\,dA = 0$, or:

$$0 = \int xy\,dA = \sin\alpha\cos\alpha \int y'^2 dA + (\cos^2\alpha - \sin^2\alpha) \int x'y'dA$$

$$- \sin\alpha\cos\alpha \int x'^2 dA \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

$$\therefore \quad \tan 2\alpha = \frac{2 \int x'y'dA}{\int x'^2 dA - \int y'^2 dA}.$$

Hence, since $\tan 2\alpha = \tan (180 + 2\alpha)$, there will *always* be two principal axes 90° apart.

Now, if $\int x'y'dA = 0$, while no other condition is imposed, $\tan 2\alpha = 0$. This makes $\alpha = 0$ or 90° ; *i.e.*, $X'\,Y'$ are the principal axes.

If, however, $\int x'y'dA = 0$, while α is neither 0 nor 90°, Eq. (1a) becomes :

$$\int y'^2 dA - \int x'^2 dA = 0;$$

or:

$$tan\ 2\alpha = \frac{0}{0},\ i.e.,\ \text{indeterminate.}$$

This shows that any axis is a *principal* axis ; also, that :

$$\int x^2dA = \int y^2dA = \int x'^2dA = \int y'^2dA.$$

Hence the surface is completely isotropic in reference to its moment of inertia ; or, *its moment of inertia is the same about every axis lying in it and passing through its centre of gravity.*

It has been seen that this condition exists where there are two different rectangular systems, for which

$$\int xy\,dA = \int x'y'dA = 0;$$

but the first of these holds true if either x or y is an axis of symmetry, and the latter, if either x' or y' is an axis of symmetry.

Hence, *if the surface has two axes of symmetry not at right angles to each other, its moment of inertia is the same about all axes passing through its centre of gravity and lying in it.*

Eqs. (1*a*) and the two preceding it also show that the same condition obtains, *if the moments of inertia about four axes at right angles to each other, in pairs, are equal.*

In the case of such a surface, therefore, it will only be necessary to compute the moment of inertia about such an axis as will make the simplest operation.

Since a column fails partly by flexure, it is manifest that *the moment of inertia of its cross section should be the largest possible about an axis passing through its centre of gravity, and normal to the plane of flexure.*

Box Column of Plates and Angles.

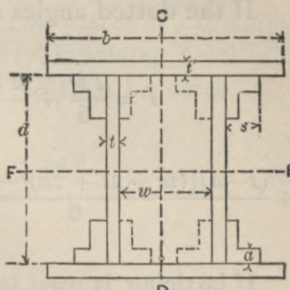


Fig. 1.

Fig. 1 shows the cross section of a box column composed of 4 plates and 4 or 8 equal legged ∟s. *FB* and *CD* intersect at the centre of gravity of the cross section.

If the 4 ∟s shown in dotted lines are omitted, the moment of inertia about *FB* will be:

$$I = \frac{bt'^3}{6} + bt'\frac{(d + t')^2}{2} + \frac{(s + t)\,d^3}{6}$$

$$- \left[\frac{(s - a)\,(d - 2a)^3 + a\,(d - 2s)^3}{6}\right] \quad . \quad . \quad . \quad (1)$$

If the dotted ∟s are not omitted:

$$I = \frac{bt'^3}{6} + bt'\frac{(d + t')^2}{2} + \frac{(2\,s + t)\,d^3}{6}$$

$$- \left[\frac{(s - a)\,(d - 2a)^3 + a\,(d - 2s)^3}{3}\right] \quad . \quad . \quad . \quad (2)$$

If the 4 ∟s shown in dotted lines are omitted, the moment of inertia about *CD* will be:

$$I = \frac{t'b^3}{6} + \frac{a\,(w + 2t + 2s)^3}{6} + \frac{(s - a)\,(w + 2t + 2a)^3}{6}$$

$$+ \frac{(d - 2s)\,(w + 2t)^3}{12} - \frac{dw^3}{12} \quad . \quad . \quad . \quad . \quad (3)$$

If the dotted angles are not omitted:

$$I = \frac{t'b^3}{6} + \frac{a\,[(w + 2t + 2s)^3 - (w - 2s)^3]}{6}$$

$$+ \frac{(s-a)\,[(w + 2t + 2a)^3 - (w - 2a)^3]}{6} + \frac{(d - 2s)\,[(w + 2t)^3 - w^3]}{12} \quad . \quad (4)$$

If latticing is used instead of the two plates bt', t' becomes equal to zero, and the first term in the second member of each of the above equations disappears.

If A represents the area of the cross section, and r the radius of gyration:

$$r^2 = \frac{I}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Box Column of Plates and Channels.

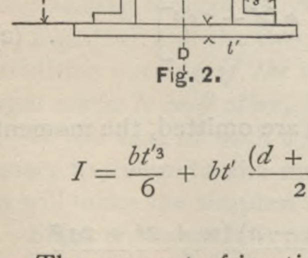


Fig. 2.

Fig. 2 shows a normal cross section of this column. FB and CD intersect in the centre of gravity of the cross section. As in the preceding Fig., these lines are lines of symmetry. The moment of inertia about FB is:

$$I = \frac{bt'^3}{6} + bt'\,\frac{(d + t')^2}{2} + \frac{(s + t)\,d^3 - s\,(d - 2a)^3}{6} \quad . \quad (6)$$

The moment of inertia about CD is:

$$I = \frac{t'b^3}{6} + \frac{2a\,(w + 2t + 2s)^3 + (d - 2a)\,(w + 2t)^3 - dw^3}{12} \quad . \quad (7)$$

If latticing takes the place of the two plates bt', all terms in the second members of Eqs. (6) and (7) involving t' will disappear. The moment of inertia about FB then becomes:

$$I = \frac{(s + t)\, d^3 - s\,(d - 2a)^3}{6}; \quad . \quad . \quad . \quad . \quad (8)$$

and that about CD:

$$I = \frac{2a\,(w + 2t + 2s)^3 + (d - 2a)\,(w + 2t)^3 - dw^3}{12}. \quad (9)$$

$(Radius\ of\ gyration)^2 = r^2 = \dfrac{I}{A}$; in which A is the area of whole section.

Eqs. (7) and (9) may also take the forms given in Eqs. (15) and (16).

Built Column of Plates and Angles.

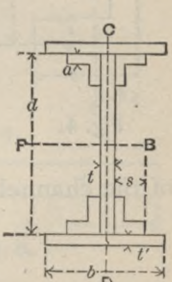


Fig. 3.

Fig. 3 shows a normal cross section of this column with the two axes of symmetry, FB and CD, intersecting at its centre of gravity. The moment of inertia about FB is:

$$I = \frac{bt'}{2}\left(\frac{t'^2}{3} + (d + t')^2\right) + \frac{(s + \frac{1}{2}t)\, d^3}{6}$$

$$- \left[\frac{(s - a)\,(d - 2a)^3 + a\,(d - 2s)^3}{6}\right] \quad . \quad . \quad . \quad (10)$$

The moment of inertia about CD takes the value:

$$I = \frac{t'b^3}{6} + \frac{a\,(2s + t)^3}{6} + \frac{(s - a)\,(2a + t)^3}{6} + \frac{(d - 2s)\, t^3}{12}. \quad (11)$$

If the two plates bt' are omitted, the terms involving t' in Eqs. (10) and (11) reduce to zero.

(*Radius of gyration*)$^2 = r^2 = \frac{I}{A}$; in which A is area of section.

False Channel Section.

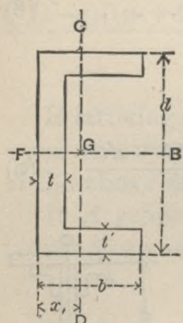


Fig. 4.

Let FB and CD intersect in the centre of gravity, G, of the section. The distance x_1 of G from the back of the channel, is:

$$x_1 = \frac{\frac{1}{2}[b^2d - (b^2 - t^2)(d - 2t')]}{A} \quad . \quad (12)$$

In which A is area of the cross section of the channel. This is usually found by taking one-tenth of the weight, in pounds, per yard of the channel. Analytically:

$$A = 2bt' + t(d - 2t') \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The moment of inertia about CD then becomes:

$$I' = \frac{2t'(b - x_1)^3 + dx_1^3 - (d - 2t')(x_1 - t)^3}{3} \quad . \quad (14)$$

About FB, it has the value:

$$I = \frac{bd^3 - (b - t)(d - 2t')^3}{12} \quad . \quad . \quad . \quad (14a)$$

(*Radius of gyration*)$^2 = r^2 = \frac{I}{A}$.

The line CD can be very quickly and accurately located by balancing the section, cut out of manilla paper, on a knife edge.

Eqs. (7) and (9) may now take the forms:

$$I = \frac{t'b^3}{6} + 2I' + 2A\left(\frac{w}{2} + x_1\right)^2 \quad . \quad . \quad . \quad (15)$$

$$I = 2\left[I' + A\left(\frac{w}{2} + x_1\right)^2\right] \quad . \quad . \quad . \quad . \quad . \quad (16)$$

In Eqs. (15) and (16) A represents the area of one channel section.

Angle Iron Section.

Fig. 5 represents this section with the two axes taken parallel to the legs, passing through the centre of gravity G. The area of cross section is usually found from the weight per yard. Analytically:

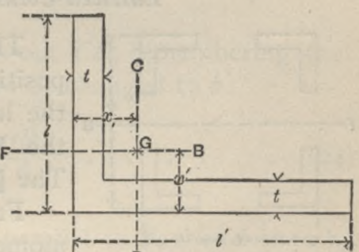


Fig. 5.

$$A = lt + (l' - t)\,t = (l + l' - t)\,t \quad . \quad . \quad . \quad (17)$$

Again:

$$x_1 = \frac{\frac{1}{2}[ll'^2 - (l'^2 - t^2)(l - t)]}{A} \quad . \quad . \quad . \quad (18)$$

$$x' = \frac{\frac{1}{2}[l^2l' - (l' - t)(l^2 - t^2)]}{A} \quad . \quad . \quad . \quad (19)$$

The moment of inertia about CD is:

$$I = \frac{t\,(l' - x_1)^3 + lx_1^3 - (l - t)\,(x_1 - t)^3}{3} \quad . \quad . \quad (20)$$

About FB:

$$I = \frac{t\,(l - x')^3 + l'x'^3 - (l' - t)\,(x' - t)^3}{3} \quad . \quad . \quad (21)$$

If the angle iron is equal legged, l becomes equal to l'.

$(Radius\ of\ gyration)^2 = r^2 = \dfrac{I}{A}$.

As in the case of the [, x_1 and x' may easily and accurately be found by balancing a model of the L section on a knife edge.

Latticed Column of Four Angles.

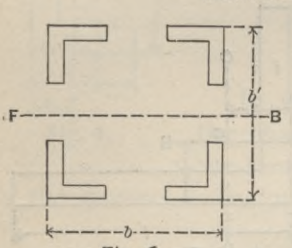


Fig. 6.

The four Ls are held in the relative positions shown in Fig. 6 by latticing, the latter being riveted to the legs of the Ls, but not shown in the Fig. The Ls are equal legged.

From either Eq. (20) or (21), the moment of inertia of the section of any one L, about an axis passing through its centre of gravity and parallel to b, is:

$$I_1 = \frac{t\,(l - x_1)^3 + lx_1^3 - (l - t)\,(x_1 - t)^3}{3}.$$

Hence the moment of inertia of the column section of Fig. 6, about FB, is:

$$I^1 = 4I_1 + A\left(\frac{b'}{2} - x_1\right)^2 \quad . \quad . \quad . \quad . \quad . \quad (22)$$

A is the area of the column section, or four times the area of one L section.

If b is different from b', the moment of inertia of the column section about an axis passing through its centre and parallel to b' will be found by simply changing b' to b in Eq. (22).

$$(Radius\ of\ gyration)^2 = r^2 = \frac{I'}{A}.$$

Latticed Columns of Plates and Angles.

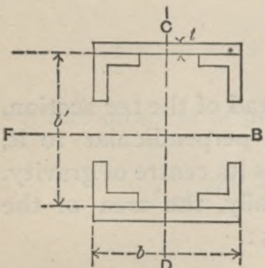


Fig. 7.

Fig. 7 represents a normal section of one of these columns. By the aid of Eq. (22), the moment of inertia of the section about FB may be written:

$$I = I' + \frac{bt}{2}\left(\frac{t^2}{3} + (b' + t)^2\right); \quad (23)$$

and that about CD, remembering that in I', b' is to be changed to b:

$$I = I' + \frac{tb^3}{6}; \quad \ldots \ldots \quad (24)$$

If the plates are on the sides parallel to b', then b is to be changed to b' and b' to b in Eqs. (23) and (24).

Fig. 8 represents the normal section of the other of these columns, in which there is no latticing, the column being perfectly closed.

Again, using Eq. (22), the moment of inertia about the axis FB is:

$$I = I' + \frac{bt}{2}\left(\frac{t^2}{3} + (b' + t)^2\right) + \frac{t'(b' + 2t)^3}{6} \quad \ldots \quad (25)$$

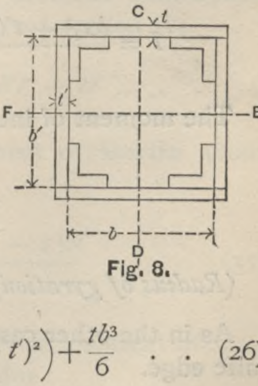


Fig. 8.

The moment of inertia about CD is:

$$I = I' + \frac{t'(b' + 2t)}{2}\left(\frac{t'^2}{3} + (b + t')^2\right) + \frac{tb^3}{6} \quad . \quad . \quad (26)$$

In the I' in Eq. 26, b' is to be changed to b. Ordinarily, $b = b'$ and $t = t'$.

(*Radius of gyration*)$^2 = r^2 = \frac{I}{A}$, A being area of cross-section.

Tee Section.

The axis FB is taken parallel to the head of the tee section, and CD perpendicular to it, while G is its centre of gravity. Analytically, the area of the section is:

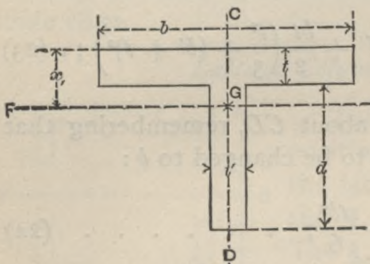


Fig. 9

$$A = bt + dt' \quad . \quad . \quad . \quad (27)$$

The area may also be taken from the weight in the usual manner.

$$x_1 = \frac{\frac{1}{2}\left[(b - t')t^2 + (d + t)^2 t'\right]}{A} \quad . \quad . \quad . \quad (28)$$

The moment of inertia about FB is:

$$I = \frac{bx_1^3 + t'(d + t - x_1)^3 - (b - t')(x_1 - t)^3}{3} \quad . \quad . \quad (29)$$

The moment of inertia about CD is:

$$I = \frac{tb^3 + dt'^3}{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

(*Radius of gyration*)$^2 = r^2 = \frac{I}{A}$.

As in the other cases, FB may be located by balancing on a knife edge.

False Eye Section.

If the area is not taken from the weight per yard, it may be written :

$$A = bd - (b - t')(d - 2t) \quad . \quad . \quad (31)$$

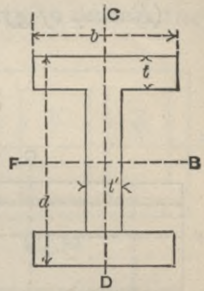


Fig.10

The moment of inertia about CD is:

$$I = \frac{2tb^3 + (d - 2t)t'^3}{12} \quad . \quad . \quad . \quad (32)$$

About FB it has the value:

$$I = \frac{bd^3 - (b - t')(d - 2t)^3}{12} \quad . \quad . \quad . \quad (33)$$

$$(\textit{Radius of gyration})^2 = r^2 = \frac{I}{A}.$$

Star Section.

Fig. 11 shows this section with the different dimensions. The area of cross section is :

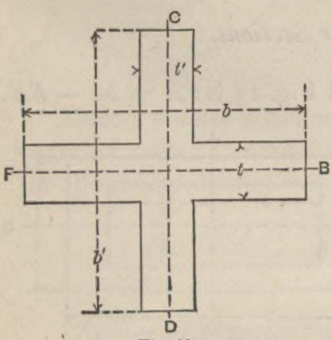


Fig.11

$$A = bt + b't' - tt' \quad . \quad . \quad . \quad (34)$$

The moment of inertia about FB is :

$$I = \frac{t'b'^3 + (b - t')t^3}{12} \quad . \quad . \quad . \quad (35)$$

About CD the moment of inertia has the value:

$$I = \frac{tb^3 + (b' - t)t'^3}{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Ordinarily, $t = t'$.

$(Radius\ of\ gyration)^2 = r^2 = \frac{I}{A}$.

Solid Rectangular Section.

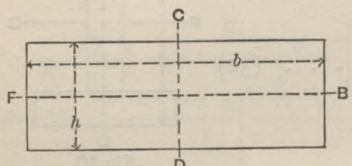


Fig. 12.

In Fig. 12 $A = bh$.

The moment of inertia about FB is:

$$I = \frac{bh^3}{12}; \quad . \quad . \quad (37)$$

and about CD:

$$I = \frac{hb^3}{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

$(Radius\ of\ gyration)^2 = r^2 = \frac{I}{A} = \frac{h^2}{12}$ or $\frac{b^2}{12}$.

If the rectangular section is square, $b = h$.

Hollow Rectangular Sections.

The area of the section shown in Fig. 13 is: $A = bh - b'h'$. The moment of inertia about FB is:

$$I = \frac{bh^3 - b'h'^3}{12}; \quad . \quad . \quad (39)$$

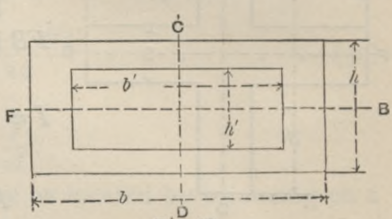


Fig. 13.

and that about CD is:

$$I = \frac{hb^3 - h'b'^3}{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

$$(Radius\ of\ gyration)^2 = r^2 = \frac{I}{A}.$$

All the equations of this case (except Eq. (40)), just as they stand, apply directly to the rectangular cellular section of Fig. 14, considered in reference to the axis *FB*. If there were n cells instead of 3, the space between any adjacent two would have the width $\frac{b'}{n}$.

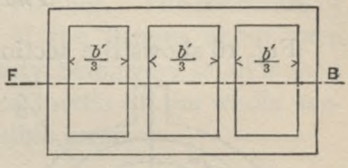


Fig. 14

Solid and Hollow Circular Sections.

First consider a solid cylindrical column whose cross section has the radius r_2, as shown in Fig. 15. The moment of inertia about any diameter is:

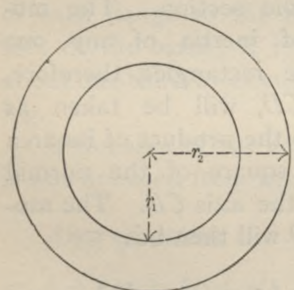


Fig. 15.

$$I = \frac{\pi r_2^4}{4} \quad . \quad . \quad . \quad (41)$$

$$(Radius\ of\ gyration)^2 = \frac{\pi r_2^4}{4\pi r_2^2}$$

$$= \frac{r_2^2}{4} = r^2.$$

Next consider a hollow circular column whose interior and exterior radii are r_1 and r_2 respectively. The moment of inertia about any diameter is:

$$I = \frac{\pi(r_2^4 - r_1^4)}{4} = \frac{A(r_2^2 + r_1^2)}{4};\ (A = \text{area}) \quad . \quad . \quad (42)$$

$$(Radius\ of\ gyration)^2 = \frac{I}{\pi(r_2^2 - r_1^2)} = \frac{r_2^2 + r_1^2}{4} = r^2.$$

As tables of circular areas are very accessible, it may be convenient to write:

$$r^2 = \frac{\pi r_2^2}{12.566}; \text{ or } r^2 = \frac{\pi(r_2^2 + r_1^2)}{12.566}.$$

Phœnix Section.

Fig. 16 shows the section of a 4 segment Phœnix column. Let *CD* represent any axis taken through the centre of the column. The moments of inertia of the rectangles *bl* about axes through their centres of gravity and parallel to *CD* will be very small indeed compared with the moment of inertia of the whole section. The moment of inertia of any one of these rectangles, therefore, about *CD*, will be taken as equal to the product of its area by the square of the normal distance from its centre of gravity to the axis *CD*. The moment of inertia of the section about *CD* will then be:

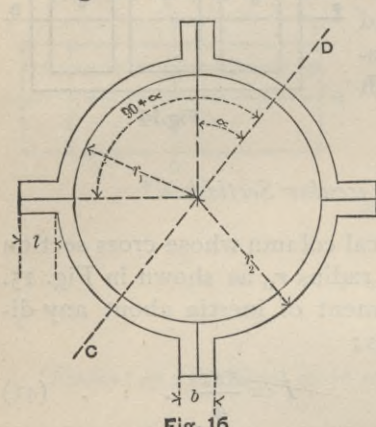


Fig. 16

$$I = \frac{\pi(r_2^4 - r_1^4)}{4} + 2bl\left[\left(r_2 + \frac{l}{2}\right)^2 \sin^2\alpha + \left(r_2 + \frac{l}{2}\right)^2 \cos^2\alpha\right]$$

$$\therefore I = \frac{\pi(r_2^4 - r_1^4)}{4} + 2bl\left(r_2 + \frac{l}{2}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

The moment of inertia is thus seen to be the same about all axes, a result of the general principle established in the first part of this Article.

The area of the cross section is:

$$A = \pi(r_2^2 - r_1^2) + 4bl. \quad . \quad . \quad . \quad . \quad . \quad (43a)$$

$$(Radius\ of\ gyration)^2 = r^2 = \frac{I}{A}.$$

The moments of inertia of six and eight segment columns may be found in precisely the same manner. The moments of inertia of the rectangular sections of the flanges about axes passing through their centres of gravity, being very small indeed when compared with the moment of inertia of the whole section, may be neglected without sensible error.

True Eye Section.

Let $r = \frac{2s}{b - t_1}$; r is then the batter, or slope, of the under side of each flange to the top or bottom of the beam; it ranges from about one-third to essentially nothing.

If the area of the cross section is not deduced from the weight:

Area of section

$$= A = 2bt + ht_1 + s(b - t_1) \ . \ (44)$$

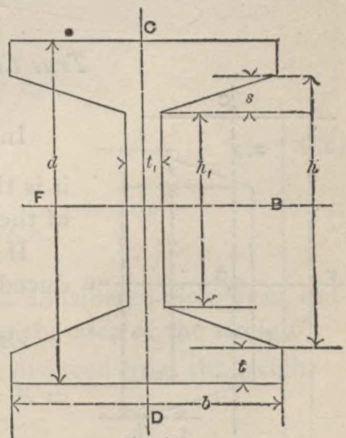


Fig. 17

The moment of inertia about CD is:

$$I = \frac{2tb^3 + h_1t_1^3}{12} + \frac{r(b^4 - t_1^4)}{48} \quad . \quad . \quad . \quad . \quad (45)$$

If t_1 is very small as compared with b, remembering that $\frac{b}{2}r$ is then essentially equal to s, there will result:

$$I = \frac{(2t + \frac{1}{2}s)\,b^3 + h_1 t_1^3}{12} \quad . \quad . \quad . \quad . \quad . \quad (46)$$

This formula is sufficiently accurate for all wrought-iron and steel beams.

The moment of inertia about FB is:

$$I = \frac{bd^3 - \frac{1}{4r}\,(h^4 - h_1^4)}{12} \quad . \quad . \quad . \quad . \quad . \quad (47)$$

In any of these three cases:

$$(Radius\ of\ gyration)^2 = \frac{I}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

True Channel Section.

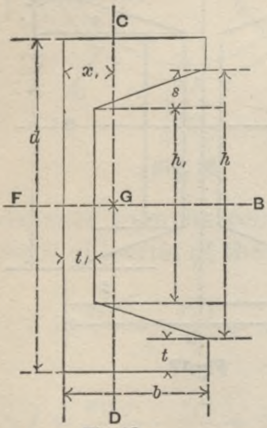


Fig. 18

In Fig. 18 let $r = \frac{s}{b - t_1}$; as before, it is the batter or slope of the under side of the flange.

If the area of the section is not deduced from the weight:

Area of section

$$= A = 2bt + ht_1 + s(b - t_1) \quad . \quad . \quad . \quad . \quad (49)$$

The centre of gravity, G, can be found by balancing a manilla, or other, pattern on a knife edge; or, analytically:

$$x_1 = \frac{b^2 t + \frac{1}{2}\,ht_1^2 + \frac{1}{3}\,s(b - t_1)\,(b + 2t_1)}{A} \quad . \quad . \quad (50)$$

The moment of inertia about CD is:

$$I = \frac{2tb^3 + h_1 t_1^3 + \frac{1}{2} r(b^4 - t_1^4)}{3} - Ax_1^2 \quad . \quad . \quad (51)$$

If t_1 is very small compared with b, and remembering that br is then essentially equal to s; this last equation will become:

$$I = \frac{(2t + \frac{1}{2} s)b^3 + h_1 t_1^3}{3} - Ax_1^2 \quad . \quad . \quad . \quad . \quad (52)$$

The moment of inertia about FB is:

$$I = \frac{bd^3 - \frac{1}{8r}(h^4 - h_1^4)}{12} \quad . \quad . \quad . \quad . \quad . \quad (53)$$

In any of these three cases:

$$(Radius\ of\ gyration)^2 = \frac{I}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

Deck Section.

The head of this section will be considered circular in outline, as shown in Fig. 19. Let a be the area of the circle C.

If the area of the section is not deduced from the weight:

Area of section

$$= A = a + (d - h)t_1 + (b - t_1)(t + \tfrac{1}{2} s) \quad . \quad . \quad (55)$$

If the centre of gravity, G, is not found by balancing a pattern on a knife edge, there will result, analytically:

$$x_1 = \frac{a(2d - h) + t_1(d - h)^2 + (b - t_1)(t^2 + st + \frac{1}{3} s^2)}{2A} \quad . \quad (56)$$

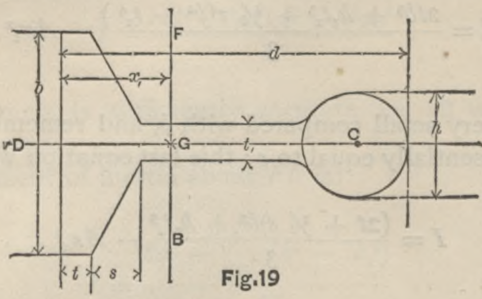


Fig. 19

The moment of inertia about FB is:

$$I = a\left\{\frac{h^2}{16} + \left(d - \frac{h}{2}\right)^2\right\} + \frac{t_1(d-h)^3}{3}$$

$$+ \frac{(t+s)^4 - t^4}{6r} - Ax_1^2; \quad . \quad . \quad . \quad . \quad . \quad (57)$$

in which equation $r = \dfrac{2s}{b - t_1}$.

The moment of inertia about CD is:

$$I = \frac{\frac{3}{4}ah^2 + t_1^3(d - h - t - s) + tb^3 + \frac{r}{8}(b^4 - t_1^4)}{12} \;. \quad (58)$$

If t_1 is small as compared with b, so that essentially $\dfrac{br}{2} = s$:

$$I = \frac{3ah^2 + 4t_1^3(d - h - t - s) + (4t + s)b^3}{48} \quad . \quad (59)$$

In all cases:

$$(Radius\ of\ gyration)^2 = \frac{I}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

Angle Section about Oblique Axis.

The angle iron is here supposed to be equal legged, and the axis about which the moment of inertia is taken, passes through the centre of gravity (before found in this Art.) and cuts the sides l at an angle of 45°. In Fig. 20, G is the centre of gravity and HK the axis.

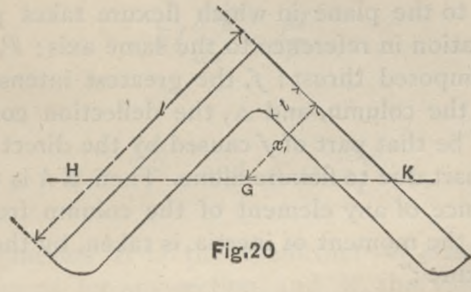


Fig. 20

The moment of inertia about HK is:

$$I = \frac{2\{x_1^4 - (x_1 - t)^4\} + t\{l - (2x_1 - \tfrac{1}{2}t)\}^3}{3} \quad . \quad (61)$$

If A is the area of cross section:

$$(Radius\ of\ gyration)^2 = \frac{I}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

If a long column has the same degree of fixedness or freedom in all directions, the *least* value of the square of the radius of gyration must be taken for insertion in Gordon's formula, because in the plane of that radius the column will offer the least resistance to flexure.

Art. 50.—Gordon's Formula for Long Columns.

Since flexure takes place, if a long column is subjected to a thrust in the direction of its length, the greatest intensity of stress in a normal section of the column may be considered as composed of two parts. In fact, the condition of stress in any normal section of a long column is that of a uniformly varying system composed of a uniform stress and a stress couple. In order to determine these two parts let S represent the area of the normal cross section; I, its moment of inertia about an axis normal to the plane in which flexure takes place; r, its radius of gyration in reference to the same axis; P, the magnitude of the imposed thrust; f, the greatest intensity of stress allowable in the column, and $\triangle$, the deflection corresponding to f. Let p' be that part of f caused by the direct effect of P, and p'' that part due to flexure alone. Then, if h is the greatest normal distance of any element of the column from the axis about which the moment of inertia is taken, by the "common theory of flexure:"

$$c'P\triangle = \frac{p''I}{h} \; ; \; \therefore \; p'' = \frac{c'P\triangle h}{I} \quad . \quad . \quad . \quad . \quad (1)$$

If the column ends are round, $c' = 1$; but if the ends are fixed, the value of c' will depend upon the degree of fixedness. Also,

$$p' = \frac{P}{S} \; ; \; \therefore \; p' + p'' = f = \frac{P}{S}\left(1 + \frac{c'S\triangle h}{I}\right) \quad . \quad . \quad (2)$$

Hence,

$$P = \frac{fS}{1 + \dfrac{c'S\triangle h}{I}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Eq. (3) may be considered one form of Gordon's formula.

Before deducing the more common and useful form, it will be necessary to show that $\Delta = a\frac{l^2}{h}$; in which expression a is considered constant.

Let p be the greatest intensity of bending stress in any section, whose greatest value in the column is p''. By the "common theory" (taking the origin of co-ordinates at the centre of gravity of the cross section at one end of the column, and the axis of x along the centre line before flexure):

$$EI\frac{d^2y}{dx^2} = \frac{pI}{h}.$$

Also,

$$p = \frac{Mh}{I}, \text{ and } p'' = \frac{M_0 h}{I}; \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in which equations E is the coefficient of elasticity M the bending moment for any section, and M_0 the value of M corresponding to p''.

Hence,

$$p = p''\frac{M}{M_0}, \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{p''}{Eh}\frac{M}{M_0}.$$

Consequently,

$$\Delta = \int_0^{l_0}\int_{l_0}^{x}\frac{p''}{Eh}\cdot\frac{M}{M_0}\cdot dx^2 \quad . \quad . \quad . \quad . \quad (5)$$

The section located by l_0 is that at which the deflection is greatest, and for which $\frac{dy}{dx} = 0$, while $\frac{p''}{Eh}$ is considered constant. The ratio $\frac{M}{M_0}$ is numerical, though variable, being one between quantities of the same degree. M_0 is exactly the same as M, except that x, in the latter, is displaced by l_0; there are the same number of terms in each, and those terms are multiplied by the same coefficients. Now $\int_0^{l_0}\int_{l_0}^{x} M\,dx^2$ may be so arranged as to have the same number of terms as M_0, but *the coefficients of those terms will be different*, and *the exponents of* l_0 *in the former will be greater by* 2 *than the exponents of* l_0 *in* M_0. Hence $l_0^2 = c^2l^2$ (c being some constant) will be a factor in *all* the terms of the definite double integral. From these considerations it follows that,

$$\frac{\int_0^{l_0}\int_{l_0}^{x} M\,dx^2}{M_0} = a'l^2 \; ; \quad . \; . \; . \; . \; . \; . \; . \quad (6)$$

in which a' is some constant. Consequently,

$$\Delta = \frac{a'p''}{Eh} l^2 = a_1 \frac{l^2}{h} \quad . \; . \; . \; . \; . \; . \; . \quad (7)$$

It is seen therefore that the quantity a_1 depends upon both p'' and E, and it is ordinarily considered constant.

Since $I = Sr^2$, Eqs. (1) and (7) give:

$$p'' = a_1 \frac{c'Pl^2}{I} = a \frac{P}{S}\frac{l^2}{r^2}; \; \therefore \; f = p' + p'' = \frac{P}{S}\left(1 + a\frac{l^2}{r^2}\right). \quad (8)$$

Eq. (8) shows that $a_1c' = a$.

Hence,

$$P = \frac{fS}{1 + a\dfrac{l^2}{r^2}} \quad . \; . \; . \; . \; . \; . \; . \quad (9)$$

The integration by which Eq. (7) is obtained, being taken between limits, causes everything to disappear which depends upon the condition of the ends of the column. Consequently Eq. (9) applies to all columns, whether the ends are rounded or fixed. Let the latter condition be assumed, and let it be represented in the adjoining figure. Since the column must be bent symmetrically, there must be *at least* two points of contraflexure. Two such points, only, may be supposed, since such a supposition makes the distance between any two adjacent points the greatest possible and induces the most unfavorable condition of bending for the column.

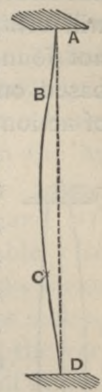


Fig. 1

If B and C are the points of contraflexure supposed, then BC will be equal to a half of AD, for each half of BC must be in the same condition, so far as flexure is concerned, as either AB or CD. Also, the bending moment at the section midway between B and C must be equal to that at A or D. Consequently, the free or round end column BC must possess the same resistance as the fixed or flat end column AD. In Eq. (9), therefore, let $l = 2BC = 2l_1$:

$$P = \frac{fS}{1 + 4a\frac{l_1^2}{r^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Eq. (10) is, consequently, the formula for free or round end columns with length l_1.

The flat, or fixed end column AD, is also of the same resistance as the column AC, with one end flat and one end free or round. Hence in Eq. (9) let there be put $l = \frac{4}{3} AC = \frac{4}{3}l'$, and there will result, nearly,

$$P = \frac{fS}{1 + 1.8\,a\frac{l'^2}{r^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Eq. (11) is, then, the formula for a column with one end flat and the other round. A slight element of approximation will ordinarily enter Eq. (11) on account of the fact that C is not found in the tangent at A just as Eqs. (9) and (10) are based on the supposition that A and D lie exactly in the line of action of the imposed load.

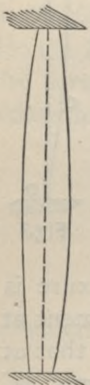


Fig. 2

If the column is swelled, as shown in Fig. 2, the the moment of inertia I, and distance, h, become variable. Hence:

$$p = \frac{Mh}{I}, \quad \text{and} \quad p'' = \frac{M_0 h_0}{I_0}.$$

Consequently,

$$p = p'' \frac{I_0}{M_0 h_0} \frac{Mh}{I};$$

and,

$$\Delta = \int_0^{l_0} \int_{l_0}^{x} \frac{p''}{Eh_0} \frac{I_0}{M_0} \frac{M}{I} dx^2 \quad . \quad . \quad . \quad . \quad (12)$$

If, in the reasoning applied to Eq. (5), there be written $\frac{M}{I}$ for M, and $\frac{M_0}{I_0}$ for M_0, it will at once be seen that Eq. (12) will give precisely the same general *form* of result as Eq. (5), but the coefficient a will have a different value. Farther, since $I_0 \div I$ can never be less than unity, but is in general greater, it follows that, for swelled columns, a is greater than for columns that are not swelled. Although these considerations show that the value of a will be different in the two classes of columns, yet they also show that the general *form* for the breaking weight P, whatever may be the condition of the ends, will be precisely the same whether the columns are swelled or straight.

Since the swelling of a column will give it a greater resistance to bending, p'' will take a correspondingly less value, while

P and S remain the same. Eq. (8), then, shows that if f and S are unchanged, P must be increased. In other words, a swelled column will sustain a greater load than one not swelled but possessing the same kind and area of cross section. This is indeed true of solid columns, but may not be, and usually is not, for reasons to be assigned hereafter, true for built columns of shape iron. These reasons are not introduced in the hypothesis on which the formulæ are based.

Although the quantities f and a, in Eqs. (9), (10) and (11), are usually considered constant, they are strictly variable. Eq. (7) shows that a is a function of $p'' \div E$. It is by no means certain that p'' is the same for different forms of cross section, or even for different sections of the same form, and the very variable character of the coefficient of elasticity is well known. It (the latter) is known not only to vary with the products of different iron mills, but even with the different products of the same mill.

Again, the greatest intensity of stress, f, which can exist in the column varies not only with different grades of material, but there is some reason to believe that it must also be considered as varying with the length of the column. The law governing this last kind of variation, for many sections, still needs empirical determination. It is clear, therefore, that both f and a must be considered *empirical variables.*

The expense necessarily attending experimental researches on the ultimate resistance of long columns built of American material, has prevented the attainment of many desirable results. Yet much very valuable work of this kind has been done.

In the "Report of Progress of Work," etc., made by Thomas D. Lovett, consulting and principal engineer to the trustees of the Cincinnati Southern Railway, Nov. 1, 1875, are found the records of some valuable experiments on wrought-iron long columns. The results of these experiments will be used in fixing values of a and f.

If the number of experiments were sufficiently great, the results should be combined by the " Method of Least Squares." In the present instance, however, the use of the method is altogether impracticable in consequence of the small number of experiments of any given class. It will be seen, however, that the combination of the experimental results is not altogether of a random nature.

Since f and a are to be considered variable quantities, let y take the place of f and x that of a; also, let $p = \frac{P}{S}$ represent the mean intensity of stress. Eq. (9) then takes the form:

$$p = \frac{y}{1 + cx}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

in which $c = l^2 \div r^2$. For round or free end columns x will take the place of $4a$, and of $1.8a$ for columns with one end round and one end flat.

In Eq. (13) there are two unknown quantities, y and x, consequently two equations are required for their determination. If two columns of different ultimate resistances per unit of section, and with different values of c, are broken in a testing machine, and the two sets of data thus established separately inserted in Eq. (13), two equations will result which will be sufficient to completely give y and x. Those two equations may be written as follows:

$$y = p' \ (1 + c'x) \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$y = p'' (1 + c''x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The simple elimination of y gives:

$$x = \frac{p'' - p'}{c'p' - c''p''}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Either Eq. (14) or (15) will then give y.

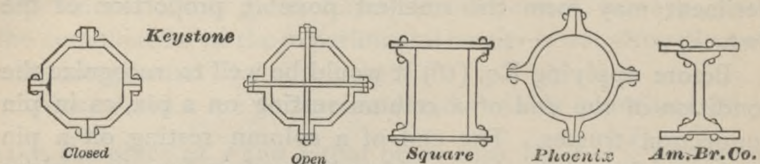
In selecting experimental results for insertion in Eq. (16), care should be taken to make the differences $p'' - p'$ and $c' - c''$ as large numerically as possible, in order that the errors of experiment may form the smallest possible proportion of the first.

Before applying Eq. (16) it would be well to recognize the condition of the end of a column resting on a pin, as in pin connection trusses. The end of a column resting on a pin might, at first sight, be considered round or free in a plane normal to the axis of the pin. The compressive strains existing in the vicinity of the surface of contact between the pin and soffit of the pin hole, produce a considerable surface on which the frictional resistance to any relative movement is very great. This resistance to movement is not sufficient to produce a "flat" or "fixed" condition of the column end, but causes a degree of constraint intermediate between the flat and round condition; so that a column with two "pin ends" gives an ultimate resistance approximating to that of a column with one round end and one fixed end. The following two cases will then hereafter be recognized:

Two Pin Ends,
One Pin End and one Flat End.

All the necessary data for the treatment of the experiments given in the report of Mr. Lovett, are found in the following table. The column "Area" gives the areas of normal cross sections in square inches. The column r^2 gives the squares of the radii of gyration, in inches, about axes normal to the plane of bending. It is inferred from the table and the report under consideration that the radii of gyration for swelled columns belong to sections at middle of columns. The c column contains the squares of l divided by r, both being taken in the same unit; it is a matter of indifference what that unit may be.

The quantities x, y and p are *found by the formulæ* (16), (15) or (14) *and* (13). The column headed *Exp.* contains the ultimate resistances in pounds per square inch, *determined by experiment.*



An "open" column is one in which the flanges of the segments that compose it are separated by an open space; a closed column is one in which no such spaces are found. All the columns treated in the table are closed except Nos. 2, 3, 4, 8, 25, 31, 24, 26, 30, and 5.

The columns 13 and 19 failed about axes giving the *greatest* moments of inertia or radii of gyration. This was probably due to some cause equivalent to a less degree of constraint at the ends than was intended. For this reason those two results are not used in determining x and y. They will be noticed again.

An examination of the table shows that the flat end swelled and open straight Keystone columns give about the same ultimate resistance, by experiment, per square inch, so long as c remains the same, though the straight columns give the largest results by a little. Hence p' was taken as an arithmetical mean of the experimental results of Nos. 4, 25, 31, 24, 26, and 30, and c' at 9,208. In the same manner p'' was taken a mean of the experimental results for Nos. 3 and 8, and c'' at 3,060. The arithmetical means mentioned are $p' = 25,517$ pounds, and $p'' = 32,850$ pounds. Substitutions in Eqs. (16) and (15) or (14), then give:

$$x = 0.00005455 \text{ and } y = 38,300.00 \text{ pounds.}$$

Wrought Iron Columns.

NO.	LENGTH.	AREA.	r^2	c.	x.	y.	p.	Exp.		
		□″								
2	5′ 00″	14.25	11.044	326	.00005455 or $\frac{1}{18300}$	36,000	35,350	33,600	Flat ends, swelled.........	Keystone.
3	15′ 00″	14.84	10.834	2,991	" "	36,000	30,900	28,800	Flat ends, swelled	Keystone.
4	27′ 00″	12.96	10.883	9,646	" "	36,000	23,600	24,100	Flat ends, swelled	Keystone.
8	15′ 00″	14.80	10.353	3,130	" "	36,000	30,700	36,900	Flat ends, swelled	Keystone.
25	27′ 00″	18.83	11.424	9,189	" "	36,000	24,000	21,100	Flat ends, swelled	Keystone.
31	27′ 00″	15.13	11.464	9,157	" "	36,000	24,000	25,400	Flat ends, swelled	Keystone.
24	27′ 00″	19.20	12.041	8,718	" "	39,500	26,700	25,000	Flat ends, open ... — Straight......	Keystone.
26	27′ 00″	14.49	11.178	9,391	" "	39,500	26,100	27,500	Flat ends, open ... — Straight	Keystone.
30	27′ 00″	15.13	11.464	9,157	" "	39,500	26,300	30,000	Flat ends, open ... — Straight	Keystone.
7	15′ 00″	14.62	9.206	3,519	" "	39,500	33,100	30,000	Flat ends, closed.. — Straight	Keystone.
9	15′ 00″	23.67	7.833	4,136	" "	39,500	32,200	32,000	Flat ends, closed.. — Straight	Keystone.
27	27′ 00″	18.83	9.798	10,714	" "	39,500	24,900	27,800	Flat ends, closed.. — Straight	Keystone.
5	27′ 00″	13.12	10.945	9,591	$\frac{1}{15000}$	36,000	21,950	22,000	Pin ends, swelled.........	Keystone.
22	26′ 00″	13.60	9.347	10,414	$\frac{1}{35000}$ or .0000286	39,000	30,000	30,000	Flat ends..................	Square Column.
23	24′ 00″	13.70	11.628	7,133	" "	39,000	32,400	33,200	Flat ends	Square Column.
32	27′ 00″	26.05	10.909	9,623	" "	39,000	30,600	30,200	Flat ends	Square Column.
21	25′ 9″	13.60	11.000	8,680	.0000588 or $\frac{1}{17000}$	39,000	25,800	25,500	Pin ends	Square Column.
6	15′ 00″	14.09	8.536	3,796	.00002 or $\frac{1}{50000}$	42,000	39,000	37,500	Flat ends..................	Phœnix.
10	27′ 00″	13.70	8.935	11,749	" "	42,000	34,000	31,000	Flat ends	Phœnix.
28	28′ 00″	13.58	8.935	12,635	" "	42,000	33,500	34,800	Flat ends	Phœnix.
29	28′ 00″	13.58	8.935	12,635	" "	42,000	33,500	36,600	Flat ends	Phœnix.
11	27′ 00″	13.89	8.935	11,749	.00008	42,000	21,650	21,700	Round ends...............	Phœnix.
15	30′ 00″	14.97	5.388	24,053	.0000217 or $\frac{1}{46000}$	36,000	23,700	23,700	Flat ends..................	Am. Br. Co.
18	20′ 00″	20.10	8.653	6,657	" "	36,000	31,500	31,500	Flat ends	Am. Br. Co.
13	26′ 00″	25.05	18.215	5,344	$\frac{1}{11500}$	36,000	24,600	24,000	Special round ends........	Am. Br. Co.
16	20′ 00″	12.50	5.479	10,513	.000047 or $\frac{1}{21500}$	36,000	24,100	26,700	Pin ends	Am. Br. Co.
17	20′ 00″	19.90	8.733	6,596	" "	36,000	27,400	26,500	Pin ends	Am. Br. Co.
14	26′ 00″	20.72	8.733	11,147	" "	36,000	23,600	22,000	Pin ends	Am. Br. Co.
19	27′ 00″	20.10	13.51	7,770	" "	36,000	26,400	27,800	Special pin ends...........	Am. Br. Co.

y is taken at the nearest hundred. These values of x and y, placed in Eq. (13), give the values in column "p."

Since, however, the resulting values of "p" were a little too large for the swelled columns, and a little too small for the straight open ones, x was allowed to remain as determined, and y was made 36,000.00 for swelled, and 39,500.00 for straight open columns. The resulting values of "p" are given in the table.

The differences between the results in the columns "p" and "*Exp.*" are not greater than experimental differences.

Since x depends on the condition of the ends of the columns, as well as on the character of the iron, it is reasonable to give it the same value for all flat end Keystone columns. Then taking y at 39,500.00 pounds, it will be seen that Eq. (13) gives results agreeing, as nearly as could be expected, with those of experiment for straight closed Keystone columns with flat ends.

No. 5 is the only experiment with a pin end Keystone column. As it was also swelled, y has been taken at 36,000.00 and x at $\frac{1}{15000}$, so that p would be a little less than the result of experiment. As these values depend on one pin end experiment only, they should not be considered very satisfactory. At the same time corresponding values for other columns show that they cannot be very erroneous.

Precisely the same general principles and considerations governed the selection of x and y for the several remaining classes of columns shown in the table. The agreement between the columns p and *Exp.* is as close as could be expected.

The extraordinary character of Nos. 13 and 19 has already been noticed. No. 13 was intended to be a pin end column, but the plane of flexure contained the axis of the pin. Now if it be considered a round end column in the plane of failure, x will have the value $4 \times \frac{1}{46000} = \frac{1}{11500}$, and the resulting value of p will be 24,600.00 pounds. The result of experiment was 24,000.00 pounds. Again, No. 19 was intended to be a flat end column, but it failed in the direction of its greatest radius of gyration. Using the values of x and y for pin ends, there will result $p = 26{,}400.00$ pounds. The result of experiment was 27,800.00. The effect of defective fitting, etc., would therefore seem to be the lessening of the end constraint by what may be termed one degree.

Expressing all the results in concise formulæ, they may be written:

Keystone Columns.

Flat Ends—Swelled .... $p = \dfrac{36000}{1 + \dfrac{1}{18300}\dfrac{l^2}{r^2}}$; . . . (17)

Flat Ends— { Open. } Straight.. { Closed } .... $p = \dfrac{39500}{1 + \dfrac{1}{18300}\dfrac{l^2}{r^2}}$; . . . (18)

Pin Ends—Swelled......... $p = \dfrac{36000}{1 + \dfrac{1}{15000}\dfrac{l^2}{r^2}}$; . . . (19)

Square Columns.

Flat Ends .... $p = \dfrac{39000}{1 + \dfrac{1}{35000}\dfrac{l^2}{r^2}}$; . . . (20)

Pin Ends.................. $p = \dfrac{39000}{1 + \dfrac{1}{17000}\dfrac{l^2}{r^2}}$; . . . (21)

Phœnix Columns.

Flat Ends . . . . $p = \dfrac{42000}{1 + \dfrac{1}{50000}\dfrac{l^2}{r^2}}$; . . . (22)

Round Ends . . . . $p = \dfrac{42000}{1 + \dfrac{1}{12500}\dfrac{l^2}{r^2}}$; . . . (23)

Pin Ends (hypothetical) . . . . $p = \dfrac{42000}{1 + \dfrac{1}{22700}\dfrac{l^2}{r^2}}$; . . . (24)

American Bridge Co. Columns.

Flat Ends . . . . $p = \dfrac{36000}{1 + \dfrac{1}{46000}\dfrac{l^2}{r^2}}$; . . . (25)

Round Ends . . . . $p = \dfrac{36000}{1 + \dfrac{1}{11500}\dfrac{l^2}{r^2}}$; . . . (26)

Pin Ends . . . . $p = \dfrac{36000}{1 + \dfrac{1}{21500}\dfrac{l^2}{r^2}}$; . . . (27)

The pin end formula for the Phœnix column is based on the hypothesis that the relation between the values of x for flat and pin ends is the same as that existing in the American Bridge Co. columns, which last is shown by experiment. This is a very unsatisfactory method, and should not be implicitly relied upon.

All values of x for round end columns are found by multiplying the corresponding flat end quantities by 4, according to Eq. (10).

Eqs. (17) to (28), inclusive, give the ultimate resistances of the various classes of columns. With great variations of stress a safety factor as high as six or eight may be used, or it may be as low as three or four if the condition of stress is uniform or essentially so.

For a complete account of the details of the foregoing experiments, the original " Report " must be consulted. The consideration of the shades of influence exerted by the different devices to produce a given end condition have here been neglected on the ground that such degrees of influence are too small to be involved in a practical formula.

Some important deductions bearing on built columns of all forms of cross section may be drawn from the results of these experiments. It has already been noticed that the swelled columns Nos. 2, 3, 4, 8, 25, 31, do not give as great ultimate resistances as similar straight ones; a result perhaps not to be expected, though the explanation is simple. Both internal tensile and compressive stresses are induced in the originally straight segments when they are sprung to their proper curvature in the swelled column. Consequently this internal compressive stress causes a portion of the material to reach its ultimate resistance much sooner than would be the case if the columns were straight. Again, a slight increase of direct compressive stress is caused by the inclination of the segments of the column to its axis. If the segments could be prepared for the column without initial internal stress, the ultimate resistance would probably be considerably increased.

A consideration of these experiments would also seem to indicate that a closed column is somewhat stronger than an open one. This is undoubtedly due to the fact that the edges of the segments are mutually supporting if they are brought in contact and held so by complete closure, but not otherwise.

Thus the crippling or buckling of the individual parts of the column is delayed, and the ultimate resistance increased.

The general principles which govern the *resistance* of built columns may, then, be summed up as follows:

The material should be disposed as far as possible from the neutral axis of the cross section, thereby increasing r;

There should be no initial internal stress;

The individual portions of the column should be mutually supporting;

The individual portions of the column should be so firmly secured to each other that no relative motion can take place, in order that the column may fail as a whole, thus maintaining the original value of r.

These considerations, it is to be borne in mind, affect the *resistance* of the column only; it may be advisable to sacrifice some elements of resistance in order to attain accessibility to the interior of the compression member, for the purpose of painting. This point may be a very important one, and should never be neglected in designing compression members. It may be observed, however, that the sole object is to prevent oxidation in the interior of the column, and if the column is *perfectly* closed this object is attained. Phœnix columns which have been in the most exposed situations (in one case submerged in water at one time for several hours) during periods varying from twelve to twenty years, without the slightest oxidation in the interior of the columns, have come within the observation of the writer. Different results, however, in other cases have been found.

In the experiments detailed in Mr. Lovett's report it is to be noticed that all deduced values of y are less than the ultimate resistance of wrought iron in short blocks, and some, though not nearly all, would seem to indicate that this difference increased slightly with the length of the column. Further experiments, therefore, may show that the quantity f has some such value as the following:

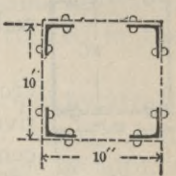
$$f = Cf\left(\frac{1}{l}\right).$$

C being a constant quantity, and f a function of the reciprocal of the length.

In connection with the experiments already detailed, Mr. G. Bouscaren, C.E., has given an account, in the Trans. of the Am. Soc. of Civ. Engs. for Dec., 1880, of other experiments, the results of which are given in the table below.

Column No. 33 was composed of four angle irons,

$$2\frac{1}{4}'' \times 2\frac{1}{4}'' \times \frac{5}{16}'',$$



arranged as shown in the figure. It was swelled from $8\frac{1}{4}'' \times 8\frac{1}{4}''$ at the ends to $10'' \times 10''$ at the centre. There was only one experiment with this form of column, consequently the values of x and y in Eq. (13) could not be determined. The angle irons, however, were of the same manufacture as the iron of which the Am. Br. Co.'s columns were built. As a mere matter of trial, therefore, y is taken at 36,000.00 pounds, and x is then found to be $\frac{1}{43000}$. This result seems to indicate considerable advantage in such a form of column, but one experiment alone furnishes insufficient basis for such a deduction.

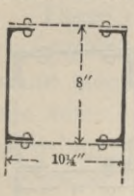
The columns 35 and 36 illustrate the effect of repeated stress.

The columns 37 to 43, inclusive, were intended to furnish information in regard to the distance between the rivets in the zigzag bracing and the thickness of the metal, in order that the column may fail as a whole and not by "local buckling."

Columns 39 and 40 were each composed of a single short piece of channel bar; the others were composed of two channel bars held together by zigzag bracing.

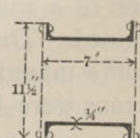
NO.	LENGTH.	AREA.	r^2.	c.	x.	y.	p.	*Exp.*	
		□″							
33	28′ 6″	5.68	20.07	5,828	$\frac{1}{43000}$	36,000	31,700	31,700	Pin Ends.
35	34′ 0″	7.48	8.73	——	——	——	——	20,053	" "
36	34′ 0″	7.48	8.73	——	——	——	——	23,128	" "
43	26′ 7″	6.50	5.95	——	——	——	——	18,000	" "
37	27′ 6″	12.08	19.98	——	——	——	——	29,600	Flat Ends.
38	23′ 00″	13.48	20.69	——	——	——	——	32,300	" "
39	24″	6.6	0.7	——	——	——	——	35,400	" "
40	19″$\frac{5}{16}$	6.6	0.7	——	——	——	——	35,700	" "
41	27′ 6″	13.74	20 79	——	——	——	——	32,400	" "
42	27′ 6″	11.05	21.26	——	——	——	——	32,300	" "

The following forms of cross section, and observations, are taken from Mr. Bouscaren's account:

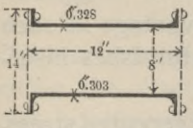


No. 35.—Gave way by pin crushing and splitting web of channel. Column not injured otherwise.

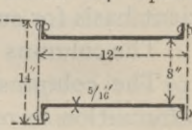
No. 36. — Column No. 35 tested again after crushed ends had been cut off and thickening plates riveted on with pin holes 34 feet from centre to centre. Column failed by deflection.



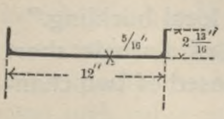
No. 43.—Failed by bending sideways at right angles to pins, without buckling of metal. Bracing 1¾″ × ¼″; rivets 20″ apart in same flange and 10″ in opposite flanges.



No. 37.—Webs buckled in both directions, in middle and one end of column; column did not bend. Bracing 2″ × $\frac{5}{16}$″; rivets 24″ apart in same flange and 12″ in opposite flanges.



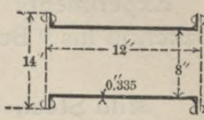
No. 38.—Failed in same manner as No. 37 and by deflection, simultaneously. Bracing and rivets same as in No. 37.

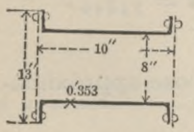


No. 39.—Failed by buckling of web and flanges.

No. 40.—Same as No. 39. Failed by buckling of web and flanges.

No. 41.—Column same as No. 37 with rivets spaced 20″, in same flange, instead of 24″. Failed by buckling of web and bending in both directions, simultaneously.





No. 42.—Failed by buckling in plane of latticing, without buckling of metal.

From these experiments Mr. Bouscaren concluded that, for the ratio of length to diameter used, "the thickness of metal should not be less than $\frac{1}{30}$ of the distance between supports transversely, ... and that the distance between rivets longitudinally should be such that the length of channel spanning it, considered as a column, ... shall give the same resistance per square inch of area as the column itself, treated in the same manner with the same constant "f"," (y).

These conclusions are agreeable to that reached by Mr. B. B. Stoney: "When the length of a rectangular wrought-iron tubular column does not exceed 30 times its least breadth, it fails by the bulging or buckling of a short portion of the plates, not by the flexure of the pillar as a whole." (Theory of Strains, 2d Edit., Art. 334.)

It should be stated that the experiments whose results have been given were made in hydraulic machines in which the forces were not weighed, consequently the results involve the "packing" friction, which was probably not great, however.

In applying Eqs. (9), (10), and (11) to solid cast-iron columns, there may be taken, approximately:

$$f = 80000.00 \text{ pounds, and } a = \tfrac{1}{6400}.$$

For solid wrought-iron columns, approximately:

$$f = 36000.00 \text{ pounds, and } a = \tfrac{1}{36000}.$$

Experiments on steel columns are still lacking. Mr. B. Baker, in his "Beams, Columns, and Arches," gives for

Mild Steel, $f = 67000.00$ pounds, and $a = \frac{1}{22400}$,
Strong Steel, $f = 114000.00$ pounds, and $a = \frac{1}{14400}$.

These, however, must be considered only loose approximations for the ultimate resistance.

In the "Trans. of Am. Soc. Civ. Engrs.," for Oct. 1880, are given the following formulæ for ultimate resistance of wrought-iron columns, designed several years since by C. Shaler Smith, C.E.:

Square Column.

FLAT ENDS.	ONE PIN END.	TWO PIN ENDS.
$p = \dfrac{38500}{1 + \dfrac{1}{5820}\dfrac{l^2}{d^2}}$.	$p = \dfrac{38500}{1 + \dfrac{1}{3000}\dfrac{l^2}{d^2}}$.	$p = \dfrac{37500}{1 + \dfrac{1}{1900}\dfrac{l^2}{d^2}}$.

Phœnix Column.

$$p = \frac{42500}{1 + \frac{1}{4500}\frac{l^2}{d^2}}. \qquad p = \frac{40000}{1 + \frac{1}{2250}\frac{l^2}{d^2}}. \qquad p = \frac{36600}{1 + \frac{1}{1500}\frac{l^2}{d^2}}.$$

American Br. Co. Column.

$$p = \frac{36500}{1 + \frac{1}{3750}\frac{l^2}{d^2}}. \qquad p = \frac{36500}{1 + \frac{1}{2250}\frac{l^2}{d^2}}. \qquad p = \frac{36500}{1 + \frac{1}{1750}\frac{l^2}{d^2}}.$$

Common Column.

$$p = \frac{36500}{1 + \frac{1}{2700}\frac{l^2}{d^2}}. \quad p = \frac{36500}{1 + \frac{1}{1500}\frac{l^2}{d^2}}. \quad p = \frac{36500}{1 + \frac{1}{1200}\frac{l^2}{d^2}}.$$

The formula for "square columns" may be used, without much error, for the common chord section composed of two channel bars and plates, with the axis of the pin passing through the centre of gravity of the cross section.

Compression members composed of two channels connected by zigzag bracing, may be treated by the same formula after putting 36,000.00 for 39,000.00 in Eqs. (21) and (22).

Art. 51.—Experiments on Phœnix Columns,* Latticed Channel Columns and Channels.

In May and July, 1873, some experiments were made at Phœnixville, Penn., on full sized Phœnix columns, by the Phœnix Iron Co. The results of these experiments are given in column headed "*Experiment*," while the column headed "p" contains the results of the application of the formula established in the preceding Article:

$$p = \frac{42000}{1 + \frac{1}{50000}\frac{l^2}{r^2}}, \text{ or } = \frac{42000}{1 + \frac{4}{50000}\frac{l^2}{r^2}}; \quad . \quad . \quad . \quad (1)$$

* The preceding Article was written as a lecture and read to the Class in Civil Engineering at the Rensselaer Polytechnic Institute nearly a year before this Article was written; it has, therefore, been allowed to stand without change.

according as the ends are "flat" or "round." All columns are "4 segment" ones.

TABLE I.

DATE.	ENDS.	AREA.	LENGTH.	r^2.	$l^2 \div r^2$.	*Experiment.*	*p*.
		Sq. Ins.	Feet.			Lbs.	Lbs.
May 3, 1873...	Flat ...	5.84	23.81	4.10	19,950	30,274.00	30,000.00
May 3, 1873...	Round..	5.95	24.00	4.10	20,230	16,387.00*	16,040.00
May 3, 1873...	Flat ...	10.21	23.38	8.68	9,065	36,419.00	35,600.00
May 3, 1873...	Flat ...	8.50	22.71	8.00	9,282	38,235.00	35,430.00
July 19, 1873..	Flat ...	13.31	23.20	8.47	9,151	32,742.00	35,500.00
July 19, 1873..	Flat ...	12.85	23.20	8.47	9,151	35,408.00	35,500.00

In applying the formula the length was reduced to inches, in order to bring it to the same unit as that in which the radius of gyration, r, is expressed.

The columns "*Experiment*" and "*p*" are each, of course, per square inch.

It is seen that the experimental results, and those by Gordon's formula, give a very close and satisfactory agreement. It is also seen that the analytical relation between flat and round ends is a true one.

The square of the radius of gyration, 4.10, was taken the same for the first and second columns because their normal sectional areas are so nearly the same. The value 4.10 belongs to a 4 segment column, whose area is 5.88 sq. ins.

The same observation applies to the last two columns. The value 8.47 belongs to a column whose area of cross section is 13.08 square inches.

A most valuable and instructive set of experiments on Phœnix columns was also made in the large testing machine at the

U. S. arsenal at Watertown, Mass., under the direction of Messrs. Clark, Reeves & Co., the results of which were presented to the American Society of Civil Engineers at the 13th annual convention, June 15, 1881. The value of these experiments is enhanced by the fact that they were made on full sized columns, such in reality as are used in ordinary bridge construction.

In the following table are given the results of these experiments, as well as those of several formulæ presently to be explained.

The following is a portion of the notation:

l = length in inches;
r = radius of gyration in inches;
$E. L.$ = elastic limit in pounds per square inch;
$Exp.$ = ultimate resistance in pounds per square inch.

TABLE II.

NO.	LENGTH.	AREA.	r^2.	$l \div r$.	$l^2 \div r^2$.	$E. L.$	$Exp.$	p_1.	p'.	p''.
	Feet.	Sq. in.	Ins.			Lbs.	Lbs.	Lbs.	Lbs.	Lbs.
1	28	12.062	8.94	112	12,544	——	35,150	32,550	34,488	——
2	28	12.181	8.94	112	12,544	——	34,150	32,550	34,488	——
3	25	12.233	8.94	100	10,000	27,960	35,270	34,000	35,040	——
4	25	12.100	8.94	100	10,000	——	35,040	34,000	35,040	——
5	22	12.371	8.94	88	7,744	——	35,570	35,420	35,592	——
6	22	12.311	8.94	88	7,744	——	34,360	35,420	35,592	——
7	19	12.023	8.94	76	5,776	——	35,365	36,800	36,144	——
8	19	12.087	8.94	76	5,776	29,290	36,900	36,800	36,144	——
9	16	12.000	8.94	64	4,096	——	36,580	38,130	36,696	——
10	16	12.000	8.94	64	4,096	——	36,580	38,130	36,696	——
11	13	12.185	8.94	52	2,704	28,890	36,857	39,400	37,248	——
12	13	12.069	8.94	52	2,704	——	37,200	39,400	37,248	——
13	10	12.248	8.94	40	1,600	26,940	36,480	40,700	37,800	——
14	10	12.339	8.94	40	1,600	28,360	36,397	40,700	37,800	——
15	7	12.265	8.94	28	784	29,350	38,157	42,200	38,352	40,360
16	7	11.962	8.94	28	784	29,590	43,300	42,200	38,352	40,360
17	4	12.081	8.94	16	256	——	49,500	44,770	——	46,300
18	4	12.119	8.94	16	256	28,050	51,240	44,770	——	46,300
19	8 ins.	11.903	8.94	2.7	7.29	——	57,130	69,600	——	57,140
20	8 ins.	11.903	8.94	2.7	7.29	——	57,300	69,600	——	57,140
21	25′ 2.65″	18.300	19.37	68.8	4,733	——	36,010	37,600	36,666	——
22	8′ 9″	18.300	19.37	24	576	29,510	42,180	42,840	——	42,160

In determining r^2 for Nos. 1 to 20, inclusive, a column whose area of cross section was 12.23 square inches was taken. The areas of the actual cross sections varied so little from this quantity, that the resulting value of r^2 was assumed to belong to all of the first 20 columns. All the columns were tested with flat ends.

An application of Eq. (1) to these columns reveals considerable discrepancies between the results of that formula and the quantities given in the column "*Exp.*" of the table, when the values of $l \div r$ become comparatively small, as was anticipated in the preceding article. Instead of the constant 42,000 in the numerator of Gordon's formula, these experiments show that a variable quantity must be used, which shall increase as $l \div r$ decreases, or as $r \div l$ increases.

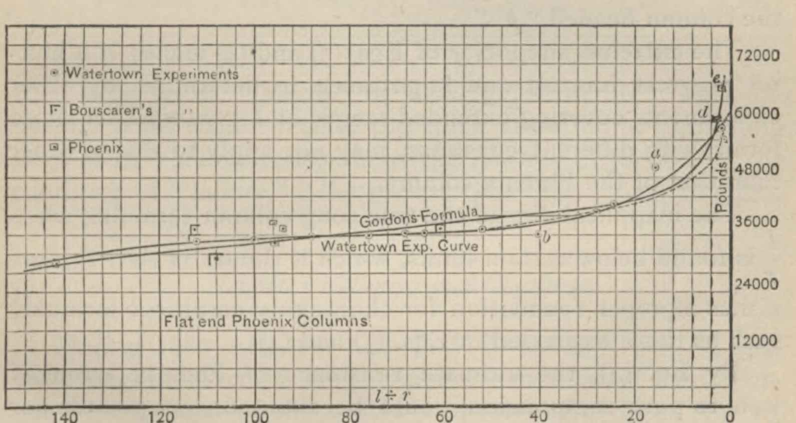
After several trials it was found that the following modified form of Gordon's formula would give tolerable results throughout the entire range of the experiments:

$$p_1 = \frac{40000\left(1 + \frac{2r}{l}\right)}{1 + \frac{1}{50000} \cdot \frac{l^2}{r^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The results of Eq. (2) are given in the column of the table headed p_1. The agreement between the two columns is not as close as could be desired, yet the discrepancies are not sufficiently great to vitiate the safe use of the formula.

In the following figure, the Watertown experiments, as well as those of Mr. Bouscaren and the Phœnix Iron Co. (given in this and the preceding Article), are shown by diagram. The different classes of experiments are indicated as shown. The experimental curve is drawn with particular reference to the Watertown experiments, for it is then found to be properly located in reference to the others. The other curve expresses Gordon's formula according to Eq. (2). It would not be difficult to find an equation which would fit the experimental curve very closely throughout the range of the experiments, but it would not be as simple as Eq. (2), or as two others to be shortly given.

It is interesting and important to observe that each experimental value in the diagram (which is a mean of two, belonging to columns of the same length, in the table), lies on or exceedingly close to the curve, with the exceptions of those shown at *a* and *b*. *a* corresponds to a mean of Nos. 17 and 18, and is abnormally high; *b* shows the mean of Nos. 13 and 14, and is abnormally low.



It may be observed that the experimental curve is nearly a straight line from a point just above *b* to the extreme left of the diagram. For that portion of the curve, therefore, the following formula applies very closely:

$$p' = 39{,}640 - 46 \frac{l}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The results of this formula are given in the column headed "p'." The table, in connection with the diagram, shows that this formula may be used with accuracy for values of $l \div r$ lying between 30 and 140, and further experiments may possibly show that it is applicable above the latter limit.

For values of $l \div r$ less than 30, the following formula will be found to give results approximating very closely to the experimental curve:

$$p'' = 64{,}700 - 4{,}600 \sqrt{\frac{l}{r}} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The results of the application of this formula are given in the column headed "p''."

The extreme simplicity of Eqs. (3) and (4) makes it a matter of great interest and importance to determine, by other experiments covering extended ranges of $l \div r$, whether those forms, with different constants, may not apply to shapes other than that of the Phœnix column.

The inapplicability of the true long column formulæ, when $\frac{l}{r}$ is found below certain limits, which is shown in Art. 25, furnishes a proper foundation for thoroughly empirical formulæ, such as those expressed in Eqs. (3) and (4).

By Eq. (4), the ultimate resistance of Phœnix wrought iron to pure compression would be about 60,000 pounds per square inch.

The results of the application of Eqs. (3) and (4) to Bouscaren's and the Phœnix experiments are not given, but the diagram shows clearly that they would be satisfactory. Data sufficient for the application are given in this and the preceding article.

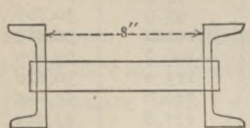
The following is the record of the Phœnix tests of the very short columns shown at *c*, *d* and *e* in the diagram. It is a question whether the degree of distortion which accompanied the extremely high result of 65,867 pounds per square inch, was not considerably greater than that which would characterize

NO.	l.	AREA.	r^2.	$l \div r$.	$l^2 \div r^2$.	EXP.	p_1.	p''.
	Ins.	Sq. in.						
1	8	6.98	4.11	3.95	15.6	60,573	51,500	55,500
2	8	6.98	4.11	3.95	15.6	60,387	51,500	55,500
3	4	5.63	2.37	2.6	6.76	65,867	55,800	57,300
4	4	5.63	2.37	2.6	6.76	65,867	55,800	57,300
5	4	2.93	2.25	2.67	7.13	56,839	55,500	57,200
6	4	2.93	2.25	2.67	7.13	55,555	55,500	57,200

the condition of "failure" in an actual structure. This important point cannot receive too much attention in connection with short column tests, where the *relative* distortion, in the condition of "failure," is far greater than that in long columns.

Latticed Columns and Channels.

During 1880 and 1881 Col. T. T. S. Laidley, U.S.A., tested a large number of long columns composed of two channel bars latticed in the ordinary manner (Ex. Doc. No. 12, 47th Cong. 1st Session). These columns were furnished with 3½-inch pin ends, and were tested at Watertown, Mass., in the large government machine. The adjoining figure shows the relative positions of the channels and pin. 6″, 8″, 10″ and 12″ [s were employed, and all the columns, the results of whose tests are given in Table III., preserved the uniform distance of 8 inches between the channels.



The radius of gyration, r, of the cross section, given in that table, is in reference to the axis of the pin.

All the posts were single latticed, and the pitch of the latticing (the distance apart of rivets in the same flange of a [) was 18 inches for the 6 and 8-inch channels, and 22 inches for the 10 and 12-inch. $2'' \times \frac{1}{4}''$ latticing was used for the 6-inch [s; $2'' \times \frac{3}{8}''$ for the 8 and 10-inch, and $2\frac{1}{4}'' \times \frac{3}{8}''$ for the 12-inch.

The area of cross section for the [s of the same depth in different columns varied slightly, consequently about an average area was taken.

TABLE III.

Pin Ends.—$3\frac{1}{2}''$ Pin.

NO.	[.	AREA OF SECTION IN SQ. INCHES (2 [s).	LENGTH IN INCHES.	RADIUS OF GYRATION, INS.	LENGTH OVER RADIUS; OR $l \div r$.	p.
	Inches.					Pounds.
1	8	7.65	160	3.00	53.3	35,025
2	10	9.70	200	3.65	54.8	33,920
3	6	4.65	144	2.35	61.3	34,450
4	6	4.65	150	2.35	63.9	34,130
5	8	7.65	200	3.00	66.7	33,790
6	10	9.70	250	3.65	68.5	33,770
7	6	4.65	180	2.35	76.7	34,180
8	8	7.65	240	3.00	80.0	32,375
9	12	12.00	360	4.44	81.0	31,475
10	10	9.70	300	3.65	82.2	33,015
11	6	4.65	210	2.35	89.5	31,935
12	8	7.65	280	3.00	93.3	31,800
13	10	9.70	350	3.65	95.9	30,780
14	6	4.65	240	2.35	102.2	30,085
15	8	7.65	320	3.00	106.7	29,600
16	6	4.65	270	2.35	115.0	30,820
17	8	7.65	360	3.00	120.0	25,885
18	6	4.65	300	2.35	127.8	24,355
19	6	4.65	330	2.35	140.6	21,330
20	6	4.65	360	2.35	153.4	15,320

"p" is the ultimate resistance per square inch, in pounds.

All these columns failed as *wholes*, and each result is a mean of two. Other columns of the same set, and tested at the same time, failed by buckling of the channels; they cannot, consequently, be classed among long columns which are so constructed as to fail as *wholes*.

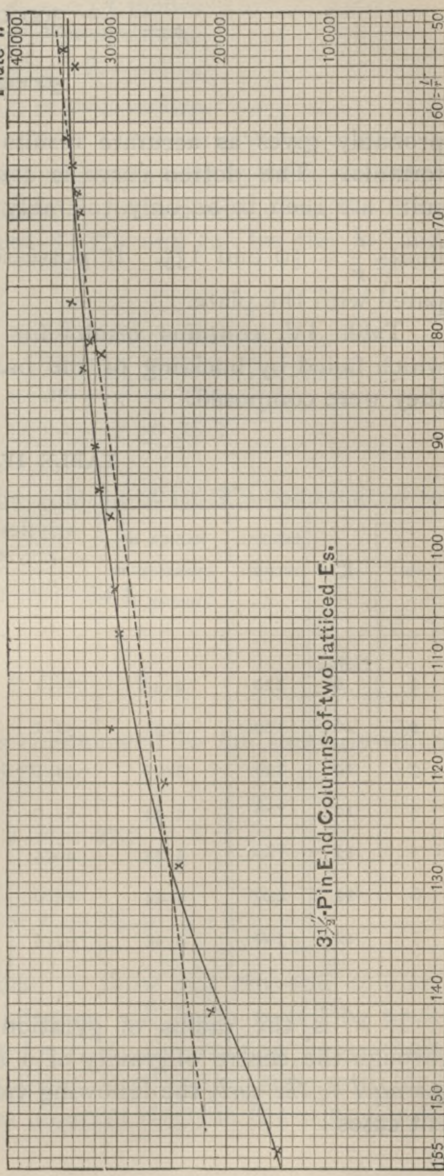
The values of p in Table III. are shown graphically in Plate I. The ratio $l \div r$ is laid off along the horizontal line and the ultimate intensity p on the vertical line, as shown. The full curved line is then the experimental curve and possesses great value of a practical nature. Within the limits of the diagram, when the ratio

$$l \div r$$

is known, the ultimate resistance of the column per square inch (p) can be at once accurately read from the plate without calculation or scale.

The following equation:

Plate I.



$$p = \frac{39000}{1 + \frac{1}{30000} \frac{l^2}{r^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

probably gives as accurate results as any form of Gordon's formula. The dotted curve is constructed from it. Its results are seen to be only tolerably approximate between the limits $\frac{l}{r} = 50$ and 135. It possesses little value when compared with the plate.

Table IV. contains results for columns of the same set which failed by buckling of the individual channels of which they were composed.

TABLE IV.

NO.	[.	LENGTH, INCHES, l.	RADIUS OF GYRATION IN INCHES, r.	$\frac{l}{r}$.	ULT. p IN LBS. PER SQ. INCH.	CONDITION OF ENDS.
	Inches.					
1	6	120	2.35	51.1	36,025	Flat.
2	6	120	2.35	51.1	33,740	One flat ; one pin.
3	10	126	3.65	34.5	35,450	Pin.
4	12	120	4.44	27.0	34,245	Pin.
5	12	180	4.44	40.5	34,660	Pin.
6	12	240	4.44	54.0	33,985	Pin.
7	12	300	4.44	67.5	33,590	Pin.

If r' is the radius of gyration in reference to an axis through the centre of gravity of a single channel section, and *parallel to the web*, the following values will hold for the present cases :

$$6'' \sqsubset; \quad r' = 0.58 \text{ inches.}$$
$$8'' \sqsubset; \quad r' = 0.48 \text{ inches.}$$
$$10'' \sqsubset; \quad r' = 0.69 \text{ inches.}$$
$$12'' \sqsubset; \quad r' = 0.87 \text{ inches.}$$

Although the lattice rivets were alternate in the same channel, each flange was unsupported for a distance equal to the pitch, *i.e.*, 18″ for the 6″ and 8″ ⊏s, and 22″ for the 10″ and 12″ ⊏s. Hence :

$$\text{For } 6'' \sqsubset; \quad 18 \div r' = 31.0$$
$$\text{For } 8'' \sqsubset; \quad 18 \div r' = 37.4$$
$$\text{For } 10'' \sqsubset; \quad 22 \div r' = 31.9$$
$$\text{For } 12'' \sqsubset; \quad 22 \div r' = 25.3$$

Table IV. shows that the column of 10″ ⊏s commenced to fail by buckling of the ⊏s when

$$l \div r = 34.5,$$

and when

$$22 \div r' = 31.9;$$

that the column of 12″ ⊏s commenced to fail similarly when the length became so small that

$$l \div r = 27.0,$$

while

$$22 \div r' = 25.3.$$

These results would seem to show that pin end columns with single but alternate latticing will begin to fail by buckling of the channels when $l \div r$, for the column as a whole, becomes so small that it is about equal to the same ratio for a single channel between two adjacent rivets in the same flange.

Nos. 1 and 2 of Table IV. show that if the ends possess a greater degree of fixedness, the value of $l \div r$ is much greater when buckling begins to take place, but the number of experiments is not sufficient to indicate the exact amount.

As would be anticipated under the circumstances, p maintains about the same value for all the columns in Table IV. Hence when $l \div r$ becomes so small that buckling takes place, the ultimate resistance of the column is independent of the length.

The graphical representations of the results given in this Article show that the curve of ultimate resistances has a very sharp declivity for small values of $l \div r$, but that it becomes nearly straight and horizontal for larger values, and that it again increases in declivity with a still father increase in that ratio. These phenomena seem to be much more pronounced in the tubular variety of columns. They find a simple and obvious explanation in the fact that in columns of moderate length the deflection at the centre of the column about keeps pace (in the same direction) with the movement of the centre of pressure at the ends.

Plate I. shows (what was to be anticipated) that this effect is also much less pronounced with pin ends than with flat ones, it being borne in mind that the phenomena here considered do not produce the horizontal straight line which would be seen if Plate I. included less values of $l \div r$ than 50. The latter represents the buckling of the individual parts of the column, and not the failure of the column as a whole.

A few experiments by Col. Laidley with columns of the same [s as the above, but with pins only three inches in diameter, gave uniformly less ultimate resistance than those with three and a half inch pins. Although this result was to be expected, the number of experiments was not sufficient to justify any quantitative conclusions ; it can only be stated that the smaller the pin the less will be the ultimate resistance.

TABLE V.

Flat End ⊏*s.*

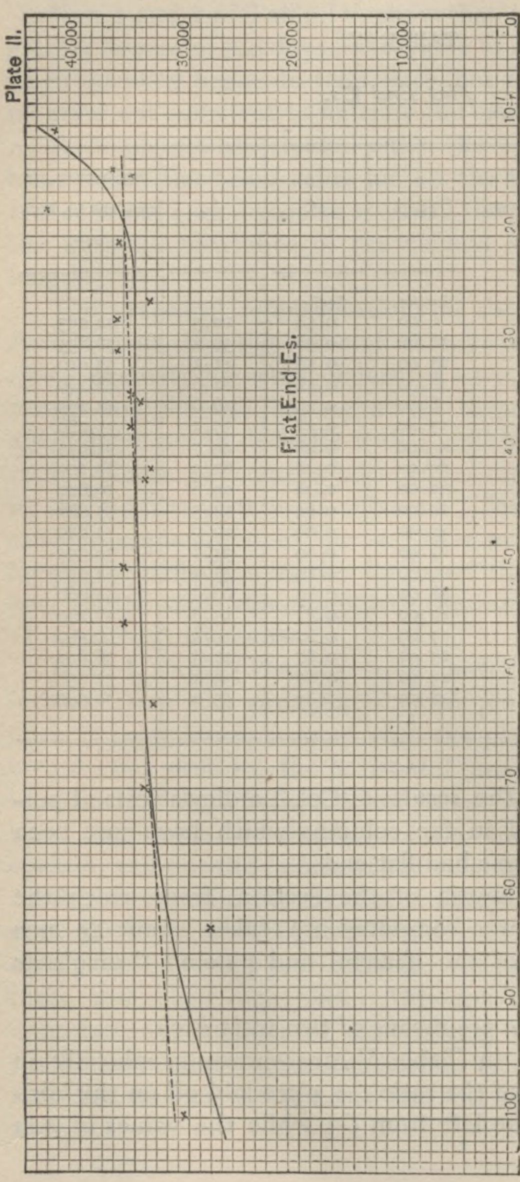
NO.	⊏.	AREA OF SECTION IN SQ. INCHES.	l.	r'.	$\frac{l}{r'}$.	ULT. RESIST., IN LBS. PER SQ. INCH $= p$.
	Inches.		Inches.	Inches.		
1	6	2.33	6.00	0.58	10.3	42,293
2	6	2.33	17.60	0.58	30.3	36,835
3	6	2.33	23 90	0.58	41.1	33,910
4	6	2.33	48.00	0.58	82.6	28,140
5	8	3.80	8.00	0.48	16.6	43,295
6	8	3.80	17.90	0.48	37.2	35,280
7	8	3.80	23.90	0.48	49.7	35,975
8	8	3.80	29.90	0.48	62.2	33,400
9	8	3.80	48.00	0.48	99.8	30,620
10	10	4.85	10.00	0.69	14.5	35,080
11	10	4.85	17.90	0.69	26.0	33,820
12	10	4.85	23.90	0.69	34.7	34,355
13	10	4.85	29.90	0.69	43.4	34,050
14	10	4.85	48.00	0.69	69.6	34,080
15	12	6.00	12.00	0.87	13.8	37,240
16	12	6.00	17.80	0.87	20.5	36,590
17	12	6.00	23.90	0.87	27.5	36,695
18	12	6.00	29 90	0.87	34.4	35,150
19	12	6.00	48.00	0.87	55.2	36,040

Table V. contains the results of Col. Laidley's tests of portions of the ⊏s used in the columns which have just been treated. These portions had flat ends.

The moment of inertia of the section, from which the radius of gyration r' was computed, was taken about an axis parallel to the web of the channel and passing through its centre of gravity.

Many of the results are means of two tests each.

The results given in Table V. are shown graphically in Plate II. The values of the ratio $l \div r$ are laid off on the horizontal base line, to the left from O; while the values of p in



pounds per square inch are laid off vertically from O, as shown. The full curve then represents with great accuracy the experimental results.

The dotted curve represents the following form of Gordon's formula for the ultimate resistance in pounds per square inch:

$$p = \frac{36000}{1 + \frac{1}{63000}\frac{l^2}{r'^2}} \quad (6)$$

This formula is sufficiently accurate for all ordinary purposes, between the limits

$$l \div r' = 15$$

and

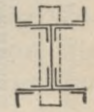
$$l \div r' = 90,$$

but does not compare in value with the experimental (full) curve.

Addendum to Art. 51.

Since the issue of the first edition of this book, the series of tests of full sized columns, of which Table III. gives the results of the first 20, has been continued at Watertown, Mass., and the test records are given in "Ex. Doc. No. 5, Senate, 48th Congress, 1st Session," and "Ex. Doc. No. 35, Senate, 49th Congress, 1st Session." Table VI. shows the digested records put in shape to be of some value to engineers. These columns had $3\frac{1}{2}$ inch pin ends, and the results belong to failures in the plane normal to the pin axes.

Columns 21, 22, 23, 24, 25, 26 and 37 to 48 inclusive were closed box columns composed of two channels and two plates; the remaining columns, except those of the Wilson section, were composed of two channels latticed together in the usual manner. The word "built" in the Table indicates that the channels were built of plates and angles; otherwise they were rolled. The Wilson column is that used so frequently by Jas. M. Wilson, C.E., formerly Eng'r Bridges and Buildings Penna. R. R. It has the section shown by the sketch in the margin. In these columns the pin was always placed parallel to the plate between the channels, *i. e.*, normal to the webs of the channels and as shown by the broken lines.



In columns 25, 26, 39, 40, 41, 42, 47 and 48, the pins were placed through (*i. e.*, normal to) the webs of the channels, as shown in the Fig. on page 455; in all the other channel columns the pins were placed parallel to the webs of the channels.

The results given in Tables III., and VI. are shown graphically on Plate A. All results are brought together on one plate in order to obtain the most probable curve for ordinary wrought iron columns with $3\frac{1}{2}$ inch pin ends.

TABLE VI.

$3\frac{1}{2}$ *Inch Pin End Columns.*

NO.	[INCHES.	SECTION SQ. INS.	LENGTH IN INCHES.	RADIUS OF GYRATION. INCHES.	LENGTH OVER RADIUS. $l \div r$.	ULTIMATE IN LBS. PER SQ. IN. p.	REMARKS.
1	8	7.6	160	4.5	36	33,910	
2	8	8.1	160	4.5	36	36,580	
3	8	7.6	160	5.23	31	34,340	Swelled.
4	8	7.6	160	5.23	31	33,530	"
5	10	11.9	200	4.6	44	33,740	
6	10	12 3	200	4.6	44	34,670	
7	10	12.4	200	5.98	34	31,130	Swelled.
8	10	12.7	200	5.98	34	31,990	"
9	8	7.5	240	5.23	46	33,390	"
10	8	7.5	240	5.23	46	34,390	"
11	8	7.6	240	4.5	53	34,120	
12	8	7.6	240	4.5	53	33,410	
13	10	12.1	300	4.6	65	33,630	
14	10	12.2	300	4.6	65	32,440	
15	10	11.9	300	5.98	50	32,830	Swelled.
16	10	11.9	300	5.98	50	32,740	"
17	8	7.7	320	4.5	71	31,610	
18	8	7.7	320	4.5	71	29,870	
19	8	7.7	320	5.23	61	30,840	Swelled.
20	8	7.7	320	5.23	61	30,770	"
21	8	16.2	320	3.78	85	28,020	Built
22	8	16.3	320	3.78	85	27,910	"
23	8	21.0	320	3.8	84	25,770	"
24	8	20.6	320	3.8	84	25,950	"
25	8	17.9	320	2.7	119	26,480	
26	8	19.4	320	2.7	119	25,290	
27	6	9.8	120	1.87	64	30,220	
28	6	10.2	120	1.87	64	31,380	
29	6	10.0	180	1.87	96	25,160	
30	6	10.0	180	1.87	96	21,050	
31	8	16.1	240	2.44	98	26.430	Wilson column.
32	8	16.3	240	2.44	98	22,540	
33	6	9.7	240	1.87	128	19,380	
34	6	9.8	240	1 87	128	16,220	
35	8	16.2	320	2.44	131	19,700	
36	8	16.1	320	2.44	131	17,570	
37	5.6	9.21	180	2.9	62	31,650	Built.
38	5.6	9.44	180	2.9	62	30,720	"
39	6.0	11.42	180	2.73	66	33,205	
40	6.0	11.42	180	2.73	66	32,329	
41	8.0	17.8	240	3.6	67	32,077	
42	8.0	17.2	240	3.6	67	32.253	
43	8.0	12.65	240	3.6	76	37,668	
44	8.0	12.76	240	3.6	76	30,596	
45	5.6	9.24	240	2.9	83	28,950	Built.
46	5.6	9.36	240	2.9	83	29,879	"
47	6.0	11.42	240	2.73	88	29.947	
48	6.0	11.31	240	2.73	88	29,186	
49	8.0	15.34	160	4.2	38	30,965	Wilson column.
50	8.0	15.40	160	2.5	64	31,494	

TABLE VI*a*.

Flat End Channel Columns.

NO.	⊏ INS.	SECTION. SQ. INS.	LENGTH IN INCHES.	RADIUS OF GYRATION. INCHES.	LENGTH OVER RADIUS. $l \div r$.	ULTIMATE IN LBS. PER SQ. IN. p.	REMARKS.
1	8	17.0	168	3.6	47	34,950	
2	8	17.8	168	3.6	47	35,595	
3	7.2	21.0	168	3.26	51.5	33,682	Built channels.
4	7.2	21.5	168	3.26	51.5	33,061	
5	7.5	15.7	248	3.32	75	33,003	
6	7.5	15.6	248	3.32	75	34,505	
7	7.2	21.2	248	3.26	76	33,019	
8	7.2	21.5	248	3.26	76	33,943	
9	10	17	308	4.02	76.6	34,279	These columns latticed on one side. All above are complete box columns.
10	10	17.4	308	4.02	76.6	33,333	
11	8	12.6	248	3.16	78.5	32,666	
12	8	12.7	248	3 16	78.5	33,862	
13	6	4.8	121	2.3	53	36,720	Rolled channels latticed both sides.
14	6	4.7	121	2.3	53	35,330	

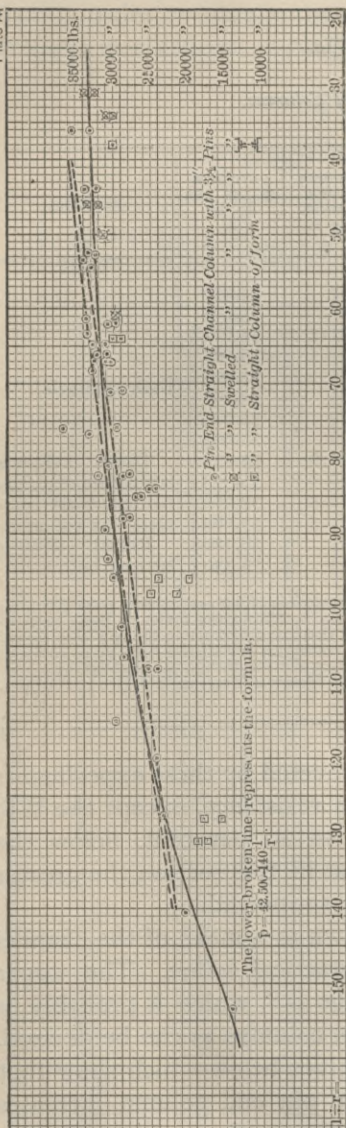
The results given in this table are a digest of the records of flat end column tests given in "Senate Ex. Doc. No. 35, 49th Congress, 1st Session." Within the limits of these tests, a comparison with the 3½ inch pin end results shows that the difference in the end conditions exerts no influence on the ultimate compressive resistance per square inch up to at least $l \div r = 80$. The number of tests in this Table is quite insufficient to establish any law between pin and flat end columns of this character. Hence no diagram is drawn or formula given.

The various kinds of columns covered by these experiments are seen to possess about the same resistance, except the Wilson column, which falls from 10 to 20 per cent. below the others. This is due to the fact that in this section the greater portion of the metal is but slightly supported.

The full line on the Plate is drawn as a mean of the channel columns only, and is of great practical value. The upper broken line expresses Eq. (5) of page 458, which is probably as good a pin end formula for channel columns as can be devised.

Results of experiments will be given below which show that the resistance per square inch of a pin end column increases with the diameter of the pin, and inasmuch as pins ordinarily

Plate A.



used in columns of the dimensions of those tested, usually considerably exceed $3\frac{1}{2}$ inches, the mean value of these tests may probably be a little too low for ordinary bridge practice.

Tables VII. and VIII. contain the mean of a large number of most valuable tests of full size iron and steel angle, tee, channel and beam struts with the various end conditions indicated, by James Christie Esq., Supt. of the Pencoyd Iron Co. The detailed account of this complete series of tests should be carefully consulted; it may be found in the "Trans. of The Am. Soc. of C. E.," Vol. XIII., 1884. All sizes of angles and tees up to 4 inches by 4 inches by $\frac{3}{8}$ inch and over 15 feet in length were used in these tests. The "hinged ends" were either one inch or two inch pins in semi-cylindrical bearings or one inch or two inch balls in sockets. The "round ends" were the above described balls resting on flat or plane surfaces.

The "flat ends" were secured by simply resting the carefully faced ends of the struts

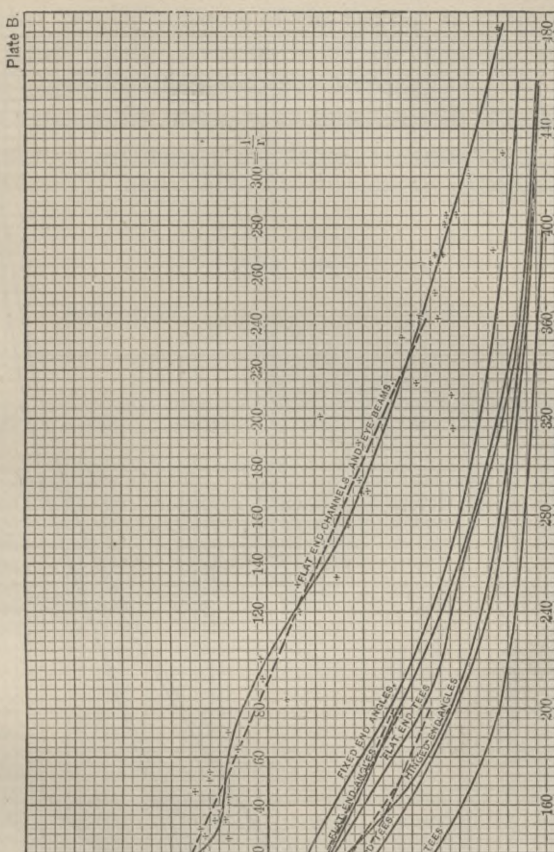


Plate B.
IRON STRUTS.
FLAT END CHANNELS AND EYE BEAMS
FIXED END ANGLES
FLAT END ANGLES
FLAT END TEES
HINGED END ANGLES
HINGED END TEES
ROUND END TEES
50000 lbs.
40000 "
30000 "
20000 "
10000 "
A

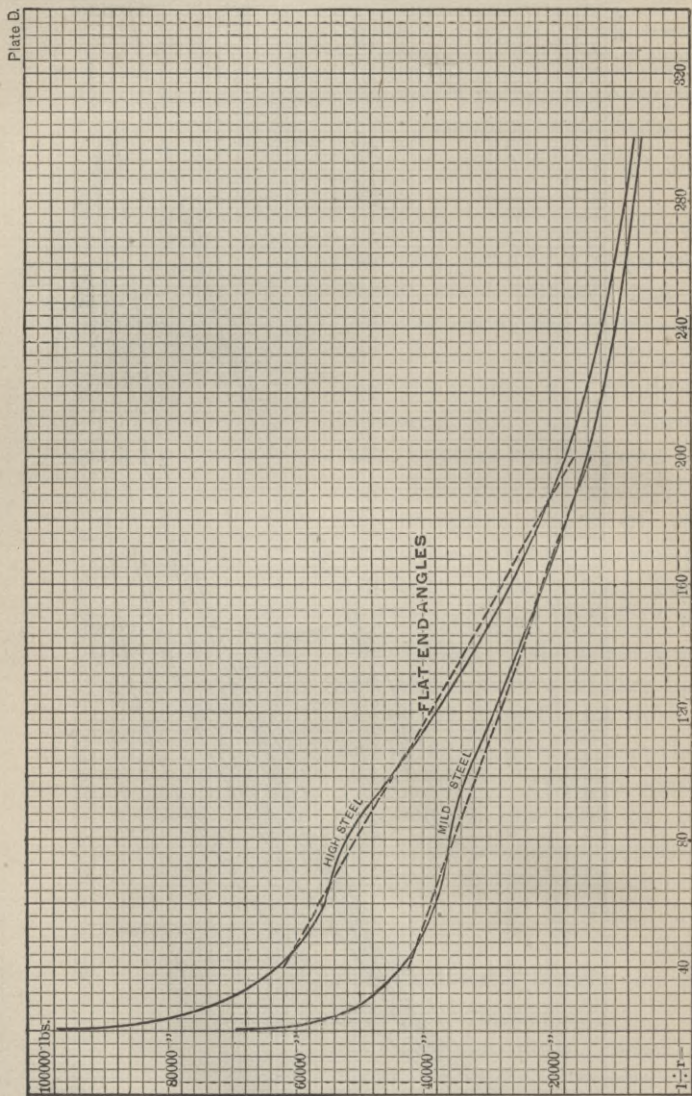


Plate D.
FLAT END ANGLES
HIGH STEEL
MILD STEEL
100000 lbs.
80000 "
60000 "
40000 "
20000 "
l ÷ r =
40
80
120
160
200
240
280
320

on the plane bearing surfaces of the testing machine, while the "fixed" strut "ends" were obtained by clamping the ends of the struts rigidly to those bearing surfaces.

Every imaginable means was taken by Mr. Christie to secure the utmost accuracy in all details of these tests.

In Tables VII. and VIII., l is the length of strut and r the

TABLE VII.

Mean Results of Wrought Iron Strut Tests.

$\frac{l}{r}$	FLAT END ANGLES.	HINGED END ANGLES.	FIXED END ANGLES.	FLAT END TEES.	HINGED END TEES.	ROUND END TEES.	FLAT END CHANNELS AND BEAMS.
20	49,000	45,000	45,000	49,000	47,000	44,000	38,000
40	40,000	40,000	38,000	42,000	41,000	36,500	35,000
60	35,000	36,000	34,000	38,000	36,000	30,500	34,000
80	32,000	32,000	32,000	35,000	31,000	25,000	31,500
100	29,000	29,000	30,000	31,500	27,000	20,500	29,000
120	26,000	26,000	28,000	27,000	22,500	16,500	26,000
140	23,500	22,000	25,500	23,000	18,500	12,800	24,000
160	21,000	17,000	23,000	20,000	15,500	9,500	21,000
180	19,000	13,000	20,000	17,000	12,500	7,500	18,000
200	16,500	11,000	17,500	14,000	10,500	6,000	15,000
220	14,000	9,000	15,000	12,000	8,500	5,000	12,500
240	12,000	8,000	13,000	11,000	7,000	4,300	11,000
260	10,500	7,000	11,000	10,000	6,000	3,800	10,000
280	9,000	6,000	10,000	8,500	5,500	3,200	9,000
300	7,500	5,000	9,000	7,000	5,000	2,800	7,500
320	6,000	4,500	8,000	5,500	4,500	2,500	6,000
340	4,800	4,000	7,000	4,500	4,000	2,100	5,000
360	3,800	3,500	6,500	4,000	3,500	1,900	4,000
380	3,200	3,000	5,800	3,500	3,000	1,700	
400	2,900	2,500	5,200	3,000	2,500	1,500	
420	2,500	2,300	4,800	2,500	2,200	1,300	
440	2,200	2,100	4,300				
460	2,000	1,900	3,800				
480	1,900	1,700					

least radius of gyration of its normal cross section. In order to get the least radius of the angle sections, the moment of inertia was taken about an axis through the centre of gravity of the cross section and parallel to a line through the extremities of the legs.

All results in Table VII. belong to wrought iron struts, while Table VIII. belongs to struts of Bessemer steel. The "mild" steel contained from 0.11 to 0.15 per cent. carbon, but 0.36 per cent. carbon was found in the "high steel." The ultimate tensile resistance of the former varied from 60,000 to 66,000 pounds per square inch with 26 to 24 per cent. stretch

TABLE VIII.

Flat End Steel Angle Struts.

$\frac{l}{r}$	ULTIMATE RESISTANCE, POUNDS PER SQUARE INCH.		$\frac{l}{r}$	ULTIMATE RESISTANCE, POUNDS PER SQUARE INCH.	
	Mild Steel.	High Steel.		Mild Steel.	High Steel.
20	72,000	100,000	170	21,000	26,000
30	51,000	74,000	180	19,500	23,800
40	46,000	65,000	190	18,000	21,800
50	43,000	61,000	200	16,500	20,000
60	41,000	58,000	210	15,200	18,400
70	39,000	56,000	220	14,000	16,900
80	38,000	54,000	230	13,000	15,400
90	36,500	51,000	240	12,000	14,000
100	35,000	47,000	250	11,100	12,800
110	33,500	43,500	260	10,300	11,800
120	31,500	40,000	270	9,600	11,000
130	29,000	36,500	280	9,000	10,200
140	27,000	33,500	290	8,400	9,500
150	25,000	30,800	300	7,900	9,000
160	23,000	28,300			

in 8 inches, while the high steel possessed an ultimate tensile resistance of about 100,000 pounds per square inch and a stretch of about 16 per cent. in 8 inches.

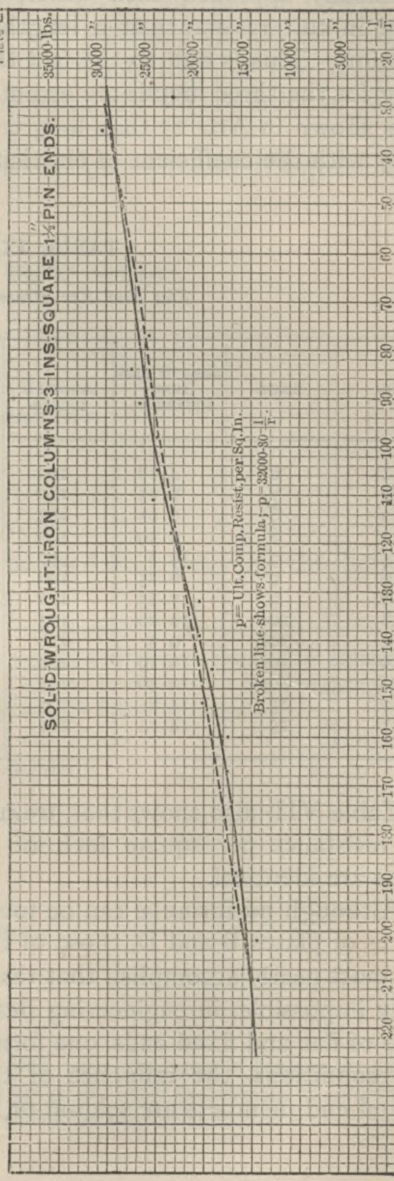
It is to be observed that up to 80 radii of gyration the resistance of the fixed end angles falls below that of both the hinged and flat end struts, but beyond that limit it exceeds them both until it reaches over double their values at and about 400 radii of gyration.

The flat and hinged end conditions approach each other in their resistances until they become nearly equal at the highest values of $l \div r$.

Plates B and D represent graphically the results given in Tables VII., VIII. and IX.; Plate D being devoted wholly to Table VIII. The curve for flat end channels and beams has been moved to the right in order to separate it from the others. This curve includes not only Mr. Christie's data, but that of Table V. and results of later tests found in "Senate Ex. Doc. No. 1, 47th Congress, 2d Session," and given in Table IX.

Plates B and D and the preceding Tables show that at and above 200 radii of gyration the iron and mild steel angle struts possess the same ultimate resistance per square inch. The iron and high steel

Plate E.



continually approach each other, and undoubtedly become equal in unit resistance at a length a little above 300 radii of gyration. This is due to the fact that the coefficients of elasticity for the three metals are essentially identical, since it has been shown in Art. 25 that long column resistance varies directly with the coefficient of elasticity.

TABLE IX.

Flat End Eye Beam Struts.

NO.	BEAM.	AREA OF SECTION. SQUARE IN.	LENGTH IN INCHES.	r LEAST RADIUS GYRATION. INCHES.	$\frac{l}{r}$	ULT. RESIST. IN LBS. PER SQ. INCH $= p$.
1	6″	4.18	120	0.6	200	24,210
2	7″	6.05	180	0.75	240	13,990
3	8″	6.65	192	0.8	241	12,540
4	8″	6.59	193	0.8	242	14,000
5	9″	14.4	57	1.24	46	37,850
6	9″	6.85	192	0.72	267	12,460
7	9″	7.15	192	0.72	267	11,920
8	10½	10.26	155	0.92	169	20,170
9	10½	9.3	216	0.92	234	16,020(?)
10	10½	10.19	264	0.93	284	11,100
11	10½	10.46	264	0.93	284	10,300
12	15	14.8	264	1.00	264	12,400
13	15	14.74	264	1.00	264	12,690

Table VII. also shows that, from 40 to 120 radii of gyration, the resistance per square inch of hinged and flat-end angle struts are identical.

Although no formulæ can be found that will exactly fit the curves of plates B and D, those of the form of Eq. (3), on page 453, most nearly accomplish that result. Inasmuch as it is generally impossible, in engineering design, to separate the conditions of flat and fixed ends, one formula only is given for these two conditions, the influence of the former predominating. Round end members are seldom or never found in engineering structures, hence a formula is given for pin or hinged end angles and tees. If round end members should be used, the table and plate will show how much the pin end resistance must be reduced for a given value of $l \div r$, in order to get the round end resistance.

The straight broken lines on plates B and D represent the following formulæ: *

Flat and fixed end iron angles and tees.

$$p = 44000 - 140 \frac{l}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hinged end iron angles and tees.

$$p = 46000 - 175 \frac{l}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Eqs. (7) and (8) are to be used only between the limits of $l \div r = 40$ and $l \div r = 200$.

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* Although the above formulæ possess great advantages, both in accuracy and simplicity, over the old Gordon or Tredgold forms, it is not amiss to state that the curved broken lines on plate B represent the following formulæ:

Flat and fixed end iron angles and tees.

$$p = \frac{40000}{1 + \frac{l^2}{r^2} \frac{1}{30000}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Hinged end iron angles and tees.

$$p = \frac{40000}{1 + \frac{l^2}{r^2} \frac{1}{20000}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

These formulæ can be used with fairly good results between the limits of $l \div r = 40$ and $l \div r = 180$. They are given simply in deference to an old usage, with the decided opinion that they should be abandoned.

Flat end iron channels and eye beams.

$$p = 40000 - 110\frac{l}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Eq. (9) is to be used only between the limits of $l \div r = 20$ and $l \div r = 240$.

Flat end mild steel angles.

$$p = 52000 - 180\frac{l}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Flat end high steel angles.

$$p = 76000 - 290\frac{l}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Eqs. (10) and (11) are to be used only between the limits of $l \div r = 40$ and $l \div r = 200$.

TABLE X.

Solid 3-*Inch Square Columns* — $1\frac{1}{2}$ *Inch Pin Ends.*

LENGTH. INCHES.	$\frac{l}{d}$	$\frac{l}{r}$	ULT. COMP. RESIST. LBS. PER SQ. IN.	LENGTH. INCHES.	$\frac{l}{d}$	$\frac{l}{r}$	ULT. COMP. RESIST. LBS. PER SQ. IN.
30	10	35	30,125	137.6	46	160	17.780
42	14	49	28,160	143.8	48	167	17,600
54	18	63	26.515	149.8	50	174	17,180
60	20	70	26.475	155.7	52	181	17,670
66	22	77	25 430	161.8	54	188	16,725
72	24	84	27,245	167.8	56	195	16,900
78	26	91	26,800	173.6	58	202	14.525
84	28	98	24,630	179.5	60	210	14,355
90	30	105	24,705				
95.5	32	111	25.050	*Flat ends.*			
101.7	34	118	23,365	89.6	30	104	26,180
107.6	36	125	21,415	119.4	40	139	22,730
113.6	38	132	?0,395	*One flat and one pin end.*			
119 6	40	139	20,430				
125.6	42	146	19,085	89.7	30	104	25 155
131.7	44	153	20,150	119 5	40	139	22,160

Plate E shows the results of tests of solid 3 inch square wrought iron columns with ends bearing on pins 1.5 inches in diameter, as given in Table X., which has been digested from the records of tests found in "Senate Ex. Doc. No. 5, 48th Congress, 1st Session." According to the usual notation, l in the Table is the length in inches; r, the radius of gyration (in inches) of a normal section, and d the length of a side (3 inches). As all bars are here 3 inches square, there is a constant ratio between d and r.

The formula shown by the broken line on Plate E is as follows: for *pin end solid wrought iron columns:*

$$\left.\begin{aligned} p &= 32000 - 80\frac{l}{r} \\ p &= 32000 - 277\frac{l}{d} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

Eq. (12) is to be used only between the limits of $l \div r = 20$ and $l \div r = 220$, or $l \div d = 6$ and $l \div d = 65$.

TABLE XI.

Three-inch Square Solid Columns.

PIN DIAM. INCHES.	LENGTH. INCHES.	$\frac{l}{d}$	$\frac{l}{r}$	ULT. REST. LBS. PER SQ. IN.
	120	40	139	16,285
$1\frac{1}{8}$	120	40	139	18,335
$1\frac{1}{2}$	120	40	139	20,430
$1\frac{7}{8}$	120	40	139	21,440
$2\frac{1}{4}$	120	40	139	22.250

The "flat end" and "one pin and one flat end" results in Table X. are both interesting and important—as showing that the resistance of the latter end condition is essentially a mean between those for pin and flat ends.

Table XI., taken from the same source as Table X., also possesses no little importance as showing the influence of pin diameter. An increase of $\frac{3}{8}$ inch in pin diameter below $1\frac{1}{2}$ inches increases the column resistance over 2,000 pounds per sq. in. Above that limit the increment of resistance for the same increase in pin diameter is continually less, although very material. As a general principle, it may be said that an increase in pin diameter will produce a corresponding increase in column resistance.

Formulæ for Engineering Practice.

If the greatest allowed working stresses in columns be taken at one one-fifth the ultimate resistance, as is usual for railway structures, the following formulæ will result:

Flat end latticed channel columns.

$$p = \frac{8000}{1 + \frac{1}{40000}\frac{l^2}{r^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Pin end latticed channel columns.

$$p = \frac{7800}{1 + \frac{1}{30000}\frac{l}{r^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Or, $$p = 8500 - 28\frac{l}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Eqs. (13), (14) and (15) should be used only between the limits of $l \div r = 40$ and $l \div r = 140$; and Eq. (13) is given only as a formula which is quite generally used among engineers, but which, as yet, has no foundation on a series of tests of full sized columns; it gives results which are probably too high.

Flat and fixed end iron angles and tees.

$$p = 8800 - 28\frac{l}{r} \quad \ldots\ldots \quad (16)$$

Hinged end iron angles and tees.

$$p = 9200 - 35\frac{l}{r} \quad \ldots\ldots \quad (17)$$

Eqs. (16) and (17) are to be used only between the limits of $l \div r = 40$ and $l \div r = 200$.

Flat end iron channels and eye beams.

$$p = 8000 - 22\frac{l}{r} \quad \ldots\ldots \quad (18)$$

Eq. (18) is to be used only between the limits of $l \div r = 20$ and $l \div r = 240$.

Pin end solid wrought iron square columns.

$$\left.\begin{aligned} p &= 6400 - 16\frac{l}{r} \\ p &= 6400 - 55\frac{l}{d} \end{aligned}\right\} \quad \ldots\ldots \quad (19)$$

Eq. (19) is to be used only between the limits of $l \div r = 20$ and $l \div r = 220$, or $l \div d = 6$ and $l \div d = 65$.

Flat end mild steel angles.

$$p = 10400 - 36\frac{l}{r} \quad \ldots\ldots \quad (20)$$

Flat end high steel angles.

$$p = 15200 - 58\frac{l}{r} \quad \ldots\ldots \quad (21)$$

Eqs. (20) and (21) are to be used only between the limits of $l \div r = 40$ and $l \div r = 200$.

For columns with one flat and one pin end, in all cases use a mean of two pin ends and two flat ends.

For hinged end angles of steel, in the absence of experimental data, the proper reduction from flat end angles of the same material may be assumed to be the same percentage, or ratio, as that between flat and hinged end iron angle columns with an equal value of $\frac{l}{r}$.

It is important to observe that the new form of column formula (Eqs. (15) to (21) inclusive) is better adapted to forms of section in which the metal is near the neutral axis, than to those in which the metal is placed at the greatest possible distance from that axis. It is yet a question whether the old Tredgold form (Eqs. (13) and (14)) is not the best for channel columns and those of similar section. The new form, on the other hand, is much the best for angles, tees, eye beams, solid sections, etc. No formula, however, which can be devised, is to be compared in value with the experimental diagram, like Plates A to E.

Steel Latticed Channel Columns.

Although some interesting tests of full sized pin end channel columns of Bessemer steel have been published by Mr. James Dagron in the Trans. of the Am. Soc. of C. E. for 1887, yet the number was only 8, and the range of $\frac{l}{r}$ too limited for the deduction of any law or formula, had the design of the columns been satisfactory. Again, it is not stated whether the rivet holes were drilled, or punched, or punched and reamed, while the resistance of the columns would probably be materially affected by those processes. Tests of full sized steel latticed columns are therefore still needed in order to positively fix their resistance.

Such tests as have been made, however, indicate that properly designed and fabricated steel columns, of metal ranging in tensile strength, in specimens, from 65,000 to 73,000 pounds per sq. in., will give a resistance from 25 to 33 per cent. in excess of that of wrought iron columns with the same value of $\frac{l}{r}$, provided that ratio does not exceed 135 to 140. The working stresses for such columns, therefore, can be found by increasing those given for wrought iron 25 per cent. for ordinary railway practice and usual lengths of span, or 33 per cent. for spans of, say, 300 feet and over. These limits represent about the general engineering practice of the present time (1887).

Details of Columns.

In addition to the data already given in another portion of this article, the tests cited in this Addendum show that the unsupported width of no plate in a compression member should exceed 30 to 35 times its thickness. These tests have usually been made with plates or metal $\frac{1}{4}$ to $\frac{1}{3}$ inch in thickness, and it is altogether probable that the above ratio of width over thickness would be increased with greater thicknesses.

In built columns, however, *the transverse distance between centre lines of rivets securing plates to angles or channels, etc., should not exceed* 35 *times the plate thickness.* If this width is exceeded, longitudinal buckling of the plate takes place, and the column ceases to fail as a whole, but yields in detail.

The same tests show that *the thickness of the leg of an angle to which latticing is riveted should not be less than* $\frac{1}{9}$ *of the length of that leg or side*, if the column is purely and wholly a compression member. The above limit may be passed, somewhat, in stiff ties and compression members designed to carry transverse loads.

The panel points of latticing should not be separated by a greater distance than 60 *times the thickness of the angle leg to which the latticing is riveted,* if the column is wholly a compression member.

The rivet pitch should never exceed 16 *times the thickness of the thinnest metal pierced by the rivet,* and if the plates are very thick it should never nearly equal that value.

Art. 52.—Euler's and Tredgold's Forms of Long Column Formulæ.

The *form* of the general formula given in the preceding Article, as will presently be shown, does not seem to be as well adapted to the expression of accurate results as that of Euler, given in Art. 25.

It has already been observed that the coefficient a (Eq. (9) of Art. 50), contains $\frac{p''}{E}$ as a factor, in which p'' is the greatest intensity of bending stress, *i.e.*, a part of the quantity "p" which is sought. The possible use of the formula is based on the fact that E is very large in respect to p''.

The existence of p'' in a is due to the redundant form of Eq. (8) of the Article cited.

Since, in that Article, $p' = \frac{P}{S}$ and $a = a_1 c' = \frac{a'c'p''}{E}$ (see Eq. (7)), Eq. (8) gives:

$$p'' = a\frac{Pl^2}{Sr^2} = \frac{a'c'p''}{E}\frac{Pl^2}{Sr^2};$$

$$\therefore \quad \frac{P}{S} = \frac{E}{a'c'}\frac{r^2}{l^2} = b\frac{r^2}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is Euler's formula as given in Eq. (6) of Art. 25. In this equation b has the analytical values $4\pi^2 E$, $\pi^2 E$ and $2.25\ \pi^2 E$ for ends fixed, rounded and one fixed one rounded, respectively, as shown in Art. 25.

It would seem, therefore, that, since Eq. (1) involves nothing variable in the second member but $r \div l$, it ought to give more accurate results than Tredgold's form of Art. 50.

It was shown, however, in Art. 25 that the common theory of flexure is analytically applicable only to fixed end columns of wrought iron, in which the ratio of length over radius of gyration is somewhat greater than 140; and to round end columns in which that ratio is somewhat greater than 70. Since the implicit assumption of an indefinitely small cross section underlies the analytical treatment of long columns, it is possible that the analytical coefficients and exponent may not obtain far above the limits indicated in Art. 25. Now, since other conditions of ends will lie between these limits, it is seen that both long column formulæ are strictly inapplicable to a large portion of the columns designed by engineers.

Fortunately, a sufficient number of experiments have been made with full sized columns to show that either *form* of formula, when holding empirical quantities properly determined, will give excellent results. This has already been shown for Tredgold's form, and it will now be seen that Euler's form may be expected to give still better results.

If, as is usual, r is the radius of gyration and l the length (both in the same unit), and if both the coefficient and exponent of $\frac{r}{l}$, in Euler's general formula, be considered variable, the following equation (see Art. 25), may be written :

$$\frac{P}{S} = p = y\left(\frac{r}{l}\right)^x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For other values (r' and l') of r and l, the mean intensity becomes:

$$p' = y\left(\frac{r'}{l'}\right)^x . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Dividing Eq. (3) by Eq. (2), then taking logarithms and solving for x:

$$x = \frac{\log\left(\frac{p'}{p}\right)}{\log\left(\frac{lr'}{rl'}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Subtracting Eq. (3) from Eq. (2) and solving for y:

$$y = \frac{p - p'}{\left(\frac{r}{l}\right)^x - \left(\frac{r'}{l'}\right)^x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

These formulæ will first be applied to results of the experiments made on Phœnix columns at Watertown, Mass. These results are contained in Table II. of the preceding Article, and the columns $\frac{\text{“}l\text{”}}{r}$ and "*Exp.*" are reproduced in Table I. of this Article. In the latter, however, the column "*Exp.*" contains the means of the various pairs of experiments whose results are given in the former.

TABLE I.

Phœnix Columns.

$\frac{l}{r}$.	*Exp.*	p.	$\frac{l}{r}$.	*Exp.*	p.
112	34,650	34,550	40.0	36,440	39,000
100	35,150	35,000	28.0	40,700	40,630
88	35.000	35,530	16.0	50,400	43,400
76	36.130	36,150	2.7	57,200	53,400
64	36,580	36,900	68.8	36,000	36,570
52	37,000	37,800	24.0	42,200	41,400

Now, let there be taken:

$$\frac{l}{r} = 28 \ldots\ldots\ldots\ldots p = 40{,}700.$$

30

$$\frac{l'}{r'} = 112 \ldots\ldots\ldots\ldots p' = 34,650.$$

Inserting these values in Eqs. (4) and (5), there will result:

$$x = 0.117 \text{ and } y = 59,723.$$

Then let there be written:

$$p = 60,000 \left(\frac{r}{l}\right)^{0.117} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The various values of $\left(\frac{l}{r}\right)$ in Table I., inserted in Eq. (6), give the results shown in columns "p" of that Table. They are seen to be much more satisfactory, as a whole, than those given by any form of Tredgold's formula in the preceding Articles; although Eq. (2) of Art. 51 is a little closer to the experimental results for values of $\frac{l}{r}$ less than 24.

So much of the curve represented by Eq. (6) as does not coincide with the experimental curve, is shown by the dotted line in the Fig. of the preceding Article.

That curve, together with the results given in Table I., shows the close agreement of Eq. (6) with experiment for all values of $\frac{r}{l}$ from 1 to $\frac{1}{112}$.

It is interesting and important to observe that when $\frac{r}{l} = 1$, Eq. (6) gives:

$$p = 60,000;$$

or about the ultimate compressive resistance of wrought iron in cubes.

An application of Eqs. (4) and (5), in the manner already shown, to the results of Bouscaren's experiments on Keystone columns, given in the large table of Art. 50, gave the following results for swelled Keystone columns:

$$x = 0.25 \quad \text{and} \quad y = 78{,}000\text{; or:}$$

$$p = 78{,}000 \left(\frac{r}{l}\right)^{\frac{1}{4}} \quad \ldots \ldots \quad (7)$$

TABLE II.

Keystone Columns.

SWELLED.			STRAIGHT.		
c.	*Exp.*	*p.*	*c.*	*Exp.*	*p.*
326	33,600	37,800	8,718	25,000	28,000
2,991	28,800	28,700	9,391	27,500	27,700
9,646	24,100	24,800	9,157	30,000	27,800
3,130	36,900	28,500	3,519	30,000	31,350
9,189	21,100	24,900	4,136	32,000	30,700
9,157	25,400	24,900	10,714	27,800	27,300

Also, for straight Keystone columns:

$$x = 0.25 \quad \text{and} \quad y = 87{,}000\text{; or:}$$

$$p = 87{,}000 \left(\frac{r}{l}\right)^{\frac{1}{4}} \quad \ldots \ldots \quad (8)$$

The results of the application of these formulæ, and the experimental results, are given in Table II. The lengths and other data can be found in the table just cited.

By the same operations with the square column results (Bouscaren's) of the same table, there were found :

$$x = 0.5, \quad \text{and} \quad y = 303{,}000; \text{ or:}$$

$$p = 303{,}000 \left(\frac{r}{l}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The following columns, "*Exp.*" and "*p*" contain the experimental square column results and those computed from Eq. (9).

c.	*Exp.*	*p.*	
10,414	30,000	30,000	Square Columns.
7,133	33,200	33,000	
9,623	30,200	30,600	

Only "flat end" experiments have been treated, for the others are utterly insufficient in number for the determination of the empirical quantities.

In fact, with the exception of the Watertown experiments on the Phœnix columns, the number of those with "flat ends" is not sufficiently great, nor the range of $l \div r$ sufficiently extended, to establish reliable formulæ.

In all cases, however, it is to be observed that the formulæ of this Article give results more nearly agreeing with the experimental ones than those computed from any form of Tredgold's or Gordon's formula. It would seem that this form of formula has not heretofore received the attention to which its importance and value entitle it.

Each of the three Eqs. (7), (8) and (9), become inapplicable when the value of $\frac{r}{l}$ is such that "p" approaches the ultimate compressive resistance per square inch of wrought iron in short blocks.

These empirical results tend to give experimental confirmation to Euler's formula, for the exponent and coefficient of $\left(\frac{r}{l}\right)$ are seen to increase very much as the lowest value of c, in the different sets of experiments, increases.

Art. 53.—Hodgkinson's Formulæ.

The detailed account of the experiments on which Eaton Hodgkinson based his various formulæ is given in the Phil. Trans. of the Royal Society of London, for 1840. His cast-iron columns were small ones, the greatest length of which was 60.5 inches. The greatest value of the length divided by the radius of gyration was:

$$\frac{l}{r} = 2 \times \frac{60.5}{0.25} = 484 ;$$

while the least value of the same ratio was:

$$\frac{l}{r} = 4 \times \frac{3.78}{0.5} = 30.2 \text{ (nearly)}.$$

The greatest diameter was about two inches.

Let d = diameter of column in inches.
Let l = length of column in feet.

Then for the breaking weight (P) of solid cylindrical cast-iron columns, when expressed in pounds, Hodgkinson's formulæ take the shape :

$$P = 33{,}379 \frac{d^{3.76}}{l^{1.7}}\text{; (for rounded ends)} \quad . \quad . \quad . \quad (1)$$

$$P = 98{,}922 \frac{d^{3.55}}{l^{1.7}}\text{; (for fixed ends)} \quad . \quad . \quad . \quad . \quad (2)$$

For hollow cylindrical columns of cast iron :

$$P = 29{,}120 \frac{D^{3.76} - d^{3.76}}{l^{1.7}}\text{; (for rounded ends)} \quad . \quad . \quad (3)$$

$$P = 99{,}320 \frac{D^{3.55} - d^{3.55}}{l^{1.7}}\text{; (for fixed ends)} \quad . \quad . \quad . \quad (4)$$

In Eqs. (3) and (4), D is the greater, or exterior, diameter of the column, while d is the interior diameter. It is to be observed that P is the total breaking weight in pounds.

The longest wrought-iron solid cylindrical column tested by Hodgkinson had a length of 90.75 inches and a diameter of about 1.02 inches. Hence the greatest ratio of length over radius of gyration was about $90.75 \times 4 = 363$.

His formulæ for the total breaking weight of solid cylindrical wrought-iron columns, in pounds, are :

$$P = 95{,}848 \frac{d^{3.76}}{l^{2}}\text{; (for rounded ends)} \quad . \quad . \quad . \quad (5)$$

$$P = 299{,}617 \frac{d^{3.55}}{l^{2}}\text{; (for fixed ends)} \quad . \quad . \quad . \quad . \quad (6)$$

In his experiments on square pillars of Dantzic oak, the greatest dimensions were : length = 60.5 inches, and side of square section = 1.75 inches.

His longest red deal pillar was 58 inches in length, and the cross sections were 1×1, 1×2 and 1×3 ; all in inches.

Hodgkinson used Lamandé's experiments on French oak in establishing a formula for that material. In those experiments, the longest pillar had a length of 76.5 inches and a normal section of 2.13 inches by 2.13 inches.

Retaining the same notation, the following are the total breaking weights, in pounds, of solid square timber pillars with flat ends:

$$\text{Dantzic oak (dry)}; \quad P = 24{,}542 \frac{d^4}{l^2} \quad . \quad . \quad . \quad (7)$$

$$\text{Red deal (dry)}; \quad P = 17{,}511 \frac{d^4}{l^2} \quad . \quad . \quad . \quad (8)$$

$$\text{French oak (dry)}; \quad P = 15{,}455 \frac{d^4}{l^2} \quad . \quad . \quad . \quad (9)$$

In Eqs. (7), (8) and (9), "d" is the side of the square section of the column in inches, while l is the length in feet.

All the preceding formulæ are to be used only in those cases in which the length exceeds 30 times the diameter or side of square, if the ends are fixed; or 15 times the length, if the ends are rounded. Between these limits and a short block, in which the length is 4 or 5 times the diameter or less, the following formula is to be used: Let C be the ultimate compressive resistance of the material, per unit of area, in short blocks, and let A be the area of the normal section of the column; then Hodgkinson's formula for these columns of intermediate lengths is:

$$P' = \frac{PCA}{P + \frac{3}{4}CA} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The small size of the columns experimented upon by Hodgkinson militates very strongly against the practical value of his formulæ, unless it should be shown experimentally that the same formulæ may be equally applicable to large and small columns.

With the greatest ratio of l over r, the ratio of the resistance of a fixed end pillar over that of one of the same length and with rounded ends was about 3.34. With the lowest value of l over r, the same ratio was about 1.63. According to Euler's formula, that ratio should have been 4. It is seen, therefore, that with these columns the common theory of flexure failed far above the limit given in Art. 25.

From his experiments Hodgkinson drew the following conclusions:

The strength of a pillar with one end round and the other flat, is the arithmetical mean between that of a pillar of the same dimensions with both ends rounded, and with both ends flat.

A long uniform pillar, with its ends firmly fixed, whether by disks or otherwise, has the same power to resist breaking as a pillar of the same diameter and half the length, with the ends rounded or turned so that the force would pass through the axis.

Long uniform cast-iron pillars with both ends round break in one place only—the middle; those with both ends flat, near each end and at the middle; those with one end round and one end flat, about one-third the length from the round end.

The resistance of solid pillars with round ends was increased about one-seventh by increasing the diameter at the middle. Flat-end pillars (solid) had their resistances increased very slightly by the same means, but hollow pillars seemed to derive no benefit at all by enlargement at the middle.

The resistance of flat-end pillars was increased slightly by the application of disks to their ends.

Irregular and imperfect fixedness of the ends may cause a loss of two-thirds, or more, of the resistance with ends perfectly fixed.

Solid square cast-iron pillars failed in diagonal planes.

The relative resistances of columns of the same length and area of cross section were about as follows:

Long, solid, round pillar	100
" " square pillar	93
" " triangular pillar	110

Art. 54.—Graphical Representation of Results of Long Column Experiments.

If the values of l over r (length over radius of gyration), for

TABLE I.

Tubes.—Flat Ends.

NO.	LENGTH.	EXT. DIA.	THICKNESS.	AREA.	$l \div r$.	ULT. RESIST. PER SQUARE INCH.
	Inches.	Inches.	Inch.	Sq. ins.		Pounds.
1	120	1.5	0.10	0.44	240	14,670
2	120	2.00	0.10	0.61	179	23,206
3	120	2.35	0.23	1.50	160	21,900
4	120	2.50	0.11	0.80	141	29,800
5	120	3.00	0.15	1.35	120	27,670
6	60	1.50	0.10	0.44	120	31,180
7	90	3.04	0.17	1.41	90	29,790
8	60	2.00	0.10	0.61	89	33,300
9	120	4.05	0.16	1.9	87	26,960
10	60	2.35	0.22	1.47	80	29,330
11	60	2.34	0.21	1.37	80	30,000
12	60	2.50	0.11	0.80	71	35,100
13	89	4.00	0.24	2.87	67	26,800
14	90	4.05	0.12	1.61	65	33,330
15	30	1.50	0.10	0.44	60	34,220
16	60	4.00	0.24	2.85	45	32,200
17	30	2.00	0.10	0.61	45	36,980
18	30	2.35	0.24	1.60	40	35,660
19	30	2.34	0.21	1.44	40	36,000
20	29	2.37	0.23	1.55	39	36,910
21	29	2.34	0.20	1.36	39	39,570
22	30	2.50	0.11	0.80	35	36,490
23	28	3.00	0.15	1.41	28	37,390
24	28	4.00	0.25	2.85	21	48,200

a series of columns which have been tested to breaking, be accurately laid off on a horizontal scale, and if the breaking weights per square inch be laid off with equal accuracy on a vertical scale, the resulting curve will represent the resistances of all columns for which l over r lies within the limits of the experiments, with far more accuracy than any simple and practicable formula that can be devised. Such a curve for the Watertown experiments on Phœnix columns has already been incidentally constructed in Art. 51.

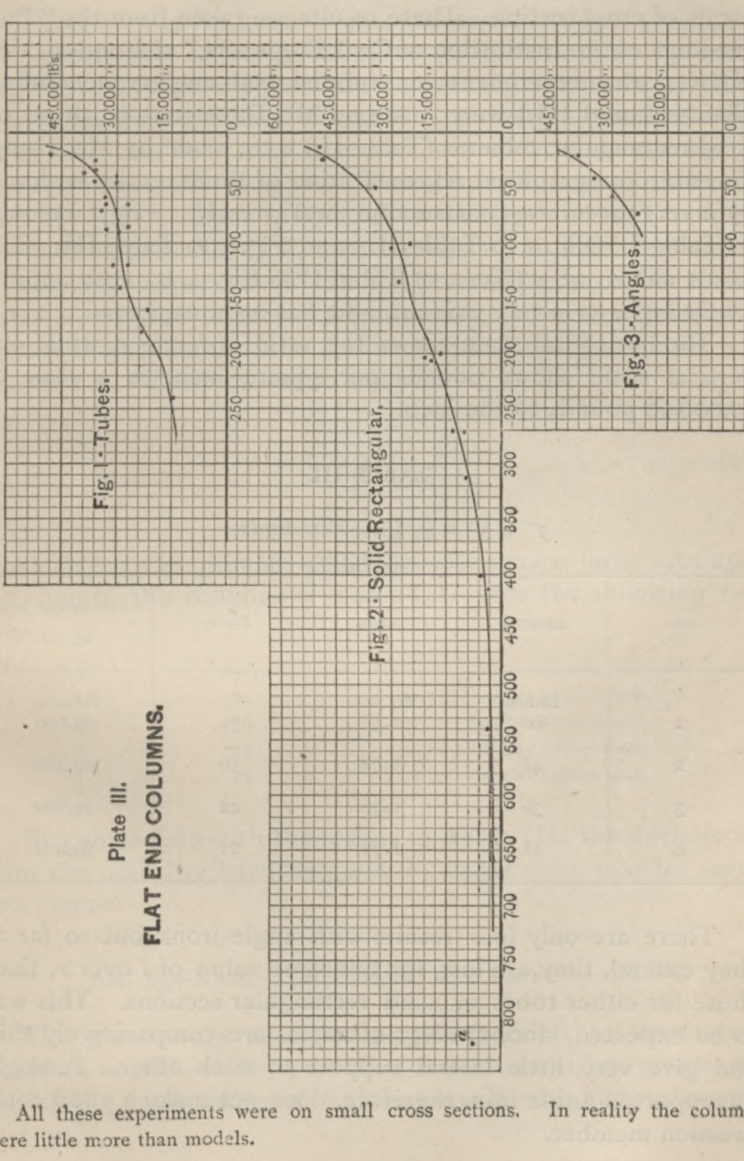
TABLE II.

Solid Rectangular Pillars.—Flat Ends.

NO.	LENGTH.	SECTION.	AREA.	$l \div r$.	ULT. RESIST. PER SQ. IN.
	Inches.	Inches.	Sq. Ins.		Pounds.
1	120	2.98 × 0.5	1.5	822	8,160
2	90	2.98 × 0.5	1.5	643	2,410
3	120	3.01 × 0.77	2.31	540	3,380
4	120	3.00 × 1.00	3.00	414	4,280
5	60	2.98 × 0.5	1.50	400	5,630
6	90	{ 5.86 × 0.99 / 3.00 × 1.0 }	3.00	311	9,600
7	90	1.02 × 1.03	1.05	300	9,750
8	120	3.00 × 1.51	4.53	272	10,170
9	60	3.01 × 0.77	2.31	270	12,970
10	60	3.01 × 0.99	2.99	207	18,070
11	60	5.84 × 1.00	5.84	207	17,700
12	30	2.99 × 0.50	1.50	206	16.850
13	90	3.00 × 1.53	4.59	204	19,990
14	60	1.03 × 1 02	1.05	200	17,270
15	30	3.01 × 0.76	2.30	135	27,770
16	30	3.00 × 1.00	3.00	104	29,660
17	30	1.02 × 1.02	1.04	100	25,330
18	15	1.02 × 1.02	1.04	50	34,550
19	7.5	1.02 × 1.02	1.04	25	48,680
20	3.8	1.02 × 1 02	1.04	13	50,400*

* Bore this without failure.

Tables I., II. and III. contain the results of some English experiments on small flat-end wrought-iron columns of different



All these experiments were on small cross sections. In reality the columns were little more than models.

forms of cross section. These results are taken from the "Proceedings of the Institution of Civil Engineers," of London, Vol. XXX. The experiments on tubular and angle-iron columns (Tables I. and III.) were made by Mr. Davies, while those on solid rectangular columns (Table II.) were made by Mr. Hodgkinson. The graphical representation of these results is shown by a very accurate construction in Plate III. Fig. 1 belongs to Table I.; Fig. 2 to Table II.; and Fig. 3 to Table III. The result shown at *a* (No. 1 of Table II.), Fig. 2, is most anomalously high, as is very evident, and has been neglected

The horizontal scale shows the ratio of l over r, while the vertical scale shows pounds per square inch, to a scale of 30,000.00 pounds to the inch.

TABLE III.

$3'' \times 3'' \times \frac{5}{16}$ L*s.—Flat Ends.*

NO.	LENGTH.	AREA.	$l \div r$.	ULT. RESIST. PER SQ. IN.
	Inches.	Sq. Ins.		Pounds.
1	60	1.78	71	23,600
2	48	1.78	56	29,480
3	36	1.78	42	35,380
4	18	1.78	21	39,400

There are only four results with angle irons, but so far as they extend, they are less, for the same value of l over r, than those for either tubes or solid rectangular sections. This was to be expected, since the legs of angles are comparatively thin and give very little lateral support to each other. A single unsupported angle iron, therefore, does not make a good compression member.

These results, in connection with those of Art. 51, show very clearly that an empirical curve (or formula) may be constructed to cover, with sufficient accuracy for practical purposes, columns of different forms of cross section, ***provided they are so built that their component parts are mutually supporting.***

As compression members of single angle irons with fixed ends are quite common in some riveted bridge and roof trusses, it would be desirable to frame a formula on an extended series of numerous experiments. In the present instance this is impossible, but the following formula may be used with safety for equal legged angle iron columns with flat or fixed ends, so long as $l \div r$ lies between 20 and 100:

$$p = 200{,}000 \sqrt{\frac{r}{l}} \quad \ldots \ldots \quad (1)$$

in which p is the ultimate resistance per square inch. An application to the columns of Table III. gives the following results:

No.	$l \div r$	p.
1	71	23,740 lbs. per square inch.
2	56	26,760 lbs. per square inch.
3	42	30,860 lbs. per square inch.
4	21	43,600 lbs. per square inch.

By comparison with the results in Table III., the deviations from the actual resistances given by experiment may be seen at a glance.

Art. 55.—Limit of Applicability of Euler's Formula.

The great range of $l \div r$ in the experimental results of Tables I. and II. of the preceding Article, furnishes means of testing the applicability of Euler's formula with high values of that ratio.

Mr. Hodgkinson determined the mean value of the compressive coefficient of elasticity for some wrought iron of presumably the same grade as that to which Table II. belongs, at about 23,250,000 pounds per square inch. That value gives:

$$4\pi^2 E = 917{,}920{,}000.$$

Taking $l \div r$ from No. 1, Table I.:

$$\left.\begin{array}{l} p = 4\pi^2 E\left(\dfrac{r}{l}\right)^2 = 16{,}000 \text{ (nearly)} \\ \textit{Experiment gave } 14{,}670 \end{array}\right\} \quad . \quad . \quad . \quad (2)$$

Taking $l \div r$ from No. 7, Table II.:

$$\left.\begin{array}{l} p = 4\pi^2 E\left(\dfrac{r}{l}\right)^2 = 10{,}200 \text{ (nearly)} \\ \textit{Experiment gave } 9{,}750 \end{array}\right\} \quad . \quad . \quad . \quad (3)$$

Taking $l \div r$ from No. 5, Table II.:

$$\left.\begin{array}{l} p = 4\pi^2 E\left(\dfrac{r}{l}\right)^2 = 5{,}740 \text{ (nearly)} \\ \textit{Experiment gave } 5{,}630 \end{array}\right\} \quad . \quad . \quad . \quad (4)$$

Taking $l \div r$ from No. 3, Table II.:

$$\left.\begin{array}{l} p = 4\pi^2 E\left(\dfrac{r}{l}\right)^2 = 3{,}150 \text{ (nearly)} \\ \textit{Experiment gave } 3{,}380 \end{array}\right\} \quad . \quad . \quad . \quad . \quad (5)$$

Taking $l \div r$ from No. 2, Table II.:

$$\left.\begin{array}{l} p = 4\pi^2 E\left(\frac{r}{l}\right)^2 = 2{,}220 \text{ (nearly)} \\ \textit{Experiment gave } 2{,}410 \end{array}\right\} \quad . \; . \; . \; . \quad (6)$$

In Eq. (2), $l \div r$ is 240, yet the result by formula is only a little too large. With $l \div r$ ranging from 300 to 643, the formula gives very satisfactory results. These tests would seem to show, therefore, that only when $l \div r$ becomes equal to about 250 for flat-end columns, does Euler's formula become applicable to wrought-iron compression members, but that above that limit it gives very satisfactory results.

This is an interesting and striking confirmation of the correctness of the formula, which, as was stated in Art. 25, is based on the supposition that the lateral dimensions are very small compared with the length.

Art. 56.—Reduction of Columns at Ends.

When columns are built of angle irons, channel bars, or I beams, it is frequently the practice to cut off, for some distance back from the ends, the flanges of bars or beams, or one of the legs of angle irons, in order to give clearance for other members of the structure. In such cases the whole compression to which the column is subjected is carried, at the ends, by the webs of the bars or beams, or legs of the angles, which are thus solid rectangular columns of great comparative breadth and little thickness, even when reinforced by plates of the same thickness as the webs or legs. In such cases, the angle iron experiments of Mr. Davies (a part of which are given in Art. 54), and a most valuable set of full sized, latticed, channel-bar column tests, made at the works of the Keystone Bridge Co., Pittsburgh, Penn. ("The American Engineer," 4th Feb., 1882), show that the full resistance of the column is not developed, but that they fail at the ends where the cutting away of the flanges and legs reduces the column to two thin, weak, rectangular columns. Columns, therefore, should never be cut away in the manner indicated unless the circumstances render it absolutely necessary, and then the ends should be reinforced by extraordinarily heavy thickening plates, so that the sum of the resistances of these rectangular columns, at each end, shall be equal to that of the column as a whole.

Art. 57.—Timber Columns.

Tests of this class of members, the results of which have been published, although of great value, have not been made with sufficiently large ratios of length to radius of gyration to produce true "long column" failures. This renders impossible the establishment of a long column formula or diagram for practical use in connection with the use of long timber columns.

Some very valuable experiments, however, have been made with full sized columns having lengths as great as fourteen feet. The first results to be given are those of a large number of tests by Prof. Lanza, of Boston, in which he used the United States testing machine at Watertown, Mass. These tests were made during 1881, on such members as are commonly used in the construction of cotton and woollen mills.

Table I. contains the results of Prof. Lanza's tests. A large majority of the columns had cores bored out of the centre, which varied in diameter from 1.5 to 2.0 inches. The absence of material did not affect, in any way, so far as could be observed, the resistance per square inch.

Column 20 had the force applied 2¼ inches out of centre at one end, and column 35, 1.9 inches. These tests were made in order to observe the effect of eccentricity in the application of loads. They show a marked decrease in ultimate resistance.

TABLE I.

Timber Mill Columns.

NO.	FORM.	LENGTH, FEET.	DIA. IN INCHES AT Top.	DIA. IN INCHES AT Mid.	DIA. IN INCHES AT Base.	SECT. AREA IN SQ. INS. AT Top.	SECT. AREA IN SQ. INS. AT Base.	ULT. RESIST., LBS. PER SQ. INCH.	ENDS.
					Yellow pine; partially seasoned.				
1	Round	12.01	9.31	10.65	10.55	65.89	85.22	4,098	Flat
2	"	12.02	8.30	9.71	10.07	51.84	77.36	3,665	"
3	"	12.01	7.54	8.88	8.99	42.38	61.21	4,719	"
4	"	12.02	6.40	7.80	7.79	29.98	45.47	4,602	"
5	Cylindrical..	12.00	10.45	——	10.45	83.75	——	4,657	"
6	" ..	11.92	8.96	——	8.96	61.19	——	4,086	"
7	" ..	11.98	7.70	——	7.70	44.72	——	4,584	"
8	" .	2.01	10.46	——	10.46	85.92	——	4,422	"
9	" ..	2.00	9.98	——	9.98	78.23	——	4,705	"
10	" ..	2.01	8.91	——	8.91	62.35	——	4,330	"
11	" ..	2.00	7.79	——	7.79	47.66	——	4,511	"
12	Square......	12.44	8.43	——	8.40	68.80	——	5,451	"
13	"	12.57	8.30	——	8.30	63.10	——	3,804	"
14	Cylindrical..	11.93	9.92	——	9.92	75.45	——	3,512	"
					Yellow pine; air seasoned.				
15	Round	13.99	7.70	——	7.90	44.56	47.01	4,488	"
16	Cylindrical..	2.00	7.70	——	7.90	44.56	——	4,892	"
					Yellow pine: dock seasoned.				
17	Cylindrical..	11.92	8.00	——	8.00	48.26	——	4,662	One flat; one round.
18	" ..	1.98	7.93	——	7.98	48.00	——	3,604	Flat.
19	Square......	11.93	8.08	——	8.08	63.28	——	3,477	One flat; one round.
20	"	12.94	8.75	——	8.92	76.04	——	3,682	One flat; one round.
21	"	12.85	10.05	——	10.13	99.79	——	5,111	Flat, pintle.
22	"	2.00	3.98	——	9.02	81.00	——	5,951	Flat.
23	"	2.01	10.20	——	10.07	102.71	——	5,452	"
					White wood; partially seasoned.				
24	Round	12.01	8.46	9.61	9.65	54.02	70.95	3,333	"
25	"	11.97	6.38	7.40	7.72	29.78	44.62	2,687	"
					White oak: partially seasoned.				
26	Round	12.01	9.13	10.15	11.01	63.28	39.01	3,003	"
27	"	12.01	8.37	9.40	10.23	52.83	80.00	3,786	"
28	"	12.01	7.55	8.75	9.05	42.58	62.14	3,758	"
29	"	12.01	6.60	7.79	8.06	32.02	48.83	3,435	"
30	Cylindrical..	12.00	10.00	——	——	76.70	——	2,478	"
31	" ..	6.33	10.00	——	——	76.70	——	2,738	"
32	" ..	2.00	9.98	——	——	78.23	——	3,132	"
33	" ..	2.00	8.18	——	——	52.55	——	3,140	"
34	" ..	1.98	7.73	——	——	46.93	——	3,303	"
35	" ..	11.93	8.20	——	——	50.92	——	1,964	"

TABLE I.—*Continued.*

NO.	FORM.	LENGTH, FEET.	DIA. IN INCHES AT			SECT. AREA IN SQ. INS. AT		ULT. RESIST., LBS. PER SQ. INCH.	ENDS.
			Top.	Mid.	Base.	Top.	Base.		
			White oak; used in mill 6½ years.						
36	Round	12.08	5.85	—	6.84	23.89	33.76	4,604	Flat.
37	"	12.08	5.85	—	6.85	24.05	34.02	6,029	"
38	"	12.11	5.87	—	6.70	23.92	32.12	4,680	"
39	"	12.05	6.02	—	6.75	25.47	32.79	2,945	"
40	"	11.72	6.10	—	6.83	26.08	33.50	3,451	"
41	"	12.01	5.97	—	6.74	25.09	32.78	4,225	"
42	"	12.07	5.75	—	6.88	22.98	34.19	3,264	"
			White oak; used in mill 25 years.						
43	Cylindrical..	13.87	10.56	—	—	84.74	—	4,602	"
44	" ..	14.00	10.54	—	—	84.83	—	4.951	"
45	" ..	13.89	10.54	—	—	84.40	—	4,266	"
46	" ..	13.80	10.50	—	—	83.75	—	3,881	"
47	" ..	13.92	10.20	—	—	79.17	—	4,674	Flat, pintle.
48	" ..	13.89	10.80	—	—	82.68	—	4,838	" "
49	Round	13.67	9.25	—	9.50	64.36	68.04	3,434	Flat.
50	Cylindrical..	13.85	9.55	—	—	68.65	—	4,618	"
51	" ..	13.65	9.40	—	—	66.56	—	3,981	"
52	" ..	13.90	9.35	—	—	65.82	—	3,266	Pintle ends.
53	Round	11.51	5.98	—	7.20	26.03	38.66	6,147	Flat.
			White oak; thoroughly seasoned; 1 year old.						
54	Cylindrical..	12.00	7.74	—	—	45.04	—	3,219	Flat.
55	" ..	11.12	10.95	—	—	91.16	—	1,865	One flat; one round.
56	" ..	2.00	10.91	—	—	95.48	—	4,450	Flat.

Although the ends of Nos. 39 and 44–51 were flat, they were not parallel.

All the columns indicated by "Round" were tapered, and they almost invariably gave way by the crushing of the fibres at the small end. In all such columns the ultimate resistance is *per square inch of the small end.*

Some of the square columns had their corners slightly beveled.

In his report to the Boston Manufacturers' Mutual Fire Ins. Co., Prof. Lanza says: "The immediate location of the fracture was generally determined by knots;" . . . but states that, whether knotty or straight grained, failure took place in the tapered columns at the small ends. Tapering a column, therefore, to the extent shown in these cases, is a source of weakness.

TABLE II.

Yellow Pine.

NO.	LENGTH, INCHES.	FORM OF SECTION.	SECTION DIMENSIONS, INCHES.	ULTIMATE RESISTANCE PER SQ. IN.	
				Lbs.	
1	20.4	Circular.	10.2 Diam.	6,676	
2	119.95	Square.	11 × 11	6,230	
3	119.90	"	11 × 11	6,552	
4	20.0	"	10.4 × 10.4	7,936	
5	16.0	"	8 × 8	8,165	
6	8.0	"	4 × 4	7,394	
7	3.0	"	1.5 × 1.5	5,533	
8	6.0	"	3 × 3	8,644	Straight grained and seasoned 20 years.
9	6.0	"	3 × 3	8,133	
10	3.0	"	1.5 × 1.5	8,389	
11	3.0	"	1.5 × 1.5	8,302	
12	3.0	"	1.5 × 1.5	6,355	
13	14.0	"	4.6 × 4.6	9,947	
14	17.2	"	4.6 × 4.6	10,250	
15	19.1	"	5.3 × 5.3	7,820	
16	180.0	Rectangular.	16 × 13.65	3,070	
17	180.0	"	16.2 × 7.0	2,795	
18	180.0	"	17 × 8.75	3,180	

Nos. 13, 14 and 15 were pine of very slow growth.
Nos. 16, 17 and 18 were very green and wet.

Tables II. and III. contain the results of Col. Laidley's tests, some of which belong to short blocks. These tests were made during 1881, and a detailed account of them is given in "Ex. Doc. No. 12, 47th Congress, 1st Session."

These experiments give some very important deductions.

In the first place, within the limits of the ratio of length to diameter, or shortest side of rectangular section, appearing in these tests, the ultimate resistance is essentially independent

TABLE III.

Spruce, thoroughly seasoned.

NO.	LENGTH, INCHES.	FORM OF SECTION.	SECTION DIMENSIONS, INCHES.	ULTIMATE RESISTANCE PER SQ. IN.
				Lbs.
1	24	Rectangular.	5.4 × 5.4	4,946
2	24	"	5.4 × 5.4	4,811
3	36	"	5.4 × 5.4	4,874
4	36	"	5.4 × 5.4	4,500
5	60	"	5.4 × 6.4	4,451
6	60	"	5.4 × 6.4	4,943
7	120	"	5.4 × 5.4	3,967
8	120	"	5.4 × 5.4	4,908
9	60	"	5.4 × 5.4	5,275
10	30	"	5.4 × 5.4	5,372
11	15	"	5.4 × 5.4	5,754
12	121.2	Circular.	12.4 Diam.	4,681

of the length. This is the result of the action of causes noticed in the consideration of wrought-iron columns composed of ['s. The ultimate resistance of any such column, therefore, is to be obtained by multiplying the area of its cross section by the ultimate resistance, per square inch, of short blocks.

In Prof. Lanza's experiments, the greatest ratio of length to radius of gyration was about 86. Below this value the general conclusion just given may be expected to hold, but probably not much above it.

In Col. Laidley's tests the greatest value of the same ratio was about 90 (No. 17 of Table II.), at which there seemed to be a little decrease in ultimate resistance.

Again, it is to be observed that Prof. Lanza's results are much less than those of Col. Laidley for the same timber. The columns of the former were of ordinary merchant material, with the usual accompaniment of knots, weak spots, crooked grain, etc., while the latter experimented with fine, straight-grained timber.

The slow growth specimens (13, 14 and 15, of Table II.), gave much the highest results, while the wet and unseasoned ones (16, 17 and 18) gave the lowest of all.

Hence, the ultimate resistance of timber columns will depend upon quality and condition of material, mode of growth, degree of seasoning, etc., etc.

Table II. also shows, what has been observed elsewhere, that smaller specimens give higher results than larger ones.

Formula of C. Shaler Smith, C. E.

During the winter of 1861–62, Mr. C. Shaler Smith conducted a series of over 1,200 tests of full size yellow pine square and rectangular columns for the Ordnance Dept. of the Confederate Government. The results of these tests have never been published, but Mr. Smith has kindly furnished the writer with the following summary:

The tests were grouped as follows:

"1st. Green, half-seasoned sticks answering to the specification, 'good merchantable lumber.'

"2d. Selected sticks reasonably straight, and air-seasoned under cover for two years and over.

"3d. Average sticks cut from lumber which had been in open air service for four years and over."

If l = length of column in inches;

d = least side of column section in inches;

and p = Ult. Comp. resistance in lbs. per sq. in.;

then the formulæ found for these three groups were:

$$\text{For No. 1:} \quad p = \frac{5{,}400}{1 + \frac{1}{250}\frac{l^2}{d^2}}.$$

For No. 2: $$p = \frac{8{,}200}{1 + \frac{1}{300}\frac{l^2}{d^2}}.$$

For No. 3: $$p = \frac{5{,}000}{1 + \frac{1}{250}\frac{l^2}{d^2}}.$$

But in order to provide against ordinary deterioration while in use, as well as the devices of unscrupulous builders, Mr. Smith recommends the formula for group No. 3 as the proper one for general application. He also recommends that the factor of safety shall be $\sqrt{\frac{l}{d}}$ until 25 diameters are reached, and *five* thenceforward up to 60 diameters. This last limit he regards as the extreme for good practice.

Mr. Trautwine computed his tables from tests of group No. 3.

Addendum to Article 57.

Tables IV. and V. have been formed by digesting the results of tests of timber columns made at Watertown, Mass., and found in " Ex. Doc. No. 1, 47th Congress, 2d Session."

Each result in both tables is usually a mean of from two to four tests, although a few belong to one test only. All timber, both of yellow and white pine, was ordinary merchantable material, with about the usual defects, knots, etc., and failure frequently took place at the latter; it was all well seasoned, and all columns were tested with flat ends.

Flat end yellow pine columns were observed to begin to fail with deflection at a length of about 22 d, d being the width or least dimension of the normal cross section. All columns were of rectangular section, and l in the following table is the length.

TABLE IV.

Yellow Pine Columns with Flat Ends.

Length.		Size of Stick. Inches.	$\frac{l}{d}$	Ult. Comp. Resistance. Pounds per Sq. In.	Length.		Size of Stick. Inches.	$\frac{l}{d}$	Ult. Comp. Resistance. Pounds per Sq. In.
Ft.	*Ins.*				*Ft.*	*Ins.*			
15	0	8.25 × 16.25	21.7	3,445	15	0	5.0 × 12	35.6	3,764
10	0	5.5 × 5.5	22	4,738	23	4	7.7 × 9.7	36.4	3,304
16	8	7.7 × 9.7	26.7	4,384	17	6	5.5 × 5.5	38.2	3,242
15	0	6.6 × 15.6	27.0	3,593	15	0	4.5 × 11.6	41	2,462
12	6	5.5 × 5.5	27.3	5,077	26	8	7.4 × 9.4	43	2,893
15	0	5.9 × 12.0	30.8	3,546	15	0	4.0 × 11.4	44	3,065
20	0	7.6 × 9.6	31.2	3,496	20	0	5.4 × 5.4	44.3	2,867
15	0	5.7 × 11.7	31.9	3,106	22	6	5.5 × 5.5	50	2,065
15	0	5.6 × 15.6	32.1	3,656	25	0	5.5 × 5.5	55	1,856
15	0	5.5 × 5.5	32.8	3,962	27	6	5.3 × 5.3	62.3	1,709

Table IV., therefore, includes no short column, *i. e.*, one which failed by compression alone with no deflection.

About sixteen of the latter were tested with the following results:

Short yellow pine columns; $l \div d$ below 22. { maximum = 5,677 lbs. per sq. in.; mean = 4,442 " " " "; minimum = 3,430 " " " "

Each of the preceding tests was made on a single rectangular stick. A number of tests, however, were made on compound columns formed by bolting together from two to three rectangular sticks, with bolts and packing or separating blocks at the two ends and at the centre. The bolts were parallel to the smaller sectional dimensions of the component sticks. As was to be expected, those compound columns possessed essentially the same ultimate resistance per square inch as each component stick considered as a column by itself, as the following results show. l is the length of the column and d the smallest dimension or width of one member of the composite column. All had flat ends.

$l \div d$.	*Number of Tests.*		
32.1	18	maximum	= 4,559 lbs. per sq. in.
		mean	= 3,841 " " " "
		minimum	= 2,756 " " " "
36	18	maximum	= 3,357 " " " "
		mean	= 3,122 " " " "
		minimum	= 2,942 " " " "

Table V. gives the results for white pine columns, and corresponds with Table IV. in that it shows only the failures with deflection, which was observed to begin with those columns at a length of 32 d. l and d possess the same signification as in Table IV., the column $l \div d$, showing the ratios between the lengths and least widths.

Thirty columns with lengths less than 32 d were tested to destruction. These sticks failed generally at knots by direct compression and without deflection. The results of these thirty tests were as follows:

Short white pine columns ; $l \div d$ below 32.	maximum	= 3.700 lbs. per sq. in.
	mean	= 2.414 " " " "
	minimum	= 1,687 " " " "

All the preceding white pine columns were single sticks, but a large number of built posts composed of two to four white pine sticks bolted together, with spacing blocks at the two ends and at the centre, were also tested with the results shown below. $l \div d$ is the ratio of length over least width of a single stick of the set forming the composite column.

$l \div d$.	*Number of Tests.*		
32.1	15	maximum	= 2,273 lbs. per sq. in.
		mean	= 1,980 " " " "
		minimum	= 1,661 " " " "
36	9	maximum	= 2,255 " " " "
		mean	= 1,999 " " " "
		minimum	= 1,804 " " " "
40	6	maximum	= 2,021 " " " "
		mean	= 1,830 " " " "
		minimum	= 1,419 " " " "

A comparison of these results with those given in Table V. shows that these composite or built columns were the same in

TABLE V.

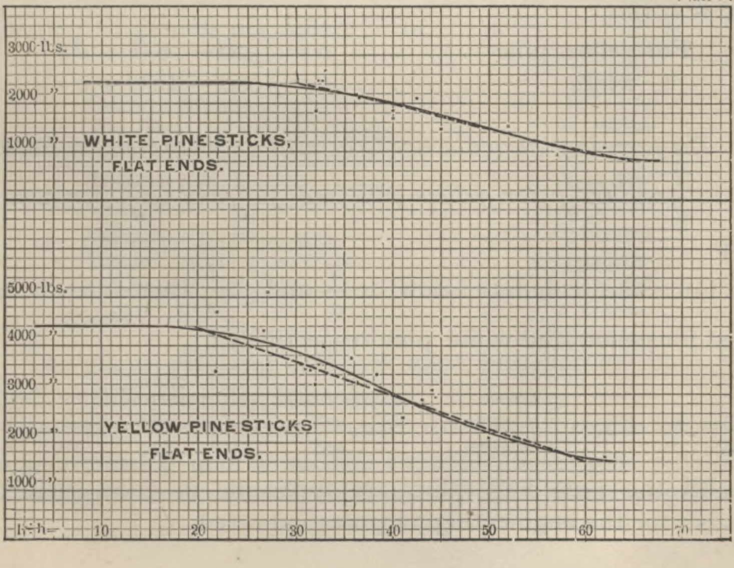
White Pine Columns with Flat Ends.

Length.	Size of Stick. Inches.	$\frac{l}{d}$	Ult. Comp. Resistance. Pounds per Sq. In.	Length.	Size of Stick. Inches.	$\frac{l}{d}$	Ult. Comp. Resistance. Pounds per Sq. In.
Ft. Ins.				*Ft. Ins.*			
15 0	5.6 × 15.6	32	1,874	17 6	5.4 × 5.4	40	1,841
20 3	7.4 × 9.3	32.4	2,448	26 8	7.5 × 9.3	42.7	2,113
15 0	5.6 × 11.5	32 7	2,432	20 0	5.3 × 5.3	45	1,455
15 3	5.4 × 5.4	33	2.744	22 6	5.2 × 5.2	52	1,501
23 4	7.7 × 9.6	36.4	2,072	25 0	5 3 × 5.3	57	952
15 0	4.5 × 11.6	40	1,672	27 6	5.4 × 5.4	62	1,081

strength per square inch with the single sticks of which they were composed, the latter being considered single columns.

All the white pine composite columns were tested with flat

Plate F.



ends and were built up with the greatest widths of individual sticks adjacent to each other.

The results in Tables IV. and V. are shown graphically in Plate F. One ordinate gives the values of $l \div d$, and the other the ultimate resistance in pounds per sq. in.

The full curved lines running into horizontal tangents at the left represent about mean lines through the points indicating the actual column tests.

The broken lines represent the following empirical formulæ; in which p is either the ultimate resistance or working stress in pounds per sq. in.

For yellow pine . . . . $p = 5800 - 70\,(l \div d)$
" white " . . . . $p = 3800 - 47\,(l \div d)$

For wooden railway structures there may be used:
For yellow pine . . . . . $p = 750 - 9\,(l \div d)$
" white " . . . . . $p = 500 - 6\,(l \div d)$

For temporary structures, such as bridge false works *carrying no traffic*:

For yellow pine . . . . $p = 1500 - 18\,(l \div d)$
" white " . . . . $p = 1000 - 12\,(l \div d)$

The preceding formulæ are to be used only between the limits of $20\frac{l}{d}$ *and* $60\frac{l}{d}$ *for yellow pine and* $30\frac{l}{d}$ *and* $60\frac{l}{d}$ *for white pine.*

For short columns below $20\frac{l}{d}$ and $30\frac{l}{d}$ there are to be used for yellow and white pine respectively:

	Ultimate.	*Railway Bridges.*	*Temporary Structures.*
Yellow pine	$p = 4400$	550	1100 lbs. per sq. in.
White "	$p = 2400$	300	600 " " " "

All the preceding values are applicable to good average lumber for the engineering purposes indicated.

CHAPTER VIII.

SHEARING AND TORSION.

Art. 58.—Coefficient of Elasticity.

It has already been shown in some of the Articles of the first portion of this book, on shearing and torsion, that the coefficients of elasticity for those two stresses are the same; and, indeed, that those two stresses are identical in character. The coefficients of elasticity, given in this Article, are then derived chiefly from experiments in torsion.

In his "Leçons de Mécanique Pratique," 1853, Gen. Arthur Morin gives the following *résumé* of the results of experiments up to that time, in which G is the coefficient of elasticity, for shearing, in pounds per square inch.

MATERIAL.	G., lbs.
Soft wrought iron	8,571,000
Iron bars	9,523,000
German steel	8,571,000
Fine cast steel	14,300,000
Cast iron	2,857,000
Copper	6,237,000
Bronze	1,523,000
Oak	571,400
Pine	618,600

The above value for cast iron must, however, be much too small, as will presently be seen.

In "Der Civilingenieur," Heft 2, 1881, the results of some very interesting and important experiments on cast-iron rods or prisms of various cross sections, by Prof. Bauschinger, are given in full detail. The rods or prisms were about 40 inches long, and were subjected to torsion, while the twisting of two sections about 20 inches apart, in reference to each other, was carefully observed. The results for four different cross sections will be given—*i. e.*, circular, square, elliptical (the greater axis was twice the less) and rectangular (the greater side was twice the less). In each case the area of cross section was about 7.75 square inches. The angle α is the angle of torsion—*i. e.*, the angle twisted or turned through by a longitudinal fibre, whose length is unity, and which is at unit's distance from the axis of the bar.

SECTION.	α.	G.
Circular	0.007 degree	7,466,000 lbs. per sq. in.
	0.07 "	6,157,000 " " " "
Elliptical	0.009 "	7,437,000 " " " "
	0.076 "	6,228,000 " " " "
Square	0.008 "	7,039,000 " " " "
	0.073 "	5,987,000 " " " "
Rectangular	0.01 "	6,996,000 " " " "
	0.08 "	5,716,000 " " " "

The formula by which G is computed, when the torsional moment and angle α are given, is the following:

$$G = \frac{M}{\alpha} \cdot c\,\frac{I_p}{A^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which M is the twisting moment; A the area of the cross section; I_p the polar moment of inertia of that cross section; and c a coefficient which has the following values:

$4\pi^2 = 39.48$ for circle and ellipse;
42.70 " square;
42.00 " rectangle;

as was shown in Art. 10.

Bauschinger's experiments show that the coefficient of shearing elasticity for cast iron may be taken from 6,000,000 to 7,000,000 pounds per square inch; also, that it varies for different ratios between stress and strain.

It has been shown in Art. 4, that if E is the coefficient of elasticity for direct stress, and r the ratio between direct and lateral strains, for tension and compression, that G may have the following value:

$$G = \frac{E}{2(1 + r)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Prof. Bauschinger, in the experiments just mentioned, measured the direct strain for a length of about 4.00 inches, and the accompanying lateral strain along the greater axis of the elliptical and rectangular cross sections, and thus determined the ratio r between the direct and lateral strains per unit, in each direction. The following were the results:

Compression.

SECTION.	r.	G.
Circular	0.22	6,541,000 lbs. per sq. in.
Elliptical	0.23	6,541,000 " " " "
Square	0.24	6,442,000 " " " "
Rectangular	0.24	6,499,000 " " " "

Tension.

Circular	0.23	6,570,000 lbs. per sq. in.
Elliptical	0.21	6,811,000 " " " "
Square	0.26	6,399,000 " " " "
Rectangular	0.22	6,527,000 " " " "

The values of E are not reproduced, but they can be calculated indirectly from Eq. (2) if desired.

It is seen that the values of G, as determined by the different methods, agree in a very satisfactory manner, and thus furnish experimental confirmation of the fundamental equations of the mathematical theory of elasticity in solid bodies.

The fact that G is essentially the same for all sections is also strongly confirmatory of the theory of torsion, in particular.

These experiments show that, for cast iron, the lateral strains are a little less than one quarter of the direct strains. If r were one quarter, then $G = \frac{2}{5} E$; or $E = \frac{5}{2} G$.

In the "Journal of the Franklin Institute," for 1873, Prof. Thurston gives the following values of G, as determined from experiments with his torsion machine.

White Pine	$G =$	220,000 pounds per sq. in.
Yellow Pine, sap	$G =$	495,000 " " " "
Yellow Pine, heart	$G =$	495,000 " " " "
Spruce	$G =$	211,000 " " " "
Ash	$G =$	410,000 " " " "
Black Walnut	$G =$	582,000 " " " "
Red Cedar	$G =$	890,000 " " " "
Spanish Mahogany	$G =$	660,000 " " " "
Oak	$G =$	570,000 " " " "
Hickory	$G =$	910,000 " " " "
Locust	$G =$	1,225,000 " " " "
Chestnut	$G =$	355,000 " " " "

The specimens were small ones, and the timber was seasoned.

Art. 59.—Ultimate Resistance.

Before considering the ultimate shearing resistance of special materials it will be well to notice the two different methods in which a piece may be ruptured by shearing.

If the dimensions of the piece in which the shearing force or stress acts are very small, *i.e.*, if the piece is very thin, the case is said to be that of "simultaneous" shearing. If the piece is thick, so that those portions near the jaws of the shear begin to be separated before those at some distance from it, the case is said to be that of "shearing in detail." In the latter case failure extends gradually, and in the former takes place simultaneously over the surface of separation. Other things being the same, the latter case (shearing in detail), will give the least ultimate shearing resistance per unit of the whole surface.

In reality no plate used by the engineer is so thin that the shearing is absolutely simultaneous, though in many cases it may be essentially so.

Wrought Iron.

The following averages (each result being an average of six tests), are from Chief Engineer Shock's experiments, in 1868, on ordinary commercial "rounds" ("Steam Boilers," by William H. Shock, Chief Engineer, U. S. N.), in which S is the ultimate shearing resistance in pounds per square inch :

	S.	
DIAM. OF ROUND.	SINGLE SHEAR.	DOUBLE SHEAR.
0.5 inch	44,150 lbs.	41,090 lbs.
0.625 inch	39,250 lbs.	38,670 lbs.
0.75 inch	39,550 lbs.	39,770 lbs.
0.875 inch	41,500 lbs.	37,890 lbs.
1.00 inch	40,700 lbs.	37,650 lbs.

Although these figures show some irregularities, the general result is unmistakable, and shows a decrease of S with an increase of diameter.

The results of experiments at Bristol, England, by Mr. Jones ("Proc. Inst. Mech. Engrs.," 1858), on punching plate iron, are as follows :

THICKNESS OF PLATE.	DIAM. OF HOLE.	*S.*
0.437 inch	0.250	54,700 lbs. per sq. in.
0.625 "	0.500	60,900 " " " "
0.625 "	0.750	52,900 " " " "
0.875 "	0.875	51,700 " " " "
1.000 "	1.000	55,100 " " " "

Mr. C. Little found the following for English "hammered scrap bars and rolled iron," *with parallel cutters or shears:*

AREA CUT.	DIRECTION.	*S.*
0.50 × 3.00 ins	Flat	49,950 lbs. per sq. in.
0.50 × 3.00 ins	Edge	51,750 " " " "
1.00 × 3.00 ins	Flat	51,750 " " " "
1.00 × 3.00 ins	Edge	50,850 " " " "
1.00 × 3.02 ins	Flat	44,350 " " " "
1.00 × 3.02 ins	Edge	46,150 " " " "
1.80 × 5.00 ins	Edge	46,150 " " " "

In these experiments the edges of the shears were always parallel to each other, thus tending to produce simultaneous shearing. In ordinary workshop practice, however, the jaws of the shears make a constant angle with each other, thus shearing successive portions of the material as the jaws approach, whatever may be the dimensions of the piece, and consequently always producing shearing in detail. In the experiments (by the same authority, *i. e.*, Mr. C. Little, "Proc. Inst. Mech. Engrs.," 1858) from which the following results were deduced, the angle between the jaws of the shears was an inclination of 1 in 8:

BARS.	FLATWAYS.	EDGEWAYS.
3 × 1.5 ins	$S = 40{,}800$	45,000 lbs. per sq. in.
4.5 × 1.375 ins	$S = 32{,}000$	40,100 " " " "
3.0 × 1.00 ins	$S = 35{,}200$	47,300 " " " "
5.25 × 1.75 ins	$S = 37{,}400$	50,600 " " " "
6.00 × 1.50 ins	$S = 33{,}600$	41,200 " " " "

As was to be expected, the "*Edgeways*" results are much the largest, as with that position of the bar the shearing approached more nearly the simultaneous condition. These results show that it is much more economical to shear a bar flatways than edgeways.

Mr. Edwin Clark ("On the Tubular Bridges") found the resistance of ⅞-inch rivet iron, in single and double shear, to vary from 49,500 to 54,100 pounds per square inch. The tensile resistance of the same iron was about 53,800 pounds per square inch.

Reviewing all these results, the ultimate shearing resistance of wrought iron may safely be taken at 0.8 of its tensile resistance, as stated by Mr. D. K. Clark.

Cast Iron.

Very few experiments on the resistance of cast iron to shearing have been made, as this metal is seldom or never used to resist such a stress.

Mr. Bindon B. Stoney ("Theory of Strains in Girders and Similar Structures," p. 357 of 2d Edit.) has found, by experiment, that the ultimate shearing resistance of the cast iron with which he experimented varied from about 17,900 to 20,200 pounds per square inch. He concluded that the shearing and tensile resistances might be taken the same.

Steel.

The results of Prof. Ricketts' shearing tests on both open hearth and Bessemer steel rounds with different grades of carbon are given in Table II. of Art. 34. The elastic limit is the point at which the metal first fails to sustain the scale beam. The double shear resistance in one case exceeds the single by over six per cent. According to these tests, the ultimate shearing resistance of mild steel may be taken at three-quarters of its ultimate tensile resistance. Each shear result is a mean of three tests.

Mr. Kirkaldy investigated the ultimate shearing resistance of four grades of Fagersta steel, and the following results are taken from " Experimental Enquiry into the Mechanical Properties of Fagersta Steel," by David Kirkaldy, 1873. The test-piece, in each case, was turned from a 2-inch square bar, to a diameter of 1.128 inches, and each result is a mean of three experiments. S is the ultimate resistance to shearing, in pounds per square inch; r is the ratio of ultimate shearing over ultimate tensile resistance of the same steel; while "d" is the detrusion or relative movement of one part of the specimen in respect to the other at the instant of separation over the entire surface.

MARK.	S.	r.	d.
1.2	61,400.00 lbs.	0.73	0.19 inch.
0.9	79,740.00 "	0.75	0.25 "
0.6	71,650.00 "	0.70	0.28 "
0.3	45,410.00 "	0.74	0.32 "

As is evident, the lower "*Mark*" numbers belong to the softer steels.

In each case two surfaces were sheared, as the "round" was a pin for three links, two of which pulled one way, and one the other.

All of Mr. Kirkaldy's experiments seem to show that the ultimate shearing resistance of steel is about three-quarters the tensile.

Table I. contains the results of the experiments of Prof. A. B. W. Kennedy, as given in "Engineering" for May 6, 1881. The tensile resistance of the same steel was given in the chapter on "Tension."

The specimens were round and of mild rivet steel. The ratio of the ultimate resistance to shearing over that to tension varied from 0.80 to 0.89.

In the "Journal of the Franklin Institute," for March, 1881, Charles B. Dudley, Ph.D., gives the results of 192 tests of rail steel, the specimens, 0.625 inch round, having been taken from rails which had been subjected to service for considerable periods of time on the Penn. R. R. The tests were made by

TABLE I.

Rivet Steel.

DIAMETER IN INCHES.	ULTIMATE RESIST. IN LBS. PER SQ. IN.	MEAN.	RATIO OF ULT. SHEAR OVER ULT. TENSION.
1.00	54,110		
1.00	54,930		
1.00	55,240		
1.00	52,830	54,550	0.89
1.00	56,660		
1.00	53,530		
0.62	60,260		
0.62	59,400		
0.62	59,600	59,640	0.87
0.62	59,220		
0.62	59,740		
0.62	53,670		
0.62	51,290		
0.62	52,670	52,450	0.80
0.62	53,000		
0.62	51,620		

Mr. J. W. Cloud, engineer of tests for the Penn. R. R. Co. The following is a summary of the results:

$$S = \begin{cases} 63{,}560.00 \text{ pounds per sq. in. } (\textit{greatest}). \\ 59{,}880.00 \text{ pounds per sq. in. } (\textit{mean}). \\ 53{,}380.00 \text{ pounds per sq. in. } (\textit{least}). \end{cases}$$

The percentages of carbon and ultimate tensile resistances are given in Table IV. of Art. 34. By reference to that table it will be observed that S is not far from three-fourths the tensile resistance.

Copper.

From some English experiments, Mr. Bindon B. Stoney concluded that the ultimate shearing resistance of copper was about two-thirds of that of wrought iron.

Timber.

In treating the shearing resistance of timber, it is very necessary to consider whether the shearing takes place *along* the fibres, or in a direction *normal* to them.

TABLE I.

Along Fibres.

KIND OF WOOD.	S. GREATEST.	S. MEAN.	S. LEAST.
Georgia Pine	934	843	713
White Pine	530	482	433
Locust	1,389	1,165	970
White Oak	1,474	1,250	1,076
Spruce	647	542	463
Hemlock	410	369	322

TABLE II.

Across Fibres.

KIND OF WOOD.	S.	KIND OF WOOD.	S.
Ash	6,280	Locust	7,176
Beech	5,223	Maple	6,355
Birch	5,595	Oak, white	4,425
Cedar, white	1,372 to 1,519	Oak, live	8,480
Cedar, Central Amer.	3,410	Pine, white	2,480
Cherry	2,945	Pine, yellow, northern	4,340
Chestnut	1,535	Pine, yellow, southern	5,735
Dogwood	6,510	Pine, yel., very resinous	5,053
Ebony	7,750	Poplar	4,418
Gum	5,890	Spruce	3,255
Hemlock	2,750	Walnut, black	4,728
Hickory	6,045 to 7,285	Walnut, common	2,830

Table I. contains the results of experiments on the shearing of small specimens *along* the fibres, by the late Mr. R. G. Hatfield ("Transverse Strains," 1877). S is the ultimate shearing resistance in pounds per square inch. There were about nine experiments for each kind of timber.

Table II. contains the results of experiments by Mr. John C. Trautwine on round specimens 0.625 inch in diameter, and *across* the fibres ("Journal of the Franklin Institute," Feb.

TABLE III.

Along Fibres.

NO.	KIND OF MATERIAL.	SHEARING AREA IN SQUARE INCHES.	ULT. SHEAR IN POUNDS PER SQ. INCH.	DIRECTIONS TO RINGS OF GROWTH.
1	Oregon pine	5.0 and 14.0	442 and 1,096	Perpendicular (2 exps.)
2	Oregon pine	10.7	820	Oblique.
3	Orégon maple	14.4	436	Perpendicular.
4	Oregon maple	10.9	1,028	Oblique.
5	California laurel	11.0 and 14.2	549 and 1,204	Oblique (2 exps.)
6	Ava Mexicana	14.8	346	Perpendicular.
7	Ava Mexicana	11.0	700	Parallel.
8	Oregon ash	14.6	443	Parallel.
9	Oregon ash	8.2	1,126	Perpendicular.
10	Mexican white mahogany	11.0 and 15.1	438 and 1,000	Oblique (2 exps.)
11	Mexican cedar	15.0	423	Perpendicular.
12	Mexican cedar	9.8	814	Parallel.
13	Mexican mahogany	15.0	566	Parallel.
14	Mexican mahogany	11.1	1,333	Perpendicular.
15	Oregon spruce	22.9 and 34.5	261 and 356	Parallel (2 exps.)
16	Oregon spruce	5 5	315	Perpendicular.
17	White pine	3.0 and 16.0	381 and 423	Perpendicular (2 exps.)
18	White pine	16.0 " 24.0	324 " 352	Parallel (2 exps.)
19	Whitewood	7 3 " 11.0	127 " 370	Oblique (2 exps.)
20	Whitewood	10.8 " 21.9	328 " 481	Parallel (2 exps.)
21	Whitewood	21.6 " 32.3	322 " 385	Perpendicular (2 exps.)
22	Yellow pine	13.0 " 13.1	317 " 399	Oblique (2 exps.)
23	Yellow pine	17.0 " 25.4	286 " 409	Perpendicular (2 exps.)
24	Ash	16.3 " 24.4	592 " 600	Parallel (2 exps.)
25	Ash	16.2 " 16.0	458 " 700	Perpendicular (2 exps.)
26	Red oak	16.0 " 23.9	743 " 745	Perpendicular (2 exps.)
27	Red oak	16.0 " 24.0	726 " 999	Parallel (2 exps.)
28	White oak	16.2 " 24.0	803 " 966	Parallel (2 exps.)
29	White oak	15.8 " 24.0	752 " 846	Perpendicular (2 exps.)
30	Yellow birch	17 0 " 17 0	563 " 815	Oblique (2 exps.)
31	Yellow birch	25.6 " 25.6	612 " 672	Perpendicular (2 exps.)
32	White maple	16.0	647	Oblique.
33	White maple	15.9 and 24.0	399 and 537	Perpendicular (2 exps.)
34	Spruce	15.8 " 23.8	235 " 347	Parallel (2 exps.)
35	Spruce	16.0 " 24.0	316 " 374	Perpendicular (2 exps.)

1880). As before, S is the ultimate shearing resistance in pounds per square inch.

Table III. has been condensed from the results of Col. Laidley's tests at the Watertown Arsenal (Ex. Doc. No. 12, 47th Congress, 1st Session). Usually, two such results have been selected as will give a correct idea of the resistance. In all cases except Nos. 19, 20, 23 and 33, the smaller resistance belongs to the larger shearing surface. In No. 33 the smaller resistance belongs to an unsatisfactory experiment.

Art. 60.—Torsion.

Coefficients of Elasticity.

The coefficients of elasticity for torsion or shearing have been given in Art. 58, and need not be repeated here.

Ultimate Resistance and Elastic Limit.

Wrought Iron.

In 1866 Mr. Kirkaldy tested four hammered Swedish iron bars turned to a diameter of 1.5 inches for a length of seven diameters. The average ultimate moment of torsion was produced by a weight of 2,677 pounds with a leverage of 12 inches; hence, in Eq. (83) of Art. 10; $M = 2{,}677 \times 12 = 32{,}124$. Putting $2r_0 = d = 1.5$ inches in that equation, there will result:

$$T_m = 5.1\,\frac{M}{d^3} = 48{,}540 \text{ pounds per square inch.}$$

This is the greatest intensity of torsional shear in the section.

If T_m be taken at 48,000 the diameter of a wrought-iron shaft required to resist an ultimate moment M, will be:

$$d = 0.047 \sqrt[3]{M} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the working moment be taken at one-eighth the ultimate, then the diameter required will be:

$$d = 0.047 \sqrt[3]{8 M_1} = 0.094 \sqrt[3]{M_1} \quad . \quad . \quad . \quad (2)$$

in which M_1 is the working moment.

If H is the number of horse powers per minute to be transmitted by the shafting, and n the number of revolutions which it is to make:

$$M_1 = \frac{12 \times 33{,}000}{2\pi} \cdot \frac{H}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Putting this value in Eq. (2):

$$d = 3.742 \sqrt[3]{\frac{H}{n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This value of d will be much too small in the case of long shafting required in the distribution of power, in consequence of the bending caused by the belting.

The mean torsional moment at the elastic limit, in Mr. Kirkaldy's four experiments, was about 0.4 the ultimate.

In 1846 Major Wade ("Experiments on Metals for Cannon") tested three wrought-iron circular cylinders about 1.9 inches in diameter and 15 inches long, with the following results:

$$T_m = \frac{5.1M}{d^3} = 28,325 \text{ lbs. per sq. in.}$$
$$= 27,525 \text{ " " " "}$$
$$= 27,800 \text{ " " " "}$$
$$\overline{83,650}$$

Mean = 27,900 (nearly).

If the mean be taken at 28,000:

$$d = 0.056 \sqrt[3]{M} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It is seen that Major Wade found T_m much less than Kirkaldy's value for Swedish iron, and d in Eq. (5) is correspondingly greater. If H and n carry the same signification as before, and if 8 is the safety factor:

$$d = 4.49 \sqrt[3]{\frac{H}{n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In all these results, the moments are supposed to be given in *inch-pounds*, and the resulting values of d are consequently in inches.

CAST IRON.

Major Wade also made tests on circular cylinders of cast iron about 1.9 inches in diameter and 15 inches long.

If d is the diameter $= 2r_0$ in Eq. (83) of Art. 10, he found the following results with the grades of iron shown:

2d fusion	T_m =	31,500 pounds per square inch.
3d fusion	T_m =	44,775 " " " "
2d and 3d fusion	T_m =	49,735 " " " "
2d fusion	T_m =	40,020 " " " "
3d fusion	T_m =	53,380 " " " "
2d fusion	T_m =	49,526 " " " "
3d fusion	T_m =	46,230 " " " "
	Mean =	45,000 (nearly).

Hence the diameter in inches, for the ultimate moment M in *inch-pounds* is:

$$d = \sqrt[3]{\frac{5.1}{45{,}000} M} = 0.048 \sqrt[3]{M} \quad . \quad . \quad . \quad . \quad (7)$$

These values of T_m are very high, because the iron with which Major Wade experimented was evidently of a special character and extraordinarily strong.

The same experimenter tested some square sections, for which, by Eq. (73) of Art. 10:

$$T = 5 \frac{M}{b^3} ; \ (b = \textit{side of square}) \quad . \quad . \quad . \quad . \quad (8)$$

The following are from Major Wade's results:

$b = 1.00$ inches ; $M = 8{,}750$ inch-pounds ; $T_m = 43{,}750$ pounds.
$b = 1.42$ inches ; $M = 23{,}000$ inch-pounds ; $T_m = 40{,}210$ pounds.
$b = 1.75$ inches ; $M = 54{,}000$ inch-pounds ; $T_m = 50{,}370$ pounds.

The mean of these results is: $T = 44{,}800$ (nearly).
Hence for the ultimate moment in *inch-pounds :*

$$b = \sqrt[3]{\frac{5M}{44{,}800}} = 0.0481 \sqrt[3]{M} \quad . \quad . \quad . \quad . \quad (9)$$

It is to be observed that, according to these experiments, T_m is the same for circular and square sections; a result very different from that of Prof. Bauschinger's experiments, as will presently be seen.

Four of Major Wade's experiments on hollow circular cylinders are next to be given.

Since $T_{xr} = 0$, in Eq. (78) of Art. 10, the resisting moment of such a cylinder, if d is the external and d_1 the internal diameter, will be:

$$M = \frac{T_m d^3 - T'_m d_1^3}{5.1} = \frac{T_m}{5.1d}(d^4 - d_1^4). \quad . \quad . \quad (10)$$

$$\therefore \quad T_m = \frac{5.1dM}{d^4 - d_1^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

For the first case:

$d = 3.25$ ins.; $d_1 = 2.61$ ins.; $M = 95{,}000$ in.-lbs.

Substituting in Eq. (11):

$T_m = 24{,}170$ lbs. per sq. in. (nearly).

For the second case:

$d = 2.21$ ins.; $d_1 = 1.54$ ins.; $M = 49{,}500$ in.-lbs.

Substituting in Eq. (11):

$T_m = 30{,}610$ lbs. per sq. in. (nearly).

For the third case:

$d = 1.81$ ins.; $d_1 = 0.91$ in.; $M = 37{,}250$ in.-lbs.

Substituting in Eq. (11):

$T_m = 34{,}220$ lbs. per sq. in. (nearly).

For the fourth case:

$d = 1.30$ ins.; $d_1 = 0.65$ in.; $M = 13{,}000$ in.-lbs.

Substituting in Eq. (11):

$$T_m = 32{,}180 \text{ lbs. per sq. in. (nearly).}$$

These results indicate that T_m decreases as the thickness of the wall of the hollow cylinder decreases and as the exterior diameter increases.

Professor Bauschinger (Der Civilingenieur, 1881, heft 2) tested cylinders about 40 inches long, and with the following cross sections and approximate dimensions:

Circle...........	Diameter =	3.25 inches.
Ellipse..........	Diameters =	2.30 inches. 4.40 inches.
Square..........	Sides =	3.00 inches. 3.00 inches.
Rectangle	Sides =	2.04 inches. 4.10 inches.
Rectangle	Sides =	1.02 inches. 4.10 inches.

The ultimate twisting moments substituted in Eqs. (83), (41), (73), (75), and (77) of Art. 10, give:

For Circle.................. T_m = 27,730 pounds per square inch.
For Ellipse................. T_m = 36,120 pounds per square inch.
For Square .... T_m = 37,160 pounds per square inch.
For Rectangles (sides 2 to 1).. T_m = 36,370 pounds per square inch.
For Rectangles (sides 4 to 1).. T_m = 37,090 pounds per square inch.

These experiments give T_m considerably less value for the circular cross section than for the others.

The U. S. Board, however, found the following values for four cast-iron cylinders one inch long and 0.565 inch in diameter:

$$T_m = 35{,}980;\ 34{,}110;\ 34{,}280, \text{ and } 33{,}770 \text{ lbs. per sq. in.}$$

Elas. Lim. = 60; 55; 64, and 62 per cent. of T_m, respectively.

STEEL.

In connection with the torsional resistance of steel, tests of circular cylinders only are to be considered. Those to first receive attention were made by Mr. Kirkaldy on English steel, in 1866–1870, and the results have been deduced from his data.

As the sections are all circular, Eq. (83) of Art. 10 is the only one needed:

$$T_m = \frac{5.1M}{d^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

In this equation T_m is the greatest intensity of torsional shear, in any section, in pounds per square inch; "d" the diameter of the shaft or cylinder in inches; and M the twisting moment in *inch-pounds*.

In all the following experiments the lever arm of the twisting couple was 12 inches; hence, if P is the twisting force, $M = 12P$, and Eq. (12) becomes

$$T_m = \frac{61.2P}{d^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The mean of four experiments with Bessemer steel gave for the ultimate resistance

$$P = 2,307 \text{ lbs., with } d = 1.25 \text{ inches;}$$

$$\therefore T_m = 72,298 \text{ lbs. per sq. in.} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The length was 10 inches.

The mean of some results with Krupp's cast steel in specimens 1.25 inches in diameter, and 2.5 inches for torsion length, gave:

$$P = 2,867 \text{ lbs.} \quad \therefore \quad T_m = 89,847 \text{ lbs.} \quad . \quad . \quad (15)$$

The following set of results were obtained from 2-inch square bars turned down to 1.382 inches in diameter for a length of 11 inches, and gives the means of the number of tests indicated.

SPECIMENS.		P (ULTIMATE).	T_m (ULTIMATE).	ELASTIC STRAIN.
5 Hammered tires,	Bessemer steel.	3,450 lbs.	80,006 lbs.	0.014
5 " axles,		3,293 "	76,365 "	0.011
4 " rails,		3,248 "	75,321 "	0.012
4 Rolled tires, axles and rails.		3,226 "	74,811 "	0.008
5 Hammered tires,	Crucible steel.	3,562 "	82,603 "	0.014
4 " axles,		3,786 "	87,797 "	0.013
1 " rail,		4,054 "	94,012 "	0.016
1 Rolled rail.		3,002 "	69,616 "	0.012

. . (16)

The elastic strain is the fraction of a complete turn made by the specimen at the elastic limit.

The mean of the Bessemer steels in (14) and (16) give:

$$\text{Mean } T_m = 75{,}760 \text{ lbs. per sq. in.}$$

Hence, if M is the breaking moment of the twisting couple in *inch-pounds*, the following will be the diameter of the shaft in inches:

$$d = \sqrt[3]{\frac{5.1M}{75{,}760}} = 0.0407\sqrt[3]{M}\,; \quad . \quad . \quad . \quad . \quad (17)$$

Or, if n is the safety factor, and M_1 the greatest working moment:

$$d = 0.0407\sqrt[3]{nM_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The mean of the crucible steel results in (16), with the exception of the last, is:

$$\text{Mean } T_m = 88{,}140 \text{ lbs.}$$

Hence the diameter (in inches) of the shaft which will just sustain the breaking moment M, in *inch-pounds*, is:

$$d = \sqrt[3]{\frac{5.1M}{88,140}} = 0.0387\sqrt[3]{M} \quad . \quad . \quad . \quad . \quad (19)$$

Or, if n is the safety factor, and M_1 the greatest working moment:

$$d = 0.0387\sqrt[3]{nM_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

In all the preceding experiments the elastic limit varied from 40 to 47 per cent. of T_m (ultimate) as given in (14), (15) and (16).

In 1873 Mr. Kirkaldy made some experiments on specimens of Fagersta steel which possessed a length of about 9 inches and a diameter of 1.128 inches, the length of the twisting lever being still 12 inches. Eq. (13) then gives the following results, each being a mean of three tests:

MARK.	P (ULTIMATE).	T_m (ULTIMATE).	STRAIN.
1.2	2,120 lbs.	90,397 lbs.	0.29
0.9	2,336 "	99,607 "	0.79
0.6	2,261 "	96,409 "	1.02
0.3	1,520 "	64,813 "	3.22

The "*strain*" is the number of complete turns made by the specimen at the place and instant of rupture.

The specimens with the higher "mark" numbers were the higher steels.

The elastic limit varied from 46 to 58 per cent. of the ultimate T_m.

The diameter of a shaft for any of these grades may readily be computed by the use of these values of T_m in equations similar to Eqs. (17) to (20).

The following values were determined by the Committee on Chemical Research of the U. S. Board, "Ex. Doc. 23, House of Rep., 46th Congress, 2d Session," with specimens 1 inch long turned to diameters of 0.625 and 0.565 inch, and tested in a Thurston machine:

T_m.	ELAS. LIMIT IN PER CENT. OF T_m.	ULT. ANGLE OF TORSION.	
100,990 lbs. per sq. in.	34	149°.0	
95,230 " " " "	34	142°.3	
110,260 " " " "	33	68°.4	
115,780 " " " "	42	56°.1	
52,375 " " " "	34	278°.2	
71,420 " " " "	45	220°.8	
88,210 " " " "	39	99°.5	
55,885 " " " "	35	165°.0	
119,040 " " " "	40	84°.9	
75,430 " " " "	44	180°.7	
91,690 " " " "	39	53° to 113°	Tool steels; annealed.
96,450 " " " "	36	48° to 84°	
109,010 " " " "	29	61° to 143°	
107,315 " " " "	32	42° to 123°	
109,590 " " " "	32	73° to 141°	

Each of the last five results is a mean of eight tests.

The first portion of these results would possess more value if the test specimens had been larger.

With these values of T_m, the diameter of a shaft, with the torsion moment M in inch-pounds, becomes:

$$d = \sqrt[3]{\frac{5.1M}{T_m}} = 1.721 \sqrt[3]{\frac{M}{T_m}}.$$

COPPER, TIN, ZINC, AND THEIR ALLOYS.

The following values of T_m have been computed by the aid of Eq. (12) from data determined by Prof. R. H. Thurston, and given by him in the works already cited in connection with tension and compression. The test specimens were 0.625 inch in diameter, with a torsion length of 1.00 inch, and were tested in his torsion machine. The ultimate shearing resistances of these alloys in torsion are thus seen to vary as widely as their tensile resistances.

TABLE I.

COMPOSITION.		ULTIMATE TORSIVE SHEAR, T_m.	ELASTIC LIMIT; PER CENT. OF T_m.	ULTIMATE TORSION ANGLE.
Cu.	Sn.			
		Pounds.		Degrees.
100	00	35,910	35	153.0
100	00	28,430	40	52 to 154
00	100	3,196	45	557.0
00	100	3,297	33	691.0
90	10	43,943	41	114.5
80	20	47,671	62	16.3
70	30	4,407	100	1.5
62	38	1,770	100	1.0
52	48	686	100	1.0
39	61	5,881	100	1.7
29	71	5,257	100	2.34
10	90	5,761	63	131.8
90	10	25,027	49	57.2
90	10	31,851	57	72.6

T_m is in pounds per square inch.

Table I. relates to alloys of copper and tin, and Table II. to alloys of copper and zinc.

None but specimens with circular sections were tested.

With the preceding values of T_m, the following expression for the diameter in inches may be written, if M is given in *inch-pounds :*

$$d = \sqrt[3]{\frac{5.1M}{T_m}} = 1.721 \sqrt[3]{\frac{M}{T_m}}.$$

TABLE II.

PERCENTAGE OF		ULTIMATE TORSIVE SHEAR, T_m.	ELASTIC LIMIT; PER CENT. OF T_m.	ULT. TORSION ANGLE.
Copper.	Zinc.	Pounds.		Degrees.
90.56	9.42	35,100	17.2	458.0
81.90	17.99	41,575	27.5	345.0
71.20	28.54	41,000	24.0	269.0
60.94	38.65	48,520	29.4	202.0
55.15	44.44	52,320	32.7	109.0
49.66	50.14	43,154	36.0	38.0
41.30	58.12	4,588	100.0	1.8
32.94	66.23	7,241	100.0	1.2
20.81	77.63	16,374	100.0	0.8
10.30	88.88	22,500	85.6	7.1
0.00	100.00	9,186	38.1	141.5

TIMBER.

In the July, 1873, number of Van Nostrand's Magazine, Prof. Thurston gave the results of some experiments on timber test specimens of circular section, 0.875 inch in diameter. Eq. (12) may be written as follows:

$$M = \frac{T_m}{5.1} d^3 = Cd^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Prof. Thurston determined the values of C, and the values of $T_m = 5.1C$ have been computed from them:

	T_m (per sq. in.)
White pine	1,530 pounds.
Yellow pine, sap	2,142 "
Yellow pine, heart	2,448 "
Spruce	1,836 "
Ash	2,632 "
Black walnut	3,366 "

	T_m (per sq. in.).
Red cedar	1,958 pounds.
Spanish mahogany	3,978 "
Oak	3,244 "
Hickory	5,202 "
Locust	4,896 "
Chestnut	2,142 "

It is presumed that the axis of torsion was parallel to the fibres, which would cause the shear to take place across the latter.

It is interesting to observe that T_m is generally considerably less than the ultimate resistance to simple shear as given in Table II. of Art. 59.

If d is in inches and M in *inch-pounds*, there may again be written :

$$d = \sqrt[3]{\frac{5.1M}{T_m}} = 1.721 \sqrt[3]{\frac{M}{T_m}}.$$

If M is given in *foot-pounds*, $12M$ is to be written for M. If M_1 is the greatest working moment, and n the safety factor, nM_1 is to be written for M.

Relation between Ultimate Resistances to Tension and Torsion.

In the "Trans. Am. Soc. of Civ. Engrs.," Vol. VII., 1878, Prof. Thurston gave the results of some of his experiments which were made with a view to the determination of this relation. If M is the ultimate torsional moment in *foot-pounds* of specimens one inch long and 0.625 inch in diameter ; θ the angle of torsion corresponding to this greatest moment M; and T the ultimate tensile resistance in pounds per square inch ; he deduced from a large number of steel specimens of wide range in grades the following formula :

$$T = M\left(\frac{900 - \theta}{3}\right).$$

No experiments were made in which θ was greater than 300°.

T is thus seen to increase as M increases and as θ decreases.

CHAPTER IX.

BENDING OR FLEXURE.

Art. 61.—Coefficient of Elasticity.

The coefficient of elasticity, as determined by experiments in flexure, can scarcely be considered other than a conventional quantity. If the coefficients of elasticity for pure tension and compression were exactly equal to each other, and if all the hypotheses involved in the common theory of flexure were true, then, indeed, the coefficient of elasticity for flexure would possess actual existence, and be the same as that for either tension or compression.

These conditions, however, never exist, and the quantities found in this chapter under the name "coefficient of elasticity" possess value chiefly as empirical factors which enable the deflections in the different cases to be estimated with sufficient accuracy for all ordinary purposes.

The formulæ to be used in the determination of the coefficients of elasticity for flexure have already been established, and their use will be shown in succeeding Articles.

Art. 62.—Formulæ for Rupture.

As with the formulæ for the coefficient of elasticity, so with the formulæ for rupture in bending; they are all deductions from the common theory of flexure, and, strictly speaking, are subject to all the limitations involved in it.

If K and K' are the greatest intensities of stress in the section of rupture and at the instant of rupture; y the variable normal distance of any fibre from the neutral surface; y_1 and y' the greatest values of y; b the variable width of the section (normal to y); and M the resisting moment at the instant of rupture; then the general formula for rupture by bending, as given by Eq. (1) of Art. 27, is:

$$M = \frac{K}{y_1}\int_0^{y_1} y^2 b\, dy + \frac{K'}{y'}\int_{-y'}^{0} y^2 b\, dy \quad . \quad . \quad . \quad . \quad (1)$$

This equation is based on the supposition that the coefficients of elasticity for tension and compression are not equal. Although this supposition is strictly true, yet equality is almost invariably assumed; particularly in the treatment of solid beams. Fortunately, this assumption is not far wrong in those materials which are most valuable to the engineer.

Eq. (1), however, will hereafter be applied to some cast-iron flanged beams.

If the tensile and compressive coefficients of elasticity are equal, $\frac{K}{y_1} = \frac{K'}{y'}$. Or, if K is the greatest intensity of stress in the section which exists in the fibre at the greatest normal distance, d_1, from the neutral surface, then $\frac{K}{y_1} = \frac{K}{d_1}$, and Eq. (1) becomes:

$$M = \frac{KI}{d_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This is Eq. (14) of Art. 18, and is the one almost invariably used in engineering practice.

In Eq. (2) I is the moment of inertia of the cross section of the beam about its neutral axis. By introducing the value of I for each particular shape of section, simple working forms of Eq. (2) may easily be obtained. This will be done for two sections in the following Article.

Art. 63.—Solid Rectangular and Circular Beams.

While the rectangular form of cross section almost invariably characterizes timber beams, similar ones of iron, steel and other metals are only occasionally seen. Beams of iron and steel with circular cross sections, however, are quite common as pins in pin connection bridges.

If ΣPx represents the moment of the external forces about the neutral axis of any section, Eq. (2) of the preceding Article becomes:

$$\Sigma Px = \frac{KI}{d_1} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The following are the values of I and d_1 for rectangular and circular sections, h being the side of the rectangle normal, and b that parallel to the neutral axis, while r is the radius of the circular section, and A the area in each case:

Rectangular:
$$\begin{cases} I = \dfrac{bh^3}{12} = \dfrac{Ah^2}{12} . \\ d_1 = \dfrac{h}{2} . \end{cases}$$

Circular:
$$\begin{cases} I = \dfrac{\pi r^4}{4} = \dfrac{Ar^2}{4} . \\ d_1 = r. \end{cases}$$

If the beams are supported at each end and loaded by a weight W at the centre of the span (or distance between supports), which may be represented by l, then the moment at the centre of the beam becomes:

$$\Sigma Px = M = \frac{Wl}{4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

There will then result from Eq. (1):
For rectangular sections:

$$M = \frac{Wl}{4} = \frac{Kbh^2}{6} = \frac{KAh}{6} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For circular sections:

$$M = \frac{Wl}{4} = \frac{\pi Kr^3}{4} = \frac{KAr}{4} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The quantity K is called the *modulus of rupture for bending*, and if experiments have been made, so that W is known, Eq. (3) gives:

$$K = \frac{3}{2}\frac{Wl}{Ah} = \frac{3}{2}\frac{Wl}{bh^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

And Eq. (4):

$$K = \frac{Wl}{Ar} = \frac{Wl}{\pi r^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If the rectangular section is square, $bh^2 = b^3 = h^3$.

Wrought Iron.

If the beam is simply supported at each end and carries a load W at the centre, while E is the coefficient of elasticity and w the deflection at the centre, Eq. (28) of Art. 24 gives:

$$w = \frac{Wl^3}{48EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If, in any given experiment, w is measured, E may then be found by the following form of Eq. (7):

$$E = \frac{Wl^3}{48wI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If the section is rectangular:

$$E = \frac{Wl^3}{4wbh^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Mr. Edwin Clark tested a one-inch square wrought-iron bar with the following results at the "elastic limit:"

$l = 12$ inches. $W = 2,636.00$ lbs.
$w = 0.09$ inch. $b = h = 1$ inch.

Eq. (9) then gives:

$E = 12,652,809.00$ pounds per square inch.

The mean for 2 one and a half inches square bars was as follows:

$l = 36$ inches. $W = 2,766.00$ lbs.
$w = 0.305$ inch. $b = h = 1.5$ inches.

$\therefore \quad E = 20,894,600.00$ pounds per square inch.

A mean of 4 two inches square bars of Swedish iron, tested by Mr. Kirkaldy, in 1866, gave the following results at the "elastic limit:"

$l = 25$ inches. $W = 6,625.00$ lbs.
$w = 0.082$ inch. $b = h = 2$ inches.
$E = 19,725,000.00$ pounds per square inch.

By "weighting" these results in proportion to the number of tests of which each is a mean, the mean of all becomes:

$$E = 19,049,000.00 \text{ pounds per square inch.}$$

It is very probable that if w had been measured for smaller loads, E would have been materially increased.

Mr. Kirkaldy tested the same four square Swedish iron bars to rupture. By the aid of Eq. (5), and the data given above, the greatest, mean, and least results were as follows:

	W.	K.	FINAL DEFLECTION.
Greatest......	15,885 lbs......	74,475 lbs. per sq. in......	5.85 ins.
Mean........	14,516 lbs......	68,044 lbs. per sq. in......	5.35 ins.
Least........	13,338 lbs......	62,522 lbs. per sq. in	4.98 ins.

The ultimate tensile resistance of the same iron was about 45,000 pounds per square inch. These experiments would seem to show that K, for square bars under similar circumstances of span and depth, may be taken about 1.5 times the ultimate resistance to tension.

The results in the following table were computed by the aid of Eq. (6), for some circular beams of "Burden's Best" iron, which were tested at the Rensselaer Polytechnic Institute in November, 1882. As beams cannot be actually broken under such circumstances, the "ultimate" value of K was taken with a final deflection of one to one and quarter the diameter.

The "elastic limit" is taken at that point beyond which the metal "flows," and is indicated by the incapability of the specimen to hold up the scale beam beyond it, under a small increase of stress; in other words, it is that point at which the specimen "breaks down."

These experiments show conclusively that "ultimate" K decreases as the ratio of span over diameter increases, but they

Circular Beams of "Burden's Best" Wrought Iron.

KIND.	DIAMETER.	SPAN.	*W.*		*K.*	
			Elastic.	Ultimate.	Elastic.	Ultimate.
	Ins.	Ins.	Lbs.	Lbs.	Lbs.	Lbs.
Turned ...	1.25	12	3,000	6,000	46,950	93,900
Turned ...	1.25	8	4,400	10,500	45,900	109,500
Turned ...	1.25	12	——	——	54,760	93,870
Turned ...	1.25	8	——	——	52,150	114,700
Rough....	1.00	12	——	——	55,000	91,700
Rough....	1.00	8	——	——	57,000	101,900
Turned ...	1.00	12	——	——	55,000	91,600
Turned ...	1.00	8	——	——	——	107,000
Rough....	1.00	12	1,700	3,000	51,950	91,680
Rough....	1.00	8	2,800	4,800	57,000	97,800
Turned ...	0.75	12	700	1,100	47,100	74,050
Turned ...	0.75	8	1,200	1,900	53,880	85,310
Turned ...	0.75	12	700	1,100	47,100	74,050
Turned ...	0.75	8	1,300	1,900	58,370	85,310

are not sufficiently extended to establish the limits of application of the observation.

Cast Iron.

All the following results for cast-iron beams are found from Major Wade's experiments ("Strength and other Properties of Metals for Cannon," 1856). His test bars were nearly two inches square in section or two inches in diameter, and were twenty-four inches long. They were loaded at the centre, and the distance between supports was twenty inches. The following table gives results for square bars. K is given in pounds per square inch, and is found by the aid of Eqs. (5) and (6). "*Def.*" is the final deflection.

Although Major Wade made many other experiments of the same kind, these may be considered representative ones.

Bars with square section, Eq. (5).

KIND OF IRON.	HOURS IN FUSION.	*W.*	*K.*	DEF.
		Lbs.	Lbs.	In.
Richmond iron ; 2d fusion........	0	11,587	42,130	0.156
	1	12,487	45,110	0.152
	2	15,019	52,870	0.152
	2	15,525	55,930	0.147
Stockbridge iron ; 2d fusion.......	½	11,812	42,760	0.162
	1	14,512	52,670	0.195
	1½	16,481	60,500	0.202
	2	19,462	69,680	0.230
Franklin iron ; 3d fusion	0	12,987	49,070	0.250
	0	13,365	50,120	0 217
	1	15,363	57,330	0.220
	1	14,616	54,550	0.195
	2	13,788	48,730	0.152
	2	14,850	50,720	0.170
	3	16,056	56,050	0.175
	3	16,722	60,410	0.170

Bars with circular section, Eq. (6).

KIND OF IRON.	HOURS IN FUSION.	*W.*	*K.*	DEF.
Franklin iron ; 3d fusion..........	½	10,437	70,600	0.237
	1½	8,665	57,720	0.166
	3	11,112	70,740	0.254
	3¾	10,606	71,740	0.240
Franklin iron ; 2d fusion..........	1	7,920	52,360	——
	2	9,270	63,670	0.240
	3	9,481	64,820	0.262
	4	7,920	52,360	——

It is both interesting and important to observe that K and the final deflection are materially larger for circular beams than for square ones.

By comparing these values of K with the ultimate tensile resistances found by Major Wade, and which have been given under the head of "Tension," it will be seen that no great error will be involved if K is taken at *twice the ultimate tensile* *resistance for square bars*, and *two and a quarter times the same quantity for bars with circular section.*

Whether these ratios will hold for iron of inferior quality to that used by Major Wade, can only be determined by farther experimenting.

Steel.

Some circular Bessemer steel beams with 12 and 8-inch spans were tested at the Rensselaer Polytechnic Institute in Nov., 1882, with the results which are given in the next table. The "elastic limit" is that point at which the specimen "breaks down." The "ultimate" value was that for which the deflection was equal to one or one and a quarter the diameter.

Circular Bessemer steel beams, Eq. (6).

KIND.	DIAM.	SPAN.	*W.*		*K.*	
			Elastic.	Ultimate.	Elastic.	Ultimate.
	In.	Ins.	Lbs.	Lbs.	Lbs.	Lbs.
Turned	1.00	12	——	——	86,000	146,750
"	1.00	8	——	——	85,300	152,800
"	1.00	12	2,500	4,500	76,400	137,520
"	1.00	8	3,750	7,500	76,400	152,800
"	0.75	12	1,150	1,800	77,400	122,200
"	0.75	8	1,800	3,300	80,800	148,200
"	0.75	12	1,150	1,700	77,400	114,400
"	0.75	8	1,800	3,300	80,800	148,200

The "ultimate" K is seen to decrease as the ratio of length over diameter increases.

The following table contains results computed from the experiments of the "Steel Committee" of the British Institution of Civil Engineers; the experiments were made in 1868.

The bars were 1.9 inches square in section, and the distance between supports was twenty inches.

Bessemer Steel, Eq. (5).

KIND AND NUMBER OF TESTS.	K.	ELASTIC OVER ULTIMATE.	FINAL DEFLECTION IN INCHES.
	Lbs. per sq. in.		
5, tires, hammered..............	129,030	0.573	3.82
5, axles, "	129,325	0.615	4.08
4, rails, "	125,900	0.612	3.94
4, tires, axles, rails ; rolled	115,120	0.563	4.03
Crucible Steel, Eq. (5).			
5, tires, hammered..............	143.530	0.574	3.32
4, axles, "	152,055	0.539	3.35
1, rail, "	175,470	0 436	3.65
1, axle, rolled..................	118,160	0.538	3.84

Each result is an average of the number of tests shown in the left column.

The ratio "elastic over ultimate" is the value of K at the "elastic limit" divided by its ultimate value as given in the table.

Table IX. of Art. 34 gives the ultimate tensile resistances of these same steels. That table, taken in connection with the results just given, shows that *K is about* 1.66 *times the ultimate tensile resistance for square Bessemer steel bars, and about* 1.85 *times the same quantity for square crucible steel bars.*

Mr. J. W. Cloud, of the Penn. R. R. Co., made bending tests of the Bessemer rail steel whose ultimate tensile resistances are given in Table IV. of Art. 34. His test pieces were 12 inches long, 1.5 inches wide, and 0.5 inch thick. The load was applied in the direction of the thickness, and midway between supports 10 inches apart. The greatest, mean and least results of the 18 means of the groups shown in Table IV., Art. 34, are the following:

	W.	*K.*
Greatest	3,349 lbs	133,960 lbs. per sq. in.
Mean	3,026 lbs	121,040 lbs. per sq. in.
Least	2,765 lbs	110,600 lbs. per sq. in.

With these rectangular specimens of Bessemer rail steel, supported flatwise, therefore, K may be taken about 1.6 the ultimate tensile resistance.

The following table contains the results of Mr. Kirkaldy's experiments on square bars of Fagersta steel. These bars were 1.9 inches square in section, and rested on supports 20 inches apart. W is the breaking weight at centre, and K is

Fagersta Steel Square Bars.

MARK.	*W*, POUNDS.		*K*, LBS. PER SQ. IN.		ELASTIC OVER ULTIMATE.		FINAL DEFLECTION.
							Ins.
1 2	30,496	Mean = 32,589.	133,380	Mean = 144,000.	0.669	Mean = 0.66.	0 75
1.2	32,896		143,880		0.695		0.72
1.2	35,376		154,710		0.616		0.87
0.9	43,820	Mean = 43,833.	191,640	Mean = 191,700.	0.500	Mean = 0.496.	1.46
0.9	44,552		194,850		0.476		1.62
0.9	43,128		188,640		0.512		1.38
0.6	40,260	Mean = 38,145.	176,100	Mean = 167,040.	0.467	Mean = 0.48.	3.15
0.6	36,200		158,310		0.491		3.56
0.6	38,120		166,740		0.482		3.22
0.3	24,420	Mean = 25,283.	106,800	Mean = 110,580.	0.561	Mean = 0.623.	5.22
0.3	23,280		101,820		0.653		5.05
0.3	28,150		123,120		0.654		5 05

computed by the aid of Eq. (5). The column "*Elastic over ultimate*" contains the ratios of the values of K at the "elastic limit" divided by the ultimate values given in the table.

The "*Mark*" shows the character of the steel; 1.2 is the hardest, and 0.3 the softest.

K is about 1.6 times the ultimate tensile resistance for the grades 1.2 and 0.6, and 1.8 times the same quantity for the grades 0.9 and 0.3.

Combined Steel and Iron.

In Sept., 1881, some interesting and valuable experiments on the transverse resistance of pins (solid circular beams) were made at Phœnixville, Pa., by the Phœnix Iron Co.

The pins were of combined iron and steel, the core of the pin being of steel, and the outside of iron. In such a pin the iron seems to change gradually to the steel, but the shell of iron may perhaps be considered one quarter to one half an inch thick.

These pins are supported at each end and loaded in the centre. The results of the experiments are given in the following table:

D = diameter of pin.
l = length in inches between supports.
W = weight (pounds) at centre.
K' = intensity of stress per sq. in. on extreme fibre, in general.
K = intensity of stress per sq. in. on extreme fibre, at rupture.

K is the greatest value of K' for any one pin. Either K or K', by Eq. (6), has the value:

$$K \text{ or } K' = \frac{Wl}{Ar}.$$

PIN.	D.	l.	W.	K′ OR K.	REMARKS.
	Ins.				
1	$4\frac{1}{16}$	24	36,000	32,815.	——
1	$4\frac{1}{16}$	24	60,000	54,692.	Elastic limit.
1	$4\frac{1}{16}$	24	100,000	91,154	Not broken. Deflection = $\frac{5}{8}$ ins.
2	$4\frac{1}{2}$	24	60,000	40,241.	——
2	$4\frac{1}{2}$	24	84,000	56,337.	Elastic limit.
2	$4\frac{1}{2}$	24	148,000	99,260. = K	Broken.
3	$4\frac{1}{2}$	20.5	68,000	38,955.	——
3	$4\frac{1}{2}$	20.5	84,000	48,121.	Slight permanent set.
3	$4\frac{1}{2}$	20.5	252,000	144,360. = K	Broken.

The mean of the two values of K is :

$$K = \frac{99,260 + 144,360}{2} = 121,810.00 \text{ pounds.}$$

Copper, Tin, Zinc, and their Alloys.

In the following table are given the data and the results of the experiments of Prof. R. H. Thurston, as contained in his various papers, to which reference has already been made. The distance between the points of support was twenty-two inches, while the bars were about one inch square in section, and of cast metal.

The modulus of rupture, K, is found by Eq. (5), in which, however, in many of these cases, W is the weight applied at the centre, added to half the weight of the bar. When K is large and the specimens small, this correction for the weight of the bar is unnecessary ; otherwise, it is advisable to introduce it.

The coefficient of elasticity, E, is found by Eq. (9), in which W is the centre load added to five-eighths of the weight of the bar.

The manner in which both these corrections arise, is completely shown in *Case* 2 of Art. 24.

E, for any particular bar, has a varying value for different degrees of stress and strain. Those given in the table may be considered average values within the elastic limit.

As usual, "*elastic over ultimate*" is the ratio of K at the elastic limit over its ultimate value.

An examination of the ultimate tensile and compressive resistances of these same alloys, as given in preceding pages, shows that the ratio of K over either of those resistances is very variable. It is usually found between them, but occasionally it exceeds both.

Square Bars.

PERCENTAGE OF			K, LBS. PER SQ. IN.	ELASTIC OVER ULTIMATE.	FINAL DEFLECTION.	E, LBS. PER SQ. IN.
Cu.	Sn.	Zn.			Ins.	
100	0.00	0.00	29,850	——	8.00	9,000,000
100	0.00	0.00	25,920	0.14 to 0.41	1.38 to 8.00	10,830,600
100	0.00	0.00	21,251	0.346	2.31	13,986,600
100	0.00	0.00	29,848	0.140	Bent.	10,203,200
90	10.00	0.00	49,400	0.400	Bent.	14,012,135
90	10.00	0.00	56,375	0.41	3.36	——
80	20.00	0.00	56,715	0.657	0.492	13,304,200
70	30.00	0.00	12,076	1.00	0.062	15,321,740
61.7	38.3	0.00	2,761	1.00	0.032	9,663,990
48.0	52.0	0.00	3,600	1.00	0.019	17,039,130
39.2	60.8	0.00	8,400	1.00	0.060	12,302,350
28.7	71.3	0.00	8,067	0.583	0.121	9,982,832
9.7	90.3	0.00	5,305	0.25	Bent.	7,665,988
0.00	100	0.00	3,740	0.273	Bent.	6,734,840
0.00	100	0.00	4,559	0.267	Bent.	5,635,590
80.00	0.00	20.00	21,193	——	3.27	11,000,000
62.50	0.00	37.50	43,216	——	3.13	14,000,000
58.22	2.30	39.48	95,620	——	1.99	11,000,000
55.00	0.50	44.50	72,308	——	——	——
92.32	0.00	7.68	21,784	0.30	Bent.	13,842,720
82.93	0.00	16.98	23,197	0.41	Bent.	14,425,150
71.20	0.00	28.54	24,468	0.51	Bent.	14,035,330
63.44	0.00	36.36	43,216	0.53	Bent.	14,101,300
58.49	0.00	41.10	63,304	0.48	Bent.	11,850,000
54.86	0.00	44.78	47,955	0.39	Bent.	10,816,050

Square Bars.—Continued.

PERCENTAGE OF			K, LBS. PER SQ. IN.	ELASTIC OVER ULTIMATE.	FINAL DEFLECTION.	E, LBS. PER SQ. IN.
Cu.	Sn.	Zn.				
					Ins.	
43.36	0.00	56.22	17,691	1.00	0.0982	12,918,210
36.62	0.00	62.78	4,893	1.00	0.0245	14,121,780
29.20	0.00	70.17	16,579	1.00	0.0449	14,748,170
20.81	0.00	77.63	22,972	1.00	0.1254	14,469,650
10.30	0.00	88.88	41,347	0.73	0.5456	12,809,470
0.00	0.00	100.00	7,539	0.57	0.1244	6,984,644
70.22	8.90	20.68	50,541	——	0.4019	14,400,000
56.88	21.35	21.39	2,752	——	0.0146	14,800,000
45.00	23.75	31.25	6,512	——	0.0150	7,000,000*
66.25	23.75	10.00	8,344	——	0.0162	12,000,000*
10.00	50.00	40.00	21,525	——	Bent.	9,000,000
58.22	2.30	39 48	95,623	——	2.000	10,600,000
60.00	10.00	30 00	24,700	——	0.1267	14,500,000
65.00	20.00	15.00	11,932	——	0.0514	17,000,000
70.00	10.00	20.00	36,520	——	0.1837	15,000,000
75.00	5.00	20.00	55,355	——	Bent.	13,000,000
80.00	10.00	10.00	67,117	——	Bent.	13,500,000
55.00	5.00	44.50	72,308	——	Bent.	11,000,000
60.00	2.50	37.50	69,508	——	1.500	13,000,000
72.52	7.50	20.00	51,839	——	Bent.	12,000,000
77.50	12.50	10.00	61,705	——	0.705	13,500,000
85.00	12.50	2.5	62,405	——	Bent.	12,500,000

* These bars were about half the length of the others.

Timber Beams.

As timber beams are always rectangular in section, Eq. (3) only will be needed. Retaining the notation of that equation, if the beam carries a single weight W at the centre of the span l:

$$W = \frac{2}{3}\frac{KAh}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If the total load W' is uniformly distributed over the span :

$$W' = \frac{4}{3}\frac{KAh}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

As K is supposed to be expressed in pounds per square inch, all dimensions in Eqs. (10) and (11) must be expressed in inches.

In the use of timber beams it is usually convenient to take the span l in feet, and the breadth (b) and depth (h) in inches. Placing $12l$ for l, therefore, in Eqs. (10) and (11);

$$W = \frac{KAh}{18l} \; ; \quad \text{and,} \quad W' = 2\,\frac{KAh}{18l} \quad . \quad . \quad . \quad (12)$$

in which formulæ l must be taken in feet and A and h in inches.

If B be put for $\frac{K}{18}$, Eq. 12 becomes:

$$W = B\frac{Ah}{l} \; ; \quad \text{and,} \quad W' = 2B\,\frac{Ah}{l} \quad . \quad . \quad . \quad (13)$$

Hence, when W and W' have been determined by experiment:

For single load W at centre:

$$B = \frac{Wl}{Ah} \therefore h^2 = \frac{Wl}{Bb} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

For total load W' uniformly distributed:

$$B = \frac{W'l}{2Ah} \therefore h^2 = \frac{W'l}{2Bb} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

If the beam has a section one inch square and is one foot long, $B = W = \frac{W'}{2}$. B, therefore, may be considered *the unit of transverse rupture*; it is sometimes called *the coefficient for centre breaking loads.*

Table I. is a condensed statement of the result of experiments by the late R. G. Hatfield, a complete account of which may be found in his "Transverse Strains," 1877. All the test

TABLE I.

MATERIAL.	B.	K = 18B.	MATERIAL.	B.	K = 18B.
	Lbs.	Lbs.		Lbs.	Lbs.
Georgia Pine........	850	15,300	Ash................	900	16,200
Locust..............	1,200	21,600	Maple..............	1,100	19,800
White Oak..........	650	11,700	Hickory...........	1,050	18,900
Spruce..............	550	9,900	Cherry............	650	11,700
White Pine.........	500	9,000	Black Walnut......	750	13,500
Hemlock...........	450	8,100	Canadian Oak......	590	10,600
Whitewood.........	600	10,800	New England Fir...	370	6,610
Chestnut...........	480	8,640			

specimens were of American woods with cross dimensions varying from one to two inches and span of 1.6 feet.

Table II. contains the results of experiments on specimens of American timber, given by Prof. R. H. Thurston in the "Journal of the Franklin Institute," Oct., 1879. The test specimens were 3 inches square and 4.5 feet between supports. The coefficient of elasticity is in pounds per square inch, and is found by Eq. (9).

Later experiments by Prof. Thurston ("Jour. Frank. Inst.," Sept., 1880), on a great variety of yellow pine specimens, both in respect to dimensions and degree of seasoning, induced him to draw the following conclusions in regard to that timber:

The elasticity of yellow pine timber as used in construction is very variable, the coefficient varying from one to three millions, the average being about two millions for small sections, and a little above one and a half millions of large timber.

TABLE II.

Specimens 3 *ins.* × 3 *ins.* × 4.5 *ft.*

MATERIAL.	ELASTIC LIMIT.	*K.*	*B.*	DEFLECTION IN INCHES.		COEFFICIENT OF ELASTICITY.
				Elastic.	Ultimate.	
	Lbs.	Lbs.	Lbs.			Lbs.
White Pine...	4,320	5,280	293	0.86	1.28	883,636
Yellow Pine..	12,720	16,740	930	0.84	1.96	3,534,727
Locust.......	8,400	13,680	760	0.82	2 70	2,046,315
Black Walnut.	5.640	7,440	413	0.50	0.72	1,944,000
White Ash ...	6,360	9,72c	540	1.50	2.50	1,080,000
White Oak...	7,200	9,840	547	0.90	1.76	1,620,000
Live Oak.....	9,040	11,280	627	0.94	1.38	1,851,428

The highest values are as often given by green as by seasoned timber. . . . .

The density of the wood does not determine the coefficient ; . . . .

A high coefficient usually accompanies high tenacity and great transverse strength, but it is not invariably the fact that maximum ultimate strength is accompanied by initial stiffness. . . . .

K varies from 10,000 to 17,000 pounds per square inch (or *B*, from 556 to 944) with a mean value of about 13,000 (or about 722 for *B*).

In "Van Nostrand's Magazine" for Feb., 1880, Mr. F. E. Kidder, B.C.E., gives the following results of experiments with 5 yellow pine specimens about 1.25 inches square in section and 8 white pine specimens about 1.5 inches square; all on supports 40 inches apart:

Yellow Pine.

	GREATEST.	MEAN.	LEAST.
Coefficient of elasticity...	1,926,160 lbs....	1,821,630 lbs....	1,707,282 lbs.
K....................	14,654 lbs....	13,048 lbs....	12,280 lbs.
B....................	813 lbs....	725 lbs....	682 lbs.

White Pine.

Coefficient of elasticity ...	1,461,728 lbs	1,388,497 lbs....	1,251,252 lbs.
K....................	9,440 lbs....	8,297 lbs....	7,578 lbs.
B....................	524 lbs....	461 lbs....	421 lbs.

Table III. contains values of B which have been computed from data determined by MM. Chevandier and Wertheim (" Mémoire sur les Propriétés Mécaniques du Bois," 1848). The timber was from the Vosges. The great variations in the length of span and dimensions of beam render these especially valuable.

TABLE III.

Vosges Timber.

	BREADTH.	DEPTH.	SPAN.	W AT CENTRE.	B.
	Ins.	Ins.	Ft.	Lbs.	Lbs.
Fir.	11.4	12.8	42.64	14,120	339
Fir.	10.0	11.2	36.08	11,867	356
Fir.	8.8	9.6	29.52	7,584	287
Fir.	6.7	7.7	29.52	4,580	355
Fir.	3.65	4.85	29.52	1,137	415
Fir.	9.7	2.16	9.91	2,017	445
Fir.	9.5	1.11	9.91	581	500
Oak.	9.2	10.9	18.04	17,356	293
Oak.	8.6	9.3	18.04	15,816	392
Oak.	7.6	8.6	18.04	11,495	376
Oak.	6.3	7.4	18.04	12,155	643
Oak.	5.4	6.3	18.04	4,895	421
Oak.	3.26	3.2	9.84	1,188	354
Oak.	3.07	3.16	8.20	1,617	433
Oak.	11.50	2.15	18.04	957	343
Oak.	5.64	1.66	9.84	825	532
Oak.	9.5	1.11	9.84	715	614

The weights of the beams were allowed for in the manner already shown in that section of this Art. which is headed "*Copper, Tin, Zinc, and their Alloys.*"

TABLE IV.

Laslett's Tests.

Sections 2 × 2 inches with span of 6 feet.

KIND OF TIMBER.	W, IN LBS.	B, IN LBS.	FINAL DEFLECTION.	COEFFICIENT OF ELAS., OR E.
			Inches.	Pounds.
Oak, English	562	422	5.10	——
Oak, English	407	305	3.95	——
Oak, English	813	610	7.71	902,600
Oak, French	877	658	6.00	1,536,800
Oak, French	831	623	7.58	1,440,000
Oak, Tuscan	758	569	7.66	605,000
Oak, Sardinian	758	569	6.50	871,400
Oak, Dantzic	474	356	6.46	——
Oak, Spanish	562	422	6.62	——
Oak, American, white	804	603	8.83	1,184,600
Oak, American, Baltimore	723	542	7.13	1,547,200
Oak, African (or teak)	1,108	831	5.14	1,010,880
Teak, Moulmein	913	685	3.38	1,378,500
Teak, Moulmein	843	632	6.49	1,172,400
Iron wood, Burmah	1,273	955	4.25	2,369,300
Chow, Borneo	975	731	2.83	2,472,300
Greenheart, Guiana	1,333	1,000	4.62	1,057,900
Sabicu, Cuba	1,293	970	3.75	2,369,300
Mahogany, Spanish	856	642	3.45	1,882,800
Mahogany, Honduras	802	602	4.06	1,187,100
Mahogany, Mexican	783	587	3.92	2,021,800
Eucalyptus, Australia :				
Tewart	1,029	772	4.75	1,791,000
mahogany	686	515	4.71	——
iron-bark	1,407	1,055	3.81	2,420,000
blue-gum	712	534	4.21	1,805,100
Ash, English	862	647	8.63	1,404,000
Ash, Canadian	638	479	7.37	——
Elm, English	393	295	5.29	——
Rock elm, Canada	920	690	8.79	1,299,700
Fir, Dantzic	877	658	5.14	1,395,400
Fir, Riga	600	450	3.63	1,763,200
Fir, spruce, Canada	670	503	5.19	1,849,200
Larch, Russia	626	470	4.33	——
Cedar, Cuba	560	420	4.37	——
Red pine, Canada	653	490	4.63	——
Yellow pine, Canada	627	470	4.66	——
Yellow pine, Canada	483	362	3.39	——
Yellow pine, Canada	304	228	3.45	——
Pitch pine, American	1,049	787	4.79	2,030,800
Pitch pine, American	930	698	4.67	1,834,300
Pitch pine, American	744	558	4.42	1,602,000
Kauri pine, New Zealand	719	539	4.00	1,636,300

E has been computed only for those cases in which W exceeds 700.

In the cases of the fir specimens, B increases very considerably as the depth of the beam decreases, and with little irregularity. The same general result seems to hold with the oak specimens, although there are very marked irregularities. On the whole, therefore, these experiments would seem to show unmistakably that B or K has much larger values for small depths of beam than large.

The modulus of rupture, K, may of course be found by taking $18B$, but its values are not given in the table.

Tables IV., V., VI. and VII. contain values of B and E which have been computed from data determined by the English experiments of Messrs. Laslett, Maclure, Fincham, Edwin Clark and G. Graham Smith. These experiments are among the latest and most valuable ever made.

In all these tables W is the total load applied, including the weight of the beam, wherever that correction is made.

In Table IV. the coefficient of elasticity is computed, in all cases, for a centre load of 390 pounds. In Table V. the centre load for the same computation is 1,680 pounds; and in Table VII. the elastic load had different values for different beams.

In all cases, except the four noted in Table VII., the applied loads were placed at the centre of the span.

Although these experiments do not embrace a great variety of cross section for all kinds of timber, yet Tables IV., VI. and VII. give much larger values of B for small depths of pine and fir beams than for large ones. This is a very important consideration in connection with the ultimate resistance of beams, and probably obtains for all kinds of timber. In fact, Table III., as has been observed, indicates the same results for Vosges fir and oak.

These experiments also showed that the coefficient of elasticity, E, varied materially in the same specimen for different deflections, and that values among the greatest may be found with large deflections; also that the "elastic limit" for flexure in timber beams is more conventional than real, since with

TABLE V.

Fincham's Tests.

3 × 3 inches, section ; 4 feet span ; very dry timber.

KIND OF TIMBER.	*W.*	*B.*	COEFFICIENT OF ELASTICITY.
	Pounds.	Pounds.	Pounds.
Riga fir................	4,530	670	2,293,760
Red pine................	3,780	559	1,593,000
Yellow pine	2,756	408	1,550,000
Norway fir..............	3,292	487	1,850,000
Scotch pine	2,520	373	925,000
Kauri pine..............	4,110	608	1,977,400

TABLE VI.

Maclure's Tests.

Specimens of Memel Fir.—1849.

BREADTH.	DEPTH.	SPAN.	*W.*	*B.*	FINAL DEFLECTION, INCHES.
Inches.	Inches.	Feet.	Pounds.	Pounds.	
1	1	$1\frac{1}{3}$	483	644	0.75
1	1	$1\frac{1}{3}$	450	600	0.75
2	2	$2\frac{2}{3}$	1,910	637	1.00
2	2	$2\frac{2}{3}$	1,311	437	1.125
3	3	9	1,104	368	3.5
3	3	9	1,482	494	4.5
6	12	12	34,720	482	2.0
9	12	12	38,080	353	2.5
12	12	12	61,600	428	3.25

TABLE VII.

Tests by Edwin Clark and G. Graham Smith.

TIMBER.	BREADTH.	DEPTH.	SPAN.	*W.*	*B.*	FINAL DEFLECTION.	COEFFICIENT OF ELAS.	
	Inches.	Inches.	Feet.	Pounds.	Lbs.	Inches.	Pounds.	
American red pine.....	12	12	15	33,497	291	4.00	1,443,830	E. Clark.
American red pine.....	12	12	15	29,908	260	3.10	1,155,100	E. Clark.
American red pine	6	6	7.5	7,370	256	1.68	1,015,900	E. Clark.
Memel fir...............	13.5	13.5	10.5	Distrib'd 68,560	293	——	2,150,500	E. Clark.
Memel fir...............	13.5	13.5	10.5	Distrib'd 68,560	293	——	1,561,300	G. Graham Smith.
Baltic fir................	6	12	12.25	19,145	271	1.11	1,573,400	G. Graham Smith.
Baltic fir................	6	12	12.25	23,625	335	1.93	1,442,300	G. Graham Smith.
Pitch pine	6	12	12.25	23,030	326	1.31	3,125,000	G. Graham Smith.
Pitch pine	6	12	12.25	23,700	336	1.31	1,431,300	G. Graham Smith.
Pitch pine	14	15	10.5	134,400	448	1.14	1,935,400	G. Graham Smith.
Pitch pine	14	15	10.5	132,610	442	——	1,693,400	G. Graham Smith.
Red pine	6	12	12.25	16,800	238	——	1,247,000	G. Graham Smith.
Red pine	6	12	12.25	19,040	270	1.94	1,247,000	G. Graham Smith.
Quebec yellow pine....	14	15	10.5	Distrib'd 68,600	229	——	1,329,750	G. Graham Smith.
Quebec yellow pine....	14	15	10.5	Distrib'd 68,600	229	——	1,329,750	G. Graham Smith.
Quebec yellow pine....	14	15	10.5	85,792	286	——	1,270,000	G. Graham Smith.
Quebec yellow pine....	14	15	10.5	76,160	254	——	——	G. Graham Smith.

loads about half the breaking weight, not only the deflection but the "set" varied with the time.

The quantity ordinarily termed the load at the "elastic limit" may be taken from 0.5 to 0.6 the breaking weight. In Table VII. it varied from 0.50 to 0.78.

The latest experiments on timber beams are those of Col. Laidley and Prof. Lanza; both experimented during 1881. Col. Laidley's results are given in Table VII*a*.

As was to be expected, in accordance with conclusions already drawn, the sticks of Oregon pine with the smallest depths gave values of K and B considerably larger than the others. These results emphasize the fact that for large beams K or B must be taken from tests on beams equally large if accurate computations are to be made. With these consider-

TABLE VII*a*.

Seasoned Sticks, Loaded at Centre.

NO.	KIND OF WOOD.	SPAN.	WIDTH.	DEPTH.	$K = 18B$. LBS. PER SQ. INCH.	B.	REMARKS.
		Ins.	Ins.	Ins.			
1	Oregon pine...............	44	3.48	3.48	11,900	661	
2	Oregon pine...............	22	1.22	1.23	13,210	734	Cross grained.
3	Oregon pine...............	22	1.21	1.20	16,570	921	
4	Oregon maple.....	44	3.63	3.63	10,560	587	
5	California laurel...........	44	3.58	3.58	8,920	496	Worm eaten.
6	Ava Mexicana.............	44	3.69	3.69	9,930	552	
7	Oregon ash................	44	3.64	3.64	8,460	470	Cross grained.
8	Mexican white mahogany.	44	3.77	3.77	9,610	534	
9	Mexican cedar............	44	3.75	3.75	7,935	441	
10	Mexican mahogany........	44	3.75	3.75	15,830	879	

TABLE VII*b*.

Seasoned Spruce Beams.

NO.	SPAN.	WIDTH.	DEPTH.	MANNER OF LOADING.	$K = 18B$. LBS. PER SQUARE INCH.	B.
	Feet.	Inches.	Inches.			
1	15.00	2.00	12.00	At centre.	5,526	307
2	6.60	2.00	9.00	" "	5,389	299
3	15.00	2.00	12.00	" "	5,237	291
4	6.67	2.75	9.00	" "	4,082	226
5	4.00	3.00	9.00	" "	3,285	183
6	10.00	3.00	9.00	" "	4,508	250
7	15.00	3.00	9.00	" "	5,651	314
8	20.00	3.90	12.00	" "	4,253	237
9	10.00	2.50	13.50	" "	3,787	210
10	16.00	3.75	12.00	4.5 feet from one end.	3,271	182
11	7.00	7.00	2.00	At centre.	8,748	486
12	7.00	1.75	6 75	" "	7,562	420
13	6.67	3.00	9.00	" "	4,931	274
14	6.67	3.00	9 00	At 4 points, 16 ins. apart.	4,961	276
15	16.00	3.90	12.00	4.5 feet from one end.	5,218	289

ations in view, Prof. Lanza's results for large spruce beams, which are given in Table VII*b*., possess great value.

With the exception of Nos. 11 and 12 the material was common merchantable lumber.

Timber Beams of Natural and Prepared Wood.

Table VII*c*. contains the results of some experiments by A. M. Wellington, C.E. ("R. R. Gazette," Dec. 17, 1880) on small specimens 1¼ inches square and 15 inches between supports. "All the woods, except as specified, had been cut six to eight months and were partially seasoned."

TABLE VII*c*.

Specimens 1.25 *inches square*, 15 *inches long*.

KIND OF TIMBER.	NATURAL.		PREPARED.		LOSS, PER CENT.
	W, in Lbs.	*B*, in Lbs.	*W*, in Lbs.	*B*, in Lbs.	
White oak, well seasoned..	989	633	——	——	——
White ash	926	593	825	527	11.2
Beech	864	553	801	513	7.2
Elm.......................	763	489	763	489	0.0
Pin oak...................	941	602	755	482	20.0
White oak, green.........	747	479	——	——	——
Soft maple	742	476	643	411	13.7
Black ash..................	685	439	640	409	6.9
Sycamore	628	401	550	332	17.2

The "prepared" specimens had been treated by the Thilmeny (sulphate of baryta) process; and all specimens of the same kind of wood were cut from the same stick.

The column "*Loss*" is the per cent. of loss caused by the preservative process employed.

Cement, Mortar and Concrete.

Table VIII. and Table IX. contain values of K computed from data given by Gen'l Gillmore in his "Limes, Hydraulic Cements and Mortars," 1872. All the prisms were 2 inches square in cross section and 8 inches long, and were broken by the weight W, which was applied at the centre of a 4-inch span. K is computed by Eq. (5), all dimensions being in inches. The composition is shown in the tables. The pure mortars of Table VIII. were kept 24 hours in a damp place, and then immersed in salt water until broken. Nos. 1, 2, 3 and 4 were 59 days old; the others, 320. As a rule, those which set under pressure were considerably stronger than the others.

In Table IX., all the prisms set under a pressure of 32 pounds per square inch, and were kept in sea water, after the first 24 hours, until broken.

Many reliable experiments, such as those which follow, show that when masonry is built in a strictly first-class manner, its transverse resistance is very considerable.

Table X. is taken from a paper entitled "Notes and Experiments on the Use and Testing of Portland Cement," by Wm. W. Maclay, C.E., in the "Trans. Am. Soc. of Civ. Engrs.," 1877.

The concrete prisms were six inches square in cross section and two feet long, and rested on supports one foot apart. W was applied at the centre of the span. If W_1 is the weight of the prism whose length is equal to the span, Eq. (5) becomes:

$$K = \frac{3}{2}\frac{(W + \frac{1}{2}W_1)l}{bh^2} \quad \ldots\ldots \quad (16)$$

in which b, h and l are to be taken in inches.

TABLE VIII.

Section of Prism 2 inches square. Supports 4 inches apart.

NO.	KIND OF CEMENT	COMPOSITION OF MORTAR.	PRESSURE PER SQUARE INCH WHILE SETTING.	W, AT CENTRE.	K.	N.
				Pounds.	Pounds.	
1	James River	4 vols. dry cement, 2.6 vols. water	0.00	281.5	211	8
2	James River	4 vols. dry cement, 1.4 vols. water	32.00	497.5	373	5
3	James River	4 vols. dry cement, 1.4 vols. water	0.00	288.8	217	4
4	James River	4 vols. dry cement, 2 vols. water	0.00	250.8	188	6
5	James River	Pure cement and water, thin	32.00	409.8	308	4
6	James River	Pure cement and water, stiff	32.00	392.4	294	5
7	Rosendale, Hoffman Brand	Pure cement and water, thin	0.00	646.0	485	2
8	Rosendale, Hoffman Brand	Pure cement and water, very thin	0.00	400.0	300	2
9	Rosendale, Hoffman Brand	Pure cement and water, thin	32.00	692.5	519	4
10	Rosendale, Hoffman Brand	Pure cement and water, stiff	32.00	635.5	477	3
11	Rosendale, Delafield & Baxter	Pure cement and water, thin	0.00	613.0	459	2
12	Rosendale, Delafield & Baxter	Pure cement and water, very thin	0.00	418.0	314	6
13	Rosendale, Delafield & Baxter	Pure cement and water, stiff	32.00	871.5	654	8

"K" is a mean of the number of experiments shown in column N.

TABLE IX.

Section of Prisms 2 inches square. Supports 4 inches apart.

KIND OF CEMENT.		K, POUNDS PER SQUARE INCH.		
		Pure cement.	1 vol. cement. 1 vol. sand.	1 vol. cement. 2 vols. sand.
English Portland (artificial)	320 days old.	1,152	945	713
Cumberland, Md	320 days old.	716	690	419
Newark and Rosendale	320 days old.	631	420	375
Delafield and Baxter (Rosendale)	320 days old.	627	519	399
" Hoffman," Rosendale	320 days old.	637	456	——
" Lawrence," Rosendale	320 days old.	583	——	——
Round Top, Md	320 days old.	——	450	——
Utica, Ill	320 days old.	549	567	422
Sheperdstown, Va	320 days old.	560	464	338
Akron, N. Y.	320 days old.	573	489	453
Kingston and Rosendale	320 days old.	540	417	375
Sandusky, Ohio	320 days old.	416	348	——
James River, Va	320 days old.	——	468	479
Lawrenceville Manf. Co. (Rosendale)	1 year old.	——	683	——
Sandusky, Ohio	1 year old.	602	——	——
Kensington, Conn.	1 year old.	716	532	380
Lawrence Cement Co., " Hoffman " Brand	1 year old.	656	684	——
Round Top, Md	1 year old.	——	630	——

In Mr. Maclay's experiments, since the span was twelve inches and the ends overhung six inches, K was computed by the formula:

$$K = \frac{3}{2}\frac{Wl}{bh^2} = \frac{W}{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Table XI. contains the results of some French experiments cited by Gen. Gillmore in his " Limes, Hydraulic Cements and Mortars." The concrete prisms were of Boulogne Portland cement, about 5.91 inches square in section, and broken by a weight (W) at the centre of a span of about 31.5 inches. K was computed by Eq. (16).

Table XII. gives the results of trials of concrete prisms by Gen. Totten, in June, July and August, 1837, the prisms hav-

TABLE X.

Concrete Prisms 6″ × 6″ × 2′. *Supports* 1 *foot apart.*

T.	T'.	DISPOSITION OF PRISMS AFTER BEING MADE		W.	K.
Fahr.	*Fahr.*			Pounds.	Pounds.
18°	40°	Placed in North River.	Age of all prisms 1 mo.	525	44
18°	40°	Exposed outside......		775	60
18°	40°	Kept indoors.........		1,125	94
18°	98°	Placed in North River.		175	15
18°	98°	Exposed outside......		325	27
18°	98°	Kept indoors.........		750	63
24°	40°	Exposed outside......		1,800	150
24°	97°	" "		800	67
32°	40°	" "		1,475	123
32°	98°	" "		700	58

All prisms were of Portland cement concrete; 1 vol. cement, 2 vols. sand, 5 vols. small broken stone.

T = temperature of air when concrete was mixed.

T' = temperature of concrete when mixed.

ing been made in Dec., 1836. The cement was from Ulster Co., N. Y. The lime (slightly hydraulic) was from Fort Adams, R. I., where the tests were made. W (the centre breaking weight) and K are in pounds.

These experimental results on the flexure of solid beams in cement, cement mortar and concrete, in connection with those of Gen. Gillmore on the adhesion of bricks and cement or cement mortar, show that masonry beams may have considerable transverse resistance; and such resistance may be an important element of strength in some arches or similar masonry structure. It should be borne in mind, however, that such a conclusion is implicitly based on the assumption of perfect manipulation of the cement and mortar and the most conscientious care in laying the masonry. These ends were attained in the test specimens, but it is probably safe to say that such is not the case even in what is termed first-class masonry.

TABLE XI.

Portland Cement of Boulogne-Sur-Mer.

MORTARS.				CONCRETES.			RESISTANCE OF CONCRETE.						
Composition in vols.			Volume of mortar produced.	Comp. in vols.		Volume of concrete produced.	W, in pounds.			W_1.	K, in pounds.		
Cement.	Sea sand.	Water.		Mortar.	Wet pebbles.		After 10 days.	After 20 days.	After 60 days.	Pounds.	After 10 days.	After 20 days.	After 60 days.
1	1	0.62	1.69	1	1	1.56	1,087	1,093	1,376	92	261	262	328
				$\frac{1}{2}$	1	1.03	800	1,322	1,504	99	196	339	360
				$\frac{1}{3}$	1	1.	856	1,065	1,480	95	208	257	352
				$\frac{1}{4}$	1	1.	492	646	481	92	123	160	122
$\frac{1}{2}$	1	0.43	1.24	1	1	1.45	778	889	1,294	90	190	215	299
				$\frac{2}{3}$	1	1.11	778	954	1,016	92	190	231	235
				$\frac{1}{2}$	1	1.00	492	668	906	92	123	165	219
$\frac{1}{3}$	1	0.38	1.12	1	1	1.40	404	448	430	88	103	113	109
				$\frac{2}{3}$	1	1.11	315	463	633	92	83	117	157
				$\frac{1}{2}$	1	1.03	271	359	600	88	73	93	148
$\frac{1}{4}$	1	0.35	1.05	1	1	1.40	227	346	542	90	63	90	135
				$\frac{2}{3}$	1	1.14	149	289	437	92	45	77	111
				$\frac{1}{2}$	1	1.01	163	240	404	92	48	66	104
$\frac{1}{5}$	1	0.34	1.	1	1	1.45	141	304	392	88	42	80	101
				$\frac{2}{3}$	1	1.13	114	218	326	88	37	60	85
				$\frac{1}{2}$	1	1.03	114	202	306	88	37	56	77
$\frac{1}{6}$	1	0.32	0.96	1	1	1.45	191	192	381	90	55	55	98
				$\frac{2}{3}$	1	1.13	136	196	337	90	42	56	88
				$\frac{1}{2}$	1	1.03	176	181	370	90	51	52	96

TABLE XII.

Prisms 6 *ins.* × 6 *ins.* × 12 *ins.—Supports* 9 *ins. apart.*

COMPOSITION OF CONCRETE.	COMPOSITION OF MORTAR.								
	Cement, 1.00 Sand, 0.00 Lime, 0.00	Cement, 1.00 Sand, 0.50 Lime, 0.00	Cement, 1.00 Sand, 0.50 Lime, 0.25	Cement, 1.00 Sand, 1.00 Lime, 0.00	Cement, 1.00 Sand, 1.00 Lime, 0.25	Cement, 1.00 Sand, 1.50 Lime, 0.00	Cement, 1.00 Sand, 1.50 Lime, 0.25	Cement, 1.00 Sand, 2.00 Lime, 0.00	Cement, 1.00 Sand, 2.00 Lime, 0.25
Granite fragments...	W = 4,973	4,142	2,778	3,989	2,721	2,045	2,056	—	1,574
1 measure of mortar.	K = 311	260	174	251	171	129	130	—	99
Granite fragments...	W = 4,068	4,983	5,064	4,088	5,366	1,547	3,537	1,643	1,972
2 measures of mortar.	K = 255	312	317	313	336	—	—	—	—
Brick fragments. ...	W = 3,242	2,117	—	4,127	3,254	1,788	2,136	1,567	3,649
1 measure of mortar.	K = 204	133	—	259	205	113	134	98	229
Brick fragments.....	W = 2,805	5,047	2,826	4,232	1,178	3,655	3,856	2,320	4,803
2 measures of mortar.	K = 176	316	277	265	74	229	242	146	301
Stone gravel........	W = 1,097	1,049	1,240	1,256	1,066	—	—	—	—
1 measure of mortar.	K = 69	66	78	79	67	—	—	—	—
Stone gravel........	W = 2,347	4,247	2,655	1,295	3,351	—	—	—	—
2 measures of mortar.	K = 147	267	167	82	210	—	—	—	—
Brick gravel.... ...	W = 5,437	6,183	3,088	—	4,726	—	—	—	—
1 measure of mortar.	K = 341	387	194	—	296	—	—	—	—
Brick gravel........	W = 6,025	5,712	5,480	3,142	2,699	—	—	—	—
2 measures of mortar.	K = 377	358	343	197	169	—	—	—	—
Stone fragments	W = 3,278	1,846	2,012	1,158	1,178	—	—	—	—
grouted..........	K = 206	116	127	73	74	—	—	—	—
Stone fragments	W = 1,634	2,305	2,869	2,726	2,770	—	—	—	—
grouted..........	K = 103	145	180	171	111	—	—	—	—

Stone Beams.

But few experiments have been made on the transverse resistance of the different kinds of stone. The following values of K have been computed from the experiments of R. G. Hatfield ("Transverse Strains") and Gen. Gillmore ("Building Stones").

	$B.$	$K = 18B.$	
Blue Stone Flagging	200 lbs.	3,600 lbs.	Hatfield.
Sandstone	59 lbs.	1,062 lbs.	Hatfield.
Brick, common	33 lbs.	594 lbs.	Hatfield.
Brick, pressed	37 lbs.	666 lbs.	Hatfield.
Marble, Eastchester	147 lbs.	2,646 lbs.	Hatfield.
Granite, Millstone Point (doubtful)	133 lbs.	2,390 lbs.	Gillmore.
Marble, Eastchester	128 lbs.	2,300 lbs.	Gillmore.
Granite, Keene, N. H	103 lbs.	1,860 lbs.	Gillmore.

All beams were broken by centre weights. The last three tests were with prisms 2 ins. × 2 ins. × 6 ins., over a span which was taken at 3 inches.

Practical Formulæ for Solid Beams.

The quantities B, K and E, which have been established, form a practical basis on which the deflection and ultimate resistance of solid beams are to be computed.

Breaking weight (in pounds) at centre of circular beam, Eq. (6):

$$W = \frac{KAr}{l} = \frac{\pi r^3 K}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

If W is a uniform load:

$$W = \frac{2KAr}{l} = \frac{2\pi r^3 K}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

In Eqs. (18) and (19), A (the area), r (the radius) and l (the span) are to be taken in inches.

Breaking weight (in pounds) at centre of rectangular beams, Eq. (5):

$$W = \frac{2}{3}\frac{KAh}{l} = \frac{2}{3}\frac{bh^2K}{l} \quad . \; . \; . \; . \; . \quad (20)$$

If W is a uniform load:

$$W = \frac{4}{3}\frac{KAh}{l} = \frac{4}{3}\frac{bh^2K}{l} \quad . \; . \; . \; . \; . \quad (21)$$

In Eqs. (20) and (21), A (the area), b (the breadth), d (the depth) and l (the span), are to be taken in inches.

If l is expressed in feet, and all other dimensions in inches, Eq. (20) becomes:

$$W = B\frac{Ah}{l} = B\frac{bh^2}{l} \quad . \; . \; . \; . \; . \quad (22)$$

and Eq. (21):

$$W = 2B\frac{Ah}{l} = 2B\frac{bh^2}{l} \quad . \; . \; . \; . \; . \quad (23)$$

Deflection (in inches) at centre of circular beams:

$$w = \frac{(W + \frac{5}{8}pl)l^3}{12E\pi r^4} = \frac{(W + \frac{5}{8}pl)l^3}{12EAr^2} \quad . \; . \; . \quad (24)$$

Deflection (in inches) at the centre of rectangular beams:

$$w = \frac{(W + \frac{5}{8}pl)l^3}{4Ebh^3} = \frac{(W + \frac{5}{8}pl)l^3}{4EAh} \quad . \; . \; . \quad (25)$$

In Eqs. (24) and (25), W is the centre load, and pl the total uniform load, expressed in pounds; while A (area), l^3 (cube of span), r (radius), b (breadth), and d (depth), are to be taken in inches. If there is no uniform load, pl is zero; and if there is no centre load, W is zero.

Comparison of Modulus of Rupture for Bending with Ultimate Resistances.

The experiments on solid beams which have been cited, show the somewhat remarkable result that, in general, K has neither the value of the ultimate resistance to tension nor of that to compression; nor, indeed, in some cases, is there anything like a well defined relation between those quantities. If those ultimate resistances have widely different values, K may be found between them; in other cases it may considerably exceed either. In no case, however, it may safely be asserted, will it be found less than both. These investigations show that K varies with the *kind* of cross section, and it is altogether probable that it also varies with *varying proportions* of the same kind of cross section. Experimental data for the determination of this point, however, are still lacking.

In the absence of experiments conducted in a manner proper to the solution of this problem, it is difficult to assign confidently the reason for the facts as they appear.

The explanation will probably be found in the effects of the following causes, while it is borne in mind that with the small ratios of span to depth usually found in connection with solid beams, the common theory of flexure is only loosely approximate, and hence, that the greatest intensity shown by the common formulæ is probably considerably different from the actual.

The varying intensity in adjacent fibres prevents perfect freedom in lateral strains, and causes a corresponding increase in resistance. In the experiments which have been made, the place of greatest intensity of stress is exceedingly small, thus placing the part first ruptured somewhat in the condition of a very short specimen. Again, after the elastic limit is passed, in consequence of the flow of the material, it is highly probable that the law of the variation of stress intensity changes and becomes such that, with the same greatest intensity at the surface of the solid beam, the *resisting moment* is considerably increased.

Finally, it has been shown that the experimentally determined ultimate resistances to tension and compression are, in reality, mean intensities, and not the greatest which the material is capable of exerting at any one point, or along any one line, as in the extreme fibres of a bent beam. On this ground alone, K ought to be considerably greater than either T or C, as determined from the usual cross sections.

Art. 64.—Flanged Beams with Unequal Flanges.

In the beams which are to follow, the material is distributed in a much more advantageous manner, in respect of its resisting moment, than in the solid beams which have been heretofore treated. In these beams, it will be found, in almost all cases, that the ultimate intensity of bending stress, at the point which first ruptures, is equal either to the ultimate resistance to tension or compression. In this respect, at least, therefore, the ultimate load for flanged beams is more easily and exactly determined than for solid ones.

In Fig. 1 is shown a "flanged beam." The "flanges" are the two horizontal parts above and below; the "web" is the vertical part uniting the two flanges so as to form the perfect beam.

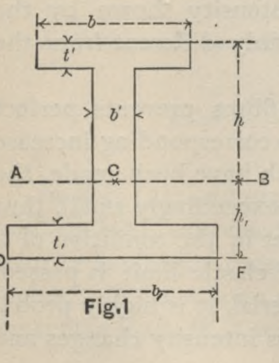


Fig. 1

In order that there may be economy of material in the beam, neither flange must begin to fail before the other; in other words, the two exterior layers of fibres, above and below, must begin to fail at the same time.

The intensities, then, in these two

layers must, at the instant of rupture, equal the ultimate resistances to tension and compression in bending.

Equal Coefficients of Elasticity.

By the common theory of flexure, if the two coefficients of elasticity are equal, it has been shown that if C is the centre of gravity of the cross section, the neutral axis of the section will pass through that point. Let it now be supposed that the lower flange is in tension while the upper is in compression. Also let T represent the ultimate resistance to tension in bending, and let C represent the same quantity for compression in bending. Then, since intensities vary directly as distances from the neutral axis,

$$\frac{h_1}{h} = \frac{T}{C}; \qquad \therefore \qquad h_1 = h\frac{T}{C} = n'h \quad . \quad . \quad . \quad (1)$$

The ratio between ultimate intensities is represented by n'. If $d = h + h_1$ is the total depth of the beam, and hence $h = d - h_1$:

$$h_1 = n'(d - h_1) = \frac{dn'}{1 + n'} = \frac{\frac{T}{C}d}{1 + \frac{T}{C}} \quad . \quad . \quad . \quad . \quad (2)$$

If, as an example, for mild steel there be taken:

$$n' = \frac{T}{C} = 0.75; \ h_1 = \frac{4}{7} \cdot \frac{3}{4} d = \frac{3d}{7}.$$

The relation between h and h_1 shown in Eq. (2) is entirely independent of the form of cross section. But according to the principles just given, in order that economy of material shall obtain, *the cross section should be so designed that h and h_1 shall represent the distances of the centre of gravity from the exterior fibres.*

The analytical expression for the distance of the centre of gravity from DF is:

$$x_1 = \frac{\frac{1}{2}b'd^2 + (b - b')t'(d - \frac{1}{2}t') + \frac{1}{2}(b_1 - b')t_1^2}{b'd + (b - b')t' + (b_1 - b')t_1} \quad . \; . \; (3)$$

The meaning of the letters used is fully shown in the figure. In order that the beam shall be equally strong in the two flanges, the various dimensions of the beam must be so designed that

$$x_1 = h_1 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; (4)$$

It would probably be found far more convenient to cut sections out of stiff manilla paper and balance them upon a knife edge.

The moment of inertia about the axis AB, thus determined, is:

$$I = \frac{1}{3}[bh^3 + b_1h_1^3 - (b - b')(h - t')^3 - (b_1 - b')(h_1 - t_1)^3].$$

This value is to be substituted in Eq. (2) of Art. 62, now changed to

$$M = \frac{CI}{h} = \frac{TI}{h_1}.$$

For various beams whose lengths are l and total load W, the greatest value of M becomes:

Cantilever uniformly loaded:

$$M = \frac{Wl}{2}.$$

Cantilever loaded at end:

$$M = Wl.$$

Beam supported at each end and uniformly loaded:

$$M = \frac{Wl}{8} = \frac{pl^2}{8}.$$

Beam supported at each end and loaded at centre:

$$M = \frac{Wl}{4}.$$

The last two cases combined:

$$M = \frac{l}{4}\left(W + \frac{pl}{2}\right).$$

Sometimes the resistance of the web is omitted from consideration. In such a case the intensity of stress in each flange is assumed to be uniform and equal to either T or C. At the same time the lever arms of the different fibres are taken to be uniform, and equal to h for one flange and h_1 for the other, *h and h_1 now representing the vertical distances from the neutral axis to the centres of gravity of the flanges, while $d = h + h_1$.*

On these assumptions, if f is the area of the upper flange, and f' that of the lower, there will result:

$$M = fC \cdot h + f'T \cdot h_1 \quad \ldots \ldots \quad (5)$$

But since the case is one of pure flexure:

$$fC = f'T \quad \ldots \ldots \ldots \ldots \quad (6)$$

$$\therefore \quad M = fC(h + h_1) = fCd = f'Td \quad \ldots \quad (7)$$

Also, from Eq. (6):

$$\frac{f}{f'} = \frac{T}{C} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Or, *the areas of the flanges are inversely as the ultimate resistances.*

Unequal Coefficients of Elasticity.

All these results presuppose equality between the coefficients of elasticity for tension and compression. In some cases this presumption is not permissible. To the formulæ of Art. 27 resort must then be made.

The neutral surface must first be located. If d is the total depth of the beam, $h_1 = d - h$; h, then, must be found. Eq. (5) of Art. 27, when applied to Fig. 1; becomes:

$$E'\left[\frac{bh^2}{2} - \frac{(b - b')(h - t')^2}{2}\right] = E\left[\frac{b_1(d - h)^2}{2} - \frac{(b_1 - b')(d - h - t_1)^2}{2}\right];$$

E' representing the coefficient of elasticity for compression, and E that quantity for tension.

Performing the operations indicated and reducing, writing n for $E' \div E$:

$$(n - 1)b'h^2 + 2[nt'(b - b') + t_1(b_1 - b') + b'd]h$$

$$= nt'^2(b - b') + (2d - t_1)(b_1 - b')t_1 + b'd^2. \quad . \quad (9)$$

h is to be measured on the compression side of the beam.

This is a quadratic equation of condition for the determination of h. It is best to leave it as it is until the numerical substitutions are made and then to solve it. h_1 immediately results from the equation $h_1 = d - h$.

Frequently there is no compression flange, the section being like that shown in Fig. 2. In such a case b is equal to b', or t' is equal to zero; hence the two terms $nt'(b - b')$ and $nt'^2(b - b')$ in Eq. (9) disappear. No other change occurs.

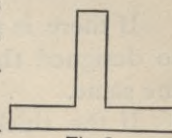


Fig. 2

Eq. (1) of Art. 27 then gives the following resisting moment of the section:

$$M = \frac{C}{3h}\left(bh^3 - (b - b')(h - t')^3 + \frac{b_1 h_1^3}{n} - \frac{(b_1 - b')(h_1 - t_1)^3}{n}\right) \quad . \; . \; . \; . \; . \quad (10)$$

C is the greatest intensity of stress in the section of the same kind as E'.

If the section is like Fig. 2, b again equals b' and the term $(b - b')$ $(h - t')^3$ in Eq. (10) disappears, but nothing else is changed.

If T is the greatest stress on the other side of the neutral surface from C:

$$M = \frac{T}{3h_1}[nbh^3 - n(b - b')(h - t')^3 + b_1 h_1^3 - (b_1 - b')(h_1 - t_1)^3] \quad . \; . \; . \; . \; . \; . \quad (11)$$

In order that the beam may be equally strong in the two flanges, the ratio between h and h_1, as determined by Eq. (9), should be the same as that determined by the following process. If u is the rate of strain at units' distance from the neutral surface:

$$\left.\begin{array}{r} E'uh = C \\ Euh_1 = T \end{array}\right\} \therefore \frac{h}{h_1} = \frac{CE}{TE'} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If there is no waste of material, the cross section must be so designed that the ratios given by Eqs. (9) and (12) will be the same.

If the thicknesses of the flanges t' and t_1 are small compared with the depth d of the beam, and if b' also is small, *i. e.*, if the flanges are assumed to give the *whole* resistance to bending while the web takes up the shear, Eqs. (10) and (11) may be much simplified.

Making, therefore, $b' = 0$ in Eq. (10), putting $\frac{C}{hn} = \frac{T}{h_1}$ and then expanding the quantities $(h - t')^3$ and $(h_1 - t_1)^3$:

$$M = Cbt'\left(h - t' + \frac{t'^2}{3h}\right) + Tb_1t_1\left(h_1 - t_1 + \frac{t_1^2}{3h_1}\right).$$

Under the conditions taken, $Cbt' = Tb_1t_1$; also, $\frac{t'^2}{3h}$ and $\frac{t_1^2}{3h_1}$ are very small and may be neglected. Hence,

$$M = Cbt'(d - t' - t_1) = Tb_1t_1(d - t' - t_1) \quad . \quad (13)$$

But both of these approximations have made M *too small*. As an approximate compensation, therefore, $-\left(\frac{t' + t_1}{2}\right)$ may be written for $-(t' + t_1)$. The moment then becomes:

$$M = Cbt'\left(d - \frac{t' + t_1}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The quantity within the parenthesis of the second member of this equation is evidently the distance between the centres of gravity of the flanges, while the quantity $Cbt' = Tb_1t_1$, is simply the flange stress. Eq. (14) is, then, identical with Eq. (7), as was to be anticipated. The equality of flange stresses gives:

$$\frac{bt'}{b_1t_1} = \frac{T}{C};$$

a relation identical with Eq. (8).

If desirable, an approximate correction for the neglect of the web may be introduced in Eq. (14). It has been seen that that equation is precisely the same as if E' were equal to E, *i.e.*, as if the two coefficients of elasticity were equal. Now, it will be shown in the next Article that if $E' = E$, the resistance of the web to bending is equal to that of one-sixth of its area of normal section concentrated in each flange. Hence, if a is the area of the normal section of the web, since bt' and b_1t_1 are areas of the normal sections of the upper and lower flanges, there may be approximately written:

$$M = C\left(bt' + \frac{a}{6}\right)\left(d - \frac{t' + t_1}{2}\right)$$

$$= T\left(b_1t_1 + \frac{a}{6}\right)\left(d - \frac{t' + t_1}{2}\right) \quad . \quad . \quad . \quad (15)$$

Values of C and T may be determined by experiment.

In the case of solid beams, it has been seen that if r and r' are certain ratios, $K = rT$ or $r'C$. Hence, since the web of a flanged beam is really a solid beam subjected to flexure, Eq. (15) may be written:

$$M = TD\left(a' + \frac{ra}{6}\right) = CD\left(a'' + \frac{r'a}{6}\right) \quad . \quad . \quad . \quad (16)$$

In which,

$D = d - \frac{t' + t_1}{2}$ = depth between flange centres;

$a' = b_1 t_1$ = area of bottom flange;

$a'' = bt'$ = area of top flange.

Cast-Iron Flanged Beams.

In the preceding Article it has been seen that r is equal to about 2 for a solid bar with square cross section. This would make $r \div 6 = \frac{1}{3}$. A few imperfect experimental indications, however, seem to indicate a decrease of r for a greater ratio of depth to breadth. Let, therefore, $r \div 6 = 0.25$. Eq. (16) then becomes:

$$M = TD\left(a' + \frac{a}{4}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

If W = centre breaking load in pounds;

W_1 = total uniform breaking load in pounds;

l = span in feet;

$12\,l$ = span in inches:

$$\frac{W \cdot 12l}{4} = 3Wl = TD\left(a' + \frac{a}{4}\right):$$

$$\therefore \quad W = \frac{TD\left(a' + \frac{a}{4}\right)}{3l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

In the same manner

$$W_1 = 2\,\frac{TD\left(a' + \frac{a}{4}\right)}{3l} \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Or, if pl is the weight of the beam, supposed uniformly distributed,

$$\left(W + \frac{pl}{2}\right) = \frac{TD\left(a' + \frac{a}{4}\right)}{3l} \quad . \quad . \quad . \quad . \quad (20)$$

It has been shown under the head of "Tension" that T varies from 15,000 pounds per square inch, for ordinary castings, to 30,000 for those of extra quality. In Eqs. (18), (19) and (20),

D must be taken in inches ;
a and a' in square inches ; and
l in feet.

Those equations have been verified in a most satisfactory manner by the numerous English experiments of Hodgkinson and Cubitt (" Experimental Researches," etc., by Eaton Hodgkinson, F.R.S., 1846), and Berkley (" Proc. Inst. of Civil Engineers," Vol. XXX.), as is shown by the following table. This table gives the actual centre breaking weights W, of the different beams, together with the values of W computed by the formula of Mr. D. K. Clark (" Rules, Tables and Data "), which is essentially identical with Eq. (18); Mr. Clark taking the total depth minus the depth of the lower flange instead of "D," and "$0.28a$," or "$0.29a$," instead of "$0.25a$."

As the results are given to confirm the accuracy of the formulæ under consideration, they are stated in tons of 2,240 pounds. Nos. 17, 27 and 34 were of the form shown in Fig. 2 ; the others had sections like Fig. 1. The results for those three beams are not satisfactory, and Eq. (10) should therefore be used in all such cases where anything more than a very loose approximation is desired. In that Eq. n may be taken equal to unity, on account of the great irregularities in the ratio of the two coefficients of elasticity. Since, in this case (see Fig. 1), $b = b'$ Eq. (10) becomes :

$$M = \frac{C}{3h}\left[bh^3 + b_1h_1^3 - (b_1 - b')(h_1 - t_1)^3\right] \quad . \quad . \quad (21)$$

Cast-Iron Flanged Beams.

NO.	SPAN.	CENTRE DEPTH.	PROPORTION, UPPER FLANGE TO LOWER.	COMPUTED W (TONS).	ACTUAL W (TONS).
	Feet.	Inches.			
1	4.5	5.125	1 to 1	2.47	2.98
2	4.5	5.125	1 to 2	3.27	3.29
3	4.5	5.125	1 to 4	3.83	3.69
4	4.5	5.125	1 to 4	3.87	3.64
5	4.5	5.125	1 to 4.5	4.68	4.79
6	4.5	5.125	1 to 4	6.45	6.46
7	4.5	5.125	1 to 5.5	7.85	7.47
8	4.5	5.125	1 to 3.2	6.49	6.71
9	4.5	5.125	1 to 4.3	8.04	7.54
10	4.5	5.125	1 to 5.6	9.56	8.68
11	4.5	5.125	1 to 6	10.98	11.65
12	4.5	5.125	1 to 7	11.00	10.40
13	4.5	5.125	1 to 6.7	9 02	9.40
14	7.0	6.93	1 to 6	10.26	9.90
15	7.0	4.10	1 to 6	5.41	6.05
16	9.0	10.25	1 to 8.3	13.28	12.80
17	4.5	5.125	none	3.83	3.93
18	4.5	5.125	1 to 4	9.67	10.00
19	4.5	5.125	1 to 4	9.67	10.00
20	4.5	5.125	1 to 5.5	11.85	11.75
21	4.5	5.125	1 to 5.5	11.85	11.85
22	4.5	5.125	1 to 7	16.47	14.25
23	4.5	5.125	1 to 7	17.08	18.00
24	18.0	17.0	1 to 4.6	24.93	25.00
25	11.67	9.0	1 to 1.33	21.24	20.00
26	27.4	30.5	1 to 2.1	94.64	76.60
27	23.1	36.1	none	330.00	153.00
28	15.0	7.15	1 to 3.6	7.75	7.00
29	15.0	7.17	1 to 3.6	7.96	7.13
30	15.0	10.75	1 to 2.3	11.02	11.50
31	15.0	10.75	1 to 2.3	11.71	12.00
32	15.0	12.75	1 to 2.7	11.95	10.25
33	15.0	12.8	1 to 2.25	14.89	15.75
34	15.0	14.0	none	18.39	12.38
35	15.0	17.25	1 to 2.2	19.39	16.00
36	7.5	7.15	1 to 3.4	15.63	15.63
37	7.5	10.75	1 to 2.25	21.76	23.87

If the weight of the beam is taken into consideration, as in Eq. (20):

$$M = \left(W + \frac{pl}{2}\right)3l.$$

A mean of three of Mr. Hodgkinson's beams of 4.5 feet span, 5.125 inches depth, gave:

$$W + \frac{pl}{2} = 8{,}766 \text{ lbs.,} \quad \text{and} \quad C = 45{,}700 \text{ lbs.}$$

One of Mr. Cubitt's beams of 15 feet span and 14 inches depth, gave:

$$W + \frac{pl}{2} = 28{,}100 \text{ lbs.,} \quad \text{and} \quad C = 30{,}850 \text{ lbs.}$$

The bottom flange of this beam was unsound:

C must necessarily depend upon the span, since that portion of the web which is subjected to compression is somewhat in the condition of a long column. This, indeed, is true of the compression flange of any flanged beam, but the effects resulting from such a condition are much more marked in the class of beams shown in Fig. 2.

If, then, W is the centre breaking weight and W_1 the total uniform breaking load (not including the weight of the beam), Eq. (21) becomes:

$$W = \frac{W_1}{2} = \frac{C}{9lh}\left[bh^3 + b_1h_1^3 - (b_1 - b')(h_1 - t_1)^3\right] - \frac{pl}{2}. \quad (22)$$

In this equation, l must be taken in feet and other dimensions in inches.

For 5 feet span C may be taken at 45,000 lbs.
For 15 feet span C may be taken at 35,000 lbs.

In order that a beam with top and bottom flanges may give the best result, *i.e.*, reach its ultimate resistance in each flange at the same time, Mr. Hodgkinson found that the area of the lower flange section should equal about six times that of the upper. That relation has been anticipated in Eq. (8).

Deflection of Cast-Iron Flanged Beams.

If W is the centre load in pounds, l and w the span and centre deflection, respectively, in inches, and I the moment of inertia of the cross section, Eq. (8) of Art. 24 gives:

$$E = \frac{Wl^3}{48wI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Or, if l is in feet, which is more convenient:

$$E = \frac{36Wl^3}{wI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

A mean of two of Mr. Berkley's beams gave:

$l = 4.5$ feet; $w = 0.284$ inch; $W = 20{,}160$ lbs.:

$I = 18.74$. Hence: $E = 12{,}424{,}600$ lbs.

A mean of two of Mr. Cubitt's beams gave:

$l = 15$ feet; $w = 0.465$ inch; $W = 11{,}200$ lbs.;

$I = 227.03$. Hence: $E = 12{,}886{,}720$ lbs.

The four preceding beams had top and bottom flanges, as in Fig. 1. Another of Mr. Cubitt's beams, without top flange, as in Fig 2, gave:

$l = 15$ feet; $w = 0.41$ inch; $W = 13,440$ lbs.;

$I = 373$. Hence: $E = 10,679,400$ lbs.

This last beam had a defective bottom flange, hence there may be taken without essential error:

$$E = 12,000,000 \text{ lbs.}$$

Taking l in feet, Eq. (24) now gives for the centre deflection:

$$w = \frac{3W'l^3}{1,000,000I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

in which W' is either the centre load, or five-eighths (⅝ths) the total uniform load, as the case may be.

The formula by which I is to be computed is the one which immediately follows Eq. (4).

Wrought-Iron **T** *Beams.*

The wrought-iron **T** beam is the most important beam of that material with unequal flanges. In the case of wrought iron the two coefficients of elasticity are essentially equal to each other; consequently the axis about which the moment of inertia of the section is to be taken passes through the centre of gravity of the latter.

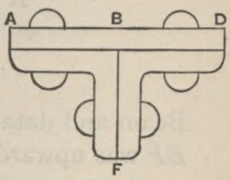


Fig.3

All the experiments cited in this section are those of Sir William Fairbairn, given in his "Useful Information for Engineers," first series.

Experiment I.

A section of the beam is shown in Fig. 3. It was composed of two $2\frac{1}{2}$-inch Ls riveted to a $5 \times \frac{1}{4}$-inch plate. AD was horizontal, and the flange, BF, downward; hence F was in tension.

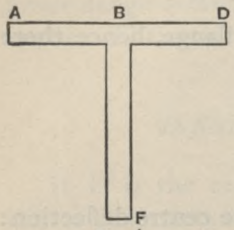


Fig. 4

W = centre breaking weight = 3,409 lbs.

I, by Eq. (29) of Art. 49, = 1.738.

x_1 = distance of centre of gravity from F = 1.91 inches.

Span = l = 7 ft. = 84 inches.
$K = T'$ = apparent intensity of tensile stress at F.
Hence:

$$K = T' = \frac{Wlx_1}{4I} = 78{,}400 \text{ lbs.}$$

Experiment II.

Beam and data the same as before, except:

$$W = 7{,}750 \text{ lbs.}$$

$$l = 27 \text{ inches.}$$

Hence:

$$K = T' = \frac{Wlx_1}{4I} = 57{,}344 \text{ lbs.}$$

Experiment III.

Beam and data the same as before, except:
BF was upward, causing compression at F.

$$W = 10{,}777 \text{ lbs.}$$

$$l = 27 \text{ inches.}$$

$K = C'$ = apparent intensity of compressive stress at F.

Hence:

$$K = C' = 78{,}400 \text{ lbs.}$$

Experiments II. and III. were made by testing portions of the same beam used in Experiment I.

Experiment IV.

A section of the beam is shown in Fig. 4., but it was tested with the rib or web upward, as shown in Fig. 2.

$$AD = 2.85 \text{ inches.} \qquad BF = 2.5 \text{ inches.}$$

$$\text{Thickness of rib} = 0.29 \text{ inch.}$$

$$\text{Thickness of flange} = 0.375 \text{ inch.}$$

$$W = 3{,}019 \text{ lbs.} \qquad l = 48 \text{ inches.}$$

x_1 distance of centre of gravity from $F = 1.86$ inches.

$$I = 0.989.$$

Hence:

$$K = C' = \frac{Wlx_1}{4I} = 68{,}100 \text{ lbs.}$$

Experiment V.

Beam and data same as for IV., except:
Rib was downward, as shown in Fig. 4:

$$W = 3{,}153 \text{ lbs.}$$

Hence:

$$K = T' = 71{,}000 \text{ lbs.}$$

In all these experiments half the weight of the beam was included in W.

These results show that the apparent ultimate intensities of resistance to compression and tension in bending of **T** beams may be taken equal to each other; also that there may be taken:

$$K = C' = T' = 70{,}000 \text{ lbs. per sq. in.}$$

The ultimate tensile resistance (T) of this iron probably ranged from 45,000 to 50,000 pounds per square inch. Hence, nearly:

$$K = \frac{3T}{2}.$$

From the equality of C' and T', it follows that the beam is equally strong whether the web or rib is up or down.

*Deflection of Wrought-Iron **T** Beams.*

If w is the centre deflection of a beam loaded with the centre weight W, E the coefficient of transverse elasticity, and l the span, then, as has been seen:

$$w = \frac{Wl^3}{48EI}. \quad \ldots \ldots \ldots \quad (26)$$

or,

$$E = \frac{Wl^3}{48wI}. \quad \ldots \ldots \ldots \quad (27)$$

A mean of the experiments II. and III. gave:

$$W = 4{,}040 \text{ lbs.}, \qquad w = \frac{0.17 + 0.18}{2} = 0.175 \text{ inches.}$$

$$I = 1.738.$$

Hence:

$$E = \frac{Wl^3}{48wI} = 5{,}446{,}500.$$

This is a small value for E, but is due to the fact that the beam was a built one.*

A mean of the experiments IV. and V. give:

$$W = 1{,}400 \text{ lbs.}, \quad w = \frac{0.135 + 0.17}{2} = 0.15025 \text{ in.}, \quad I = 0.989.$$

Hence:

$$E = 21{,}706{,}000.$$

This last value of E is about four times as large as the other. Hence the rolled beam would deflect only one-quarter as much as the built one. All values of W were within the elastic limit.

These values of E, inserted in Eq. (26), will give the deflection for a load W (including five-eighths the weight of the beam) at the centre. If W_1 is the total uniform load, $\frac{5}{8} W_1$ is to be put for W in the equation. Eq. (26) requires l, w and I to be in inches.

If, however, l is in feet and other dimensions in inches:

$$w = \frac{36 Wl^3}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

The foregoing formulæ, both for breaking weight and deflec-

* It is probable that the riveting was done by hand. The improved modern machine riveting would make a much stiffer beam.

tion, may be used for the bending of angle irons with sufficient accuracy for all ordinary purposes.

Art. 65.—Flanged Beams with Equal Flanges.

Nearly all the flanged beams used in engineering practice are composed of a web and two equal flanges. It has already been seen that the ultimate resistances, T and C, of wrought iron, to tension and compression are essentially equal to each other; the same may be said also of its coefficients of elasticity. While these observations may not be applied with precisely equal force or truth to the milder forms of steel now working their way, to a considerable extent, into engineering construction, they certainly hold without essential error.

In Fig. 1 is represented the normal cross section of an equal-flanged beam. It also represents what may be taken as the section of any wrought iron or steel **I** beam. Although the thickness t' of the flanges of such beams is not uniform, such a mean value may be taken as will cause the transformed section of Fig. 1 to be of the same area as the original section.

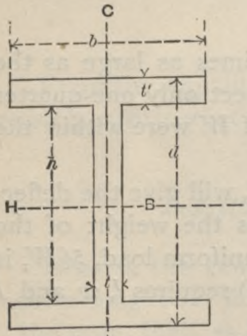


Fig. 1

Unless in very exceptional cases where local circumstances compel otherwise, the beam is always placed with the web vertical, since the resistance to bending is much greater in that position. The neutral axis HB will then be at half the depth of the beam. Taking the dimensions as shown in Fig. 1, the moment of inertia of the cross section about the axis HB, is:

$$I = \frac{bd^3 - (b - t)h^3}{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

while the moment of inertia about CD has the value:

$$I_1 = \frac{2t'b^3 + ht^3}{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

With these values of the moment of inertia, the general formula, $M = \frac{KI}{d_1}$, becomes (remembering that $d_1 = \frac{d}{2}$ or $\frac{b}{2}$):

$$M = C\frac{bd^3 - (b - t)h^3}{6d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Or;

$$M' = C\frac{2t'b^3 + ht^3}{6b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

C is written for K, since $K = T = C$.

Eq. (3) is the only formula of much real value. It will be found very useful in making comparisons with the results of a simpler formula to be immediately developed.

Let $d_1 = \frac{1}{2}(d + h)$. Since t' is small compared with $\frac{d}{2}$, the intensity of stress may be considered constant in each flange without much error. In such a case the total stress in each flange will be: $Cbt' = Tbt'$, and each of those forces will act with the lever arm $\frac{1}{2}d_1$. Hence the moment of resistance of both flanges will be:

$$Cbt' \,.\, d_1 \,.$$

The moment of inertia of the web will be: $\frac{th^3}{12}$. Consequently, its moment of resistance will have very nearly the value:

$$\frac{Cth^2}{6} \,.$$

The resisting moment of the whole beam will then be:

$$M = C\left(bt'd_1 + \frac{th^2}{6}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

A still further approximation is frequently made by writing d_1h for h^2; then if each flange area $bt' = f$, Eq. (5) takes the form:

$$M = Cd_1\left(f + \frac{th}{6}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Eq. (6) shows that *the resistance of the web is equivalent to that of one-sixth the same amount concentrated in each flange.*

If the web is very thin, so that its resistance may be neglected:

$$M = Cfd_1 = Cbt'd_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Or:

$$f = \frac{M}{Cd_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7a)$$

Cases in which these formulæ are admissible will be given hereafter. It virtually involves the assumption that the web is used wholly in resisting the shear, while the flanges resist the whole bending and nothing else. In other words, the web is assumed to take the place of the neutral surface in the solid beam, while the direct resistance to tension and compression of the longitudinal fibres of the latter is entirely supplied by the flanges.

Again recapitulating the greatest moments in the more commonly occurring cases:

Cantilever uniformly loaded:

$$M = \frac{Wl}{2} = \frac{pl^2}{2}.$$

Cantilever loaded at the end:

$$M = Wl.$$

Beam supported at each end and uniformly loaded:

$$M = \frac{Wl}{8} = \frac{pl^2}{8}.$$

Beam supported at each end and loaded at centre:

$$M = \frac{Wl}{4}.$$

Beam supported at each end and loaded both uniformly and at centre:

$$M = \frac{l}{4}\left(W + \frac{pl}{2}\right).$$

In all cases W is the total load or single load, while p, as usual, is the intensity of uniform load, and l the length of the beam.

In "Useful Information for Architects, Engineers and Workers in Wrought Iron," issued by the Phœnix Iron Co. of Phœnixville, Penn., are the record of some experiments by which the value of C or T may be determined. These will now be used.

Example I.

A 7-inch **I** was subjected to successive loads at the centre of the span, the ends being simply supported. The beam weighed 60 pounds per yard; consequently the area of the cross section was 6 square inches. The span was 21 feet, or 252 inches. The dimensions represented in Fig. 1 are the following:

$t = 0.36$ inches.

$h = 5.95$ " $\therefore\ h^3 = 210.63.$

$d = 7.00$ " $\therefore\ d^3 = 343.$

$b = 3.67$ "

$(b - t) = 3.31$ "

$t' = 0.525$ "

$d_1 = \frac{1}{2}(d + h) = 6.475$ inches.

$f = bt' \quad = 1.927$ "

The following table gives all the recorded results.

CENTRE LOAD.	DEFLECTION.	PERMANENT SET.	REMARKS.	$w = \frac{l^3}{48EI}\left(W + \frac{5}{8}pl\right).$
Lbs.	Ins.	Ins.		Ins.
2,000	0.468			$w = 0.537$
3,000	0.743			$w = 0.775$
4,000	1.020			$w = 1.012$
5,000	1.298	0.029	Weight removed...	$w = 1.250$
6,000	1.578	0.030	" "	The coefficient of elasticity,
7,000	1.887	0.060	" "	E, is taken at 30,000,000
8,000	2.300	0.183	" "	lbs.
9,000	3.540			
9,500	5.298			
10,000				

With the load of 10,000 pounds at the centre the "beam sunk slowly, top flange yielding." The beam, therefore, may be considered as essentially failing with a load of 10,000 pounds at its middle point. As the top flange yielded, the ultimate resistance to compression, or C, will be given by the experiment.

In reality, the beam carried a uniform load of 20 pounds per foot (its own weight), besides the single load of 10,000 pounds at the centre. Hence, Eq. (22) of Art. 24 will give the value of M. It is as follows:

$$M = \frac{l}{4}\left(W + \frac{pl}{2}\right).$$

But $\frac{pl}{2} = 20 \times 21 \div 2 = 210$; $W = 10{,}000$, and $l = 252$. l is taken in inches because the dimensions of the cross section are in the same unit. These values give:

$$M = 643{,}230.$$

Also the data given above, placed in Eq. (3), give:

$$M = C \times 13.37.$$

Equating these values:

$$C = 643{,}230 \div 13.37 = 48{,}110 \text{ pounds} \quad . \quad . \quad . \quad (8)$$

Again, the proper data inserted in Eq. (6), the approximate formula, give:

$$M = C \times 14.79.$$

Hence:

$$C = 643{,}230 \div 14.79 = 43{,}490 \text{ pounds} \quad . \quad . \quad . \quad . \quad (9)$$

The first permanent set was observed with a centre load of 5,000 pounds. This gives a bending moment at centre of

$$M = \frac{l}{4}\left(5{,}000 + \frac{pl}{2}\right) = 328{,}230.$$

Hence:

$$C = 328{,}230 \div 13.37 = 24{,}550 \text{ pounds.}$$

As the permanent set with this load was very small, and as there was none at all observed with a centre load of 4,000 pounds (nearly corresponding to $C = 20{,}000$ pounds), the limit of elasticity may be taken at about:

$$\frac{20{,}000 + 24{,}000}{2} = 22{,}000.00 \text{ pounds.}$$

In the right hand column of the table are calculated the deflections by Eq. (21) of Art. 24, the coefficient of elasticity being taken at 30,000,000 pounds. By Eq. (1), using the data already given:

$$I = 46.795.$$

Hence:

$$l^3 \div 48EI = 0.0002375.$$

Also:

$$\frac{5}{8}pl = 262.5.$$

These values inserted in the formula give the results shown in the table. The experimental quantities are seen to increase much more rapidly than the results given by the formula. The agreement, however, is sufficiently close for ordinary purposes.

Example II.

The second example, derived from the same source as the first, is that of a 9-inch I, 87 pounds per yard. The data to be used in connection with Fig. 1 are as follows:

$$t' = 0.72 \text{ inches.}$$
$$b = 4.00 \text{ ''} \qquad \therefore\ f = bt' = 2.88.$$
$$t = 0.39 \text{ ''}$$
$$d = 9.00 \text{ ''} \qquad \therefore\ d^3 = 729.000.$$
$$h = 7.56 \text{ ''} \qquad \therefore\ h^3 = 432.581.$$
$$(b - t) = 3.61 \text{ ''}$$

$l = 21$ feet $= 252$ inches ; $p \doteq 29$ pounds per foot.
$d_1 = 8.28$ inches. $W = 17{,}500$ pounds.

The bending moment at centre, as before, is :

$$M = \frac{l}{4}\left(W + \frac{pl}{2}\right) = 1{,}121{,}683.5.$$

The above data inserted in Eq. (3) give :

$$M = C \times 25.08.$$

Hence :

$$C = 1{,}121{,}683.5 \div 25.08 = 44{,}724 \text{ pounds.} \quad . \quad . \quad (10)$$

Again the approximate formula Eq. (6) gives :

$$M = C \times 27.92.$$

Hence :

$$C = 1{,}121{,}683.5 \div 27.92 = 40{,}175 \text{ pounds.} \quad . \quad . \quad (11)$$

The results of this experiment are given in the following table, exactly as in Ex. I.

CENTRE LOAD.	DEFLECTION.	PERMANENT SET.	REMARKS.	$w = \frac{l^3}{48EI}\left(W + \frac{5}{8}l\right)$.
Lbs.	Ins.	Ins.		Ins.
2,000	0.228			0.257
4,000	0.474			0.454
6,000	0.720			0.651
8,000	0.962			0.848
10,000	1.201	0.048	Weight removed.	1.045
12,000	1.432	0.050	" "	E is taken at 30,000,000
13,000	1.580	0.117	" "	lbs.
14,000	1.863	0.269	" "	
16,000	3.256			
17,000	5.233			
17,500	5.602			

The beam may be considered as having yielded, in failure, with a centre load of 17,500 pounds. That number was consequently taken above in the greatest value of M.

If it be assumed that permanent set was just at the point of beginning with the centre load of 9,000 pounds, which cannot be far wrong, the corresponding moment will be:

$$M = \frac{l}{4}\left(9{,}000 + \frac{pl}{2}\right) = 586{,}152;$$

$$\therefore \quad C = 586{,}152 \div 25.08 = 23{,}370 \text{ pounds (limit of elas.).}$$

Taking a mean of the results of the two examples:

By exact formula [Eq. (3)]:

$$C = 46{,}417 \text{ pounds.}$$

By app. formula [Eq. (6)]:

$$C_1 = 41{,}833 \text{ pounds.}$$

For the limit of elasticity:

$$C_e = 22{,}700 \text{ pounds (nearly).}$$

These results may be considered accurate for the Phœnix Iron Co.'s beams. These experiments were made in 1858.

It is interesting to notice that these beams failed in the compression flanges.

It is also important to observe that the ultimate resistance, C, is fully equal to the ultimate tensile resistance of good wrought iron in large bars. This serves to confirm the opinion that the ultimate tensile and compressive resistances of wrought iron are not far, at most, from being equal to each other, and that these quantities may be used for C or K in the formulæ for flanged beams. If the approximate formula, Eq. (6), is used, however, according to these results C or K should be taken about 0.90 (nine-tenths) of the value used in the exact formula, Eq. (3).

The last column of the second table is calculated by the formula, as shown, taking E at 30,000,000 pounds. The same general observations apply to these results as in the preceding example.

Example III.

The data for this example are taken from the hand-book for 1881 published by the N. J. Steel and Iron Co., Trenton, N. J., where the beams were broken. The breaking weight is the mean of two results for light 6-inch wrought iron Is.

$$d = 6.00 \text{ ins.} \qquad t = 0.25 \text{ in.} \qquad t' = 0.456 \text{ in.}$$

$$l = 12 \text{ ft.} = 144 \text{ ins.} \qquad I = 23.815, \text{ by Eq. (1).}$$

Since the beam weighed 40 pounds per yard :

$$W = 14{,}000 + 80 = 14{,}080 \text{ lbs. (\textit{centre breaking load}).}$$

Hence :

$$C = \frac{Md}{2I} = 63{,}840 \text{ lbs. per square inch.}$$

By approximate formula :

$$\frac{th}{6} = 0.21. \qquad f = 1.368 \quad \therefore \quad \frac{th}{6} + f = 1.578.$$

$$d_1 = 5.544 \text{ ins.} \qquad M = 506{,}880.$$

Hence, by Eq. (6) :

$$C_1 = 57{,}930 \text{ lbs. per square inch.}$$

Example IV.

A 9-inch heavy Trenton beam, 85 pounds per yard. The data are taken from the same source as were those in Ex. III.

$d = 9.00$ ins. $t = 0.38$ in. $t' = 0.68$ in.

$l = 15$ ft. $= 180$ ins. $I = 108.47$, by Eq. (1).

$$W = 32{,}000 + 212 = 32{,}212 \text{ lbs. } (\textit{at centre}).$$

Hence :

$$C = \frac{Md}{2I} = 60{,}120 \text{ lbs. per square inch.}$$

By approximate formula :

$$\frac{th}{6} = 0.484. \qquad f = 2.72 \quad \therefore \quad \frac{th}{6} + f = 3.204.$$

$d_1 = 8.32$ ins. $M = 1{,}449{,}540$.

Hence by Eq. (6) :

$$C_1 = 54{,}370 \text{ lbs. per square inch.}$$

Taking the means of these two sets of results :

By exact formula [Eq. (3)]:

$$C = 61{,}980.$$

By app. formula [Eq. (6)]:

$$C_1 = 56{,}150.$$

All the conclusions reached in connection with Exs. I. and II. are confirmed by the results of Exs. III. and IV.

C and C_1 are much larger, however, for the Trenton than for the Phœnix beams, and both are very high for beams of such length of span with no lateral support for the compression flange.

In calculating the deflection of rolled wrought-iron beams E may be taken from 28,000,000 to 30,000,000.

The exact formulæ of this Article are strictly applicable to rolled beams only, but the approximate formula finds extensive application in cases of built beams.

Experiments by U. S. Test Board.

Table I. contains the results of a valuable series of tests by the U. S. Board, "Ex. Doc. 23, House of Rep., 46th Congress, 2d Session."

The values of K and E at elastic limit are computed from data contained in that document in the manner already shown in detail, and which it is not necessary to repeat. It is both interesting and important to observe the considerable, though irregular, increase of the intensity of stress in the exterior fibre, at the elastic limit, with the decrease of depth. E is seen to vary from 26,099,400 to 36,664,400, with a mean value of 31,128,260. As a general result, E is slightly larger for the smaller beams than for the larger.

TABLE I.

DEPTH IN INCHES.	COMPUTED AREA OF CROSS SECTION IN SQUARE INCHES.	WEIGHT PER YARD IN POUNDS (ACTUAL).	MOMENT OF INERTIA, OR I.	SQUARE OF RADIUS OF GYRATION.	K, IN POUNDS PER SQUARE INCH AT Elas. Lim.	K Final Load and Deflection.		FINAL CENTRE LOAD.	E, IN POUNDS PER SQUARE INCH.	SPAN IN FEET.
					Pounds.	Pounds.	Inches.	Pounds.		
15	14.88	145.73	536.56	36.06	18,210	19,120	0.56	22,351	34,328,500	20
10½	12.88	130.71	221.86	17.23	29,886	35,840	2.5	22,436	26,099,400	22
10½	10.52	104.47	174.75	16.61	26,097	26,097	1.14	12,741	28,903,000	22
10½	10.52	105.64	174.75	16.61	28,721	32,200	1.32	15,501	29,915,000	22
10½	9.13	92.12	154.90	16.96	25,191	43,100	Buckled lat.	19,888	31,167,000	21
9	8.88	89.14	106.53	12.00	28,713	32,430	1.43	11,311	32,628,000	22
9	8.88	88.89	106.53	12.00	29,478	31,100	0.5	18,647	35,409,000	13
9	8.88	89.04	106.53	12.00	29,500	31,360	0.37	22,330	30,720,000	11
8	6.33	66.03	62.34	9.85	30,508	38,280	2.05	8,802	31,859,000	22
8	6.33	65.31	62.34	9.85		37,300	1.79	7,567	29,270,200	22
8	6.33	64.29	62.34	9.85	33,957	37,700	0.93	13,581	26,863,750	14
8	6.33	65.04	62.34	9.85	30,000	37,100	0.89	13,851	28,816,710	14
7	5.74	62.00	44.12	7.69		30,150	1.79	5,856	32,834,200	22
7	5.74	58.88	44.12	7.69	27,413	32,330	2.04	6,343	30,836,000	22
7	5.74	59.44	44.12	7.69		32,400	1.97	6,366	33,333,800	22
7	5.74	59.08	44.12	7.69	32,575	36,800	1.01	10,961	30,128,700	14
7	5.74	59.66	44.12	7.69		36,800	1.03	10,961	30,099,300	14
6	4.47	43.46	24.58	5.49	34,393	34,393	2.08	4,189	33,799,500	22
6	4.47	44.62	24.58	5.49	33,207	40,300	1.08	7,749	34,057,600	14
6	4.47	44.64	24.58	5.49	36,564	39,700	1.10	7,749	32,453,400	14
5	3.21	32.63	12.85	4.00	34,740	39,600	0.92	6,129	34,064,800	11
5	3.21	32.20	12.85	4.00	38,116	41,170	0.90	6,364	31,631,800	11
5	3.21	33.16	12.85	4.00	38,116	38,120	0.74	5,877	30,976,000	11
4	2.90	30.64	7.42	2.56	37,750	41,640	0.94	4,639	36,064,400	11
4	2.90	29.63	7.42	2.56		41,640	1.05	4,639	32,687,000	11

In this table, I is exact. Hence K is computed by the exact formula : $K = \frac{Md}{2I}$. All beams were loaded at centre, and deflections were taken at same point.

A chemical analysis of six specimens from these beams gave the following results.

These experiments were conducted by Gen'l Wm. Sooy Smith, who kindly gave to the writer the final centre loads and deflections.

PERCENTAGES OF							
Sulphur.	Phosph'us.	Silicon.	Total Carb.	Manganese.	Côpper.	Cobalt.	Nickel.
0.010	0.436	0.189	0.031	0.031	0.012	0.029	0.029
0.008	0.447	0.190	0.038	0.028	0.008	0.021	0.023
0.010	0.453	0.203	0.037	0.028	0.010	0.015	0.018
0.012	0.423	0.182	0.039	0.022	0.022	0.010	0.015
0.005	0.271	0.177	0.027	0.028	0.052	0.031	0.026
0.011	0.375	0.197	0.039	0.028	0.010	0.018	0.016

Col. T. T. S. Laidley, U. S. A., has also completed the tests of a few beams to failure, the results of which are given in "Ex. Doc. 12, 47th Congress, 1st Session." Table II. contains values of K at failure, computed from Col. Laidley's data. Beams No. 1 carried a load uniform from end to end by

TABLE II.

NO.	DEPTH, INS.	SPAN IN FEET.	MEAN OF	LOAD.	LATERAL SUPPORT.	TOTAL LOAD IN POUNDS.	K IN LBS. PER SQ. IN.
1	15.0	28.5	3 Exps.	Uniform.	Uniform.	118,000	54,260
2	15.0	28.0	2 Exps.	Centre.	None.	28,650	25,890
3	10.5	17.0	2 Exps.	Centre.	None.	21,020	32,210
4	10.5	17.0	1 Exp.	Centre.	None.	22,020	38,070

means of brick masonry arches, which thus also gave to them a uniform lateral support. This lateral support produced a very high value of K, *i.e.*, 54,260 pounds, which fell to 25,890 with no lateral support. In the latter case nothing prevented the compression flange yielding laterally like a column. The 10.5-inch beams were much shorter, and the long column influence less marked; consequently the values of K are correspondingly higher. The tests are not sufficiently numerous to fix the law of the decrease of K with the increase of span.

Beams Nos. 1 and 2 weighed 200 pounds per yard, with a moment of inertia (I) equal to 706.6. Beam No. 3 weighed 105 pounds per yard, and gave $I = 174.75$; while No. 4 weighed 92 pounds per yard, with $I = 154.9$.

Art. 66.—Built Flange Beams with Equal Flanges.—Cover Plates.

A "built beam" is a beam built up of plates and angles like that shown in Fig. 1. As shown in that figure the web is composed of a single plate, called the "web plate," supported by "stiffeners," if necessary, as is usually the case. These stiffeners are vertical pieces of Ls or ⊥s riveted to the web plate, in accordance with principles to be shown hereafter. The flanges, as shown by the heavy lines, are composed of Ls and plates so arranged as to give the requisite area of cross section at any point.

The method of designing such a beam, and the calculation of the elements of its resistance, will be given in detail. The beam is supposed to be of wrought iron, and one of a system for a double track railway bridge; the stringers under the two tracks, which rest on the beam, are placed at A and B, and D and H. The weight of the beam, taken uniformly distributed, is 5,600 pounds. The concentrated load at each of the points A, B, D and H, composed of the train weight added to that of the stringers, is 42,000 pounds.

The following are some of the dimensions of the beam:

Span $RR' = 28.0$ feet. Depth of web plate $= 48$ inches. $R'H = RA = 4.50$ feet. $DH = AB = 6.00$ feet. $BD = 7$ feet.

The web plate will be taken $\frac{7}{16}$ inch thick. The method of determining this thickness will be shown hereafter.

In this case resistance to flexure of the web will be neglected; *the web will be assumed to resist the shear only*, as is assumed in Eqs. (7) and (7a) of Art. 65. The "depth," d_1, of

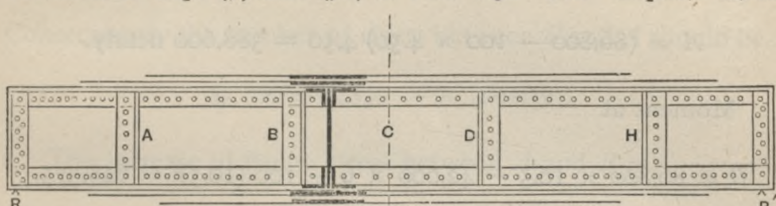


Fig. 1.

beam will then be the vertical distance between the centres of gravity of the sections of the flanges, and each flange is to be considered as composed of two ∟s and the "cover" plate or plates only; *no part of the web is to be included.* Strictly speaking, then, the depth is variable; but this variation is so slight that no essential error will be committed if it be considered constant and equal to the depth of web plate, or 48 inches. This procedure, which saves much labor and time, is always permissible where cover plates are used, and the bending resistance of the web plate neglected. The next example will exhibit a case in which they are not used.

The direct stresses of tension and compression existing in the flanges must be carried through the rivets which unite the flanges to the web; hence the necessary number of those rivets will first be determined.

The reaction at R, using the data already given, will be:

$$R = 2 \times 42{,}000 + \frac{5{,}600}{2} = 86{,}800 \text{ pounds.}$$

The weight per lineal foot of floor beam is:

$$\frac{5,600}{28.0} = 200 \text{ pounds} = w.$$

The bending moments for the two sections A and B will next be found.

Moment at

$$A = (86,800 - 100 \times 4.50)\ 4.50 = 388,600 \text{ nearly}.$$

Moment at

$$B = 86,800 \times 10.5 - 42,000 \times 6 - 100\ (10.5)^2 = 648,375.$$

Since the depth of the beam is 4 feet:

Flange stress at

$$A = 388,600 \div 4 = 97,100 \text{ pounds}.$$

Flange stress at

$$B = 648,375 \div 4 = 162,100 \text{ pounds}.$$

The allowable intensity of pressure between the rivet and its hole (see Art. 73) will be taken at 10,000 pounds. The diameter of rivets is a matter of judgment; it will be taken at $\frac{7}{8}$ inch. Rivets for built beams usually range from $\frac{5}{8}$ to 1 inch in diameter.

The selection of the Ls for the flanges is also, to some extent, a matter of judgment. In the present instance, 5″ × 4″ Ls, 60 pounds per yard, will be taken. These will be found to answer the purpose.

The effective bearing surface between each rivet and the web plate will then be:

$$\frac{7}{8} \times \frac{7}{16} = 0.383 \text{ square inch.}$$

Hence each rivet may carry:

$$0.383 \times 10{,}000 = 3{,}830 \text{ pounds.}$$

Consequently the number of rivets between R and A should be:

$$97{,}100 \div 3{,}830 = 26 \text{ (nearly).}$$

The increase of flange stress between A and B is:

$$162{,}100 - 97{,}100 = 65{,}000 \text{ pounds.}$$

Hence the number of rivets required between A and B is:

$$65{,}000 \div 3{,}830 = 17 \text{ (nearly).}$$

Since 26 rivets are required between R and A, the corresponding pitch would be but a little more than two and one-tenth inches, which is somewhat too small. With a $\frac{7}{8}$-inch rivet, a three-inch pitch is about the least advisable. If the rivets be placed at a pitch of three inches between R and B, forty-two will thus be located, and this is sufficiently near the desired number. The four-inch leg of the angle is placed against the web plate, but if necessary the five-inch leg could be so placed and still more rivets staggered in. In such methods as these, nearly the full number of rivets required between R and A may be supplied, while the two or three lacking will be found, without danger to the beam, adjacent to A on the side towards B. Three or four in excess of the number required will be found between A and B.

No central bending moment at C has been computed, because the only difference between such a one and that at either B or D is due to the weight of the beam only. This difference is essentially nothing. The proper support of the **L**s in compression, however, requires that the rivets be pitched at about six inches between B and D. In ordinary floor beams a proper bond between the flanges and web requires that the pitch should never be greater than about six or eight inches.

The shearing of the rivets is not considered, because they sustain double shear in the flanges, and their bearing capacity is by far the least of the two.

The rivets, of course, should be pitched alike in both top and bottom flanges.

The greatest allowable intensity of tensile stress in the bottom flange will be taken at 8,000 pounds per square inch, and an equal intensity will be taken for the compressive stress in the upper flange. The area required in the bottom flange at A is:

$$\frac{97,100}{8,000} = 12.1 \text{ sq. ins. (nearly).}$$

That required at B is:

$$\frac{162,100}{8,000} = 20.3 \text{ sq. ins. (nearly).}$$

The area of the two $5'' \times 4''$ **L**s, 60 pounds per yard, is 12.00 square inches. The thickness of the angle iron where it is pierced by the rivets binding it to the web is about 0.6 inch. Hence the area of metal taken out by one rivet is:

$$0.875 \times 0.6 \times 2 = 1.05 \text{ sq. in.}$$

Or, the effective area of the **L**s at A is:

$$12.00 - 1.05 = 10.95 \text{ square inches.}$$

Now, since the weight of the beam itself is small, compared with the weight of the train, the flange stress, or moment, varies almost uniformly from R to A. Hence, an increased section is first needed at

$$(10.95 \div 12.1) \times 4.50 = 4.1 \text{ feet (nearly)},$$

from R. Since, however, the cover plate to be added must take its stress through the rivets which bind it to the **L**s, it should overlap the necessary distance by one and a half to twice its width. In the present case, then, instead of beginning the cover plate at just 4.1 feet from R, a $12'' \times \frac{9}{16}''$ cover plate will begin at 30 inches from R and extend along the beam to a point at the same distance from R'. The length of this cover plate will then be $28.0 - 5.0 = 23$ feet. This cover plate will be bound to the angle irons by $\frac{7}{8}''$ rivets, which should, so far as possible, be pitched half way between the $\frac{7}{8}''$ rivets in the other legs of the angle irons. The effective area of this cover plate, for tensile stress, will then be:

$$(12 - 2 \times 1) \times \tfrac{9}{16} = 5.6 \text{ sq. in. (nearly)}.$$

The available area of two **L**s and one cover plate is, since two rivets now pierce each angle:

$$9.90 + 5.6 = 15.50 \text{ sq. ins.}$$

For the reason already given, the moment, or flange stress, varies nearly uniformly between A and B, but at a different rate than between R and A. Since AB is 6.00 feet, the point at which another increase of section must begin is at the distance

$$[(15.50 - 12.1) \div (20.3 - 12.1)] \times 6.00 = 2.5 \text{ feet (nearly)}$$

from A. Again, as in the previous instance, a second cover plate, $12'' \times \frac{1}{2}''$, will be put on, and it will begin, not at 2.5

feet from A, but at one foot from that point. The available area of this plate will be:

$$(12 - 2.0) \times \tfrac{1}{2} = 5.00 \text{ sq. ins.}$$

The total area at the centre of the beam available for tension will then be:

$$15.50 + 5.00 = 20.50 \text{ sq. ins.}$$

It is to be observed that in deducting metal taken out by a rivet in a tension flange, a diameter greater by an eighth of an inch than that of the rivet has been assumed. This should always be done, for the punch is always larger than the rivet, and the punched hole is still larger on the die side of the plate, and for the further reason that the metal is injured for some distance around the hole. In the compression flange no deduction need be made for rivets, as the latter completely fill the holes. Otherwise the method of designing the compression flange is precisely that just followed, and the two flanges will be taken alike.

The number of rivets required in a cover plate is yet an important question. Since all stress carried by the cover plates must be given to them by the rivets, *the number of rivets between the end of any cover plate and that point at which a further increase of flange section is necessary, must be sufficient to carry all the stress in the cover plate itself.*

Applying this principle to the first cover plate found necessary: The load which each $\frac{7}{8}''$ rivet in the $12'' \times \frac{9}{16}''$ cover may carry is:

$$0.875 \times \tfrac{9}{16} \times 10{,}000 = 5{,}000 \text{ pounds.}$$

The total tensile stress carried by the $12'' \times \frac{9}{16}$ cover is: $5.6 \times 8{,}000 = 45{,}000$ pounds. Hence the number of rivets required is:

$$45,000 \div 5,000 = 9.$$

According to the design it is 4.5 feet from the end of this cover to a point 2.5 feet from A toward B, where the next increase in section is required; and over this 4.5 feet these 9 rivets must be distributed. But in order that a proper bond between the component parts of the flange may be obtained, it is seldom advisable to make the pitch over 6″, and at the end of the cover plate this pitch should be halved for about three rivets. Proceeding in this manner, that part of the bottom of the beam at the end nearest R in Fig. 1, which includes the 4.5 feet of cover under consideration, will present the appearance of the sketch in Fig. 2. RG is 2.5 feet and GF 4.5 feet. In

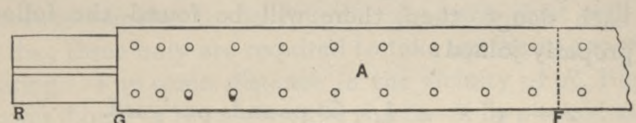


Fig. 2

this manner 20 rivets are introduced instead of 9, but it is advisable to put in the extra number.

In the compression flange other considerations appear besides the simple bearing capacity of the shaft of the rivet. *Between any two consecutive rivets the cover plate forms a solid rectangular column with essentially fixed ends, whose length is the pitch of rivets.* The pitch, therefore, must not be sufficiently great to allow the existence of any material amount of long column flexure. Unless plates, therefore, are very heavy, the greatest pitch should not exceed about six or eight inches.

The bearing capacity of a $\frac{7}{8}''$ rivet in $12'' \times \frac{1}{2}''$ cover is:

$$0.875 \times 0.5 \times 10,000 = 4,375 \text{ pounds.}$$

The full tensile capacity of the cover plate is:

$$5.00 \times 8,000 = 40,000 \text{ pounds.}$$

Hence, the number of rivets required is:

$$40,000 \div 4,375 = 9 \text{ (nearly).}$$

The end of the cover plate, as designed, is one foot from A towards B; and the nine rivets are nearly all required between that end and B, a distance of 5 feet. Hence, if the rivets are pitched in this cover plate, near the ends, as shown in Fig. 2 for the other cover, and at six inches over the intervening space, more than the number just determined will be introduced. For the reasons already given, however, the number will really be not too great.

In each flange, then, there will be found the following pieces properly joined:

2 — 5″ × 4″ Ls, 60 pounds per yard.
1 — 12″ × $\frac{9}{16}$″ plate, 23 feet long.
1 — 12″ × $\frac{1}{2}$″ plate, 17 feet long.

At the ends of the beams R and R', Fig. 1, provision must be made for the reaction. In this example the reaction is 86,800 pounds. The transverse shearing resistance of the web should at least equal this at the ends. The area of a transverse section of the web is:

$$48 \times \tfrac{7}{16} = 21.00 \text{ sq. ins.}$$

If the greatest allowable shearing intensity in the web be taken at 5,000 pounds, its shearing resistance will be:

$$21.00 \times 5,000 = 105,000 \text{ pounds.}$$

This result is about 20 per cent. greater than is required.

Hence safety, so far as shearing is concerned, is amply secured. But the end of the beam is also subject to an upward pressure of 86,800 pounds, which must also be provided for. Two 6″ × 4″ × ½″ ⅃s will be riveted to the ends as shown in Fig. 1, one on each side of the web, and the 6″ legs lying against it. By pitching ⅞″ rivets at 3″ (nearly), in a zigzag manner, 20 rivets can be introduced to hold these 4″ × 6″ × ½″ ⅃s to the web. The carrying capacity of each ⅞″ rivet against the web plate has already been found to be 3,830 pounds. These 20 rivets therefore will carry 3,830 × 20 = 76,600 pounds. Since the area of the cross section of two 4″ × 6″ × ½″ ⅃s is about 10 *sq. ins.*, the bearing of the rivets against the web plate is all that need be considered in this connection.

A proper bearing for the difference 86,800 − 76,600 = 10,-200 pounds remains to be found. As each rivet will carry 3,830 lbs., three only are required to take up the 10,200 pounds remaining. For some distance in the vicinity of *R*, Fig. 1, in the lower flange of the girder, the rivets will be pitched at three inches. Since some portion of these ends must rest on shoes or brackets, three of the rivets near the ends may be utilized to carry the 10,200 pounds in question. It is to be remembered that in such an instance as this, *the lower ends of the* 4″ × 6″ ⅃s *must bear fairly and truly against the angle irons composing the lower flange*, in order that they may take up their proper amount of the reaction.

In some cases the ends of the beam are to be secured to vertical surfaces without any supporting shoe or bracket. The entire reaction of such a beam must be carried by the vertical angles at the ends. The number of ⅞″ rivets required to hold these angle irons to the web would then be 86,800 ÷ 3,830 = 23 (nearly). By simply making two rows, these could easily be worked into the longer legs of the 6″ × 4″ × ½″ ⅃s. 12 rivets would then be put through each of the two 4″ legs and the vertical surface to which the beam is secured.

No account has heretofore been taken of the shearing resistance of the rivets, because that has been much greater than their bearing capacity, but instances may occur in which such a condition of things does not exist. Hence the shearing and bearing capacities should always be estimated, and security taken in reference to that which is least. As an example: at 7,500 pounds per *sq. in.* the shearing resistance of a $\frac{7}{8}''$ rivet is $(0.875)^2 \times 0.7854 \times 7{,}500 = 4{,}600$ pounds (nearly); while the bearing capacity of the same rivet in the $6'' \times 4'' \times \frac{1}{2}''$ L is only:

$$0.875 \times 0.5 \times 10{,}000 = 4{,}375 \text{ pounds.}$$

Precisely the same operations are required in determining the number of rivets in the vertical Ls at A and B, Fig. 1, as in those at the ends of the beam; consequently it is not necessary to repeat them.

Thus, there is completed the operation of designing the beam, with the exception of finding the thickness of the web, which will be given hereafter.

In general two or three things are to be observed. *The number of rivets actually required by these calculations should always be, as they just have been, somewhat exceeded.* In the best of riveted work the rivets will not exactly fill the holes, and the beam will not act perfectly as one continuous whole.

Again, stress is given to the flanges along the line of the rivet holes, which is some distance from the centre of gravity of the cross section of the flange. Consequently, some bending will be induced in both flanges, and this necessitates some extra material. This excess may be estimated if desirable, but ordinarily it is entirely unnecessary. The existence of this bending demonstrates the advisability of putting on as few cover plates as possible. It is far better to use heavier Ls with a little waste of material at the ends.

It is also better to use one heavy cover plate than two thin ones having an equal combined thickness, even though the use of the former entails a little waste; for the heavy plate between two consecutive rivets will resist far more bending as a column than the two others each of half the thickness.

If the end of the beam were made as shown in Fig. 3, no web plate would be required between R and A, for all shear would be carried by the inclined flange.

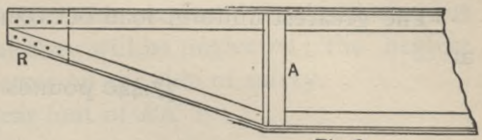


Fig. 3

The upper flange, being in compression, would require riveting, but none would be needed in the lower, except in the immediate vicinity of R. The flange stresses between A and R would also be uniform, instead of uniformly varying as in Fig. 1.

Art. 67.—Built Flanged Beams with Equal Flanges.—No Cover Plates.

The flanged beam represented in Fig. 1 is supposed to carry a portion of the floor of a highway bridge. In this case, also, the bending resistance of the web plate will be neglected. The beam proper is the portion $RR'\ R'R$, supported at RR and $R'R'$; while the portions ARR and $HR'R'$ form cantilevers for the support of the sidewalks.

The following are the dimensions:

$$AR = HR' = 6 \text{ feet.} \qquad RR' = 28 \text{ feet.}$$
$$AH = 40 \text{ feet.} \qquad RR = R'R' = 31 \text{ inches.}$$
$$RB = BM = MF = FR' = 7 \text{ feet.}$$

The depth RR has been taken at 31 inches, so that the effective depth to be used in finding the flange stresses will be about 2.5 feet.

The weight of the beam proper, $RR'R'R$, added to the flooring which it supports, is taken at

14,650 pounds.

The greatest uniform load between R and R' will be taken at

37,440 pounds.

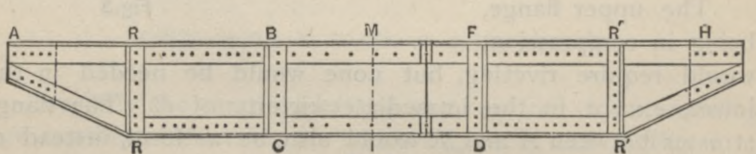


Fig. 1

Hence the total uniform load to which the beam is subjected is:

$$37{,}440 + 14{,}650 = 52{,}090 \text{ pounds.}$$

The weight of one cantilever, with the flooring which it supports, will be taken at

3,100 pounds

The total moving load on AR, or HR', will be taken at

8,640 pounds.

The total load, therefore, carried by one cantilever is:

$$3{,}100 + 8{,}640 = 11{,}740 \text{ pounds.}$$

The beam proper, RR', may sustain its greatest load when the sidewalks carry nothing but their fixed weight. This condition of things will cause the greatest compression in the upper flange and tension in the lower, and will be assumed in designing the beam.

The fixed weight of a cantilever will cause stresses in the flanges of opposite kinds to those produced in the beam, but of such small amount that they will be neglected ; the neglect originating a very small error on the side of safety.

The total load per linear foot of RR' is :

$$52{,}090 \div 28 = 1{,}860 \text{ pounds.}$$

The flange stress in the beam at R will be nothing; it will be found at the two points B and M. Strictly, the " depth " to be used should be the vertical distance between the centres of gravity of the flanges. It will not be far wrong to take this depth at 2.5 feet, since the web plate is 31 inches deep. The reaction at R is :

$$52{,}090 \div 2 = 26{,}045 \text{ pounds.}$$

The flange stress at B is:

$$(26{,}045 \times 7 - 1{,}860 \times (7)^2 \div 2) \div 2.5 = 54{,}700 \text{ pounds.}$$

The flange stress at the centre M is :

$$(52{,}090 \times 28 \div 8) \div 2.5 = 72{,}926 \text{ pounds.}$$

If, as in the preceding Article, the greatest allowable stress in the flanges is 8,000 pounds per square inch, a flange area of 9.115 square inches is required in the present case. If each flange is composed of 2—4" × 6" × ½" Ls, 51 pounds per yard, there will be a very little excess of flange area, as there should be; these Ls will then be taken for the flange, the 4″ legs being riveted to the web plate; ⅞″ rivets will be used in riveting the flanges to the web. Where pierced by the rivets, the legs of the Ls are about ½″ thick. Hence a rivet hole will cut out $2 \times \frac{1}{2} \times 1.00 = 1.00$ sqare inch. There will then still remain $10.2 - 1.00 = 9.20$ square inches of effective area, which is a little in excess of the 9.115 required.

A web plate ⅜″ thick will be assumed. Taking 10,000 pounds per square inch as the greatest allowable intensity of pressure between shaft of rivet and plate, the bearing capacity of each rivet will be:

$$0.875 \times 0.375 \times 10{,}000 = 3{,}280 \text{ pounds.}$$

In this case all the moving load rests upon the *top* of the beam, and since the edge of the web plate is only 0.375″ wide, that moving load must be taken as resting on the Ls of the upper flange, and hence indirectly on the rivets. Also, since nearly the whole of the fixed load rests upon the upper flange, *the entire load of the beam will be taken as resting on that flange.* Consequently, between R and B the rivets will be subjected to the action of a vertical force equal to $1{,}860 \times 7 = 13{,}020$ pounds, and a horizontal one equal to 54,700 pounds. The resultant force will then be:

$$\sqrt{(13{,}020)^2 + (54{,}700)^2} = 56{,}230 \text{ pounds.}$$

Between B and M the vertical force will then be the same, but the horizontal one will be

$$72{,}926 - 54{,}700 = 18{,}226 \text{ pounds.}$$

The resultant, therefore, is:

$$\sqrt{(13{,}020)^2 + (18{,}226)^2} = 22{,}400 \text{ pounds.}$$

Hence the number of rivets required between R and B is:

$$56,230 \div 3,280 = 18 \text{ (nearly)}.$$

The number between B and M is:

$$22,400 \div 3,280 = 7 \text{ (nearly)}.$$

If, therefore, commencing at R or R', the rivets be pitched at 3 inches for a distance of 4.5 feet, then at 6 inches to the centre M, about 36 or 37 rivets will be found in each half of each flange. This number is in excess of that required, but for the reasons given in the preceding Article, it is probably not too many. Thus the flanges are designed without the use of cover plates.

In this case the beam will be suspended from hanger loops at R and R', which carry resting plates or shoes for the beam at their lower extremities.

The total reactions at the lower R and R' will be half the total weight of the entire beam with the moving load, or:

$$\text{Reaction} = (52,090 + 23,480) \div 2 = 37,785 \text{ pounds}.$$

At R and R' 2—$4'' \times 4'' \times \frac{1}{2}''$ ∟s will be riveted to the beam as shown. The *lower* ends of these angles should abut firmly and squarely against the angles of the lower flange.

Since the greatest allowable pressure between a rivet and the web plate is 3,280 pounds, the number of rivets required at each end of the beam in each pair of vertical ∟s is:

$$37,785 \div 3,280 = 12 \text{ (nearly)}.$$

If, consequently, these rivets be pitched at 3 inches, a sufficient number will be obtained, if it be remembered that three or four of the lower flange rivets near R or R' may be available for bearing.

The pitch in the stiffners ($3'' \times 3'' \times \frac{5}{16}''$ Ls) at C and D may be taken at $6''$, with an extra rivet at each end.

The horizontal flange stress for the cantilevers at R and R' is:

$$(11{,}740 \times 3) \div 2.5 = 14{,}088 \text{ pounds.}$$

The secant of the angle which the inclined flange makes with the horizontal is about 1.05. Hence the inclined flange stress is:

$$14{,}088 \times 1.05 = 14{,}800 \text{ pounds.}$$

Hence, if in each flange at A there are

$$14{,}800 \div 3{,}280 = 5 \text{ (nearly)}$$

rivets, securing the flanges to the piece of plate shown, ample security will be obtained.

The cantilever flanges possess a large excess of material.

Calculations on the shearing of the rivets between the web and flange have not been made, because the resistance of a rivet to double shear is much in excess of its bearing capacity.

The excess of material in the Ls of the flanges is not as much as it really should be, because the line of horizontal stress along the rivet holes is somewhat below or above the centre of gravity of the flange, and some bending is consequently induced. This bending, however, is not as great as if cover plates had been used, and the neglect of the bending resistance of the web plate is somewhat of an offset. Besides, as has already been stated, in this particular case, the fixed weight of the cantilevers relieves a little of the flange stress of the beam as actually found.

Since the transverse section of the web plate has an area of

0.375 × 30 = 11.25 square inches, transverse shearing at the points of support is more than provided for.

If either a railway or highway floor beam has a variable depth, the operations are in no manner changed. The depth, however, to be used in finding the flange stress at any point must be the vertical depth at that point. The stress thus determined must be multiplied by the secant of the inclination to the horizontal at the same point for the inclined flange.

Art. 68.—Box Beams.

The class of beams known as box beams in engineering practice are represented in Figs. 1 and 2. In Fig. 1 the upper and lower flanges are each composed of a plate whose thickness is t' and two Ls whose lengths of legs and thickness are s and a, respectively. If it be assumed that the web plates,

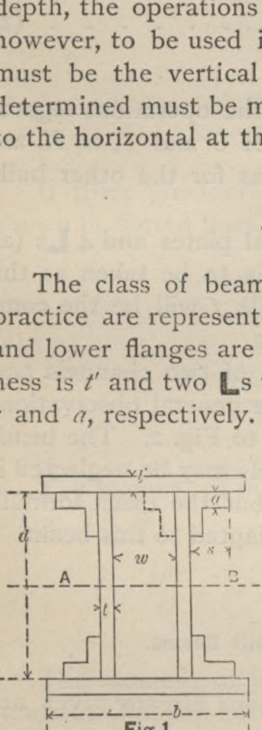


Fig. 1

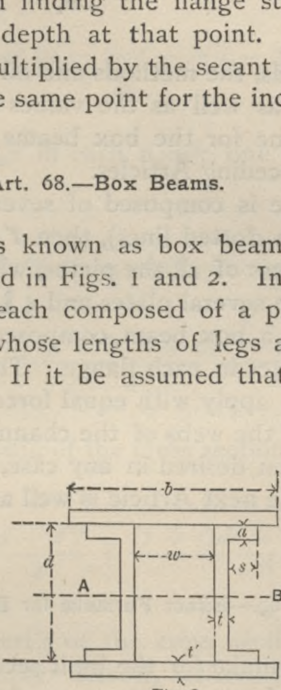


Fig. 2

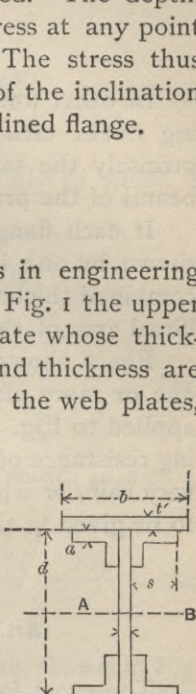


Fig. 3

the thickness of each of which is t, offer no resistance to bending, then the effective depth of the box beam will be the vertical distance between the centres of gravity of the flanges. If f is the area of one of these flanges, and d_1 this effective depth, the resisting moment of the beam, as has already been shown, will be :

$$M = Cfd_1; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which $C = K =$ intensity of stress at the distance $\frac{1}{2}d_1$ from the neutral surface. If the flange area is desired:

$$f = \frac{M}{Cd_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In other words, the methods and all the operations regarding rivets, etc., as well as the values of C and T, or K, are precisely the same for the box beams as for the other built beams of the preceding Articles.

If each flange is composed of several plates and 4 Ls (as shown by one in dotted lines), then t' is to be taken as the combined thickness of all the plates, while f will be the combined area of the several plates and 4 Ls.

Fig. 2 shows a box beam composed of two channels and one or more plates in each flange. The general observations applied to Fig. 1 apply with equal force to Fig. 2. The bending resistance of the webs of the channels may be neglected if very thin, or when desired in any case, but the exact formula to be given in the next Article is well adapted to this beam.

Art. 69.—Exact Formulæ for Built Beams.

The exact formulæ for the built sections already given are simply the special forms of the general formula :

$$M = \frac{KI}{d_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The moment of inertia, I, is to be taken about a horizontal line through the centre of gravity of the normal section, *i.e.*, about a line parallel to the side b in the three Figs. of the preceding Article.

In Fig. 1 of that Article the moment of inertia of the cross section about AB is:

$$I = \frac{bt'^3}{6} + bt' \frac{(d+t')^2}{2} + \frac{(s+t)d^3}{6}$$

$$- \left[\frac{(s-a)(d-2a)^3 + a(d-2s)^3}{6} \right] \quad . \quad . \quad . \quad (2)$$

If there are four **L**s in each flange, one only of which is shown in dotted lines:

$$I = \frac{bt'^3}{6} + bt' \frac{(d+t')^2}{2} + \frac{(2s+t)d^3}{6}$$

$$- \left[\frac{(s-a)(d-2a)^3 + a(d-2s)^3}{3} \right] \quad . \quad . \quad . \quad (3)$$

The moment of inertia of the cross section shown in Fig. 2, about AB, is:

$$I = \frac{bt'^3}{6} + bt' \frac{(d+t')^2}{2} + \frac{(s+t)d^3 - s(d-2a)^3}{6} \quad . \quad (4)$$

The moment of inertia of the cross section shown in Fig. 3, about AB, can be either directly written from an examination of that Fig., or derived from Eq. (2) by simply writing $\frac{t}{2}$ for t. It has the value:

$$I = \frac{bt'}{2} \left(\frac{t'^2}{3} + (d+t')^2 \right) + \frac{(s + \frac{1}{2}t)d^3}{6}$$

$$- \left[\frac{(s-a)(d-2a)^3 + a(d-2s)^3}{6} \right] \quad . \quad . \quad . \quad (5)$$

If the plates are omitted from the flanges in Fig. 3, as in the Article on built beams without cover plates, $t' = 0$, and

$$I = \frac{(s + \frac{1}{2}t)d^3}{6} - \left[\frac{(s - a)(d - 2a)^3 + a(d - 2s)^3}{6}\right] . \quad (6)$$

In all these cases $d_1 = \frac{1}{2}d + t'$ or $\frac{1}{2}d$, according as there are or are not cover plates.

These several values of I and d_1, substituted in Eq. (1), will give the resisting moments for the various sections. It is an open question, however, what degree of accuracy may be expected to result in the application of these formulæ. It is to be remembered that the very best of riveted work does not secure that degree of continuity presupposed by the Eq. (1). It may be stated, however, that Eq. (4) is better applicable to its cross section than the others, for there is perfect continuity between the web and a part of the flange.

Art. 70.—Examples of Built Beams Broken by Centre Weight.

Example I.—Wrought Iron Beam.

This beam was tested by Sir William Fairbairn ("Useful Information for Engineers," first series), and was composed of four 2-inch Ls riveted to a 7 by ¼-inch web plate. The distance between supports was 7 feet or 84 inches.

A section of the beam is shown by the section, only, of Fig. 1 in Art. 67; there were no cover plates.

The Ls in the bottom flange were a very little heavier than those in the upper, but the difference was so small that it has been neglected; or, rather, the small excess has been assumed to supply the loss caused by the rivet holes.

Centre breaking weight = 24,380 + 80 = 24,460 lbs.

$$l = 7 \text{ feet} = 84 \text{ inches.} \quad \therefore \quad M = \frac{Wl}{4} = 513{,}660.$$

Referring to Eq. (6) of Art. 65:

$$d = 6.5 \text{ ins.} \qquad f = 2.083. \qquad \frac{th}{6} = 0.30.$$

$$\therefore \quad d_1 \left(f + \frac{th}{6} \right) = 15.5.$$

$$\therefore \quad C = \frac{513{,}660}{15.5} = 33{,}140 \text{ lbs. per sq. in.}$$

d_1 was taken as the depth (nearly) between the roots of the Ls.

The beam gave way in the top, or compression, flange by the twisting of the Ls at a comparatively low compressive intensity. This indicates that the discontinuous riveted connection between the web and flange, although the pitch of the rivets was only 4.5 inches, fails to give such perfect support to the top flange, as a column, as the perfect continuity of the connection in a rolled beam.

The condition of the top flange, as a column, in a built beam, therefore, exercises a very important influence on the ultimate resistance of the beam, and should not be neglected.

It is probable, however, that the high compressive resistance of American wrought iron of the present day would give a much higher value of C under the same circumstances.

When the centre load added to five-eighths the weight of the beam was 8,400 pounds, the centre deflection, or w, was 0.18 inch. Hence the coefficient of elasticity was:

$$E = \frac{(W + \frac{5}{8}pl)l^3}{48wI} = 12{,}321{,}000 \text{ pounds.}$$

l^3 must be taken in inches. I was computed by Fairbairn at 46.77.

Example II.—Steel Beam.

The data for this beam were given by Albert F. Hill, C. E., in "Steel in Construction," Engineers' Soc. of West. Penn., April, 1880. Each flange was composed of two $2\frac{1}{2} \times 2\frac{1}{2} \times \frac{5}{16}$ steel angles, and one $5\frac{1}{2} \times \frac{3}{16}$ cover plate. The web was a $12 \times \frac{3}{16}$ "0.50 C" rolled steel plate. The clear span was 5 feet; pitch of rivet, 4.5 inches; total effective area of section, 8.51 square inches. The rivet holes were drilled $\frac{9}{16}$ inch in diameter.

Referring to Eq. (6) of Art. 65:

$$d^1 = 12 \text{ ins.}, \qquad f = 3.13 \text{ sq. ins.}, \qquad \frac{th}{6} = 0.375 \text{ sq. ins.}$$

$$W = \text{centre weight} = 130{,}000 + 70 = 130{,}070 \text{ pounds.}$$

$$M = 3Wl \; (l \text{ in feet}) = 1{,}951{,}050.$$

Hence:

$$C = \frac{1{,}951{,}050}{42.06} = 46{,}387 \text{ pounds.}$$

The centre load did not break the beam, but caused a deflection of 0.9375 inch, and permanent set of 0.50 inch, with beginning of side deflection.

Very closely approximate, $I = 252$. Hence, with l in feet and a centre load of 70,000 pounds with the corresponding deflection of 0.25 inch:

$$E = \frac{36 \times 70{,}000 \times 125}{0.25 \times 252} = 5{,}000{,}000 \text{ pounds.}$$

This low value of E is undoubtedly due to the fact that the beam was a built one.

The results of all the tests of built beams given in this chapter show that they are much less stiff than rolled ones of the same section. In fact, in computing deflections with the best designs and best quality of riveted work, E should probably never be taken at more than about half its value for similar rolled sections, or say at 12,000,000 to 15,000,000.

After E is determined, the deflection at once results from the usual formula:

$$w = \frac{36(W + \frac{5}{8}pl)l^3}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

l^3 is here in feet. W is the load at centre, and pl the uniform load (*i. e.* weight) of or on the beam.

Art. 71.—Loss of Metal at Rivet Holes.

As has been indicated in all examples, the metal punched or drilled from parts of beams in tension should always be deducted from the total tension area in order to obtain the effective area for computation of the ultimate resistance. In estimating this loss the actual diameter, as punched or drilled, should be taken, and not that of the cold rivet before driving, since the latter is always at least one sixteenth inch less than the former.

In the compression portions of the beam, if the work is done in a first-class manner, no deduction need be made.

Art. 72.—Thickness of Web Plate.

The following approximate method of determining the thickness of the web plate in a flanged beam is based upon the principles established in Art. 28.

It was shown in that Article that on two planes which make angles of 90° with each other and 45° with the neutral surface, and whose intersection forms the neutral axis at the section considered, there exists on one a tension and on the other a compression, each of whose intensities is equal to that of the longitudinal and transverse shear at the same point. It was also shown in Art. 17 (see Eq. (38)) that the intensity of these shears is $\frac{3}{2}$ the mean intensity of shear of the whole section.

No essential error is committed (especially in built beams) if it be assumed that the whole shear is taken up by the web. In the Article just cited it was shown that the intensity of shear at the top and bottom surfaces of the beam is zero, as well as $\frac{3}{2}$ the mean at the neutral surface. Now, if this shear be assumed uniform in intensity throughout the transverse section of the web, the shear will be made much too large at the top and bottom surfaces, and only two-thirds its proper value at the centre or neutral axis.

In accordance with these assumptions on one hand, and the established principles on the other, the web may be considered as composed of small columns with ends fixed (at the flanges), and with sections rectangular, whose axes lie at 45° with the neutral surface.

The assumption of the uniformity of shear in respect to these elementary columns causes two errors in opposite directions, with the resultant error, in most cases at least, on the side of safety.

In rolled beams, if t' is the mean thickness of a flange, and d the total depth, then the length of these elementary columns may be taken as:

$$l = (d - 2t') \sec 45°.$$

or:

$$l = 1.414(d - 2t'). \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In built beams, if d' is the depth from centre to centre of rivet holes, there may be taken:

$$l = 1.414d'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If S is the total shear at any transverse section, A the area of that section of the web, taking the depth at $d - 2t'$ or d', and s the mean shear, or:

$$s = \frac{S}{A};$$

then these elementary columns will be subjected to an intensity of compression equal to s. Hence if t, the thickness of a wrought-iron web, is sufficiently great, there may be taken, by Gordon's formula:

$$s = \frac{f}{1 + \frac{l^2}{3,000t^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Solving this equation for t:

$$t = l\sqrt{\frac{s}{3,000(f - s)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For the ultimate resistance of wrought-iron rectangular columns, f may be taken at 40,000. If a safety factor of 5 be taken, the value of t becomes:

$$t = 0.0183l\sqrt{\frac{s}{8,000 - s}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Eq. (5) is for wrought iron only. The empirical constants for steel yet remain to be determined.

These formulæ show that t decreases with the depth of the beam, and that it also varies in the same direction with s. If, therefore, the depth of the beam is constant, Eq. (5) need only be applied at the section where s is the greatest, *i.e.*, at or near the points of support.

If, however, the depth is variable, it may be necessary to apply the formula at a number of sections in order to find the greatest value of t.

Eq. (5) frequently gives much larger values of t than are required. It could be made an accurate and valuable formula if the empirical quantities which enter it were determined by experiments on flanged beams.

The data of Art. 66 give:

$$d' = 48 \text{ ins.}, \qquad t = \frac{7}{16} \text{ in.}, \qquad A = d't = 21 \text{ sq. ins.}$$

$$S = 86{,}800 \text{ lbs.}, \qquad \therefore \qquad s = \frac{S}{A} = 4{,}130 \text{ (nearly)}.$$

$$l = d' \times 1.414 = 67.9.$$

Hence:

$$t = 1.2 \text{ inches (nearly)}.$$

This value with a safety factor of five is evidently excessive, though it applies only to the portions RA and HR' of Fig. 1 in Art. 66. Yet the result may be accepted as indicating that the web needs support for those portions, and the necessity of the stiffening pieces shown.

The data of Art. 67 give:

$$d' = 27 \text{ inches}, \quad t = 3/8 \text{ inch}, \quad A = d't = 10.1 \text{ sq. ins.}$$

$$S = 26{,}000 \text{ pounds.} \quad \therefore \quad s = \frac{S}{A} = 2{,}600 \text{ (nearly)}.$$

$$l = d' \times 1.414 = 38.2.$$

Hence :

$$t = 0.49 \text{ inch (nearly)}.$$

The thickness taken, therefore, is probably ample, even without the aid of stiffening pieces.

The amount of assistance to be derived from stiffeners cannot be computed with any certainty. They are very essential however, and should be introduced in all large beams.

However small the built beam, or light its load, the web plate should never be less than 0.25 inch in thickness.

Before leaving this subject it may be well to observe that the excessive thickness given by Eq. (5) was, in some measure at least, to be anticipated. It has already been stated that the assumption of uniform compression throughout the length of the elementary column leads to an error on the side of safety. Again, the equal tension at right angles to the greatest compression in the material of the web, as well as the decreasing compression toward the centre of the beam, gives support to the elementary columns throughout their entire lengths. These causes give rise to an excess of safety, in the formula, whose amount can only be determined by experiment. Three-quarters of the thickness given by the formula would probably be ample.

The experiments of the late Baron von Weber showed that a very thin web will give a remarkably large supporting power.

CHAPTER X.

CONNECTIONS.

Art. 73.—Riveted Joints.

ALTHOUGH riveted joints possess certain characteristics under all circumstances, yet those adapted to boiler and similar work differ to some extent from those found in the best riveted trusses. The former must be steam and water tight, while such considerations do not influence the design of the latter, consequently far greater pitch may be found in riveted truss work than in boilers. Again, the peculiar requirements of bridge and roof work frequently demand a greater overlap at joints and different distribution of rivets than would be permissible in boilers.

Kinds of Joints.

Some of the principal kinds of joints are shown in Figs. 1 to 6. Fig. 1 is a "lap joint," single riveted; Fig. 2 is a "lap joint," double riveted; Fig. 3 is a "butt joint" with a single butt strap and single riveted; while Figs. 4, 5 and 6 are "butt joints with double butt straps, Fig. 4 being single riveted while the others are double riveted. Fig. 5 shows zigzag riveting and Fig. 6 chain riveting. All these joints are designed to resist tension and to convey stress from one single thickness of plate to another. Two or three other joints peculiar to bridge and roof work will hereafter be shown.

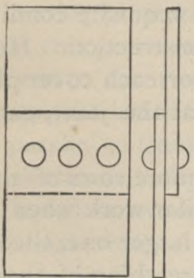


Fig. 1

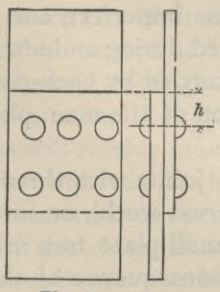


Fig. 2

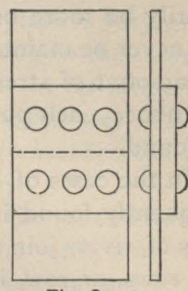


Fig. 3

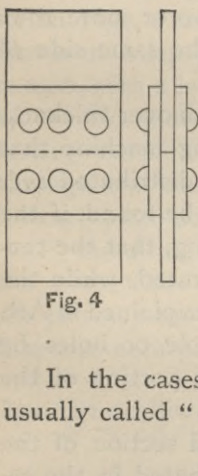


Fig. 4

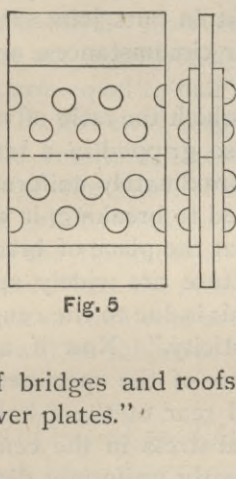


Fig. 5

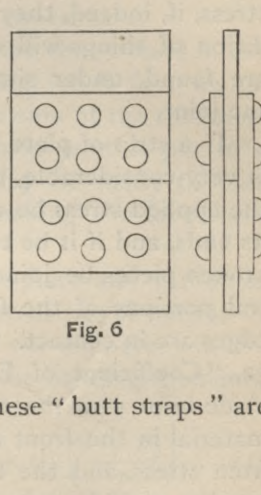


Fig. 6

In the cases of bridges and roofs these " butt straps " are usually called " cover plates."

Distribution of Stress in Riveted Joints.

A very little consideration of the question will show that only an approximate determination of the distribution of stress in a riveted joint can be reached.

In order that rivets, butt straps or cover plates, and different portions of the same main plate may take their proper portions of stress, an absolutely accurate adjustment of these different parts must be attained ; but all shop work must necessarily be more or less imperfect, and the requisite condition can never be maintained during and after construction. Hence the amount of stress carried by each rivet, or each cover plate, and hence each portion of the main plate at the joint, cannot be found.

In the cases of lap joints with three or more rows of rivets (frequently found in truss work), or in similar work when two rows of rivets join a small plate to a much larger one, the outside rows, or row, in consequence of the stretching of the material at the joint, must take far more than their portion of stress, if, indeed, they do not carry nearly all. The same condition of things will exist in butt joints if two or more rows are found, under similar circumstances, on the same side of the joint.

If a strip of plate in which the ratio of width over thickness is very considerable, be so gripped in a testing machine that the applied stress be approximately uniformly distributed over its ends, and if it be tested to breaking, it will be found, if the broken pieces be joined at the place of breaking, that the central portions of the fracture are widely separated, while the edges are in contact. This is due to the cause explained in Art. 32, "Coefficient of Elasticity." Now if a hole or holes be made in or near the centre of the specimen, a portion of the material in the front and rear of these holes will be relieved from stress, and the total stress in the central section of the specimen will be more nearly uniformly distributed in the remaining material. And again, these holes will "neck" the specimen down to a short one. The influences noticed in Art. 32, "*Ultimate Resistance and Elastic Limit*," will thus be called into action. For both these reasons the existence of the hole, or holes, *in itself*, will increase the intensity of the ultimate resistance of the plate.

On the other hand, the effect of the punch, if the hole is punched, as will presently be shown, is to decrease the resistance of the metal about the hole. If the hole is in a joint, also, the bearing pressure between the rivet and plate is very great, and as this pressure must be carried as tension to the material adjacent to the rivet hole, and through that in its immediate vicinity, the latter (*i.e.*, the material at the extremities of diamters parallel to the joint) will receive much greater tension than that in the central portion between the holes.

These last two influences tend to reduce the mean intensity of ultimate resistance of the material of the joint, and sometimes more than counterbalance the increase caused by the existence of the holes *simply as such*. In other cases the resultant effect can only be determined by experiment.

In Figs. 1 and 2 it will be observed that the stresses in the plates of a lap joint act excentrically, and, let it first be assumed, with a lever arm equal to half the sum of the thickness of the two plates. If, however, a specimen joint is put in a testing machine, the resultant stress may be made to pass through the centre of the joint, thus making the lever arm for each plate about half its thickness.

If, therefore, t is the thickness of one plate and t' that of the other, while T and T' are the mean intensities of tension in the plates, p the pitch of the rivets and d the diameter; in the first case each plate will be subjected to the bending moment:

$$M = Tt(p - d)\left(\frac{t + t'}{2}\right) = T't'(p - d)\left(\frac{t + t'}{2}\right) \quad . \quad (1)$$

And in the second:

$$M = Tt^2\left(\frac{p - d}{2}\right); \text{ or, } T't'^2\left(\frac{p - d}{2}\right) \quad . \quad . \quad . \quad (2)$$

If K is the greatest intensity of tensile bending stress, then:

$$K = \frac{tM}{2I}; \text{ or, } \frac{t'M}{2I} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The greatest intensity of tension in the plate will therefore be:

$$T + K, \quad \text{or,} \quad T' + K \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The moment of inertia I will have the value:

$$\frac{(p - d)t^3}{12}, \quad \text{or,} \quad \frac{(p - d)t'^3}{12}.$$

If each plate has the same thickness, $t = t'$ and $T = T'$; hence:

$$\text{By Eq. (1)} \quad K = 6T \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\text{By Eq. (2)} \quad K = 3T \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

These values of K are very large and appear excessive. It is to be remembered, however, that the formula used Eq. (3) is strictly applicable only within the elastic limit.

There is no reason to doubt, therefore, that within that limit the greatest intensity of tension in the plates of the joint may reach from $4T$ to $7T$.

From these considerations it is to be expected that the true elastic limit of the joint, as a whole, would be very low.

The preceding investigations in the flexure of the joint are based upon the virtual assumption that the plates remain straight after the application of external stress. In reality such a condition of things does not obtain. Even below the elastic limit the plates begin to take positions which are shown in an exaggerated manner in Fig. 7. On account of the bending, the material at the points AA stretches much more than that at the points BB (with low values of T that at the latter points may be in compression), so that the centre lines of the plates P and

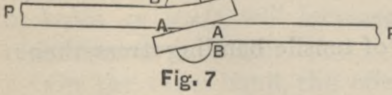


Fig. 7

P' are brought more nearly into coincidence, thus lessening the bending moment to which the joint is subjected. After the elastic limit of the material at AA is passed, a considerable increase of strain or stretch takes place at those points for the same increment of stress. Two important results follow this increase of strain between the elastic limit and failure: the joint becomes very markedly distorted, so that the plates P and P' become much more nearly in line, and the stress becomes much more nearly uniformly distributed in the sections AB, AB. This is equivalent to saying that the joint is subject to a greatly decreased bending moment.

If the plates are thin, the excess of strain at AA over that at BB, requisite to bring the plates PP' essentially into line, may easily be within the stretching capacity of the material. If, however, the plates are thick, that condition will not hold, and the material at AA will begin to fail before PP' are nearly in line. Hence, the mean intensity of stress in a thick plate, other things being equal at the instant of rupture, will be considerably less than that in a thin one. It might thus happen that a lap joint with thin plates would be found stronger, even, than one with thicker plates.

Reference will hereafter be made to experiments which verify these conclusions.

It will now be well to turn back a moment to the consideration of Eqs. (5) and (6). Those equations show the effect of bending to be dependent on T *only*, and entirely *independent of the thickness of the plates*, which apparently contradicts the conclusion just drawn. But, as has already been intimated, those equations involve the virtual assumption that the plates remain continually straight, and do not contemplate the altered conditions of the joint which exist just at and before rupture. Again, they presuppose no passage of the elastic limit. There is thus no real contradiction.

Although a single riveted lap joint only has been treated, precisely the same considerations apply to a double riveted lap joint, a butt joint with single butt strap or cover plate, and all butt straps or cover plates of butt joints. The main plates of butt joints with double cover plates are not subjected to flexure.

The rivets of all riveted joints are subjected to heavy flexure, the greatest of which usually occurs in single lap and butt joints like Figs. 1 and 3. An approximate value of the bending moment, in any case, may be found as follows:

Let n be the number of rows of rivets in one plate. In Figs. 1, 3, 4, n is 1; and 2 in Figs. 2, 5 and 6. Then if t and t' are the thickness of the two plates or of one plate and one cover, T and T' the mean intensities of tension in the same pieces, and if M be taken from Eq. (1), the approximate bending moment will be:

$$\frac{M}{n} = \frac{KAd}{8}; \text{ (From Art. 63)}; \quad . \quad . \quad . \quad . \quad (7)$$

in which A is the area of the cross section of one rivet, K the greatest intensity of tension or compression due to bending, and d the rivet diameter, as before. From Eqs. (7) and (1):

$$K = 4Tt(p - d)\frac{(t + t')}{nAd} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If $t = t'$:

$$K = 8Tt^2\frac{(p - d)}{nAd} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This equation is approximate because it is virtually assumed that the pressure on the rivet is uniformly distributed along its axis.* This is a considerable deviation from the truth, particu-

* In accordance with this assumption, strictly speaking, $\frac{1}{2}t$ (thickness of main plate) should be taken instead of t in the sum $(t + t')$ in the above formulæ foı bending, when applied to the double butt joints, Figs. 5 and 6.

larly as failure is approached. The true bending moment is much less than that given by Eq. (7) after the rivet has deflected a little.

When the joint takes the position shown in Fig. 7, it is clear that the rivet is also subject to some direct tension.

There is a very high intensity of pressure between the shaft of the rivet and the wall of the hole. This intensity is not uniform over the surface of contact, but has its greatest value at, or in the vicinity of, the extremities of that diameter lying in the direction of the stress exerted in the plate. At and near failure this intensity may be equal to the crushing resistance of the material over a considerable portion of the surface of contact.

The intricate character of the conditions involved renders it quite impossible to determine the law of the distribution of this pressure. The bending of the rivets under stress tends to a concentration of the pressure near the surface of contact of the joined plates, while the unavoidably varying "fit" of the rivet in its hole, even in the best of work, throws the pressure towards the front portion of the surface of the rivet shaft. The intensity thus varies both along the axis and around the circumference of the rivet.

If any arbitrary law is assumed, the greatest intensity of pressure is easily determined. Such laws, however, are mere hypotheses and possess no real value. All that can be done is to determine, by experiment, the mean safe working intensity on the diametral plane of the rivet which is equivalent to a fluid pressure of the same intensity against its shaft.

Thus, if f is this mean (empirically determined) intensity, d the diameter of the rivet, and t the thickness of the plate, the total pressure carried by one rivet pressing against one plate is:

$$R = fdt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

There yet remains to be considered the condition of that portion of the plate on which the pressure $R = fdt$ is applied, and which is situated immediately in front of the rivet.

This portion of the plate is really in the condition of a beam fixed at each end, with a span equal to the diameter of the rivet. The beam, however, is not a straight one. At each end of the diameter the direct bending stress will be tension; and, on account of the position of the material, its direction will be approximately, at least, that of the proper tension of the plate. At those points, therefore, the proper and bending tension will act to some extent together, and the metal will usually be more highly stressed than anywhere else. This accounts for the usual manner of tensile fracture of a joint, in which the metal begins to tear on each side of the rivets, the metal between (generally in a diagonal direction in zigzag riveting) being the last to give way.

In the interior of the joint it is quite impossible to determine the value of this tensile bending stress on each side of the rivet. On the exterior of the joint, however, an approximate result may be reached; and hence, the depth h, Fig. 2, from the centre of the outside row of rivets to the edge of the plate. The depth of the beam will be taken as $\left(h - \frac{d}{2}\right)$, and the pressure or load will be considered concentrated at the middle of the diameter or span. If t is thickness of the plate, p the pitch of the rivets and T the mean intensity of tension between the rivets, the load on the beam will be $(p - d)Tt$, and the moment of inertia of the cross section will be:

$$I = \frac{t\left(h - \frac{d}{2}\right)^3}{12}.$$

From what has been shown in the chapter on bending, the modulus of rupture in the present case may be safely taken at $\frac{3}{2}T$.

In Art. 24, the moment at the centre and end of a span fixed at each end and loaded in the centre was shown to be equal to one-eighth the load into the span.

Hence, by the usual formulæ:

$$M = \frac{d}{8}(p - d)Tt = \frac{2KI}{\left(h - \frac{d}{2}\right)} = \frac{3}{2}T\frac{t\left(h - \frac{d}{2}\right)^2}{6}.$$

$$\therefore \quad h = 0.71\sqrt{(p - d)d} + 0.5d \quad . \quad . \quad . \quad . \quad (11)$$

Reviewing the results of this section, it may be concluded that the bending of the plates about axes parallel to them, or normal to them in the interior of the joint, and the bending of the rivets, as well as the law of the distribution of pressure against them, cannot be expressed by formula with any useful degree of accuracy; but that such influences must be recognized in the empirical determination of the shearing and tearing resistances of the joint and the mean intensity of pressure against the diametral plane of the rivet.

Effect of Punching.

The effect of punching wrought-iron plates has been found to be injurious. The tensile resistance of the remaining material will be considerably less than that of the plate before punching. Yet the injurious effect of the punch does not extend far into the plate. If the punched hole is reamed, so that the diameter is increased an eighth of an inch, the remaining plate will usually give the normal resistance per unit of section, or essentially so.

It has been found by experiment that effect of the punch is less injurious as the die hole is increased in diameter, although there is probably a limit to the application of this principle.

The diameter of the die hole is usually from $\frac{1}{7}$ to $\frac{1}{5}$ larger than that of the punch. This excess should depend upon the thickness (t) of the plate, and it is sometimes taken as $0.2t$.

Numerous foreign experiments (chiefly English) by Barnaby, Stoney, Fletcher, etc., show that the loss of tensile resistance due to punching wrought-iron plates runs usually from 10 to 15, and may vary from 5 to 33 per cent. of the original resistance.

The loss of resistance due to punching and its remedy, in steel plates, have already been treated in Art. 34.

Wrought-Iron Lap Joints, and Butt Joints with Single Butt Strap.

A butt joint with single butt strap, similar to that shown in Fig. 3, is really composed of two lap joints in contact; since each half of the butt strap or cover plate with its underlying main plate forms a lap joint. It is unnecessary therefore to give it separate treatment.

From these considerations it is clear that the thickness of the butt strap or cover plate should be the same as that of the main plate.

Let t = thickness of plates.
" d = diameter of rivets.
" p = pitch of rivets (*i. e.*, distance between centres in the same row).
" T = mean intensity of tension in plates between rivets.
" T = mean intensity of tension in main plates.
" f = mean intensity of pressure on diametral plane of rivet.
" S = mean intensity of shear in rivets.
" n = number of rivets in one main plate.
" q = number of rows in one main plate.
" h = amount of extreme lap as shown in Fig. 2.

If all the dimensions are in inches, then T, T', f and S are in pounds per square inch.

The starting point in the design of a joint is the thickness t of the plate. The rivet diameter is then expressed in terms of t, and the pitch in terms of the diameter.

The thickness t of boiler plate depends upon the internal pressure, and is to be determined in accordance with the principles laid down in Art. 9, after having made allowance for the metal punched out at the holes and the deterioration or other effect caused by the punch.

In truss work the thickness depends upon the amount of stress to be carried, and the same allowances are to be made for punching and deterioration.

The relation existing between T and T' is shown by the following equations:

$$t(p-d)\,T = tpT \quad \therefore \quad \frac{T}{T'} = \frac{p}{p-d};$$

or,

$$\frac{T'}{T} = \frac{p-d}{p} = 1 - \frac{d}{p}. \qquad (12)$$

In order that the joint may be equally strong in reference to all methods of failure, the following series of equalities must hold:

$$\frac{n}{q}\,tpT' = \frac{n}{q}\,t(p-d)\,T = nfdt = 0.7854nd^2S.$$

$$\therefore \quad tpT' = t(p-d)\,T = qfdt = 0.7854qd^2S. \qquad (13)$$

It is probably impossible to cause these equalities to exist in any actual joint, but none of the intensities T', T, f or S should exceed a safe working value.

In ordinary American boiler practice d varies from $1.5t$ to

$2t$; the latter for thin plates and the former for thicker ones, the extreme limits being about $\frac{5}{8}$ inch and $1\frac{1}{8}$ inches.

The following are some rules given by the best foreign authorities for wrought iron:

Browne .... $d = 2t$ (or $1.25t$ with double covers) .... (14)

Fairbairn ...... $d = 2t$ for plates less than $\frac{3}{8}$ in .... (15)

Fairbairn ...... $d = 1.5t$ for plates greater than $\frac{3}{8}$ in.... (16)

Lemaitre....... $d = 1.5t + 0.16$ .... (17)

Antoine........ $d = 1.1\sqrt{t}$ .... (18)

Pohlig .... $d = 2t$ for boiler riveting.............. (19)

Pohlig .... $d = 3t$ for extra strength............. (20)

Redtenbacher .. $d = 1.5t$ to $2t$ .... (21)

Unwin .... $d = 0.75t + \frac{5}{16}$ to $\frac{7}{8}t + \frac{3}{8}$............. (22)

Unwin .... $d = 1.2\sqrt{t}$ .... (23)

As the results of some of his experiments on ⅜-inch steel plate joints, Prof. A. B. W. Kennedy gives in "Engineering," 10th June, 1881, the following rules for rivet diameter:

$$\left.\begin{array}{ll}\text{Single riveted lap joint....} & d = 2.25t \\ \text{Double riveted lap joint...} & d = 2.21t\end{array}\right\} \quad . \quad . \quad (24)$$

These rules are for mild steel plates and for greatest strength, but are not to be applied to plates over ½ in. thick; as the diameters would then become excessive. He therefore

THICKNESS OF PLATE.	DIAMETER OF RIVETS.						
	Lloyds' Rules.	Liverpool Rules.	English Dockyard Rules.	French Veritas.	Wilson's Rules.	Hovrez's Rules.	Hall's Rules.
In.	In.	In.	In	In.	In.	In.	In.
$\frac{5}{16}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{1}{2}$	—	$\frac{5}{8}$	$\frac{11}{16}$	$\frac{5}{8}$
$\frac{3}{8}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{11}{16}$	$\frac{3}{4}$	$\frac{11}{16}$
$\frac{7}{16}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{13}{16}$	$\frac{13}{16}$
$\frac{1}{2}$	$\frac{3}{4}$	$\frac{13}{16}$	$\frac{3}{4}$	—	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{15}{16}$
$\frac{9}{16}$	$\frac{3}{4}$	$\frac{13}{16}$	$\frac{7}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1	1
$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{7}{8}$	—	$\frac{7}{8}$	1	$1\frac{1}{16}$
$\frac{11}{16}$	$\frac{7}{8}$	$\frac{7}{8}$	$\frac{7}{8}$	$\frac{13}{16}$	$\frac{7}{8}$	—	—
$\frac{3}{4}$	$\frac{7}{8}$	$\frac{15}{16}$	1	$\frac{7}{8}$	1	—	—
$\frac{13}{16}$	$\frac{7}{8}$	1	1	—	1	—	—
$\frac{7}{8}$	1	$1\frac{1}{8}$	$1\frac{1}{8}$	1	1	—	—
$\frac{15}{16}$	1	$1\frac{3}{16}$	$1\frac{1}{8}$	—	$1\frac{1}{8}$	—	—
1	1	$1\frac{1}{4}$	$1\frac{1}{8}$	$1\frac{1}{16}$	$1\frac{1}{8}$	—	—

concluded that thicker plates than ½ in. would give proportionally less resistance.

It has been found by experiment that there is a very decided interdependence existing between the values of T and f in cases of failure by tearing. This is probably due far more to the bending action of the rivet, which was considered in detail in one of the preceding sections, than to the direct influence of the pressure between the rivet and its hole.

Table I. contains values of T and f at the instant of failure, which were tabulated by Prof. Unwin in "Engineering" for Feb. 20th, 1880. All the plate was English material. The results show very clearly the increase of T with the decrease of f. They are, however, somewhat discordant. The punched single riveted lap joints of Mr. Stoney's experiments show an apparently abnormally low value of the tenacity T for a given intensity of compression f; but the drilled holes show less disagreement.

TABLE I.

Wrought Iron.

EXPERIMENTER.	FORM OF JOINT.	f, IN LBS. PER SQUARE INCH.	T, IN LBS. PER SQUARE INCH.
Fairbairn.....	Lap, single riveted.....	83,776	39,650
	Lap, single riveted	66,860	44,580
	Lap, double riveted.....................	78,290	52,190
	Lap, double riveted.....................	76,830	48,830
	Lap, double riveted.....................	58,460	58,460
	Lap, double riveted	51,300	55,330
	Butt, double riveted, one cover	58,020	53,980
	Butt, single riveted, two covers	94,210	53,540
	Butt, single riveted, two covers	65,180	60,700
Kirkaldy	Lap, single riveted.....................	58,580	47,260
	Lap, double riveted.....................	36,740	57,340
	Butt, double riveted	74,700	43,900
	Butt, double riveted	71,750	45,570
	Butt, double riveted	63,170	45,020
	Butt, double riveted	62,610	39,200
Browne	Lap, single riveted.....	93,640	29,120
	Lap, single riveted............	86,950	27,100
	Lap, single riveted....	84,980	26,300
	Butt, single riveted	101,150	31,360
	Butt, single riveted	94,240	29,120
	Butt, single riveted	92,840	28,880
Stoney	Lap, single riveted, punched...........	66,210	31,910
	Lap, single riveted, punched....	55,660	32,930
	Lap, single riveted, punched...........	49,460	37,630
	Lap, single riveted, punched...........	47,260	35,840
	Lap, single riveted, punched...........	43,680	45,920
	Lap, single riveted, punched....	42,110	44,350
	Lap, single riveted, punched...........	38,770	40,770
	Lap, single riveted, drilled............	64,400	46,820
	Lap, single riveted, drilled.........	59,020	34,940
	Lap, single riveted, drilled....	54,650	41,440
	Lap, single riveted, drilled....	48,370	36,740
	Lap, single riveted, drilled.....	47,520	47,490
	Lap, single riveted, drilled.......	46,140	48,380
	Lap, single riveted, drilled............	45,920	48,270

Reviewing all the results, it would seem that the following values may safely be given single riveted lap joints with punched holes in first-class work :

$$f = 55{,}000 \text{ to } 60{,}000 \ldots\ldots\ldots T = 45{,}000 \text{ to } 40{,}000.$$
$$f = 55{,}000 \text{ to } 50{,}000 \ldots\ldots\ldots T = 45{,}000 \text{ to } 50{,}000.$$

The following values of f, T and S, at the instant of failure, are from the experiments (English) of Messrs. Greig and Eyth and the Master Mechanics' Association.

	f, IN LBS. PER SQ. INCH.	T, IN LBS. PER SQ. INCH.	S, IN LBS PER SQ. INCH.	
Single riveted lap joints...	64,400	46,820	40,990	..(A)
	59,490	43,650	41,300	
	59,960	43,970	41,680	
	62,400	45,760	43,340	
	66,280	47,690	38,770	

All the holes in these joints were drilled, consequently, as will hereafter be shown, S is a little low. Further, all the joints broke by simultaneous shearing of the rivets and tearing of the plates: they may therefore be considered well designed.

Now, if $f = T = 50{,}000$, which is experimentally shown to be correct in single riveted lap joints, for which $q = 1$, the second and third members of Eq. (13) give:

$$p = 2d.$$

But this pitch would scarcely give sufficient room for heading the rivets. It has just been seen that the results in group (A) belong to well proportioned joints. An examination of those results will show that f varies from $1.33T$ to $1.4T$, nearly; which is not an essential disagreement with the results of Table I. Hence, putting these values in Eq. (13):

$$p = 2.33d \quad \text{to} \quad 2.4d \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

This agrees with good ordinary practice in boiler making, which makes:

$$p = 2.3d \quad \text{to} \quad 2.75d, \text{ nearly.}$$

The preceding results are for *single riveted lap joints in wrought iron.*

TABLE II.

Wrought Iron Double Riveted Lap Joints.

EXPERIMENTER OR AUTHORITY.	MODE OF RIVETING.	HOLES MADE BY	POUNDS PER SQUARE INCH FOR	
			f.	*T.*
Sir Wm. Fairbairn............	Hand.	Punch.	68,580	51,450
" " "	"	"	70,990	53,180
" " "	"	"	60,860	45,670
" " "	"	"	69,490	52,060
" " "	"	"	58,350	58,350
" " "	"	"	51,030	54,680
David Kirkaldy............	"	"	36,710	57,270
Easton and Anderson......	Machine.	"	56,380	36,470
" " "	"	"	53,400	34,670
" " "	"	"	59,970	38,770
Greig and Eyth............	"	Drill.	57,030	45,790
" " "	"	"	34,090	49,060
R. V. J. Knight	"	Punch.	22,020	24,440
" "	"	"	21,540	23,630
" "	"	"	21,500	28,650
" "	"	"	22,220	29,610
" "	?	"	30,330	27,280
" "	?	"	31,230	28,740

In the second preceding section considerations were adduced which show that for a given value of the mean intensity of compression between the rivet and its hole, in a double riveted lap joint, an increased value of T over that for a single riveted lap joint should be expected. So far as comparison can be made, Tables I. and II. verify this conclusion, although the increase is not very great. This arises from the fact that the increased length of a double joint requires less bending at

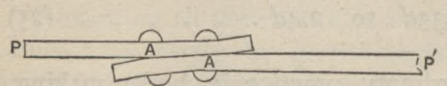


Fig. 8

A, A, Fig. 8, than a single one to bring the plates P and P' nearly into line.

The tables show that for thin plates f is equal to T, at the instant of rupture, for an intensity not far from 55,000 pounds per square inch. This will reduce somewhat the allowable ratio between f and T.

A careful examination of the results given in the tables seems to make it perfectly safe to take f from $1.1T$ to $1.25T$. These values in the second and third members of Eq. (13) give (remembering that q is here equal to 2) for *double riveted lap joints:*

$$\text{Or, say:}\qquad \left.\begin{aligned} p &= 3.2 \quad \text{to} \quad 3.5d \\ p &= 3.25 \quad \text{to} \quad 4.0d \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

The smaller values of p belong to thick plates and the larger values to thin ones, both because the increased thickness brings a greater proportional load on the rivet and because the lever arm of the bending moment is greater.

It should be stated that in some apparently good boiler practice p is sometimes taken as high even as $5d$. The ease with which a double riveted lap joint is made steam tight may tempt a decrease in expense of riveting. It is probable that the rivets of joints in which the pitch exceeds about $4d$ carry an excessive compression and a corresponding liability to weakness.

In Table II. the experiments of Mr. Knight were made on plates one inch thick, which are excessively heavy, and the values of f and T are remarkably small. It has already been demonstrated that great thickness of plates would produce results of such a character, although the sufficiency of such an explanation has been doubted. There seems little reason to doubt, however, that the cause just cited, together with the normal decrease of resistance with an increase of thickness, is a complete explanation.

It is to be observed that in the preceding deduced values of f and T, the bending of the plates about axes both parallel and normal to their surfaces, have been recognized and provided for.

If the accuracy of the experiments cited be assumed, and they are the most reliable and valuable that have ever been made, there may be taken:

For 1-inch plates, $T = 30{,}000$ to $35{,}000$ lbs. per sq. in.
For ¼-inch plates, $T = 50{,}000$ to $55{,}000$ lbs. per sq. in.

And for intermediate plates proportional values.

For single riveted lap joints, $f = 1.33$ to $1.4\ T$.
For double riveted lap joints, $f = 1.1$ to $1.25\ T$.

As f and T have been found to be dependent on the peculiar circumstances attending the use of the material in the joint, so, in the same general manner, the determination of the ultimate shearing resistance of the rivets must involve a similar recognition of environment.

It has been found by experiment, as might have been anticipated, that rivets in drilled holes offer less resistance to shearing than those in punched holes. This arises from the fact that the edges of drilled holes are much sharper than those formed by a punch.

Table III. gives the mean results of a large number of experiments by the authorities named. It has been condensed, and the results converted to pounds per square inch, from a similar one given by Prof. Unwin, in "Engineering" for 26th March, 1880.

These results are for single riveted lap joints, and therefore for single shear. They are only a very little larger than the values determined by Chief Engineer Schock for single shear, as the apparatus of the latter was essentially equivalent to a drilled hole.

For plates 0.25 inch to 0.375 inch thick, there may be taken, as is usually done, $S = 0.8\,T$. It has been seen (Table II.) that a plate an inch thick can be expected, in lap joints, to

TABLE III.

Shearing of Wrought Iron Rivets.

EXPERIMENTER OR AUTHORITY.	KIND OF HOLE.	S IN POUNDS PER SQ. IN.	RESISTANCE (TENSILE) OF PLATE OVER S.
Fairbairn	Punched.	50,180	0.783
Stoney	Punched.	42,200	0.910
Stoney	Drilled.	40,920	1.061
Fairbairn	Punched.	45,820	——
Fairbairn	Drilled.	43,610	——
Master Mechanics' Association	Drilled.	46,590	——
Greig & Eyth	Drilled.	41,280	1.071
Mean result	Punched.	46,030	0.846
Mean result	Drilled.	43,100	1.066

give T not much over 35,000, and as the thickness does not seem to appreciably affect S, for this inch plate there may be taken $S = \frac{8}{7}T$. The ratio of f over T has been seen to vary from 1.33 to $1.4T$. Let a mean value of 1.36 for this last ratio be inserted in the third member of Eq. (13); then, by inserting the other values just found in the fourth member of the same equation, there will result for *single riveted lap joints:*

$$\left.\begin{array}{l}\text{For thin plates, } d = 2.1t \\ \text{For thick plates, } d = 1.5t\end{array}\right\} \quad . \quad . \quad . \quad . \quad (27)$$

For *double riveted lap joints* these results would be diminished only slightly. Hence Eq. (27) may be taken as applicable to both single and double riveted lap joints in wrought iron.

It will be observed that Eq. (27) is included within the limits of the Eqs. (14)–(23).

A great number of results by the experimenters already cited in this chapter show that the total resistance of a single riveted lap joint, as a whole, for plates not over 0.5 inch thick, may vary from 44 to 58 per cent. of the solid plate in its normal condition, and that the mean value may be taken from 50 to 52 per cent.

In a double riveted lap joint this mean may be taken at 60 per cent. of the resistance of the original plate, for moderate thicknesses. In Mr. Knight's experiments with inch plates (double riveted), the resistance of the joint, as a whole, ranged from 33 to 36 per cent. of that of the plate.

It is clear, from the preceding investigations, that this "efficiency" of the joint must decrease as the thickness of the plate increases. In fact, Mr. Bertram found, in 1860, that some joints in $\frac{3}{8}$-inch plates were stronger than those in either $\frac{7}{16}$ or $\frac{1}{2}$-inch plates. Although such results do not involve impossibilities, they are certainly remarkable, and have not since been obtained.

As has before been observed, *all the preceding results apply directly to butt joints, in wrought iron, with single butt strap or cover plate.*

The width of overlap (h) from the centre of the outside line of rivets to the edge of the plate (see Fig. 2) may now be determined in terms of d, by the aid of Eq. (11). Since the load on the rivet is represented by $(p - d)Tt$, p must be taken in terms of d for a single riveted joint, in which $p = 2\frac{1}{3}d$ to $2\frac{3}{4}d$. As a margin of safety, and as it will, at the same time, simplify the resulting expression, let $p = 3d$.

Eq. (11) then gives:

$$h = 1.5d. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Experience has shown that this rule gives ample strength, and is about right for caulking, in boiler joints.

The distance between the rows of riveting is not susceptible of accurate expression by formulæ, although the considerations involved in the establishment of Eq. (11) would lead to an approximate value. It is evident, however, that this distance should never be as small as h. Apparently, in more than double riveted joints, this distance should increase as the centre line of the joint is receded from, in consequence of the bending action of the rivet. There are other reasons, however, besides that of inconvenience, why such a practice is not advisable.

In chain riveting the distance between the centre lines of the rows of rivets may be taken equal to the pitch in a single riveted joint, or, as a mean, at 2.5 *the diameter of a rivet.*

In zigzag riveting (Fig. 5) *this distance may be taken at three-quarters its value for chain riveting.*

Steel Lap Joints and Butt Joints with One Cover.

The general phenomena attending the tests of steel joints are precisely the same in kind with those observed in connection with riveted iron plates; they do not, therefore, need particular consideration in this section.

Table IV. contains results communicated to the "Committee of the Institution of Mechanical Engineers" by Messrs. Parker and Sharp ("Engineering," 16th April, 1880). The joints failed by tearing, and gave the values of T shown in the table. The intensity of pressure, f, existed at rupture.

The following values of T and f under precisely the same circumstances, *i.e.*, failure, were found by Prof. A. B. W. Kennedy ("Engineering," 20th May and 10th June, 1881,) for single riveted lap joints.

THICKNESS OF PLATE.	T.	f.
$\frac{3}{8}$-inch	67,060 lbs. per sq. in.	42,980 lbs. per sq. in.
$\frac{3}{8}$ "	65,310 " " " "	57,600 " " " "
$\frac{3}{8}$ "	77,050 " " " "	70,850 " " " "
$\frac{3}{8}$ "	73,030 " " " "	70,520 " " " "
$\frac{3}{8}$ "	80,920 " " " "	73,420 " " " "

TABLE IV.

Steel Joints.

JOINT.	HOLE.	THICKNESS OF PLATE.	POUNDS PER SQ. IN. FOR	
			T.	*f.*
Treble riveted (chain)	Drilled.	$\frac{3}{8}$ in.	79,220	60,010
" " "	Punched.	$\frac{3}{8}$ in.	52,280	39,380
" " "	"	$\frac{7}{16}$ in.	50,330	37,740
" " "	Drilled.	$\frac{1}{2}$ in.	73,360	57,700
" " "	"	$\frac{1}{2}$ in.	70,040	55,080
" " "	"	$\frac{1}{4}$ in.	80,890	54,470
" " "	"	$\frac{1}{4}$ in.	78,400	52,790
" " "	"	$\frac{1}{2}$ in.	66,940	52,220
" " "	"	$\frac{1}{2}$ in.	67,520	53,330
" " "	"	$\frac{1}{4}$ in.	80,380	73,830
" " "	"	$\frac{1}{2}$ in.	75,780	49,500
" " "	"	$\frac{3}{4}$ in.	68,250	47,580
" " "	?	1 in.	65,950	35,460
Quadruple riveted (zigzag)	?	$\frac{7}{8}$ in.	56,760	42,360
Double riveted butt (one cover) ...	Drilled.	?	87,920	76,200
Double riveted butt (one cover) ...	Punched.	?	97,730	83,840

The holes in these plates were all drilled, and each result is a mean of two tests.

These experiments do not present a sufficient range to show clearly the relation existing at failure between T and f. It is clear, however, that no recorded intensity f has been large enough to decrease T to any appreciable amount. In some of Prof. Kennedy's tests, in which failure took place by shearing, f was not far from $1.2T$ (with $T = 65,000$ to $75,000$), and it would appear from his experiments that such a ratio may properly be taken for thin plates in single riveted joints. *At the same time*, with the mild steel used by Prof. Kennedy, *T may be taken at* 70,000 *pounds for plates* ¼ *to* ⅜ *inch thick.*

Putting $1.2T$ for f in the third member of Eq. (13):

$$\text{Or, say,}\qquad \left.\begin{array}{l} p = 2.2d \\ p = 2.25d \end{array}\right\} \quad \ldots\ldots\ldots \quad (29)$$

for *single riveted lap joints.* It will probably be best to allow this pitch to stand for thick plates also, although experiments to verify such a conclusion are yet lacking. For very thick plates in single riveting, however, T should not be taken over 50,000 to 55,000 pounds at the highest.

Experiments on double riveted lap joints by Martell, Kirkaldy and Easton and Anderson, show that it will be essentially correct, and certainly safe, to take f and T as in the single riveted joints. With q equal to 2, Eq. (13) will then give for *double riveted steel lap joints:*

$$\text{Or, say,}\qquad \left.\begin{array}{l} p = 3.4d \\ p = 3.5d \end{array}\right\} \quad \ldots\ldots\ldots \quad (30)$$

Although relating to treble and quadruple riveted joints, Table IV. shows in a marked manner the decrease of T with the increase of thickness, and verifies the conclusion drawn in the preceding section in regard to that phenomenon.

The results cited by Prof. Unwin, in the report so frequently referred to heretofore, indicate that for treble riveted joints f may be taken essentially equal to T for thin plates, and $0.9T$ for thick ones. Hence, using Eq. (13) as before:

TREBLE RIVETING.

$$\left.\begin{array}{l} \text{Thin plates (0.25 and 0.375 in.),}\quad p = 4d \\ \text{Thick plates (0.875 and 1.00 in.),}\ p = 3.7d \end{array}\right\} \quad \ldots \quad (31)$$

Some experiments of Mr. Kirkaldy on joints with ⅞-inch Siemens steel plates quadruple riveted, seem to show that the pitch should be about the same as in treble riveted. This is undoubtedly due to the fact that with such a great number of rivets it becomes impossible to obtain even an approximately proper distribution of load among them.

In treble and quadruple riveting the tests cited show that T may be taken at 70,000 to 75,000 for thin plates, and 55,000 to 60,000 for thick ones.

In all the preceding investigations it is supposed that the holes are drilled, or that the plates are subsequently annealed if punched.

In nearly all the experiments cited by Prof. Unwin, the value of T, as found in the actual joint, exceeded the ultimate resistance of the original plate; a result which finds its explanation in the drilling of the holes and the "shortening" effect produced by their presence, aided by their equalizing effect.

Table V. gives the ultimate shearing resistance of steel rivets as determined by Sharp, Martell, Kirkaldy and Greig and Eyth. A very considerable reduction is noticed with the increase in plate thickness, due probably to increased bending and size of rivet.

Prof. Kennedy found the following values in single riveted lap joints:

RIVET DIAM.	S.
0.75 in.	54.460 lbs. per sq. in.
1.00 "	37,240 " " " "
1.00 "	38,720 " " " "
0.75 "	48,030 " " " "
0.75 "	49,450 " " " "
0.75 "	49,480 " " " "
0.75 "	49,300 " " " "
0.75 "	47,870 " " " "

Each result is a mean of two or three tests.

In Mr. Kirkaldy's four tests of ⅞-inch treble and quadruple riveted lap joints, with 1⅛-inch rivets, the ultimate shearing resistance S varied from 41,110 to 46,260 lbs. per sq. in.

TABLE V.

Shearing of Steel Rivets.

JOINT.	MEAN OF.	THICKNESS OF PLATE.	S IN POUNDS PER SQ. IN.
Single riveted	2	——	57,570
" "	6	——	53,690
Double riveted (chain)	8	——	53,310
" " "	1	——	50,650
Treble " "	7	——	60,930
" " "	1	$\frac{1}{4}$ in.	56,220
" " "	1	$\frac{1}{2}$ "	57,120
" " "	1	$\frac{3}{4}$ "	53,540
" " "	1	1 "	53,980
" " (zigzag)	1	$\frac{7}{8}$ "	43,560
" " "	1	$\frac{7}{8}$ "	46,140
Quadruple riveted (zigzag)	1	$\frac{7}{8}$ "	43,010

Four experiments by Mr. Kirkaldy on single riveted lap joints, during 1881, gave S varying from 52,106 to 54,042 lbs. per sq. in.

Prof. Kennedy's results give nearly:

$$S = 0.7 T.$$

Tables IV. and V., plates not over $\frac{1}{2}$ in. thick:

$$S = 0.8 T.$$

Mr. Kirkaldy's for treble and quadruple riveting:

$$S = 0.7 T.$$

For ordinary plates therefore in single and double riveting, for which $f = 1.2 T$ and S as a mean $= 0.75\ T$, the third and fourth members of Eq. (13) give:

$$d = 2t \text{ (nearly)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

For thick plates in treble and quadruple riveting, for which $f = 0.9T$, and $S = 0.7T$:

$$d = 1.6t \text{ (nearly)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

The rivet pitch, therefore, for steel plates, may be said to vary from $2t$ for thin plates to $1.6t$ for thick ones, with a maximum diameter of $1\frac{1}{8}$ to $1\frac{3}{16}$ inches.

Prof. Kennedy's best designed single riveted lap joints gave from 55 to 64 per cent. the strength of the solid plates.

Well designed double riveted lap joints should give from 65 to 75 per cent. the resistance of the solid plate.

Equally well constructed treble and quadruple riveted joints should have an efficiency of 70 to 80 per cent. of the solid plate.

It is therefore seen that there is little economy in more than double riveting ordinary joints.

The distance between the centre lines of the rows of rivets, and the distance from the edge of the lap to the outside centre line of holes, may be taken the same as for wrought-iron joints, according to the rules given in the last part of the preceding section.

All rivets have heretofore been supposed to be steel. In the case of steel plates and iron rivets, there may be taken, at least approximately, $0.9S$ for S, and $f = T$ for thin plates, or $0.8T$ for very thick ones. These values are to be inserted in the preceding formulæ for all steel joints, and the results for p and d taken.

Wrought-Iron Butt Joints with Double Covers.

Butt joints with double butt straps or covers differ in two respects, and advantageously, from lap joints and butt joints with a single cover; *i. e.*, in the former the rivets are in double shear and the main plates are subjected to no bending. The cover plates, however, are subjected to greater flexure than the plates of a lap joint, for there is no opportunity to decrease the leverage by stretching. As the covers form only a small portion of the total material, these, with economy, may be made sufficiently thick to resist this tendency to failure.

Let t' = thickness of each cover plate.

And let the remaining notation be the same as in the preceding section. The intensity of compression between the walls of the holes in the cover plates and the rivets, and the tension in the former, will be ignored on account of the excess in thickness of the two cover plates combined over that of the main plate. This excess in thickness is required on account of the bending in the covers noticed above.

The thickness of each cover should be from ¾ to ⅞ the thickness of the main plates, or $t' = 0.75t$ to $0.875t$.

The combined thickness of the covers will thus be from 1.50 to 1.75 that of the main plates.

The four principal methods of rupture in the main plate will then lead to the following equations, corresponding to Eq. (13):

$$\frac{n}{q}\,tpT' = \frac{n}{q}\,t(p - d)\,T = nfdt = 1.5708nd^2S.$$

$$\therefore\ tpT' = t\,(p - d)\,T = qfdt = 1.5708qd^2S \quad . \quad (34)$$

The experiments of Kirkaldy, Fairbairn, Greig and Eyth and Knight, show that in well proportioned joints $f = 1.25$ to $1.5\,T$ (the higher values belonging to the thinner plates), with a mean value of about $1.4T$. As no bending exists in the main plates, this value holds in single or double riveting.

Hence for *single riveting*, the second and third members of Eq. (34) give

$$p = 2.4d;\ \text{ or, say, }\ p = 2.5d \quad . \quad . \quad . \quad . \quad (35)$$

In *double riveting*, for which $q = 2$:

$$p = 3.8d; \quad \text{or, say,} \quad p = 4.0d \quad . \quad . \quad . \quad . \quad (36)$$

On account of the essential impossibility of even an approximately proper distribution of the load among the rivets, and the consequent liability of failure of the joint in detail, in treble riveting the pitch should probably not exceed $4.5d$, nor $5d$ in quadruple riveting.

There may be taken, according to the experiments just cited:

For punched inch plates:

$$T = 40{,}000 \text{ lbs. per square inch.}$$

For drilled ¼-inch plates:

$$T = 55{,}000 \text{ lbs. per square inch.}$$

Other thicknesses and conditions give approximately proportional values, allowing about 10 per cent. for the deterioration of the punch; *i.e.*, T, for a ⅝ punched plate, may be taken at 45,000 pounds.

It has already been observed that the value of S may be taken at $0.8\,T$ for lap joints, but the few experiments that have been made on shearing in butt joints with double covers, show that the ratio must be taken somewhat less, in consequence probably of the double shearing which takes place.

Hence, let S be taken at $0.75\,T$.

Using the third and fourth members of Eq. (34), therefore, and making $S = 0.75\,T$:

For thin plates in which $f = 1.5\,T$:

$$d = 1.3t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

For thick plates in which $f = 1.25T$:

$$d = 1.1t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

It is hardly worth while, however, to make any rivet less than ⅜ inch in diameter. Hence there may be taken the limits :

For ¼-inch plate ; $d = 0.375$ inch.
For 1-inch plate ; $d = 1.125$ inch.

These results are verified by good boiler practice.

The distance from the centre line of outside row of rivets to the edge of the cover plate, or from the edge of the main plate to the centre line of the first row of rivets in the same, may be taken at $\frac{3}{2}d$ as in lap joints, since the calculation is precisely the same. This rule frequently gives a considerable margin of safety over that of any other portion of the joint.

The distance between the centre lines of the rows of rivets may be taken at 2.5 to $3.0d$ for chain riveting, and ¾ that distance for zigzag riveting.

Steel Butt Joints with Double Cover Plates.

For the same reasons stated in the preceding section, considerations touching the stress in the cover plates will be omitted. And also, for the reasons there given, these cover plates should each possess from ¾ to ⅞ the thickness of the main plate ; or :

$$t' = 0.75 \quad \text{to} \quad 0.875t.$$

Table VI. gives the results of a large number of tests in which the joint failed by the tearing of the plates. The intensities of tension and compression, T and f, existed at failure.

TABLE VI.

Double Riveted Butt Joints.

EXPERIMENTER OR AUTHORITY.	HOLES BY	POUNDS PER SQUARE INCH FOR T.	POUNDS PER SQUARE INCH FOR f.
Henry Sharp	Drill.	96,160	83,330
" "	Punch.	87,600	75,170
Martell	Drill.	55,100	76,205
"	"	51,740	71,680
"	"	64,290	88,890
"	"	58,690	89,130
Boyd	"	55,200	76,160
"	"	51,230	70,800
"	"	64,320	88,930
Kirkaldy, annealed plates	Punch.	68,990	93,160
" " "	"	75,490	101,900
" " "	"	82,450	99,660
" " "	"	83,180	100,510
" " "	"	76,590	90,460
" " "	"	78,220	92,380
" " "	"	74,030	92,850
" " "	"	70,540	88,500
" " "	"	73,920	84,630
" " "	"	72,560	83,080
" " "	"	72,390	107,110
" " "	"	76,520	112,780
Greig and Eyth	Drill.	67,670	92,270
" " "	"	57,360	71,440
Parker	"	49,370	49,100
"	"	50,920	50,760
"	"	62,140	107,410
Kirkaldy, $\frac{7}{8}$ inch Siemens steel plates. Mean of two.	"	61,800	74,520
" $\frac{9}{16}$ " Landore " " $\frac{13}{16}$ rivets	"	66,200	112,000
" " " " " " " "	"	63,260	107,700
" " " " " " " "	Bored.	63,560	104,100
" " " " " " " "	"	69,590	117,300
" " " " " " " "	Punch.	67,540	98,630
" " " " " " " "	"	66,750	108,000
" " " " " " " "	Bored.	67,260	121,060

The first of the last set of results in the table, by Mr. Kirkaldy, was found with zigzag riveting in which the distance between the centre lines of the rows of rivets was too small.

These results are quite irregular, but it would seem to be as safe a deduction as possible to take $f = 1.25\,T$, with T equal to 70,000 to 75,000 pounds per square inch for thin plates, and 55,000 to 60,000 for thick ones.

With this value of f, and $q = 2$, the second and third members of Eq. (34) give for *double riveted butt joints with two covers :*

$$p = 3.5d. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

If the same value of f be preserved, there will result for *single riveted butt joints with two covers :*

$$p = 2.5d. \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Experiments on treble and quadruple riveting are yet lacking.

But few experiments on the shearing of rivets in butt joints with double covers have yet been made. Four tests by Messrs. Sharp and Kirkaldy give :

	THICKNESS OF PLATE.	S.
Single riveted	——	.... 42,000 lbs. per sq. in.
Double "	0.875 in..........	44,350 " " " "
" "	——	.... 53,870 " " " "
" "	0.55 in..........	42,700 " " " "
" "	0.875 in..........	44,420 " " " "

All the holes were drilled.

These values of S range about $0.7T$. Putting this ratio, therefore, in Eq. (34), and taking $f = 1.25T$, the third and fourth members of that equation give:

$$d = 1.14t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

It is probable that this is a little too small for thin plates, and a little too large for thick ones. Hence there may be taken :

$$\left.\begin{array}{l}\text{For thin plates, } d = 1\tfrac{1}{4}t \\ \text{For thick plates, } d = 1\tfrac{1}{8}t\end{array}\right\} \quad . \quad . \quad . \quad . \quad (42)$$

Double riveted butt joints designed in accordance with the foregoing deductions should give a resistance ranging from 65 to 75 per cent. of that of the solid plate.

Single riveted joints will give an efficiency somewhat less; perhaps from 60 to 65 per cent.

It is to be supposed, in applying the rules just established, that all steel plates are drilled, or subsequently annealed if punched.

As in the preceding cases, the distance between the centre lines of the rows of rivets may be taken at 2.5 to $3d$ for chain riveting, and three-quarters that distance for zigzag.

Efficiencies.

The values of the quantity which has been termed the "efficiency" of the joint, *i.e.*, the ratio of the resistance of a given width of joint over that of an equal width of solid plate, in the preceding investigations, are those actually determined by experiments with the joints themselves. They may, therefore, be relied upon. Some values which have for many years been considered as standard, but which, in reality, are of a

TABLE VII.

Butt Joints with Two Covers—1877.

NO. OF TESTS.	PLATE THICKNESS.	RIVET DIAMETER.	PITCH OF RIVETS.	HOLES.	RIVETING.	EFFICIENCY.
2	$\frac{7}{16}$ in.	$\frac{5}{8}$ in.	$2\frac{1}{8}$ in.	Punched.	Chain.	0.672
2	$\frac{7}{16}$ in.	$\frac{5}{8}$ in.	3 in.	Punched.	Zigzag.	0.669
2	$\frac{1}{2}$ in.	$\frac{3}{4}$ in.	$2\frac{1}{4}$ in.	Drilled.	Chain.	0.662
2	$\frac{1}{2}$ in.	$\frac{3}{4}$ in.	3 in.	Drilled.	Zigzag.	0.633

somewhat arbitrary nature, and at best belonging to a limited class of joints, have been disregarded.

Table VII. gives the results of Mr. Kirkaldy's experiments in reference to the comparative resistance of chain and zigzag riveting. The difference is not great, but what there is is in favor of the chain riveting.

TABLE VIII.

Kirkaldy's Tests—1872.

JOINT.	RIVETING.	HOLES.	RIVET DIAMETER IN TERMS OF t.	PITCH IN TERMS OF d.	EFFICIENCY.
Lap	Single.	Punched.	$d = 2t$	$p = 3d$	0.55
Lap	Single.	Drilled.	$d = 2t$	$p = 2\frac{5}{8}d$	0.62
Lap	Double.	Punched.	$d = 2t$	$p = 4\frac{1}{2}d$	0.69
Lap	Double.	Drilled.	$d = 2t$	$p = 4d$	0.75
Butt, 1 cover	Single.	Punched.	$d = 2t$	$p = 3d$	0.55
Butt, 1 cover	Single.	Drilled.	$d = 2t$	$p = 2\frac{5}{8}d$	0.62
Butt, 1 cover	Double.	Punched.	$d = 2t$	$p = 4\frac{1}{2}d$	0.69
Butt, 1 cover	Double.	Drilled.	$d = 2t$	$p = 4d$	0.75
Butt, 2 covers	Single.	Punched.	$d = 1\frac{1}{4}t$	$p = 3\frac{1}{4}d$	0.57
Butt, 2 covers	Single.	Drilled.	$d = 1\frac{1}{4}t$	$p = 3d$	0.67
Butt, 2 covers	Double.	Punched.	$d = 1\frac{1}{4}t$	$p = 5\frac{1}{4}d$	0.72
Butt, 2 covers	Double.	Drilled.	$d = 1\frac{1}{4}t$	$p = 4\frac{3}{4}d$	0.79

Table VIII. gives the results of the same experimenter on the relative value of punched and drilled work.

The drilled work is seen to give decidedly the greatest efficiency in every case.

The joints to which Tables VII. and VIII. belong were of wrought iron.

Experiments by Mr. Kirkaldy during 1881 show that well-designed double riveted steel butt joints with two covers may be expected to give efficiencies varying from 0.65 to 0.75.

Riveted Truss Joints.

The circumstances in which riveted joints are used in truss work, render permissible many special forms which can find no place in boiler riveting. If joints are found under the same circumstances, so far as the transference of stress is concerned, precisely the same forms would be used, except that caulking is, of course, only required in boiler work.

Fig. 9 shows a common form of chord construction in riv-

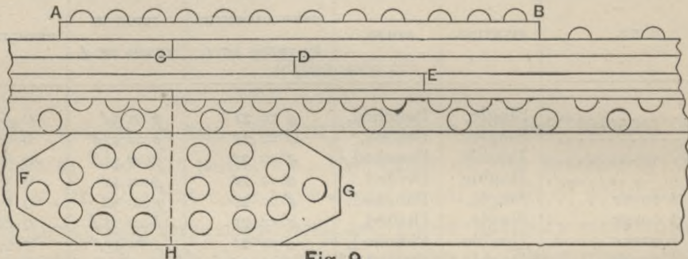
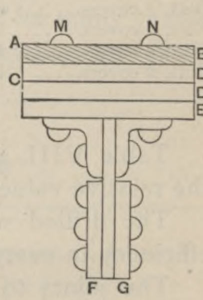


Fig. 9



eted truss work, with the relative proportions exaggerated.

The lower portion of the figure shows a section of the chord, in which the cover plates are shaded. The joint is supposed to be in tension.

AB is a horizontal cover plate, under which the horizontal component plates form lap joints at *C*, *D* and *E*. As the distance *MN* must necessarily be much greater than the allowable pitch in boiler work, these lap joints, considered in themselves, should be at least treble riveted. On the other hand, the preceding investigations show that even with treble riveting there is great disparity in the loads carried by the different rivets and consequent tendency to detailed rupture ; there would seem, therefore, to be little or no benefit in more than treble riveting.

The distance between the centres of rivets along the line of the chord—*i.e.*, along AB in the upper figure—may be taken at three diameters. The overlap $CD = DE$ (upper Fig.) would then be taken at 9 diameters, and from A, C, D or E to the centre of the first hole, at 1½ diameters. The cover AB should extend 9 diameters also on either side of C and E.

In this work the diameter of the rivet may usually be taken about the same as for boiler work. In estimating the resistance of the whole joint, however, it is to be borne in mind that the rivet holes take metal out of *all* the plates, and that they are usually punched.

It is impossible to follow the stresses in such a joint or to compute its efficiency. If tested to failure, the latter would probably be found pretty low.

The joint in the vertical plate should be formed as at FG—*i.e.*, it should be a double cover butt joint. The principles already established in a preceding section, in regard to the thickness of covers and diameter of rivets, should be observed here.

The two rows of rivets on either side of the joint may as well be chain riveted with a pitch 3½ to 4 diameters. Other rivets should then be staggered in until the group of rivet centres on each side is brought to a point, as shown in the uppart of Fig. 9. In this manner the available section of a width of plate equal to that of the cover, becomes approximately equal to the total, less the material from one rivet hole. Hence the efficiency of the joint becomes correspondingly increased.

If the joint is in compression the preceding observations hold without change, except that all covers should have the same thickness as the plates covered.

Even if the joints C, D, E and H are of planed edges, little or no reliance should be placed upon their bearing on each other, since the operation of riveting will draw them apart more or less, however well the work may be done. Melted zinc, or other similar metal, has been poured into compression joints with the intention of insuring good bearings, but the results are not satisfactory.

In the case of very wide chords, four longitudinal rows of rivets should be used in such joints as are exemplified in Fig. 9.

Unless great caution is observed and excellence of design secured, there will frequently be excessive bending in the riveted joints of trusswork, on account of the great variety of connections required.

Diagonal Joints.

It has been proposed to form riveted joints, the edges of whose plates are neither perpendicular nor parallel to the stress transferred. In this manner a greater number of rivets and a greater section of metal will resist the stress exerted in the body of the plate.

Mr. Kirkaldy made some tests on such lap joints, single riveted, with ⅜-inch plates, the joints of which lay at 45° with the applied force, with the following results :

Entire plate	100
Square joint	59.4
Diagonal joint	87.2

The diagonal joints are thus seen to give by far the best results. They are, however, much the most expensive also.

Friction of Riveted Joints.

There are not lacking experiments to show that the friction between the plates of a riveted joint is very great. This, however, cannot be relied upon to give additional resistance to the joint, since a sensible relative movement of the plates takes place in advance of its greatest resistance and essentially destroys the friction.

The experiments of Edwin Clarke, Harkort and Lavelley show that this friction may range from 8,330 to 22,400 lbs. per sq. in. of rivet section.

The specimens were prepared with one slotted plate, so that friction was the only resistance to the parting of the plates.

Hand and Machine Riveting.

Pneumatic, steam and hydraulic riveting machines have lately been brought to such a degree of perfection, that machine work is now very generally preferred to hand riveting.

The resistances of joints will vary to some extent with the method of riveting. Usually, however, the variation will not be greater than may be found for the same kind of riveting in different places and under different circumstances.

As a rule, machine riveting is much more reliable than hand, in that the hole is better filled and the rivet more quickly headed, in consequence of the great excess of pressure exerted. There is thus much less liability of loose rivets.

Many of the preceding experimental results were obtained from machine work.

Addendum to Art. 73.

The following series of valuable tests of riveted joints of both iron and steel were made in the government machine at Watertown, Mass. The results in Table IX. were taken from "Senate Ex. Doc. No. 1, 47th Cong., 2d Session," while those in Table X. are found in "Senate Ex. Doc. No. 5, 48th Cong., 1st Session." The character of plates, rivets, and holes is shown in the tables, and the intensities of tension in net sec-

TABLE IX.

Riveted Joints—Iron and Steel.

NO.	SIZE OF RIVET AND KIND.	PITCH OF RIVET.	MAX. STRESSES; POUNDS PER SQ. IN. Tension on net area of plate (T).	Compress'n on diamet'l surface (f).	Shearing on Rivets (S)	EFFICIENCY OF JOINT P. CENT.	REMARKS.
			Single riveted lap joints; ¼ inch iron plates.				
35	5/8" iron	2 Ins.	43,230	76,140	34,900	57.7	11/16" punched holes.
36	5/8" "	2 "	45,520	82,910	38,640	61.4	" " "
37	5/8" "	2 "	38,580	73,260	34,870	52.8	" drilled "
38	5/8" "	2 "	41,790	79,360	38,660	57.1	" " "
39	5/8" "	1 5/8 "	52,160	65,420	33,420	60.6	" punched "
40	5/8" "	1 5/8 "	54,930	68,890	35,200	64.0	" " "
41	5/8" steel	2 "	49,420	87,670	39,640	65.9	" " "
42	5/8" "	2 "	47,260	83,940	40,610	63.1	" " "
43	1/2" "	1 5/8 "	45,890	78,220	45,300	60.3	9/16" " "
44	1/2" "	1 5/8 "	49,720	84,660	48,420	65.5	" " "
45	7/16" iron	1 5/16 "	41,095	66,778	44,204	53.1	1/2" drilled "
46	7/16" "	1 5/16 "	37,500	60,886	42,038	48.3	" " "
			Single riveted lap joints; ¼ inch steel plates.				
426	5/8" iron	2 Ins.	46,340	82,480	37,890	53.2	11/16" punched holes.
427	5/8" "	2 "	46,010	81,780	37,860	52.8	" " "
436	5/8" steel	2 "	60,250	107,260	49,270	69.2	" " "
437	5/8" "	2 "	59,240	105,290	48,750	68.0	" " "
428	5/8" iron	2 "	40,950	77,870	36,350	48.2	" drilled "
429	5/8" "	2 "	42,370	80,200	36,710	49.6	" " "
438	5/8" steel	2 "	63,190	120,160	56,100	74.3	" " "
439	5/8" "	2 "	61,310	116,090	52,460	71.8	" " "
430	5/8" "	1 5/8 "	66,860	90.000	41.790	68.8	" " "
31	5/8" "	1 5/8 "	70,000	94,230	43,750	72.0	" " "
47	7/16" "	1 5/16 "	62,496	101.180	65,220	69.0	1/2" " "
48	7/16" "	1 5/16 "	58,338	94,800	60,382	64.8	" " "
49	5/8" "	2 "	60,184	114,603	52,742	70.6	11/16" " "
50	5/8" "	2 "	57,439	109,650	50,645	67.6	" " "
			Double riveted lap joints; ¼ inch iron plates.				
85	7/16" iron	2 Ins.	38,535	64,120	43,110	60.3	1/2" drilled holes.
86	7/16" "	2 "	41,750	69,710	41,750	65.3	" " "
617	1/2" "	1 5/8 "	50,592	42,118	28,691	65.8	9/16" punched "
618	1/2" "	1 5/8 "	49,950	41,660	28,660	65.3	" " "
			Double riveted lap joints; ¼ inch steel plates.				
432	5/8" iron.	2 Ins.	61,510	54,640	25,400	70.4	11/16" punched holes.
433	5/8" "	2 "	60,300	53,715	25.530	69.4	" " "
434	5/8" "	2 "	65,400	64,600	30,430	74.9	" " "
435	5/8" "	2 "	64,600	63,430	30,430	74.3	" " "
87	7/16" steel	2 "	56,944	94,910	57,910	76.3	1/2" drilled "
88	7/16" "	2 "	59,130	98,360	61,130	79.5	" " "
			Double welt butt joints; ¼ inch iron plates.				
615	5/8" iron	1 5/8 Ins.	53,475	67,321	16,944	62.2	11/16" punched holes.
616	5/8" "	1 5/8 "	50,959	64,138	16,719	59.3	" " "

TABLE IX.—*Continued.*

NO.	SIZE OF RIVET AND KIND.	PITCH OF RIVET.	MAX. STRESSES; POUNDS PER SQ. IN. Tension on net area of plate (T).	Compress'n on diamet'l surface (f).	Shearing on Rivets (S)	EFFICIENCY OF JOINT P. CENT.	REMARKS.
			Single riveted lap joints; ⅝ inch iron plates.				
62	11/16″ iron	2 Ins.	37,460	60,340	38,280	49.0	¾″ punched holes.
63	" "	2 "	36,130	58,150	35,520	47.2	" " "
64	" "	2 "	38,190	60,730	37,530	49.7	" drilled "
65	" "	2 "	36,210	57,530	36,050	47.1	" " "
66	" "	1¾ "	41,750	54,130	34,230	50.0	" punched "
67	" "	1¾ "	41,290	53,400	34,150	49.3	" " "
720	1″ "	2 1/16 "	61,700	52,970	26,180	60.4	1 1/16″ " "
721	1″ "	2 1/16 "	58,510	50,220	24,830	57.1	" " "
			Single riveted lap joints; ⅝ inch steel plates.				
51	11/16″ iron	2 Ins.	39,220	63,210	39,740	45.4	¾″ punched holes.
52	" "	2 "	37,700	60,760	38,190	43.6	" " "
53	" steel	2 "	55,215	89,580	56,430	64.1	" " "
54	" "	2 "	54,740	88,660	55,460	63.5	" " "
55	" "	1¾ "	63.650	80,930	50,650	66.7	" drilled "
56	" "	1¾ "	63,976	81,600	50,900	67.2	" " "
238	¾″ "	2 "	65,460	89,490	53,560	70.9	13/16″ punched "
239	¾″ "	2 "	65,210	88,990	53,600	70.6	" " "
718	1″ iron	2 5/16 "	73,394	79,510	36,614	71.4	1 1/16″ " "
719	1″ "	2 5/16 "	73,970	80,200	36,590	72.0	" " "
			Double riveted lap joints; ⅝ inch iron plates.				
68	11/16″ iron	2 Ins.	43,450	39,160	24,760	63.5	¾″ punched holes.
69	" "	2 "	50,730	41,070	26,150	66.4	" " "
58	" "	2 "	50,220	40,640	25,330	65.7	" " "
70	" "	2 "	46,255	41,480	27,550	60.5	" " "
71	" "	2 "	46,110	41,270	27,010	60.4	" " "
81	" "	3¼ "	30,920	58,700	39,130	50.4	" drilled "
82	" "	3¼ "	30,130	57,340	38,410	49.1	" " "
			Double riveted lap joints; ⅝ inch steel plates.				
57	11/16″ iron	2 Ins.	62,800	50,760	32,310	73.2	¾″ punched holes.
59	" "	2 "	64,720	52,450	32,930	75.2	" " "
60	" "	2 "	63,210	56,860	34,710	73.2	" " "
61	" "	2 "	54,930	49,530	30,830	63.8	" " "
83	" steel	3¼ "	44,660	84,460	52,750	64.4	" drilled "
84	" "	3¼ "	43,650	83,000	51,845	63.0	" " "
			Reinforced riveted lap joints; ⅝ inch iron plates. (See Fig. below.)				
244	⅝″ iron	2 Ins. joint. / 4 " welt.	38,870	59,080	40,360	67.6	11/16″ drilled hole, ⅝″ welt
245	¾″ "	" " " / " " "	43,770	56,640	34,460	74.0	13/16″ " " " "
296	¾″ "	" " " / " " "	44,840	57,910	33,890	75.7	" " " ¼″ "
297	¾″ "	" " " / " " "	42,680	55,350	31,810	71.9	" " " " "

TABLE IX.—*Continued.*

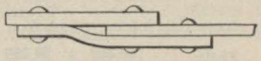
NO.	SIZE OF RIVET AND KIND.	PITCH OF RIVET.	MAX. STRESSES; POUNDS PER SQ. IN.			EFFICIENCY OF JOINT P. CENT.	REMARKS.
			Tension on net area of plate (T).	Compress'n on diamet'l surface (f).	Shearing on Rivets (S)		
							
			Reinforced riveted joints; ⅝ inch steel plates. (See above Fig.)				
246	⅞″ steel	2 Ins. joint. 4 " welt.	**62,050**	67,320	32,960	89.0	15/16″ drilled holes.
247	⅞″ "	" " " " " "	**62,880**	68,135	33,900	90.1	" " "
298	⅞″ iron	" " " " " "	61,020	**67,300**	**34,250**	87.8	" " "
299	⅞″ "	" " " " " "	61,710	**68,040**	**34,750**	88.9	" " "
			Single riveted lap joints; ½ inch iron plates.				
240	¾″ iron	2 Ins.	**31,100**	41,500	34,280	39.8	13/16″ punched holes.
241	" "	2 "	**31,395**	41,955	34,960	39.7	" " "
292	" "	2 "	32,376	47,850	**38,020**	42.9	" drilled "
293	" "	2 "	33,180	48,890	**39,220**	44.3	" " "
327	" steel	2 "	39,900	**58,880**	**47,020**	52.2	" " "
328	" "	2 "	**40,500**	**59,900**	47,830	54.2	" " "
			Single riveted lap joints; ½ inch steel plates.				
242	¾″ iron	2 Ins.	38,204	50,940	**41,100**	38.2	13/16″ punched holes.
243	¾″ "	2 "	35,915	47,890	**38,636**	35.9	" " "
294	15/16″ "	2 "	**60,210**	56,980	**36,770**	51.2	1″ " "
295	15/16″ "	2 "	**49,590**	47,060	30,540	42.2	" " "
			Double riveted lap joints; ½ inch iron plates.				
329	¾″ iron	2 Ins.	**44,320**	59,640	25,280	57.0	13/16″ punched holes.
635	¾″ "	2 "	**42,920**	57,950	24,560	55.2	" " "
			Double riveted lap joints; ½ inch steel plates.				
619	15/16″ iron	2 Ins.	**64,602**	29,354	19,670	53.8	1″ punched holes.
620	15/16″ "	2 "	**64,519**	29,371	19,644	53.8	" " "
			Single riveted lap joints; ⅝ inch iron plates.				
730	1″ iron	2⅝ Ins.	**34,680**	47,510	35,460	44.9	1 1/16″ punched holes.
731	1″ "	2⅝ "	**34,230**	46,790	34,930	42.0	" " "
			Double riveted lap joints; ⅝ inch iron plates.				
732	1″ iron	2⅝ Ins.	**43,580**	29,740	22,960	56.3	1 1/16″ punched holes.
733	1″ "	2⅝ "	**45,850**	31,310	23,670	59.3	" " "
			Single riveted lap joints; ⅝ inch steel plates.				
734	1″ steel	2⅝ Ins.	49,650	56,760	**43,490**	50.5	1 1/16″ punched holes.
735	1″ "	2⅝ "	52,770	60,150	**46,080**	53.6	" " "
			Double riveted lap joints; ⅝ inch steel plates.				
736	1″ steel	2⅝ Ins.	**69,680**	39,780	30,470	70.9	1 1/16″ punched holes.
737	1″ "	2⅝ "	**67,100**	38,300	29,340	68.3	" " "

TABLE X.

Riveted Joints—Iron and Steel.

NO.	THICKNESS OF PLATE AND KIND.	DIAM. AND KIND OF RIVET.	PITCH OF RIVET.	MAX. STRESSES; POUNDS PER SQ. IN. Tension on net area of plate (T).	Compress'n on diamet'l surface (f)	Shearing on Rivets (S)	EFFICIENCY OF JOINT P. CENT.	REMARKS.
				Single riveted iron lap joints.				
1	⅝″ iron.	11/16″ iron.	1¾ Ins.	**39,300**	50,850	33,710	47.0	¾″ punched holes.
2	" "	" "	" "	**41,000**	53,050	35,170	49.0	11/16″ " "
3	½″ "	¾″ "	2 "	**35,650**	47,350	37,300	45.6	13/16″ " "
4	⅝″ "	" "	" "	**35,150**	46,690	36,780	44.9	" " "
				Single riveted iron butt joints.				
5	⅝″ iron.	11/16″ iron.	2 Ins.	**46,360**	72,390	25,380	59.9	¾″ punched holes.
6	" "	" "	" "	**46,875**	73,050	25,450	60.5	" " "
7	½″ "	¾″ "	" "	**48,400**	61,940	24,630	59.4	13/16″ " "
8	" "	" "	" "	**46,140**	61,740	24,310	59.2	" " "
9	⅝″ "	1″ "	2⅝ "	**44,260**	60,330	23,010	57.2	1 1/16″ " "
10	" "	" "	" "	**42,350**	58,080	22,310	54.9	" " "
11	¾″ "	1⅛″ "	2.9 "	**42,310**	57,000	21,870	52.1	1 3/16″ " "
12	" "	" "	" "	**41,920**	56,540	22,140	51.7	" " "
				Single riveted steel lap joints.				
13	⅝″ steel.	¾″ iron.	1¾ Ins.	**61,270**	65,760	**40,390**	59.5	13/16″ punched holes.
14	" "	" "	" "	60,830	65,320	**39,900**	59.1	" " "
15	½″ "	15/16″ "	2 "	**47,530**	44,590	**29,390**	40.2	1″ " "
16	⅝″ "	" "	" "	**49,840**	46,960	31,070	42.3	" " "
				Single riveted steel butt joints.				
17	⅝″ steel.	11/16″ iron.	2 Ins.	**62,770**	**97,940**	31,240	71.7	¾″ punched holes.
18	" "	15/16″ "	" "	**61,210**	**95,210**	31,020	69.8	1″ " "
19	½″ "	" steel.	" "	**63,920**	62,220	20.370	57.1	" " "
20	" "	" "	" "	**66,710**	59,580	19,890	55.0	" " "
21	⅝″ "	1″ "	2⅝ "	**62,180**	**71,450**	27,750	63.4	1 1/16″ " "
22	" "	" "	" "	**62,590**	**71,930**	27,940	63.8	" " "
23	¾″ "	1⅛″ "	2½ "	**54,650**	55,610	23.190	54.0	1 3/16″ " "
24	" "	" "	" "	**54,200**	55,840	22,810	53.4	" " "

tion of plates, compression or bearing on diametral surface and shearing on rivets are those which existed at the instant of failure. The bold faced figures show the kind of failure, and when such figures are found, for the same test, in two or three columns, they show that the same two or three kinds of failure took place simultaneously.

It is important to notice that in general the highest ultimate resistances of tension and compression or bearing are found with the thin plates, and that those quantities diminish appreciably as the thickness of plate increases, both for iron and steel. This law is not so well defined in reference to the diameter of rivet, if indeed these tests show it at all, except for steel.

The length of these test joints varied from 9.75 to 13 inches.

Although the results of these Tables are somewhat irregular, they confirm the general accuracy of the relations established between the values of T, f, and S in the preceding portion of this Article, as well as other general rules and conclusions for boiler work.

Some efficiencies are much lower than given for similar joints on pages 638 and 639, but such instances can, by the aid of the Tables, be traced either to indifferent design or a phenominally low value of some one of the three resistances. In general, the results compare well with those given on the pages named.

The pitches of rivets are seen to be adapted to boiler work, and much less than are ordinarily used in bridge work; yet the corresponding resistances show what may legitimately be done and expected when rare and extraordinary conditions demand a departure from usual rules.

Before deducing from the preceding results of this Addendum working intensities for bridge construction, it is to be first explained that those results are as given in the government reports, and that the net section used is the gross section of the plate, less the actual metal removed by the punch or drill, with no allowance for deterioration by the former in the immediate vicinity of the hole. Again, in the Tables IX. and X. the diametral bearing surface and the shearing area of the rivet are taken to be those of the drill, or a mean between the punch and die in case of punched holes. In bridge work, in determining the net section, metal is deducted for a diameter equal to that of the cold rivet before driving plus one-eighth of an inch; and the shearing and bearing are computed for the section and diameter of the cold rivet before driving.

With these explanations in view, the preceding tests justify the following working stresses for the plate girder floor beams and stringers of railway bridges with machine driven rivets.

Rivet shearing	7,500 lbs. per sq. in. for iron.	10,000 " " " " " steel.
Rivet bearing	12,000 lbs. per sq. in. for iron.	15,000 " " " " " steel.
Tension in net section of plate	8,000 lbs. per sq. in. for iron.	10,000 " " " " " steel.

The apparently low bearing resistances, especially for steel, are taken for the reason that very thick plates are frequently used in bridge construction, and the ultimate bearing resistance for them is appreciably less than for the thin plates used in most of the preceding tests.

The preceding working stresses are based on steel for rivets giving from 56,000 to 64,000 pounds per square inch tensile resistance, while the steel for plates, in test specimens, should offer from 58,000 to 66,000 pounds per square inch ultimate tensile resistance.

In the government report from which Table IX. is abstracted, can be found a large number of tests made for the purpose of determining the proper minimum distance from the centres of rivet holes to the edge of plates. As a result of those tests and other experience on the same subject, it may be stated that the least distance from the centre of a rivet hole to the edge of a plate may be taken at one and one-half the diameter of the *hole* for steel and one and five-eighths the diameter of the *hole* for iron, in cases where it is important to secure the maximum resistance of the joint.

Art. 74.—Welded Joints.

At the present time the process of welding can, with proper care and material, be made to give excellent results.

Scarf welds give much better results than lap welds, on account of the bending to which the latter are subjected.

Mr. Kirtley (Institute of Mechanical engineers of Great Britain) made some experiments with small strips, 7.5 inches long and $\frac{7}{16}$ inch thick, cut across welded joints. These strips were taken out of boilers whose longitudinal joints had been welded. Twenty-three experiments with strips varying from one to one and a half inches wide, gave the following results per square inch of plate section :

	WELDED.	SOLID PLATE.
Greatest	53,310 lbs.	57,790 lbs.
Mean	46,140 "	52,860 "
Least	36,960 "	46,370 "

Art. 75.—Pin Connection.

A pin connection consists of two sets of eye bars or links, through the heads at one end of each of which a single pin passes. Fig. 1 shows a pin connection; *A*, *A*, *B*, *B*, are eye bars or links, and *P* is the pin.

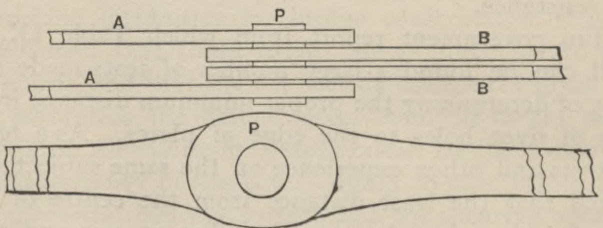


Fig. 1

The head of the eye bar (one is shown in elevation in Fig. 2) requires the greatest care in its formation. It is imperfect unless it be so proportioned that when the eye bar is tested to failure, fracture will be as likely to take place in the body of the bar as in the head—in other words, unless its efficiency is unity.

In Fig. 2 the head of the eye bar, or link, is supposed to be of the same thickness as that of the body of the bar whose width is w.

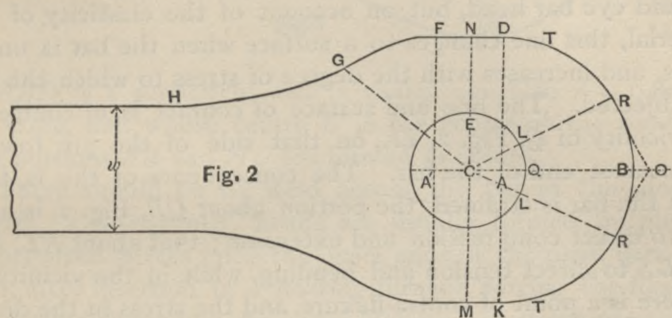


Fig. 2

If t is the thickness of the bar, so that wt is the area of its normal section, then t is almost invariably included between the limits of $\frac{1}{3}w$ and $\frac{1}{8}w$. In fact these extreme values are each too extreme for the full resistance of the bar, although they are sometimes used. These ratios, as well as the diameter of the pin in terms of w, can only be determined by experiments on full-sized bars. A large number of such experiments have been made both in this country and in Great Britain, and while the resistance of the bar as a whole depends to a considerable extent on the mode of manufacture or formation of the head, it has been found that for the best proportioned head t should range from $\frac{1}{4}w$ to $\frac{1}{6}w$, and the diameter, d, of the pin from $0.75w$ to w.

It is extremely difficult to reach more than a general idea of the condition of stress in an eye bar head, although an approximate mathematical treatment of the question may be found in the "Trans. Am. Soc. of Civ. Engrs.," Vol. VI., 1877, in which the results agree essentially with those of experiment.

Before taking a general view of the stresses which may arise in an eye bar head, it must be premised that a difference of $\frac{1}{64}'$ to $\frac{1}{100}''$ between the diameter of the pin and that of the pin-hole is exceptionally good practice. Before the eye bar is strained, therefore, there is a line of contact only between the pin and eye bar head, but on account of the elasticity of the material, this line changes to a surface when the bar is under stress, and increases with the degree of stress to which the bar is subjected. The line and surface of contact is, of course, in the vicinity of Q, Fig. 2, *i.e.*, on that side of the pin toward the nearest end of the bar. The consequence of this is, that when the bar is strained, the portion about QB, Fig. 2, is subject to direct compression and extension; that about RL, NE and GS to direct tension and bending, while in the vicinity of T there is a point of contra-flexure, and the stress in the direction of the circumference changes from compression to tension as E is approached from Q.

As a result of many of the experiments which have been made, the following mode of proportioning the head has heretofore been very extensively used: Let r represent the radius of the pin, while reference is made to Fig. 2. Then take $EN = 0.66w$. The curve $DRBK$ is a semicircle with a radius equal to $r + 0.66w$, with a centre, A, so taken on the centre line of the bar that $QB = 0.87w$. GF is a portion of the same curve, with A' as the centre ($A'C = AC$); GH is any curve with a long radius joining GF gradually with the body of the bar. HG should be very gradual in order that there may be a large amount of metal in the vicinity of CG, for there the metal is subjected to flexure as well as direct tension. FD is a straight line parallel to the centre line of the bar.

As the preceding rule gives a head whose outline causes a more expensive die than a simple circle, at the present day eye bar heads are usually formed as shown in Fig. 3.

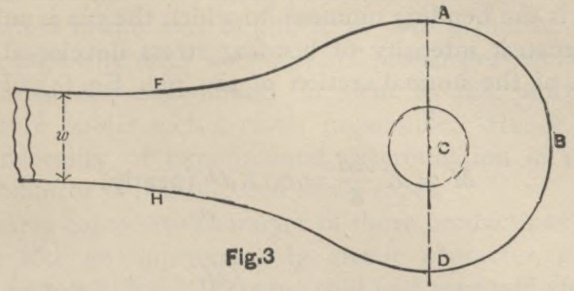


Fig. 3

ABD is a semicircle with a radius equal to $r + 0.8w$ to $r + 0.9w$, and whose centre *C* is the centre of the pin-hole. The portions *FA* and *HD* are formed as before.

There should be no weld across the bar in the vicinity of *FH*. Consequently, heads are usually formed by placing proper sized pieces upon the upset ends of the plain bars, and then, after insertion in a heating furnace, forcing the head to the desired shape in a die under hydraulic or steam pressure.

The intensity of this pressure will affect, to a considerable extent, the permissible dimensions of the head. The greater the pressure, the better will be the results.

The unfinished head is sometimes rolled on the bar, as by the Kloman process.

The thickness of the head is sometimes made greater than that of the body of the bar. If the head is circular, as in Fig. 3, the section of metal on each side of the pin (through *AC* or *CD*) should be not far from eight-tenths that in the body of the bar.

This thickening of the eye bar head is an excellent thing for the bar, but subjects the pin to a great increase of bending, and, hence, requires increased pin diameter.

In pin connections, the pin is subjected to very heavy bending.*

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* For a detailed treatment of this subject, the author's "Bridge and Roof Trusses" may be consulted.

If M is the bending moment to which the pin is subjected, K the greatest intensity of bending stress developed, and A the area of the normal section of the pin, Eq. (4) of Art. 63 gives :

$$M = K\frac{Ad}{8} = 0.1Kd^3 \text{ (nearly)} \quad . \quad . \quad . \quad . \quad (1)$$

Or :

$$d = 2.16\sqrt[3]{\frac{M}{K}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Values of K, for circular sections, may be found in Art. 63.

Art. 76.—Iron, Steel and Hemp Cables or Ropes.—Wrought-Iron Chain Cables.

The following tables of resistance and other properties of cables are those published by John A. Roebling's Sons Co.

It will be observed that the figures for hemp ropes are given in comparison with either iron or steel in each of the tables.

In considering the resistance of iron and steel cables composed of wire twisted into strands, it is of the highest importance to keep clearly in view the circumstances or conditions produced by the manner of fabrication, as they are peculiar to all classes of ropes, whether of hemp or wire.

In this class of material the fibres or strands no longer lie parallel to the direction of the stress which they carry, but the process of twisting causes each fibre or wire to take a helical form, the pitch of which is not constant for the different portions of the rope. The consequence is that if the process of fabrication were absolutely perfect, so that each wire or fibre could take its proper portion of load, the stress in that wire or fibre would be its portion of load multiplied by the secant of its inclination to the axis of the rope. As a matter of fact, however, each wire does not take its proper portion of load; the imperfections unavoidably incident to the processes of manufacture render such a result impossible. Hence the increased necessity of experimental determination of the ultimate resistances of metallic and hemp ropes.

The same composite character of these productions renders anything like an approximately elastic character, even, an essential impossibility. It is true that any rope will yield to a considerable extent while under stress, and then return nearly to its original condition, but this behavior is only apparently elastic; it is almost entirely due to the increase of helical pitch of the strands caused by the external loading. During this

Standard Hoisting Ropes with 19 *Wires to the Strand.*

IRON.

Trade No.	Circumference in inches.	Diameter.	Weight per foot in lbs. of rope with hemp cen.	Breaking strain in tons of 2,000 pounds.	Proper working load in tons of 2,000 pounds.	Circumference of hemp rope of equal strength.
1	6¾	2¼	8.00	74	15	15½
2	6	2	6.30	65	13	14½
3	5½	1¾	5.25	54	11	13
4	5	1⅝	4.10	44	9	12
5	4¾	1½	3.65	39	8	11½
5½	4⅜	1⅜	3.00	33	6½	10¼
6	4	1¼	2.50	27	5½	9½
7	3½	1⅛	2.00	20	4	8
8	3⅛	1	1.58	16	3	7
9	2¾	⅞	1.20	11½	2½	6
10	2¼	¾	0.88	8.64	1¾	5
10¼	2	⅝	0.70	5.13	1¼	4½
10½	1⅝	9/16	0.44	4.27	¾	4
10¾	1½	½	0.35	3.48	½	3½

Standard Hoisting Ropes with 19 *Wires to the Strand.*

CAST STEEL.

Trade No.	Circumference in inches.	Diameter.	Weight per foot in lbs. of rope with hemp cen.	Breaking strain in tons of 2,000 pounds.	Proper working load in tons of 2,000 pounds.	Circumference of hemp rope of equal strength.
1	$6\frac{3}{4}$	$2\frac{1}{4}$	8.00	130	26	——
2	6	2	6.30	100	21	——
3	$5\frac{1}{2}$	$1\frac{3}{4}$	5.25	78	17	$15\frac{3}{4}$
4	5	$1\frac{5}{8}$	4.10	64	13	$14\frac{1}{2}$
5	$4\frac{3}{4}$	$1\frac{1}{2}$	3.65	55	11	$13\frac{1}{2}$
6	4	$1\frac{1}{4}$	2.50	39	8	$11\frac{1}{2}$
7	$3\frac{1}{2}$	$1\frac{1}{8}$	2.00	30	6	10
8	$3\frac{1}{8}$	1	1.58	24	5	$9\frac{1}{4}$
9	$2\frac{3}{4}$	$\frac{7}{8}$	1.20	20	4	8
10	$2\frac{1}{4}$	$\frac{3}{4}$	0.88	13	3	$6\frac{1}{2}$
$10\frac{1}{4}$	2	$\frac{5}{8}$	0.70	9	2	$5\frac{1}{4}$
$10\frac{1}{2}$	$1\frac{5}{8}$	$\frac{9}{16}$	0.44	$6\frac{1}{2}$	$1\frac{1}{2}$	$4\frac{3}{4}$
$10\frac{3}{4}$	$1\frac{1}{2}$	$\frac{1}{2}$	0.35	$5\frac{1}{2}$	1	$4\frac{1}{2}$

operation the strands endeavor to place themselves more nearly parallel to the direction of stress, and give rise to a corresponding decrease in diameter. Since these influences preclude the existence of either coefficient of elasticity or elastic limit, ultimate resistances only will be given in this section.

The preceding observations evidently do not apply to suspension bridge cables which are built up of parallel wires. The operations leading to the production of such a cable are of such a refined and exact character that the total resistance of the cable may be assumed without essential error to be the sum of the resistances of all the wires taken separately: the coefficient of elasticity and elastic limit may, and usually do exist with perfect definition.

Galvanized Steel Cables for Suspension Bridges.

DIAMETER IN INCHES.	ULTIMATE STRENGTH IN TONS OF 2,000 POUNDS.	WEIGHT PER FOOT.
$2\frac{5}{8}$	220	13
$2\frac{1}{2}$	200	11.3
$2\frac{3}{8}$	180	10
$2\frac{1}{4}$	155	8.64
2	110	6.5
$1\frac{7}{8}$	100	5.8
$1\frac{3}{4}$	95	5.6
$1\frac{5}{8}$	75	4.35
$1\frac{1}{2}$	65	3.7

Transmission and Standing Ropes with 7 Wires to the Strand.

IRON.

Trade No.	Circumference in inches.	Diameter.	Weight per foot in lbs. of rope with hemp cen.	Breaking strain in tons of 2,000 pounds.	Proper working load in tons of 2,000 pounds.	Circumference of hemp rope of equal strength.
11	$4\frac{5}{8}$	$1\frac{1}{2}$	3.37	36	9	$10\frac{3}{4}$
12	$4\frac{1}{4}$	$1\frac{3}{8}$	2.77	30	$7\frac{1}{2}$	10
13	$3\frac{3}{4}$	$1\frac{1}{4}$	2.28	25	$6\frac{1}{4}$	$9\frac{1}{4}$
14	$3\frac{3}{8}$	$1\frac{1}{8}$	1.82	20	5	8
15	3	1	1.50	16	4	7
16	$2\frac{5}{8}$	$\frac{7}{8}$	1.12	12.3	3	$6\frac{1}{4}$
17	$2\frac{3}{8}$	$\frac{3}{4}$	0.88	8.8	$2\frac{1}{4}$	$5\frac{1}{4}$
18	2[illegible]	$\frac{11}{16}$	0.70	7.6	2	5
19	$1\frac{7}{8}$	$\frac{5}{8}$	0.57	5.8	$1\frac{1}{2}$	$4\frac{3}{4}$
20	$1\frac{5}{8}$	$\frac{9}{16}$	0.41	4.1	1	4
21	$1\frac{3}{8}$	$\frac{1}{2}$	0.31	2.83	$\frac{3}{4}$	$3\frac{1}{4}$
22	$1\frac{1}{4}$	$\frac{7}{16}$	0.23	2.13	$\frac{1}{2}$	$2\frac{3}{4}$
23	$1\frac{1}{8}$	$\frac{3}{8}$	0.19	1.65	—	$2\frac{1}{2}$
24	1	$\frac{5}{16}$	0.16	1.38	—	$2\frac{1}{4}$
25	$\frac{7}{8}$	$\frac{9}{32}$	0.125	1.03	—	2

Transmission and Standing Ropes with 7 Wires to the Strand.

CAST STEEL.

Trade No.	Circumference in inches.	Diameter.	Weight per foot in lbs. of rope with hemp cen.	Breaking strain in tons of 2,000 pounds.	Proper working load in tons of 2,000 pounds.	Circumference of hemp rope of equal strength.
11	$4\frac{5}{8}$	$1\frac{1}{2}$	3.37	67	16	15
12	$4\frac{1}{4}$	$1\frac{3}{8}$	2.77	55	$12\frac{1}{2}$	13
13	$3\frac{3}{4}$	$1\frac{1}{4}$	2.28	45	10	12
14	$3\frac{3}{8}$	$1\frac{1}{8}$	1.82	36	8	$10\frac{3}{4}$
15	3	1	1.50	30	$6\frac{1}{2}$	10
16	$2\frac{5}{8}$	$\frac{7}{8}$	1.12	22	5	$8\frac{1}{2}$
17	$2\frac{3}{8}$	$\frac{3}{4}$	0.88	17	$3\frac{1}{2}$	$7\frac{1}{4}$
18	$2\frac{1}{8}$	$\frac{11}{16}$	0.70	$13\frac{1}{2}$	3	$6\frac{1}{2}$
19	$1\frac{7}{8}$	$\frac{5}{8}$	0.57	10	$2\frac{1}{4}$	$5\frac{1}{2}$
20	$1\frac{5}{8}$	$\frac{9}{16}$	0.41	8	$1\frac{3}{4}$	5
23	$1\frac{3}{8}$	$\frac{1}{2}$	0.31	6	$1\frac{1}{4}$	$4\frac{3}{4}$
21	$1\frac{1}{8}$	$\frac{3}{8}$	0.19	4	1	$3\frac{3}{4}$
24	1	$\frac{5}{16}$	0.16	3	$\frac{3}{4}$	$3\frac{1}{4}$

Wrought-Iron Chain Cables.

It might at first sight be supposed that the pull which the link of a chain cable could resist would be twice that offered by a bar of round iron equal in cross section to that of one side of the link. But a weld exists at one end of the link and a bend at the other, each requiring at least one heat for the portion of the link in which it is located. These manipulations produce a considerable decrease in the resistance of the link.

The United States Committee on "Tests of Chain Cables," of which Commander L. A. Beardsley was chairman, made many experiments on the iron of which chain cables are made, as well as on the finished cables.

The following conclusions and table are taken from the report of that committee: ". . . that beyond doubt, when made of American bar iron, with cast-iron studs, the studded link is inferior in strength to the unstudded one.

Ultimate Resistance and Proof Tests of Chain Cables.

DIAM. OF BAR.	AV. RESIST. = 163% OF BAR.	PROOF TEST.	DIAM. OF BAR.	AV. RESIST. = 163% OF BAR.	PROOF TEST.
Inches.	Pounds.	Pounds.	Inches.	Pounds.	Pounds.
1	71,172	33,840	$1\frac{9}{16}$	162,283	77,159
$1\frac{1}{16}$	79,544	37,820	$1\frac{5}{8}$	174,475	82,956
$1\frac{1}{8}$	88,445	42,053	$1\frac{11}{16}$	187,075	88,947
$1\frac{3}{16}$	97,731	46,468	$1\frac{3}{4}$	200,074	95,128
$1\frac{1}{4}$	107,440	51,084	$1\frac{13}{16}$	213,475	101,499
$1\frac{5}{16}$	117,577	55,903	$1\frac{7}{8}$	227,271	108,058
$1\frac{3}{8}$	128,129	60,920	$1\frac{15}{16}$	241,463	114,806
$1\frac{7}{16}$	139,103	66,138	2	256,040	121,737
$1\frac{1}{2}$	150,485	71,550			

"That, when proper care is exercised in the selection of material, a variation of five to seventeen per cent. of the strongest may be expected in the resistance of cables. Without this care, the variation may rise to twenty-five per cent.

"That with proper material and construction the ultimate resistance of the chain may be expected to vary from 155 to 170 per cent. of that of the bar used in making the links, and show an average of about 163 per cent.

"That the proof test of a chain cable should be about 50 per cent. of the ultimate resistance of the weakest link."

The decrease of the resistance of the studded below the unstudded cable is probably due to the fact that in the former the sides of the link do not remain parallel to each other up to failure, as they do in the latter. The result is an increase of stress in the studded link over the unstudded in the proportion of unity to the secant of half the inclination of the sides of the former to each other.

From a great number of tests of bars and finished cables, the committee considered that the average ultimate resistance, and proof tests of chain cables made of the bars, whose diameters are given, should be such as are shown in the accompanying table.

Manila and Hemp Rope.

The results given below were obtained in the Govt. machine at Watertown, Mass., and are found in "Ex. Doc. No. 35, 49th Congress, 1st Session." "The rope was secured in the testing machine either with a hitch made by taking a round turn over a $4\frac{1}{2}$ inch pin held from turning, and then making fast the end to a smaller pin, or by passing the ends over $3\frac{1}{2}$ inch thimbles, and securing to the standing part by means of seizing."

KIND.	CIRCUMFERENCE. INCHES.	FASTENING.	ELONGATION JUST BEFORE FAILURE. PER CENT.	BREAKING LOAD IN POUNDS.		
				Greatest.	Mean.	Least.
Manila.	$3\frac{1}{4}$	*Hitch.*		9,600	8,620	8,250
"	$3\frac{1}{4}$	*Seizing.*	14.6 to 18.1	10,820	10,015	9,050
"	$3\frac{1}{2}$	*Hitch.*		11,400	10,650	10,050
"	$3\frac{1}{2}$	*Seizing.*	14.1 to 17.2	12,150	10,810	9,500
"	$3\frac{3}{4}$	*Hitch.*		12.200	11.350	10,550
"	$3\frac{3}{4}$	*Seizing.*	12.2 to 17.0	11,900	11,020	9,400
Hemp.	$3\frac{1}{4}$	*Hitch.*		7,950	6,990	6,000
"	$3\frac{1}{4}$	*Seizing.*	8.3 to 10.9	12,900	11,350	9,850
"	$3\frac{1}{2}$	*Hitch.*		10.100	8,315	6,500
"	$3\frac{1}{2}$	*Seizing.*	8.1 to 13.0	12,480	10,700	8,050
"	$3\frac{3}{4}$	*Hitch.*		10,500	8,535	7,300
"	$3\frac{3}{4}$	*Seizing.*	8.1 to 11.4	14,200	12,900	9,350

The rope used in these tests had never been in service, but was ten years old, having been made at the Boston navy yard, and kept in store that length of time at the Washington navy yard. All the manila rope had 3 strands, but of the hemp about half had 3 strands and the other half 4.

CHAPTER XI.

MISCELLANEOUS PROBLEMS.

Art. 77.—Resistance of Flues to Collapse.

IF a circular tube or flue be subjected to external normal pressure, such as that of steam or water, the material of which it is made will be subjected to compression around the tube, in a plane normal to its axis. If the following notation be adopted :

l = length of tube ;
d = diameter of tube ;
t = thickness of wall of the tube ;
p = intensity of excess of external pressure over internal,

then will any longitudinal section lt, of one side of the tube, be subjected to the pressure $\frac{pld}{2}$. But let a unit only of length of tube be considered. This portion of the tube is approximately in the condition of a column whose length and cross section, respectively, are πd and t.

The ultimate resistance of such a column is, Art. 25 :

$$P = \frac{\pi^2 EI}{\pi^2 d^2}.$$

As this ideal column is of rectangular section :

$$I = \frac{t^3}{12},$$

and

$$P = \frac{Et^3}{12d^2}.$$

But $P = pd$, hence :

$$p = \frac{Et^3}{12d^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is the greatest intensity of external pressure which the tube can carry. But the formulæ of Art. 25 are not strictly applicable to this ideal column. The curvature on the one hand and the pressure on the other tend to keep it in position long after it would fail as a column without lateral support. *Hence, p will vary inversely as some power of d much less than the third.*

Again, it is clear that a very long tube will be much more apt to collapse at its middle portion than a short one, as the latter will derive more support from the end attachments; and this result has been established by many experiments. Hence, p must be considered as some inverse function of the length l.

Eq. (1), therefore, can only be taken as typical in form, and as showing in a general way, only, how the variable elements enter the value of p. If x, y and z, therefore, are variable exponents to be determined by experiment, there may be written :

$$p = c\frac{t^x}{l^y d^z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which c is an empirical coefficient.

Sir Wm. Fairbairn ("Useful Information for Engineers, Second Series") made many experiments on wrought-iron tubes with lap and butt joints single riveted. He inferred from his tests that $y = z = 1$. Two different experiments would then give:

$$pld = ct^x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$p'l'd' = ct'^x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence,

$$log\,(pld) = log\,c + x\ log\,t\,;$$

$$log\,(p'l'd') = log\,c + x\ log\,t'\,;$$

in which "*log*" means "logarithm." Subtracting one of these last equations from the other, the value of x becomes:

$$x = \frac{log\,(pld) - log\,(p'l'd')}{log\,t - log\,t'} = \frac{log\left(\frac{pld}{p'l'd'}\right)}{log\left(\frac{t}{t'}\right)} \quad . \quad . \quad (5)$$

As p, l, d, t, p', l', d' and t' are known numerical quantities in every pair of tests, x can at once be computed by Eq. (5); c then immediately results from either Eq. (3) or Eq. (4). By the application of these equations to his experimental data, Fairbairn found for wrought-iron tubes:

$$p = 9{,}675{,}600\,\frac{t^{2.19}}{ld} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in which p is in pounds per square inch, while t, l and d are in inches. *Eq.* (6) *is only to be applied to lengths between* 18 *and* 120 *inches.*

He also found that the following formula gave results agreeing more nearly with those of experiment, though it is less simple:

$$p = 9{,}675{,}600\,\frac{t^{2.19}}{ld} - 0.002\,\frac{d}{t} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Fairbairn found that by encircling the tubes with stiff rings he increased their resistance to collapse. *In cases where such rings exist, it is only necessary to take for l the distance between two adjacent ones.*

In 1875 Prof. Unwin, who was Fairbairn's assistant in his experimental work, established formulæ with other exponents and coefficients (" Proc. Inst. of Civ. Engrs.," Vol. XLVI.). He considered x, y and z variable, and found *for tubes with a longitudinal lap joint :*

$$p = 7{,}363{,}000\,\frac{t^{2.1}}{l^{0.9}d^{1.16}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From *one tube with a longitudinal butt joint*, he deduced :

$$p = 9{,}614{,}000\,\frac{t^{2.21}}{l^{0.9}d^{1.16}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

For *five tubes with longitudinal and circumferential joints*, he found :

$$p = 15{,}547{,}000\,\frac{t^{2.35}}{l^{0.9}d^{1.16}} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

By using these same experiments of Fairbairn, other writers have deduced other formulæ, which, however, are of the same general form as those given above. It is probable that the following, which was deduced by J. W. Nystrom, will give more satisfactory results than any other :

$$p = 692{,}800\,\frac{t^2}{d\sqrt{l}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

At the same time, it has the great merit of more simple application.

From one experiment on an elliptical tube, by Fairbairn, it would appear that the formulæ just given can be approximately applied to such tubes by substituting for d, twice the radius of curvature of the elliptical section at either extremity of the smaller axis. If the greater diameter or axis of the ellipse is a, and the less b; then, for d, there is to be substituted $\frac{a^2}{b}$.

Art. 78.—Approximate Treatment of Solid Metallic Rollers.

An approximate expression for the resistance of a roller may easily be written. The approximation may be considered a loose one, but it furnishes a basis for an accurate empirical formula.

The following investigation contains the improvements by Prof. J. B. Johnson and Prof. H. T. Eddy on the method originally given by the author.

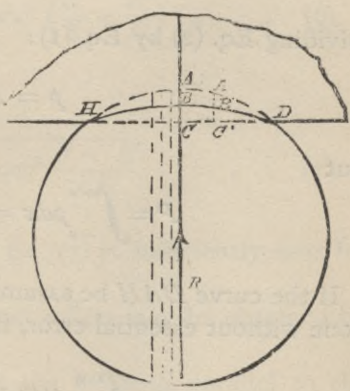

FIG. 1.

The roller will be assumed to be composed of indefinitely thin vertical slices parallel to its axis. It will also be assumed that the layers or slices act independently of each other.

Let E' be the coefficient of elasticity of the metal over the roller.

Let E be the coefficient of elasticity of the metal of the roller.

Let R be the radius of the roller and R' the thickness of the metal above it.

Let w = intensity of pressure at A.

" p = " " any other point.

" P = total weight which the roller sustains per unit of length.

" x be measured horizontally from A as the origin.

" $d = AC$.

" $e = DC$.

From Fig. 1:

$$AB = \frac{wR}{E};\ A'B' = \frac{pR}{E}.$$

$$BC = \frac{wR'}{E'};\ C'B' = \frac{pR'}{E'}.$$

$$\therefore d = AC = AB + BC = w\left(\frac{R}{E} + \frac{R'}{E'}\right);\quad . \quad . \quad . \quad (1).$$

And

$$A'C' = A'B' + B'C' = p\left(\frac{R}{E} + \frac{R'}{E'}\right).\quad . \quad . \quad . \quad (2).$$

Dividing Eq. (2) by Eq. (1):

$$p = A'C'\,\frac{w}{d}.$$

But

$$P = \int_{-e}^{+e} p\,dx = \frac{w}{d}\int_{-e}^{+e} A'C'\,dx.$$

If the curve DAH be assumed to be a parabola, as may be done without essential error, there will result:

$$\int_{e}^{+e} A'C'\,dx = \frac{4}{3}\,ed.$$

Hence:

$$P = \frac{4}{3} we \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

But:

$$e = \sqrt{2Rd - d^2} = \sqrt{2Rd} \text{ nearly.}$$

By inserting the value of d from Eq. (1) in the value of e, just determined, then placing the result in Eq. (7):

$$P = \frac{4}{3} \sqrt{2w^3R\left(\frac{R}{E} + \frac{R'}{E'}\right)} \quad . \quad . \quad . \quad . \quad . \quad (4).$$

If $R = R'$:

$$P = \frac{4}{3} R \sqrt{2w^3 \frac{E + E'}{EE'}} \quad . \quad . \quad . \quad . \quad . \quad (5).$$

The preceding expressions are for one unit of length. If the length of the roller is l, its total resistance is

$$P' = Pl = \frac{4}{3}\, l \sqrt{2w^3R\left(\frac{R}{E} + \frac{R'}{E'}\right)} \quad . \quad . \quad . \quad (6).$$

Or if $R = R'$:

$$P' = \frac{4}{3} Rl \sqrt{2w^3 \frac{E + E'}{EE'}} \quad . \quad . \quad . \quad . \quad (7).$$

In ordinary bridge practice Eq. (7) is sufficiently near for all cases.

A simple expression for conical rollers may be obtained by using Eqs. (4) or (5).

As shown in Fig. 2, let z be the distance, parallel to the axis, of any section from the apex of the cone; then consider

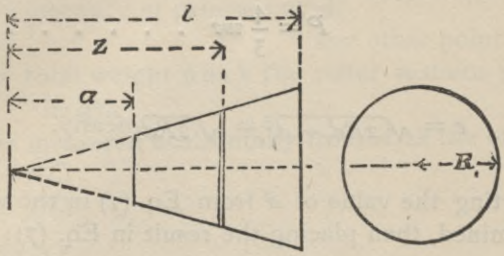

FIG. 2.

a portion of the conical roller whose length is dz. Let R_1 be the radius of the base. The radius of the section under consideration will then be

$$R = \frac{z}{l} R_1;$$

and the weight it will sustain, if $R_1 = R'$;

$$dP' = \frac{R_1}{l}\sqrt{2w^3 \frac{E + E'}{EE'}}\,.\, zdz.$$

Hence:

$$P' = \int_a^l dP' = \frac{l^2 - a^2}{2l} R_1 \sqrt{2w^3 \frac{E + E'}{EE'}} \quad . \quad . \quad . \quad (8).$$

Eqs. (6), (7), and (8) give ultimate resistances if w is the ultimate intensity of resistance for the roller.

It is to be observed that the main assumptions on which the investigation is based lead to an error on the side of safety.

If for wrought iron, $w = 12{,}000$ pounds per square inch, and $E = E' = 28{,}000{,}000$ pounds, Eq. (5) gives:

$$P = \frac{8}{3} R \sqrt{\frac{w^3}{E}} = 664 R.$$

Art. 79.—Resistance to Driving and Drawing Spikes.

Some very interesting experiments on driving and drawing rail spikes were made by Mr. A. M. Wellington, C. E., and reported by him in the "R. R. Gazette," Dec. 17, 1880. He experimented with wood both in the natural state and after it had been treated by the Thilmeny (sulphate of baryta) preserving process.

"The test blocks were reduced to a uniform thickness of 4.5 inches; this thickness being just sufficient to give a full bearing surface to the parallel sides of the spikes when driven to the usual depth, and to allow the point of the spike to project outwards. It was considered that the beveled point could add

Spikes were Standard: 5.5 *inches* × $\frac{9}{16}$ *inch.*

KIND OF WOOD.	NATURAL WOOD.				PREPARED WOOD.			
	To driving spike, pounds.		To pulling spike, pounds.		To driving spike, pounds.		To pulling spike, pounds.	
		Mean.		Mean.		Mean.		Mean.
Beech	5,216 6,743	5,980	5,673 6,282	5,978	7,288 7,656	7,472	8,873 8,267	8,420
White oak, green	5,970 5,670	5,820	7,179 5,869	6,523	——		——	
Pin oak	5,216 5,521	5,368	6,638 6,469	6,553	6,117 4,589	5,353	6,135 6,267	6,201
White ash		5,953		4,560	6,588 5,978	6,283	(Split.)	
White oak, well seasoned	6,433 6,433	6,433	5,128 3,435	4,281	——		——	
Black ash	3,996 4,202	4,090	4,408 4,868	4,638	4,453 4,453 4,301 3,380	4,147	3,340 3,028 3,300 3,493	3,290
Elm	4,453 4,758	4,606	3,536 3,843	3,690	4,453 4,148	4,300	4,148 4,202	4,175
Chestnut, green	3,996 3,386	3,691	2,730 3,790	3,260	——		——	
Soft maple	4,148 3,538	3,843	2,578 3,645	3,111	3,843 3,448	3,645	2,725 3,030	2,877
Sycamore	4,103 3,493	3,798	3,188 3,188	3,188	3,691 3,976	3,833		1,968
Hemlock		2,910		1,996	——		——	

very little to the holding power of the spike, and it was desired to press the spike out again by direct pressure after turning the block over. . . ."

The forces exerted in pulling and driving the spikes were produced by a lever. A few tests with a hydraulic press showed that the friction of the plunger varied from about 6 to 18 per cent.

The accompanying table gives the results of the experiments.

Art. 80.—Shearing Resistance of Timber behind Bolt or Mortise Holes.

Col. T. T. S. Laidley, U.S.A., made some tests during 1881 at the United States Arsenal, Watertown, Mass., on the resistance offered by timber to the shearing out of bolts or keys, when the force is exerted parallel to the fibres.

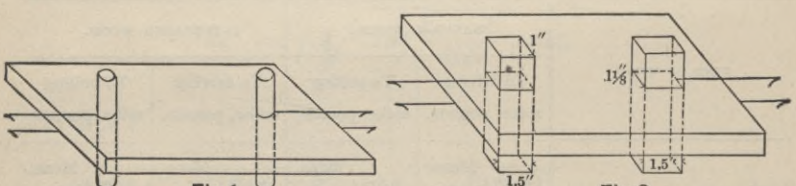

Fig.1 Fig.2

The test specimens are shown in Figs. 1 and 2. Wrought-iron bolts and square wrought-iron keys were used. All the timber specimens were six inches wide and two inches thick. The diameter of the bolts used (Fig. 1) was one inch for all the specimens. The keys were 1" × 1.5" and 1.125" × 1.5" as shown in Fig. 2. In all the latter specimens, failure took place in front of the smaller key where the pressure was greatest.

In many cases the specimen sheared and split simultaneously in front of the hole. By putting bolts through the pieces in a direction normal to the force exerted, so as to prevent splitting, the resistance was found (in most cases) to be considerably, though irregularly increased.

KIND OF WOOD.	CENTRE OF HOLE FROM END OF SPECIMEN.	TOTAL AREA OF SHEARING.	ULTIMATE SHEARING RESISTANCE PER SQUARE INCH, IN POUNDS.
	Inches.	Sq. inches.	
Spruce (bolts)	2	8	399
	4	16	359
	6	24	275
	8	32	202
White pine (bolts)	2	8	457
	4	16	611
	6	24	450
	8	32	327
Yellow pine (bolts)	2	8	607
	4	16	720
	6	24	456
	8	32	337
Yellow pine (square keys)	2	8	599
	4	16	369
	6	24	572
	7	28	438
White pine (square keys)	2	8	550
	4	16	412
	6	24	332
	7	28	236
Spruce (square keys)	2	8	410 (not thoroughly seasoned.)
	4	16	329 " " "
	6	24	242 (wet timber).
	7	28	279

Unless otherwise stated, the wood was thoroughly seasoned.

The accompanying table gives the results of Col. Laidley's tests.

Art. 81.—Bulging of Plates.

A plate offers resistance to "bulging" when it is simply supported, or firmly fixed, around its entire edge, and carries a single, or uniformly distributed load acting normal to its surface. The very complicated nature of the stresses and strains existing in a plate thus acted upon, together with the fact that its conditions just before rupture are entirely different from those accompanying the initial loading, give to the problem a character of unusual intricacy, and, indeed, preclude a solution possessing a degree of approximation commonly obtained in questions relating to the elasticity and resistance of materials.

An elegant analysis of the problem, considered as one of pure elasticity, may be found in "Die Theorie der Elasticität Fester Körper," by Clebsch. It is, however, of little value in connection with questions of ultimate resistance.

The following roughly approximate, but simple, analysis may be used to suggest the form of an empirical formula which can be completed by the aid of experiments.

Let the length, breadth and thickness of a rectangular plate simply supported around its edges, be represented by a, b and t, respectively, and let it first be loaded by a uniformly distributed pressure whose intensity (per unit of ab) is w.

If the plate is supposed to consist of two sets of small strips or beams parallel to a and b, those crossing each other at the centre must have the same deflection at middle. If, further, the uniform load w be supposed to be so divided into two parts, w_1 and w', that they would cause two rectangular beams whose spans are a and b to have the same centre deflection, the following equation (see Eq. (26) of Art. 24) must obtain:

$$\frac{5w_1a^4}{384EI} = \frac{5w'b^4}{384EI}.$$

Then, since $w' + w_1 = w$, there must result:

$$w' = \frac{a^4w}{a^4 + b^4}; \quad \text{and} \quad w_1 = \frac{b^4w}{a^4 + b^4}.$$

The bending moments at the centres of such beams would be (Eq. (27), Art. 24, and Eq. (14), Art. 18):

$$\frac{w_1 a^2}{8} = \frac{2K_1 I}{t}\,; \quad \text{and} \quad \frac{w' b^2}{8} = \frac{2K' I}{t}\,.$$

Since the beams are rectangular in section, $I = \frac{t^3}{12}$.
Hence :

$$K_1 = \frac{3w_1 a^2}{4t^2}\,; \quad \text{and } K' = \frac{3w' b^2}{4t^2}\,.$$

According to these hypothetical conditions the greatest intensity of stress at the centre of the plate will have the value:

$$K = \left(\frac{K_1^2 + K'^2}{2}\right)^{\frac{1}{2}} = \frac{3a^2 b^2 w}{4t^2\sqrt{2\,(a^4 + b^4)}} \quad . \quad . \quad . \quad (1)$$

Hence:

$$t = \frac{ab}{2}\left(\frac{3w}{K\sqrt{2\,(a^4 + b^4)}}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For square plates, $a = b$.
Hence

$$K = \frac{3a^2 w}{8t^2}\,; \quad \text{and} \quad t = 0.615a\sqrt{\frac{w}{K}} \quad . \quad . \quad . \quad (3)$$

If the edges are fixed, the greatest bending will occur along those lines; and for K_1 and K' then are to be put $\frac{2}{3}K_1$ and $\frac{2}{3}K'$.
Hence :

$$K_1 = 2\,.\,\frac{a^2 b^4 w}{4t^2\,(a^4 + b^4)}\,; \quad \text{and} \quad K' = 2\,.\,\frac{b^2 a^4 w}{4t^2\,(a^4 + b^4)}\,. \quad (4)$$

Since the greatest bending occurs along the edges, these are the expressions for the greatest intensities of stress. If ab^2 is greater than a^2b, then is K_1 greater than K'; and *vice versa.*

In the first case the expression for t is:

$$t = 0.707ab^2 \sqrt{\frac{w}{(a^4 + b^4) K_1}} \quad . \quad . \quad . \quad . \quad (5)$$

But if $K' > K_1$, or, $a^2b > ab^2$:

$$t = 0.707a^2b \sqrt{\frac{w}{(a^4 + b^4) K'}} \quad . \quad . \quad . \quad . \quad (6)$$

If the plate is square:

$$K = \frac{a^2w}{4t^2}; \quad \text{and,} \quad t = \frac{a}{2}\sqrt{\frac{w}{K}} \quad . \quad . \quad . \quad . \quad (7)$$

If a plate is loaded with a single weight P, it may be supposed to be divided in the same manner as w; so that

$$P_1 + P' = P.$$

The equation of middle deflections for ends simply supported then becomes:

$$\frac{P_1a^3}{48EI} = \frac{P'b^3}{48EI}.$$

Hence:

$$P' = \frac{a^3P}{a^3 + b^3}; \quad \text{and,} \ P_1 = \frac{b^3P}{a^3 + b^3}.$$

Proceeding in precisely the same manner as before:

$$K = 1.06 \frac{abP}{t^2 (a^3 + b^3)} \sqrt{a^4 + b^4} \quad . \quad . \quad . \quad . \quad (8)$$

and

$$t = 1.03 \left(\frac{abP}{K (a^3 + b^3)} \sqrt{a^4 + b^4} \right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (9)$$

If the plate is square :

$$K = 0.75 \frac{aP}{t^2}; \quad \text{and,} \quad t = 0.87 \sqrt{\frac{aP}{K}} \quad . \quad . \quad (10)$$

If the edges are fixed in position, the hypothetical beams are fixed at each end and loaded at the centre, and the greatest bending moments (at centre and ends alike) are thereby reduced to one-half their preceding values, or, what is the same thing, $2t^2$ is to take the place of t^2 in Eqs. (8), (9) and (10).

Hence :

$$K = 0.53 \frac{abP}{t^2 (a^3 + b^3)} \sqrt{a^4 + b^4} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$t = 0.73 \left(\frac{abP}{K (a^3 + b^3)} \sqrt{a^4 + b^4} \right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (12)$$

If the plate is square :

$$K = 0.375 \frac{aP}{t^2}; \quad \text{and} \quad t = 0.613 \sqrt{\frac{aP}{K}} \quad . \quad . \quad (13)$$

These equations are of little value as they stand, except as indicating a form of formula to which empirical coefficients are to be fitted. The hypothetical division of the plate into small beams is very far indeed from being correct. In the empirical determinations which follow, therefore, K will not be the greatest intensity of stress in the plate, but a coefficient or quantity partly analytical and partly experimental.

Circular plates have not been considered, because square ones furnish the requisite type of formula.

Experiments have thus far been made on square and circular plates only; hence, oblong rectangular plates will not again be noticed.

Kirkaldy's experiments on Fagersta steel plates and Fairbairn's on wrought-iron ones would seem to indicate that the thickness t varies about as $(w)^{0.8}$ or $(P)^{0.8}$; but the variation in diameter or side of square was not sufficient to establish any relation between t and a, while other elements remain the same. Regarding, therefore, K as an empirical quantity which may have different values for square and circular plates, Eqs. (3), (7), (10) and (13), may be written as follows:

$$K = \frac{3a^2}{8t^2} w^{1.6}; \quad \text{and} \quad t = 0.615a \frac{w^{0.8}}{\sqrt{K}} \quad . \quad . \quad (14)$$

$$K = \frac{a^2}{4t^2} w^{1.6}; \quad \text{and} \quad t = 0.5a \frac{w^{0.8}}{\sqrt{K}} \quad . \quad . \quad . \quad (15)$$

$$K = \frac{3a}{4t^2} P^{1.6}; \quad \text{and} \quad t = 0.87 \frac{\sqrt{a}\, P^{0.8}}{\sqrt{K}} \quad . \quad . \quad . \quad (16)$$

$$K = \frac{3a}{8t^2} P^{1.6}; \quad \text{and} \quad t = 0.613 \frac{\sqrt{a}\, P^{0.8}}{\sqrt{K}} \quad . \quad . \quad (17)$$

Kirkaldy made twenty experiments with mild Fagersta steel circular plates, 12 inches in diameter. He forced these through an aperture 10 inches in diameter by the pressure of a very blunt point. The edge of the aperture on which the plate rested was rounded; hence the initial diameter of aperture was somewhat more than 10 inches. Eqs. (16) are the ones to be used in connection with these experiments.

From the first member of that equation, K was computed for a number of different experiments, by substituting the numerical values of P, t and a. In this manner the following values were found to give good results:

For unannealed mild Fagersta steel circular plates:

$$K = 6,760,000,000.$$

Hence:

$$t = 0.000,010,6 \sqrt{a}\, P^{0.8} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

For annealed mild Fagersta steel circular plates:

$$K = 5,710,000,000.$$

Hence:

$$t = 0.000,011,52 \sqrt{a}\, P^{0.8} \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Eq. (16) gives:

$$P = \left(\frac{t \sqrt{K}}{0.87 \sqrt{a}} \right)^{1\frac{1}{4}}.$$

Table I. contains the results of computation by this formula and those obtained in the tests. On account of the rounded edge of the supporting ring, K was so taken that P, as computed, is a little larger than its experimental value. None of these plates were cracked, but they were bulged at the centre from 3.00 to 3.45 inches.

In "Engineering" for Sept. 28, 1877, Robert Wilson describes four experiments on unstayed flat boiler heads subjected to hydraulic pressure. These flat circular plates were

TABLE I.

Circular Plates simply Supported.

UNANNEALED.			ANNEALED.		
t, in inches.	P, in pounds.		t, in inches.	P, in pounds.	
	Experimental.	*By formula.*		*Experimental.*	*By formula.*
$\frac{5}{8}$	215,690	219,420	$\frac{5}{8}$	198,000	196,530
$\frac{1}{2}$	162,740	166,000	$\frac{1}{2}$	154,330	148,690
$\frac{3}{8}$	104,850	115,860	$\frac{3}{8}$	95,600	103,780
$\frac{1}{4}$	71,800	69,800	$\frac{1}{4}$	59,430	62,520
$\frac{1}{8}$	35,400	29,350	$\frac{1}{8}$	25,430	26,290

Each "*experimental*" result is a mean of two.

riveted to angles encircling the body of the boiler. The edges of the plates were thus fixed, and Eqs. (15) are therefore to be used. Proceeding in precisely the same manner as before, the following values were established:

For wrought-iron flat boiler heads, with fixed edges:

$$K = 11,000,000.$$

Hence:

$$t = 0.000,015 a w^{0.8} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

w was taken in pounds per square inch; it has the value:

$$w = \left(\frac{4t^2K}{a^2}\right)^{0.625}.$$

The results of the experiments, and of this formula, are :

DIAMETER, INCHES.	t, INCH.	w, IN POUNDS PER SQ. IN. Experimental.	By formula.
34.5	$\frac{9}{16}$	280	349
34.5	$\frac{3}{8}$	200	211
26.25	$\frac{3}{8}$	371	296
28.25	$\frac{3}{8}$	300	270

The agreement, in this case, is not satisfactory. It is probably due to the lack of a proper exponent of a. These plates were fractured along the lines of rivet holes in the edges.

Two means of four experiments by Fairbairn remain to be considered. His plates were square ones of wrought iron, firmly fixed to a square frame 12 inches by 12 inches in the aperture. The force was applied by a blunt point at the centre, consequently Eqs. (17) are to be used.

By precisely the same method already used, the following results were established :

For wrought-iron 12-inch square plates, with edges firmly fixed:

$$K = 390{,}000{,}000.$$

$$t = 0.000{,}031\sqrt{a}\,P^{0.8} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The expression for the indenting force is :

$$P = \left(t\sqrt{\frac{8K}{3a}}\right)^{1\frac{1}{4}}.$$

The experiments and computations are :

DIAMETER, INCHES.	t, INCH.	P, IN POUNDS. Experimental.	By formula.
12	$\frac{1}{4}$	16,780	16,350
12	$\frac{1}{2}$	37,720	38,890

The plates gave way at the centre, under the blunt point.

Some experiments by Kirkaldy, in 1875, on *wrought-iron circular plates simply supported around the edge*, show that for 12-inch plates forced through a 10-inch aperture with rounded edge, there may be safely taken :

$$t = 0.000013 \sqrt{a}\, P^{0.8} \quad . \quad . \quad . \quad . \quad . \quad (22)$$

In all the preceding formulæ, a and t are to be taken in inches; w in pounds per square inch, and P in pounds.

The investigations can only be considered provisional. Although they give, as a whole, tolerably satisfactory results, the range of the experiments is far too small for the establishment of thoroughly reliable formulæ. Experiments on which a proper exponent of a can be based, are yet wholly lacking; and as the only resort, that found in the rough analysis has been retained.

Art. 82.—Special Cases of Flexure.

There are a few cases of flexure which, while not frequently found in engineering experience, are of some practical importance, and are occasionally required. The two or three which follow involve the integration of some linear differential equations that are treated in all the advanced works on the integral calculus; consequently the operations of integration will not be given here, but the general integrals will be assumed.

Flexure by Oblique Forces

In Fig. 1 let OX represent a beam acted upon by the oblique forces P, which make angles α with the axis of X. The origin O is supposed to be taken anywhere on the axis of the beam. If right-hand

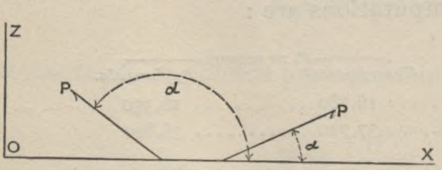

Fig. 1

moments are positive and left-hand negative, the component $P \sin \alpha$ will have the negative moment $-P \sin \alpha x$ about O. The lever arm of $P \cos \alpha$, if the deflection w is positive, is $+w$, and its moment $P \cos \alpha . w$ is positive. Hence the resultant moment of any force, P, in reference to the origin O is :

$$EI \frac{d^2w}{dx^2} = -P \sin \alpha . x + P \cos \alpha . w \quad . \quad . \quad (1)$$

If α is greater than 90°, $\cos \alpha$ is negative, so that if

$$A = \pm \frac{P \cos \alpha}{EI} \quad \text{and} \quad V = -\frac{P \sin \alpha . x}{EI},$$

the two cases may be expressed by the equation :

$$\frac{d^2w}{dx^2} + Aw = V. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If $a = +\sqrt{-A}$, and $b = -\sqrt{-A}$, the general integral of Eq. (2) is :

$$w = Ce^{ax} + C'e^{bx} + \frac{e^{ax}}{a-b}\int Ve^{-ax}\,dx - \frac{e^{bx}}{a-b}\int Ve^{-bx}\,dx ; \quad (3)$$

in which C and C' are arbitrary constants to be determined by the special conditions of any given problem, and $e = 2.71828$.

When α is less than 90° and

$$a = \sqrt{\frac{P \cos \alpha}{EI}}, \quad b = -\sqrt{\frac{P \cos \alpha}{EI}} ;$$

Eq. (3) becomes :

$$w = Ce^{ax} + C'e^{bx} + x \tan \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

C and C' yet remain to be determined by the particular circumstances of a given case.

The conditions on which the determination of these constants rest are expressed by giving known values to w and $\frac{dw}{dx}$ for values of x, also known.

If α is greater than 90°:

$$a = \sqrt{\frac{P \cos \alpha}{EI}} \cdot \sqrt{-1}, \quad \text{and} \quad b = -\sqrt{\frac{P \cos \alpha}{EI}} \cdot \sqrt{-1},$$

and Eq. (3) * becomes:

$$w = C \cos \left(\frac{P \cos \alpha}{EI}\right)^{\frac{1}{2}} x + C' \sin \left(\frac{P \cos \alpha}{EI}\right)$$

$$x - \tan \alpha \cdot x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

As before, C and C' are to be determined by the circumstances of each case to which the equation is applied; and the value of $\cos \alpha$, it is to be remembered, is to be substituted with the positive sign.

Let a column with one fixed and one free end, and with the force P acting parallel to its original axis, be considered. Since $\alpha = 180°$:

$$w = C \cos \left(\sqrt{\frac{P}{EI}} \cdot x\right) + C' \sin \left(\sqrt{\frac{P}{EI}} \cdot x\right) \quad . \quad . \quad (6)$$

With the origin of co-ordinates at the free end w must equal zero for $x = 0$; hence $C = 0$.

* The symbolical method by which Eq. (3) was established shows that if $a = -\frac{1}{2} B + \sqrt{A - \frac{1}{4} B^2}$ and $b = -\frac{1}{2} B - \sqrt{A - \frac{1}{4} B^2}$, the complete integral of the equation $\frac{d^2 w}{dx^2} + B \frac{dw}{dx} + Aw = V$ is given by Eq. (3).

The value of w then becomes :

$$w = C' \sin\left(\sqrt{\frac{P}{EI}} \cdot x\right) \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\therefore \quad \frac{dw}{dx} = C' \sqrt{\frac{P}{EI}} \cos\left(\sqrt{\frac{P}{EI}} \cdot x\right) \quad . \quad . \quad . \quad (8)$$

But if l is the length of the column, $\frac{dw}{dx} = 0$ for $x = l$. Hence :

$$\cos\left(\sqrt{\frac{P}{EI}} \cdot l\right) = 0 ;$$

or if n is any whole number from 0 to infinity :

$$\sqrt{\frac{P}{EI}} \cdot l = \tfrac{1}{2}(2n + 1)\pi \quad . \quad . \quad . \quad . \quad . \quad (9)$$

If the value of $\sqrt{\frac{P}{EI}}$ be taken from Eq. (9) and inserted in Eq. (8), there will result :

$$\frac{dw}{dx} = C' \sqrt{\frac{P}{EI}} \cos\left(\frac{x(2n + 1)}{2l}\right)\pi \quad . \quad . \quad . \quad (10)$$

Eq. (10) shows that for values of x equal to 1, 3, 5, 7 . . . times $\frac{l}{2n + 1}$, $\frac{dw}{dx} = 0$. The most dangerous supposition, *i. e.*, that which requires the greatest value of P, is $n = 0$.

This value of n in Eq. (9) gives :

$$P = \frac{\pi^2 EI}{4l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The ultimate resistance of the column is thus seen to be independent of the deflection, as was found for a different case in Art. 25. The end of the column, in this case, which carries the load is free to deflect laterally, but in Art. 25 both ends were supposed to be fixed in a lateral direction in reference to each other. In the latter case the resistance is seen to be nine times as great as in the present.

Since :

$$\cos\left(\sqrt{\frac{P}{EI}} \cdot l\right) = 0, \quad \sin\left(\sqrt{\frac{P}{EI}} \cdot l\right) = 1 .$$

Hence, if w_1 is the deflection of the free end from a vertical tangent to the fixed, Eq. (7) becomes, for $x = l$:

$$w_1 = C'.$$

In general, therefore :

$$w = w_1 \sin\left(\sqrt{\frac{P}{EI}} \cdot x\right) \quad . \quad . \quad . \quad . \quad (12)$$

For the same value of x, therefore, w varies directly as w_1, and the relative deflections may be computed by the equation :

$$\frac{w}{w_1} = \sin\left(\frac{(2n + 1)\pi x}{2l}\right) \quad . \quad . \quad . \quad . \quad (13)$$

or in the ordinary case :

$$\frac{w}{w_1} = \sin \frac{\pi x}{2l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Eq. (1) was written for one force only. If any number of forces act :

$$EI \frac{d^2w}{dx^2} = \Sigma(- P \sin \alpha \, . \, x + P \cos \alpha \, . \, w);$$

and in place of w there is to be put Σw.

General Flexure by Continuous Normal Load.

The most general case of flexure by a continuous normal load, is that in which the intensity (load per unit of length of beam) is a variable quantity. Let x be an abscissa measured along the original axis of the beam, and let w represent the deflection. Then the intensity of the load may be represented by $f(x, w)$. It was shown in Art. 20 that :

$$\frac{d^2M}{dx^2} = EI \frac{d^4w}{dx^4} = p = f(x, w) \quad . \quad . \quad . \quad . \quad (15)$$

The integration of the equation :

$$\frac{d^4w}{dx^4} = \frac{f(x, w)}{EI},$$

will depend upon the form of the function $f(x, w)$.

Let it be supposed that $f(x, w) = cx$, c being a constant. Then if A, A_1, A_2 and A_3 are constants of integration, there will result :

$$w = \frac{cx^5}{120\,EI} + \frac{Ax^3}{6} + \frac{A_1x^2}{2} + A_2x + A_3 . \quad . \quad (16)$$

Again, if $f(x, w) = cw$, c, as before, being a constant :

$$\frac{d^4w}{dx^4} = \frac{cw}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

For simplicity of notation, let :

$$a^4 = \frac{c}{EI} ;$$

then the general integral of Eq. (17) becomes :

$$w = Ae^{ax} + A_1e^{-ax} + A_2 \cos ax + A_3 \sin ax . \quad . \quad (18)$$

In Eq. (18) $e = 2.71828$ is the base of the Naperian logarithms ; while in both Eqs. (16) and (18) A, A_1, A_2 and A_3 are arbitrary constants to be determined by the circumstances of each individual case.

CHAPTER XII.

Working Stresses and Safety Factors.

Art. 83 —Definitions.

In all metallic and timber constructions the greatest (supposed) possible loads are determined from the attendant circumstances of the different cases, and then the stresses induced by these greatest loads are computed. These stresses are called the "*working stresses.*"

The ultimate resistance of any piece in a structure divided by the working stress gives a number called the "*safety factor.*" Occasionally the reciprocal of this number is called the safety factor, though but seldom.

The intensity of the ultimate resistance of any piece in a structure divided by the intensity of the working stress, will also give the safety factor. This is the more usual and convenient form, since it does not involve the cross section of the piece.

The values of safety factors depend upon many circumstances, such as kind and character of material, kind of stress, circumstances in which material is used and the amount of variation of stress in the piece, or the fatigue of the material. The safety factor is intended, also, to cover both computed stresses and others which are recognized, but are not within the reach of exact analysis. The latest practice among American engineers will be illustrated in the following Articles by extracts from specifications drawn for some first-class constructions.

Art. 84.—Specifications for the Cincinnati and Covington Bridge, 1887.

The following clauses are from specifications prepared by the Phœnix Bridge Co., and accompanied their design for the river spans of the Cincinnati and Covington bridge. Both specifications and design were adopted, and the unprecedentedly long and heavy spans were constructed in accordance therewith.

"All parts of the structure shall be so proportioned that the combined effects of temperature and all the specified loads, except the wind pressure, shall not cause the stress per square inch to exceed the following limits:

For Iron.—In tension, ten thousand (10,000) pounds. In compression, for lengths less than fifty (50) times the least radius gyration, eight thousand (8,000) pounds. In shearing across fibres, seven thousand five hundred (7,500) pounds. In bending on the extreme fibres of pins fifteen thousand (15,000) pounds. On bearing surfaces, twelve thousand (12,000) pounds. The bearing surfaces of pins and rivets shall be reckoned from the diameter and not from the semi-circle. The stress per square inch in compression for members whose length exceeds fifty (50) times their least radii of gyration, shall be reduced according to the following formulæ:

For square bearings, $R = \dfrac{8000}{1 + \dfrac{1}{36000}\left(\dfrac{l}{r}\right)^2}$

For pin bearings, $R = \dfrac{8000}{1 + \dfrac{1}{18000}\left(\dfrac{l}{r}\right)^2}$

For the flanges of rolled beams, $R = \dfrac{10000}{1 + \dfrac{1}{5000}\left(\dfrac{l}{b}\right)^2}$

For top flanges of built beams, $R = \dfrac{8000}{1 + \dfrac{1}{5000}\left(\dfrac{l}{b}\right)^2}$

where R is intensity of working stress, l length in inches of member between supports, r least radius of gyration of cross section, b breadth of top flange of girder in inches.

For Steel.—In tension on chord bars and end main diagonals, sixteen thousand (16,000) pounds. On main diagonals nearest the middle of the spans, thirteen thousand (13,000) pounds tension. For intermediate main diagonals the tensile intensities are to be directly interpolated. In shearing on rivets and pins, ten thousand (10,000) pounds. In bending on the extreme fibres of pins, twenty thousand (20,000) pounds. On bearing surfaces, fifteen thousand (15,000) pounds for rivets and eighteen thousand (18,000) pounds for pins. In compression on top chords and inclined end posts, provided that the ratio of length to least radius of gyration does not exceed fifty (50), fourteen thousand (14,000) pounds. For all other steel struts the intensities are to be found by the following formula :

$$R = \frac{14000}{1 + \frac{1}{20000}\left(\frac{l}{r}\right)^2}$$

where R is the intensity, l the length of column in inches, and r the least radius of gyration in inches. Steel struts subject to alternating stresses of compression and tension shall be proportioned by the following formula :

$$R = \frac{14{,}000\left(1 - \frac{1}{2}\,\frac{\text{min. stress}}{\text{max. stress}}\right)}{1 + \frac{1}{20000}\left(\frac{l}{r}\right)^2}$$

where R, l and r have the same signification as in the last clause.

An addition of fifty (50) per cent. to all specified intensities of working stresses shall be allowed for all wind stresses and for all combinations of wind stresses and other stresses.

The thickness of metal in compression shall not be less than one-sixteenth (1–16) of the distance between supports in line of stress, or less than one-thirtieth (1–30) of the distance between supports at right angles to the line of stress, or less than one-eighth (1–8) of the distance from the edge of plate of flange to line of support, or less than one-quarter (1–4) inch when both faces are accessible for painting, or less than five-sixteenths (5–16) of an inch when only one face is accessible for painting.

The ratio of length of strut between supporting points to its least diameter shall not exceed forty-five (45).

The limits of stress specified for shearing and for the pressure on bearing surface of holes shall determine the number and the size of the rivets.

Bed plates and bearing plates shall be truly planed on all sliding and rolling surfaces, and shall be so proportioned that the maximum pressure per square foot on masonry will not exceed thirty-six thousand (36,000) pounds. They will be securely anchored against upward and sideways motion.

The rollers shall be of steel; the pressure per lineal inch on same shall not exceed $\sqrt{540,000 \times d}$, where d is the diameter of the roller in inches. . . . All wrought iron must be tough, ductile, fibrous, and of a uniform quality for each class, straight, smooth, free from cinder pockets or injurious flaws, buckles, blisters and cracks.

As the thickness of the bar approaches the maximum that the rolls will produce, the same perfection of finish will not be required as in the thinner ones. No specific process or pro. vision of manufacture will be demanded, provided the material fulfills the requirements of this specification.

3. The tensile strength, limit of elasticity and ductility shall be determined from a standard test piece, not less than one-quarter inch in thickness, cut from the full sized bar, and planed and turned parallel; if the cross section is reduced, the tangent between shoulders shall be at least twelve times its shortest dimensions, and the area of the minimum cross section in either case shall not be less than one quarter of an inch and not more than one square inch. Whenever practicable, two opposite sides of a piece are to be left as they come from the rolls, but the finish of opposite sides must be the same in this respect. A full sized bar, when not exceeding the above limitations, may be used as its own test piece. In determining the ductility, the elongation shall be measured, after breaking, on an original length the nearest multiple of a quarter inch to ten times the shortest dimension of the test piece, in which length must occur the curve of reduction from stretch on both sides of the point of fracture, but in no case on a shorter length than five inches.

4. All iron to be used in the tensile members of open trusses, laterals, pins and bolts, except plate iron over eight inches wide and shaped iron, must show by the standard test pieces a tensile strength in pounds per square inch of:

$$52,000 - \frac{7,000 \times \text{area of original bar}}{\text{circumference of original bar}} \text{ (all in inches),}$$

with an elastic limit not less than one-half the strength given by this formula, and the elongation of 20 per cent.

Plate iron 24 inches wide and under, and more than 8 inches wide, must show by the standard test pieces a tensile strength of 48,000 pounds per square inch, with an elastic limit not less than 26,000 pounds per square inch, and an elongation of not less than 12 per cent. All plates over 24 inches in width must have a tensile strength not less than 46,000 pounds per square inch, with an elastic limit not less than 26,000 pounds per square inch.

Plates from 24 to 36 inches in width must have an elongation of not less than 10 per cent.; those from 36 to 48 inches in width, 8 per cent.; over 48 inches in width, 5 per cent.

All shaped iron and other iron not hereinbefore specified must show by the standard test pieces a tensile strength in pounds per square inch of:

$$50{,}000 - \frac{7{,}000 \text{ area of original bar}}{\text{circumference of original bar}},$$

with an elastic limit of not less than one-half the strength given by this formula, and an elongation of 15 per cent. for bars five-eighths of an inch and less in thickness, and of 12 per cent. for bars of greater thickness.

All plates, angles, etc., which are to be bent hot, in the manufacture must, in addition to the above requirements, be capable of bending sharply to a right angle at a working heat without a sign of fracture.

All rivet iron must be tough and soft, and pieces of the full diameter of the rivet must be capable of bending cold until the sides are in close contact, without sign of fracture on the convex side of the curve.

All iron specified in clause 4 must bend cold 180 degrees, without sign of fracture, to a curve the inner radius of which equals the thickness of the piece tested.

Specimens of full thickness cut from plate iron or from the flanges or webs of shaped iron, must stand bending cold through 90 degrees, to a curve, the inner radius of which is one and a half times its thickness, without sign of fracture.

. . . .

A variation in cross section or weight of rolled material of more than 2½ per cent. from that specified may be cause for rejection.

Steel.

No specific process or provision of manufacture will be demanded, provided the material fulfills the requirements of this specification. The ultimate tensile resistance of the steel to be used in tension shall be 62,500 pounds per square inch, and the ultimate tensile resistance of the steel to be used in compression shall be 68,000 pounds per square inch, the tests to be made in the following manner:

16. From one among the ingots of each cast a round sample bar, not less than three-quarters of an inch in diameter, and having a length not less than twelve diameters between jaws of testing machine, shall be furnished and tested by the manufacturer without charge. These bars are to be truly round, and shall be finished at a uniform heat and arranged to cool uniformly, and from these test pieces alone the quality and material shall be determined as follows:

17. All the above-described test bars must have a tensile strength within 4,000 pounds per square inch of that specified, an elastic limit not less than one-half of the tensile strength of the test bar, a percentage of elongation not less than 1,200,000 ÷ the tensile strength in pounds per square inch, and a percentage of reduction of area not less than 2,400,000 ÷ the tensile strength in pounds per square inch. In determining the ductility, the elongation shall be measured after breaking on an original length of ten times the shortest dimension of the test piece, in which length must occur the curve of reduction from stretch on both sides of the point of fracture.

Finished bars must be free from injurious flaws or cracks, and must have a workman-like finish, and round or square test pieces cut therefrom when pulled asunder shall have a reduction of area at the point of fracture as above specified.

Rivet steel shall have a specified tensile strength of 60,000 pounds per square inch, and test bars must have a tensile strength within 4,000 pounds per square inch of that specified, and an elastic limit, elongation and reduction of area at the point of fracture, as stated in clause 17, and be capable of bending double, flat, without sign of fracture on its convex surface of the bend.

. . . .

A variation in cross section or weight of rolled material of more than 2½ per cent. from that specified may be cause for rejection.

Cast Iron.

Except where chilled iron is specified, all castings shall be tough gray iron, free from injurious cold shuts or blow holes, true to pattern and of a workman-like finish. Sample pieces one inch square, cast from the same heat of metal in sand moulds, shall be capable of sustaining on a clear span of 4 feet 6 inches a central load of 500 pounds when tested in the rough bar.

. . . .

To determine the strength of eyes, full size eye bars or rods with eyes may be tested to destruction, provided notice is given in advance of the number and size required for the purpose, so that the material can be rolled at the same time as that required for the structure, and any lot of iron bars from which full size samples are tested shall be accepted:

1st. If not more than one-third of the bars tested break in the eye; or,

2d. If more than one-third do break in the eye and the average of the tests of those which so break shows a tensile strength in pounds per square inch of original bar, given by the formula—

$$52{,}000 - \frac{7{,}000 \times \text{area of original bar}}{\text{circumference of original bar}} - 500 \times \text{width of bar}$$

(all in inches), and not more than one-half of those which break in the eye fail at more than 5 per cent. below the strength given by the formula. Any lot of steel bars from which full sized samples are tested shall be accepted if the average of the tests shows a strength per square inch of original bar, in those which do break in the eye, within 4,000 pounds of that specified in Clause 17; but if one-half the full sized samples break in the eye, it shall be cause for rejecting the lot from which the sample bars were taken.

. . . .

In all cases where a steel piece in which the full strength is required has been partially heated, the whole piece must be subsequently annealed.

. . . .

All bends in steel must be made cold, or if the degree of curvature is so great as to require heating, the whole piece must subsequently be annealed.

Art. 85.—Specifications for the Blair Crossing Bridge.

The following specifications for this bridge are taken from the report of Geo. S. Morison, Chief Engineer, 1886.

. . . .

The steel shall be manufactured by the open hearth process; Bessemer steel will not be accepted. A small ingot shall be cast from every charge, and from this ingot a sample bar 3-4 of an inch in diameter shall be rolled; if this bar fails to meet the requirements of the laboratory tests, the whole charge will be rejected.

Steel used in the compression members, bolsters, bearing plates, pins, and rollers shall contain not less than 34-100 nor more than 42-100 of one per cent. of carbon, and not more than 1-10 of one per cent. of phosphorus. A sample test bar 3-4 of an inch in diameter shall bend 180 degrees around its own diameter without sign of crack or flaw. The same bar tested in a lever machine shall show an elastic limit of not less than 50,000 pounds, and an ultimate strength of not less than 80,000 pounds per square inch; it shall elongate at least 15 per cent. in a length of 8 inches before breaking, and shall have a reduced area of 35 per cent. at the point of fracture. It shall be incapable of tempering.

Steel for rivets and eye bars shall contain not more than 25-100 of one per cent. of carbon, and less than 1-10 of one per cent. of phosphorus. A sample bar 3-4 of an inch in diameter shall bend 130 degrees and be set back upon itself without showing crack or flaw; when tested in a lever machine it shall have an elastic limit of not less than 40,000 pounds, and an ultimate strength of not less than 70,000 pounds per square inch; it shall elongate at least 18 per cent. in a length of 8 inches, and shall show a reduction of at least 45 per cent. at the point of fracture. In full sized bars this steel shall have an elastic limit of at least 35,000 pounds, and an ultimate strength of at least 65,000 pounds per square inch; it shall elongate 10 per cent. before breaking, and for strains less than 30,000 pounds per square inch shall show a modulus of elasticity between 28,000,000 and 30,000,000 pounds.

. . . .

The steel plates for the chords and end posts shall be rolled in universal mill.

Steel for pins shall not be hammered, but rolled between gothic rolls.

The iron used in tension members shall be double refined iron, rolled twice from the puddled bar. Small samples having a minimum length of 8 inches shall be furnished by the contractor for testing as directed by the engineer; these samples shall show an elastic limit of at least 26,000 pounds, and an ultimate strength of at least 50,000 pounds per square inch; shall elongate at least 15 per cent., and shall show a reduced area of at least 25 per cent. at the point of fracture. The fracture shall be of uniform fibrous character free from crystalline appearance. When tests are made of full sized bars, a reduction of from 5 to 10 per cent., according to size of bar, from these requirements will be allowed, provided the iron is of uniform and fibrous character.

Small samples having a minimum length of 8 inches shall be furnished by the contractor from the iron used in shapes, plates, and other miscellaneous forms, as directed by the engineer; these samples shall show an elastic limit of at least 24,000 pounds, and an ultimate strength of at least 47,000 pounds per square inch; shall elongate at least 10 per cent. before breaking, and show a reduction of area of at least 15 per cent. at the point of fracture. In plates more than 30 inches wide, an elongation of 8 per cent. and a reduction of 12 per cent. at the point of fracture will be considered satisfactory.

Cast iron shall be of the best quality of tough, gray iron.

. . . .

The heads of iron eye bars, and the enlarged ends of screws in laterals and counters shall be formed by upsetting, or by die-forging with a plate welded on the side; welds in the body of the bar will not be allowed. Six extra iron eye bars, of such size as the engineer shall direct, shall be furnished by the contractor to be tested; these test bars shall meet the requirements above specified for strength of material, and at least four of them shall break in the body of the bar. Should these test bars fail to meet the requirements of the specifications, the whole lot of bars may be rejected.

The heads of steel eye bars shall be formed by upsetting and forging into shape, or by such other process as may be accepted by the engineer; no welds will be allowed. After the working is completed, the bars shall be annealed by heating them to a uniform dark red heat throughout their entire length, and allow them to cool slowly. Four sample bars of sizes required in the work shall first be manufactured by the contractor, and tested under the direction of the engineer; these bars shall meet the requirements above specified, and at least three of them shall break in the body of the bar. If the tests of these four bars are satisfactory, the contractor shall proceed with the manufacture of the full order of steel bars for the work, and from the bars so manufactured the inspector shall from time to time select six bars to be tested to breaking, which bars shall also conform to the requirements of the specifications. Should these test bars fail to meet the requirements of the specifications, the whole lot of bars may be rejected. All steel bars shall be tested to a strain of 20,000 pounds per square inch before shipment.

Art. 86.—General Specifications for Iron Railroad Bridges and Viaducts, by Theodore Cooper, C. E., 1887.

The excerpts given in this article are from the general specifications of Mr. Theodore Cooper, C. E., consulting engineer, which have secured a wide adoption in American railway practice.

Proportion of Parts.

30. All parts of the structure shall be so proportioned that the maximum loads shall in no case cause a greater tension than the following (except as per 36):

	Pounds per Square Inch.
On lateral bracing	15,000
On solid rolled beams, used as cross floor beams and stringers	9,000
On bottom chords and main diagonals (forged eye bars)	10,000
On bottom chords and main diagonals (plates or shapes), net section.	8,000
On counter rods and long verticals (forged eye bars)	8,000
On counters and long verticals (plates or shapes), net section	6,500
On bottom flange of riveted cross girders, net section	8,000
On bottom flange of riveted longitudinal plate girders, over 20 ft. long, net section	8,000

	Pounds per Square Inch.
On bottom flange of riveted longitudinal plate girders, under 20 ft. long, net section....	7,000
On floor beam hangers, and other similar members liable to sudden loading (bar iron with forged ends)......	6,000
On floor beam hangers and other similar members liable to sudden loading (plates or shapes), net section......	5,000

Angles subject to direct tension must be connected by both legs, or the section of one leg only will be considered as effective.

31. Compression members shall be so proportioned that the maximum load shall in no case cause a greater strain than that determined by the following formula (except as per 36) :

$$P = \frac{8,000}{1 + \frac{L^2}{40,000\,R^2}}$$ for square end compression members.

$$P = \frac{8,000}{1 + \frac{L^2}{30,000\,R^2}}$$ for compression members with one pin and one square end.

$$P = \frac{8,000}{1 + \frac{L^2}{20,000\,R^2}}$$ for compression members with pin bearings.

$P =$ the allowed compression per square inch of cross section.

$L =$ the length of compression member, in inches.

$R =$ the least radius of gyration of the section in inches.

No compression member, however, shall have a length exceeding 45 times its least width.

32. The lateral struts shall be proportioned by the above formula to resist the resultant due to an assumed initial strain of 10 000 pounds per square inch upon the rods attaching to them, produced by adjusting the bridge or towers.

33. In beams and girders compression shall be limited, as follows :

In rolled beams, used as cross floor beams and stringers.	8,000
In riveted plate girders used as cross floor beams, gross section....	7,000
In riveted longitudinal plate girders, over 20 ft. long, gross section	7,000
In riveted longitudinal plate girders, under 20 ft. long, gross section	6,000
In riveted lattice girders, gross section......	7,000

34. Riveted longitudinal girders shall have, preferably, a depth not less than 1–10 of the span.

Rolled beams used as longitudinal girders shall have, preferably, a depth not less than 1–12 of the span.

35. Members subject to alternate strains of tension and compression shall be proportioned to resist each kind of strain. Both of the strains shall, however, be considered as increased by an amount equal to 8–10 of the least of the two strains, for determining the sectional area by the above allowed strains (30, 31).

36. For spans exceeding 150 feet, the above allowed tension (30) on bottom chords and main diagonals, and the compression on top chord sections (31) may be increased for each member by the following amount:

$$\frac{150 \times \text{its strain from dead load}}{\text{Its strain from dead and live loads}} - 50 \text{ per cent.}$$

The strains in the chords from the assumed wind forces need not be considered, except as follows:

1st. When the wind strains on any member exceed one-quarter of the maximum strains due to the dead and live loads upon the same member. The section shall then be increased until the total strain per square inch will not exceed by more than one-quarter the maximum fixed for dead and live loads only.

2d. When the wind strain alone, or in combination with a possible temperature strain, can neutralize or reverse the tension in the end panels of the lower chord.

38. The rivets and bolts connecting the parts of any member must be so spaced that the shearing strain per square inch shall not exceed 7,500 pounds, or three-fourths of the allowed tension per square inch upon that member; nor the pressure upon the bearing surface per square inch of the projected semi-intrados (diameter × thickness of piece) of the rivet or bolt hole exceed 12,000 pounds, or one and a half times the allowed tension per square inch upon that member. In the case of field riveting the above limits of shearing strain and pressure shall be reduced one-third part. Rivets must not be used in direct tension.

39. Pins shall be so proportioned that the shearing strain shall not exceed 7,500 pounds per square inch; nor the crushing strain upon the projected area of the semi-intrados of any member (other than forged eye bars, see article 69) connected to the pin be greater per square inch than 12,000 pounds, or one and a half times the allowed tension per square inch; nor the bending strain exceed 15,000 pounds per square inch when the centres of bearings of the strained members are taken as the points of application of the strains.

40. In case any member is subjected to a bending strain from its own weight or from local loadings, such as distributed floors on deck bridges, in addition to the strain produced by its position as a member of the structure, it must be proportioned to resist the combined strains.

41. Plate girders shall be proportioned upon the supposition that the bending or chord strains are resisted entirely by the upper and lower flanges, and that the shearing or web strains are resisted entirely by the web plate; no part of the web plate shall be estimated as flange area.

42. The iron in the web plates shall not be subjected to a shearing strain greater than 4,000 pounds per square inch; but no web plate shall be less than three-eighths of an inch in thickness.

43. The webs of plate girders must be stiffened at intervals, about the depth of the girders, whenever the shearing strain per square inch exceeds the strain allowed by the following formula:

$$\text{Allowed shearing strain} = \frac{12{,}000}{1 + \dfrac{H^2}{3{,}000}}$$

where H = ratio of depth of web to its thickness.

44. No wrought iron shall be used less than 1-4 inch thick, except for lining or filling vacant spaces.

45. The compression flanges of beams and girders shall be

stayed against transverse crippling when their length is more than thirty times their width.

46. The unsupported width of any plate subjected to compresssion shall never exceed thirty times its thickness.

47. The flange plates of all girders must be limited in width so as not to extend beyond the outer lines of rivets connecting them with the angles, more than five inches or more than eight times the thickness of the first plate. Where two or more plates are used on the flanges, they shall either be of equal thickness or shall decrease in thickness outward from the angles.

48. In members subject to tensile strains full allowance shall be made for reduction of section by rivet holes, screw threads, etc.

Quality of Material.

101. All wrought iron must be tough, fibrous and uniform in character. It shall have a limit of elasticity of not less than 26,000 pounds per square inch.

Finished bars must be thoroughly welded during the rolling, and be free from injurious seams, blisters, buckles, cinder spots, or imperfect edges.

102. For all tension members double rolled bars must be used. They shall stand the following tests:

103. Full sized pieces of flat, round or square iron, not over 4 1-2 inches in sectional area, shall have an ultimate strength of 50,000 pounds per square inch, and stretch 12 1-2 per cent. in their whole length.

Bars of a larger sectional area than 4 1-2 square inches, when tested in the usual way, will be allowed a reduction of 1,000 pounds per square inch for each additional square inch of section, down to a minimum of 46,000 pounds per square inch.

104. When tested in specimens of uniform sectional area of at least 1-2 square inch for a distance of 10 inches taken from tension members which have been rolled to a section not more

than 4 1-2 square inches, the iron shall show an ultimate strength of 52,000 pounds per square inch, and stretch 18 per cent. in a distance of eight inches.

Specimens taken from bars of a larger cross section than 4 1-2 inches will be allowed a reduction of 500 pounds for each additional square inch of section, down to a minimum of 50,000 pounds.

105. The same sized specimens taken from angle and other shaped iron shall have an ultimate strength of 48,000 pounds per square inch, and elongate 15 per cent. in 8 inches.

106. The same sized specimens taken from plates less than 24 inches in width shall have an ultimate strength of 48,000 pounds, and elongate 15 per cent. in 8 inches.

107. The same sized specimens taken from plates exceeding 24 inches in width shall have an ultimate strength of 46,000 pounds, and elongate 10 per cent.

108. All iron for tension members must bend cold, for about 90 degrees, to a curve whose diameter is not over twice the thickness of the piece, without cracking. At least one sample in three must bend 180 degrees to this curve without cracking. When nicked on one side, and bent by a blow from a sledge, the fracture must be nearly all fibrous, showing but few crystalline specks.

109. Specimens from angle, plate (106) and shaped iron must stand bending cold through 90 degrees, and to a curve whose diameter is not over three times its thickness, without cracking.

When nicked and bent, its fracture must be mostly fibrous.

110. Rivets and pins shall be made from the best double-refined iron.

111. The cast iron must be of the quality of soft gray iron.

115. The timber shall be strictly first class white pine, southern yellow pine or white oak bridge timber; sawed true, and out of wind, full size, free from wind shakes, large or loose knots, decayed or sap wood, worm holes, or other defects im-

pairing its strength or durability. It will be subject to the inspection and acceptance of the Chief Engineer.

Art. 87.—Standard Specifications for Iron and Steel Railway Structures by the Phœnix Bridge Co.

The following clauses are from the standard specifications of the Phœnix Bridge Co. for iron and steel railway structures.

. . . .

Working Stresses for Iron.

The greatest working stresses in all wrought iron tensile members of railway spans 150 feet in length and under, shall be as follows:

In counter web members	8,000 lbs. per sq. in.
In long verticals	8,000 " "
In main web and lower chord members (eye bars)	10,000 " "
In suspension loops	7,000 " "
In suspension plates (net section)	7,000 " "
In tension members of lateral and transverse bracing	15,000 " "
In counter rods and long verticals of lattice girders (net sect.)	7,000 " "
In lower chords and main tension members of lattice girders (net sect.)	8,000 " "
In bottom flange of plate girders (net sect.)	8,000 " "
In bottom flange of rolled beams	8,000 " "
In angle iron lateral ties (net sect.)	12,000 " "

The greatest working stresses in wrought iron compression members of spans 150 feet in length and under, shall be the following, in which "P" is in pounds per square inch:

	Flat Ends.	Pin Ends.
Phœnix column	$P = \dfrac{8,400}{1 + \dfrac{l^2}{50,000\, r^2}}$;	$P = \dfrac{8,400}{1 + \dfrac{l^2}{30,000\, r^2}}$
Latticed or common column	$P = \dfrac{8,000}{1 + \dfrac{l^2}{40,000\, r^2}}$;	$P = \dfrac{7,800}{1 + \dfrac{l^2}{30,000\, r^2}}$
Angle iron struts	$P = 9,000 - 30\dfrac{l}{r}$;	$P = 9,000 - 34\dfrac{l}{r}$

"l" is the length of column, and "r" the radius of gyration of section, in direction of failure; both are to be taken in feet or both in inches.

Upper chords shall be proportioned by the flat end formula.

A mean between flat end and pin end results shall be used for one pin end and one flat end.

Lateral and transverse struts shall be designed by taking working stresses, equal to one and four-tenths those given by the preceding formulæ.

In spans over 150 feet in length, the greatest working tensile stresses per square inch of wrought iron, lower chord and end main web eye bars shall be:

$$8{,}000\left(1 + 0.9 \times \frac{\text{min. total stress}}{\text{max. total stress}}\right),$$

whenever this quantity exceeds 10,000.

In such spans the main web eye bars nearest the centre shall be proportioned for 10,000 pounds per square inch; and the web eye bars between the end and centre shall be found by direct interpolation between the above values.

The greatest working stresses on the upper chords and end posts of spans exceeding 150 feet in length, shall be determined by increasing the results of the above column formulæ by the same proportion that the preceding process gives to the lower chord eye bars of the same span. The proportionate increase of working stresses for the intermediate posts, shall be the same as that of the web eye bars, meeting their upper ends in through spans, and lower ends in deck spans.

In the compression flange of plate girders and rolled beams, the working stress shall not exceed 8,000 pounds per square inch of gross section. The greatest shearing stress due to combined dead and moving loads, shall not exceed 7,500 pounds per square inch, in any rivet or pin; or 10,000 pounds per square inch for wind stresses; or 9,000 pounds per square inch for wind stresses in combination with those due to moving load. A deduction of 20 per cent. from these values shall be made for field-driven rivets.

The greatest bearing stress on any rivet or pin due to combined dead and moving loads shall be taken at 12,000 pounds per square inch of diametral surface; or 17,000 pounds per square inch of diametral surface for wind stresses; or 15,000 pounds for wind stresses in combination with those due to moving load. A deduction of 20 per cent. from these values shall be made for field-driven rivets.

The bending stress of tension or compression on the extreme fibres of pins shall not exceed 15,000 pounds per square inch for combined dead and moving loads; or 20,000 pounds for wind stresses; or 18,000 pounds for wind stresses combined with those due to moving load.

Working Stresses for Steel.

The greatest allowed working stresses in steel tension members, for spans of 200 feet in length and less, shall be as follows:

In counter web members	10,500	lbs.	per sq. in.
In long verticals	10,000	"	"
In all main web and lower chord eye bars	13,200	"	"
In plate hangers (net section)	9,000	"	"
In tension members of lateral and transverse bracing	19,000	"	"
In steel angle lateral ties (net sect.)	15,000	"	"

For spans over 200 feet in length, the greatest allowed working stresses per square inch, in lower chord and end main web eye bars, shall be taken at

$$10{,}000\left(1 + \frac{\text{min. total stress}}{\text{max. total stress}}\right),$$

whenever this quantity exceeds 13,200.

The greatest allowable stress in the main web eye bars nearest the centre of such spans, shall be taken at 13,200 pounds per square inch; and those for the intermediate eye bars shall be found by direct interpolation between the preceding values.

The greatest allowable working stresses in steel plate and lattice girders and rolled beams, shall be taken as follows:

Upper flange of plate girders (gross section)	10,000	lbs. per sq. in.
Lower flange of plate girders (net section)	10,000	" "
In counters and long verticals of lattice girders (net sect.)	9,000	" "
In lower chords and main diagonals of lattice girders (net section)	10,000	" "
In bottom flanges of rolled beams	10,000	" "
In top flanges of rolled beams	10,000	" "

The greatest allowable working stresses in steel latticed or common columns, or steel Phœnix columns, for spans of 200 feet in length and less, shall be determined by taking four-thirds the values given by the formula for iron columns, on page 697.

The greatest allowable working stresses for the same kind of columns, in spans over 200 feet in length, shall be found by increasing the values established by the preceding paragraph by the same proportion, for the upper chord and end posts, as the lower chord eye bar stresses are increased for the same length of spans.

The greatest allowable working stresses in the columns nearest the centre shall remain unchanged; and those for the intermediate columns shall be found by direct interpolation.

The greatest working stresses, in pounds per square inch, allowed in steel angle struts, shall be as follows:

$$\text{Flat end steel angles} \ldots\ldots\ldots P = 12{,}500 - 44\frac{l}{r}$$

$$\text{Pin end steel angles} \ldots\ldots\ldots P = 12{,}500 - 50\frac{l}{r}$$

"l" is the length of the column, and "r" the radius of gyration of section, in direction of failure; both are to be taken in feet, or both in inches.

Upper chords shall be proportioned by the "flat end" formulæ, in all cases.

A mean between flat end and pin end results shall be used for one pin and one flat end.

Lateral and transverse struts shall be designed by taking working stresses equal to 1.4 those given by the preceding formulæ for both 200 feet pin spans and angle struts.

The greatest shearing stress on any rivet or pin, due to combined dead and moving loads, shall not exceed 10,000 pounds per square inch; or 13,000 pounds per square inch for wind stresses; or 12,000 pounds for wind stresses in combination with those due to moving load. A deduction of 20 per cent. from these values shall be made for field-driven rivets.

The greatest bearing stress on any rivet or pin, due to combined dead and moving loads, shall be taken at 16,000 pounds per square inch of diametral surface; or 22,000 pounds per square inch of the same surface for wind stresses; or 20,000 pounds for wind stresses in combination with those due to moving load. A deduction of 20 per cent. shall be made from these values for field-driven rivets.

The bending stress of tension or compression on the extreme fibres of pins shall not exceed 20,000 pounds per square inch for combined dead and moving loads; or 26,000 pounds for wind stresses; or 24,000 pounds for wind stresses combined with those due to moving loads.

General Clauses.

In case wind stresses combine with those due to dead and moving loads, no increase of section will be required, unless the wind stresses exceed one-third the sum of those caused by the dead and moving loads, in which case provision will be made for the excess, at a unit stress equal to four-thirds that allowed in the same member for the combined dead and moving loads; but, in no case, shall that unit stress exceed 15,000 pounds for iron, or 18,500 pounds for steel for tension; or 10,500 pounds for iron, or 14,500 pounds for steel for compression; or 10,000 pounds for iron, or 12,000 pounds for steel for shear; or 19,000 pounds for iron, or 25,000 pounds for steel for extreme fibre stress in pin bending.

. . . .

Art. 88.—Niagara Suspension Bridge.

In his "Report on the Renewal of the Niagara Suspension Bridge," Mr. Leffert L. Buck, C. E., has given some data and calculations, from which he deduces that the safety factor for the cables is:

$$11{,}000 \div (1{,}400 \times 1.78) = 4.41,$$

the total load between the towers being 1,400 tons, and the ultimate resistance of the four wrought-iron cables 11,000 tons, while 1.78 is the ratio between the cable tension at the top of the towers and the vertical load between the towers.

The new iron and steel stiffening truss is designed for a safety factor of 5.

Art. 89.—Specifications for Boiler and Fire Box Steel, Penn. RR. Co., 1883.

1. A careful examination will be made of every sheet, and none will be received that show mechanical defects.

2. A test strip from each sheet, taken lengthwise of the sheet, and without annealing, should have a tensile strength of 55,000 lbs. per square inch, and an elongation of 30 per cent. in section, originally 2 inches long.

3. Sheets will not be accepted if the test shows a tensile strength less than 50,000 lbs., or greater than 65,000 lbs. per square inch, nor if the elongation falls below 25 per cent.

4. Should any sheets develop defects in working, they will be rejected.

5. Manufacturers must send one test strip for each sheet (this strip must accompany the sheet in every case); both sheet and strip being properly stamped with the marks designated by this company, and also lettered with white lead to facilitate matching.

Art. 90.—The St. John Cantilever, St. John, N. B., 1885.

The general dimensions of the river spans are as follows:

Total length of structure on centres of end pins	812 ft. 6″
Length of centre opening from centre to centre of the piers	477 ft. 0″
Length of centre span	143 ft. 6″
Length of east cantilever	287 ft. 0″
Length of west cantilever	382 ft. 0″
(The two arms of each cantilever being of equal length)	
Length of panels from centre to centre of pins, about	24 ft. 0″
Depth of cantilever trusses, 27 ft. at the ends, 80 ft. at the centre for the west cantilever, and 65 ft. for the east cantilever.	
Depth of centre span	27 ft. 0″

. . . .

Under the maximum strains produced by any condition of the loads and wind pressures jointly, the strain on the steel composing the structure was limited to the following amounts per square inch:

For the upper chords	14,000 lb.	tension.
Diagonal ties	13,000 lb.	"
Centre and counter ties	12,000 lb.	"
Suspension ties	10,000 lb.	"
Wind ties	20,000 lb.	"
Floor beam or stringer flanges	12,000 lb.	"
For the lower chords and central posts	12,000 lb.	compression.
Intermediate posts	10,000 lb.	"
Wind struts	14,000 lb.	"
Floor beam or stringer flanges	12,000 lb.	"

The above values for compression being used for the value of F in Gordon's formula, the value of A in said formula being taken as 1-4500 for the lower chords and central posts, and as 1-1500 for the intermediate posts and wind struts. The tensile strains in riveted connections were to be taken ten per cent. less than the above amounts, and the shearing strains were limited to 10,000 lbs.; the bearing pressure on rivets or pin-holes not exceeding the diameter of rivet or pin multiplied by the thickness of bearing multiplied by 16,000 lbs., and the strains on the extreme fibres in pins from bending not exceeding 20,-

400 lbs. No steel of a less thickness than 5-16 inch was allowed to be used, and no rods of a less diameter than 1 inch.

Art. 91.—Penn. RR. Specifications, 1887.

The following clauses are from the standard specifications of the Penn. RR. Co.

. . . .

21. The maximum and minimum stresses in tension and compression, as found for the before-mentioned loads, are to be used in determining the permissible working stress in each piece of the structure according to following formulæ:

For pieces subject to one kind of stress only (all compression or all tension):

$$a = u\,(1 + r).$$

For pieces subject to stresses acting in opposite directions:

$$a = u\,(1 - r_1).$$

In the above formulæ:

a = Permissible stress per sq. in., either tension or compression.

$u = \begin{cases} 7{,}500 \text{ lbs. per sq. inch, for double rolled iron in tension (links or rods).} \\ 7{,}000 \text{ " " " rolled iron in tension (plates or shapes).} \\ 6{,}500 \text{ " " " " " compression.} \end{cases}$

$$r = \frac{\text{Minimum stress in piece}}{\text{Maximum stress in piece}}.$$

$$r_1 = \frac{\text{Maximum stress of lesser kind}}{2 \times \text{Maximum stress of greater kind}}.$$

22. The permissible stress "a" for members in compression is to be reduced in proportion to the ratio of the length to the least radius of gyration of the section, by the following formulæ:

For both ends fixed.......... $b = \dfrac{a}{1 + \dfrac{l^2}{36000g^2}}$

For one end hinged.......... $b = \dfrac{a}{1 + \dfrac{l^2}{24000g^2}}$

For both ends hinged........ $b = \dfrac{a}{1 + \dfrac{l^2}{18000g^2}}$

Where "a" = Permissible stress already found.
"b" = Allowable working stress per sq. in.
"l" = Length of piece in inches, centre to centre of connections.
"g" = Least radius of gyration of section in inches.

Art. 92.—Specifications for Steel Cable Wire for the East River Suspension Bridge.

". . . . .

3. The general character of the wire is as follows: it must be made of steel; it must be hardened and tempered; and, lastly, it must all be galvanized.

4. The size of the wire shall be No. 8 full, Birmingham gauge. . . . .

5. Each wire must have a breaking strength of no less than 3,400 pounds. This corresponds in wire weighing 14 feet to the pound, to a rate of 160,000 pounds per square inch of solid section. The elastic limit must be no less than $\frac{47}{100}$ of the breaking strength, or, 1,600 pounds. Within this limit of elasticity, it must stretch at a uniform rate corresponding to a modulus of elasticity of not less than 27,000,000 nor exceed 29,000,000. . . . .

Mode of Testing.

. . . .

There will be four kinds of tests.

Firstly.—One ring in every forty (40) will be tested as follows: a piece of wire sixty (60) feet long, will be cut off from either end of the ring, and it will then be placed in a vertical testing machine. An initial strain of 400 pounds is now applied, which should take out every crook and bend. A vernier gauge, capable of being read to $\frac{1}{10,000}$ of one foot, is so attached as to indicate the stretch of 50 feet of the wire. Successive increments of 400 pounds strain are then applied, and the vernier read each time, until a strain of 1,600 pounds is reached.

The conditions now are as follows: that the amount of stretch for each of these increments shall be the same, and that the total stretch between the initial and terminal strains shall not be less than $\frac{97}{1,000}$ of one foot, equal to $\frac{194}{100,000}$ of the 50 feet. And furthermore, on reducing the strain to 1,200 pounds there shall be a permanent elongation not exceeding $\frac{1}{100,000}$ of its length.

The same wire will then be subjected to a breaking strain, and the total amount of stretch noted. The minimum strength required is 3,400 pounds, equal to an ultimate strength of 160,000 pounds per square inch. The minimum stretch, when broken, shall have been 2 per cent. in 50 feet, and the diameter of the wire at the point of fracture shall not exceed $\frac{15}{100}$ of one inch.

. . . .

Fourthly.—Every ring will be subjected to a bending test by cutting off from each ring a piece of wire one foot long, and coiling it closely and continuously around a rod one half inch in diameter, when, if it breaks it will be rejected.

. . . .

Straight Wire.

9. All the wire . . . must be "straight" wire; that is to say, when a ring is unrolled upon the floor the wire behind must lie perfectly straight and neutral, without any tendency to spring back in the coiled form, as is usually the case. This

straight condition must not be produced by the use of straightening machines of any kind, as they only injure the strength and elasticity of the wire. . . . . ."

Art. 93.—Specifications for Steel Wire Ropes for the Over-Floor Stays and Storm Cables of the East River Suspension Bridge.

" . . . .

3. The steel from which the wire for these ropes is made must be of a uniform and suitable quality, and after drawn must be thoroughly and evenly galvanized throughout.

4. The galvanized wire must have an ultimate strength of 150,000 pounds per square inch of full section. When tested in lengths of five feet it must stretch no less than three and one-half per cent. of its length, and in lengths of one foot it must stretch no less than four per cent.

5. It must be capable of being bent continuously around a rod of three times the diameter of the wire, without fracture.

6. The modulus of elasticity must not vary more than 2,000,000 pounds, nor exceed 30,000,000 pounds.

7. It must have a limit of elasticity of not less than 70,000 pounds per square inch.

. . . ."

Art. 94.—Specifications for Steel Suspenders, Connecting Rods, Stirrups and Pins, for the East River Suspension Bridge.

" . . . .

All of the steel used must be of a uniform and suitable quality, known as "Mild Steel." It must have an ultimate tensile strength of 75,000 pounds per square inch of full section, and an ultimate stretch of no less than 15 per cent. in one foot of length, including the fractured section; and a reduction of no less than 25 per cent. of area at the point of fracture. It must have an elastic limit of no less than 45,000 pounds per square inch, and a modulus of elasticity between 26,000,000 and 30,000,000 pounds per square inch. Specimens turned down from full-sized rods to an area of one square inch, or less, must show a greater strength per square inch, and a greater elongation than that called for in the full section."

Art. 95.—Specifications for Certain Steel Work . . . East River Bridge, 1881.

". . . . .

All of the steel used in this work must be of a mild, uniform, elastic and ductile quality, suitable for bridge members. Siemens-Martin or open-hearth steel, or Bessemer steel under the Hay process, will be preferred. . . .

Specimens of the steel proposed to be used must be furnished by each bidder. Two specimens, direct from the rolls, each 1 inch square and 24 inches long, are required.

. . . .

All of the steel must be capable of sustaining a tensile strain in every full-sized round or flat bar of not less than 70,000,000 pounds per square inch of cross section. It must have an elastic limit in all shapes of no less than 40,000 pounds per square inch. A modulus of elasticity of not less than 26,000,000 nor more than 30,000,000 pounds per square inch.

An ultimate elongation of 10 per cent. of the full length of uniform sections, and 15 per cent. in one foot of length, inclusive of fractured section, is also required. The area of the reduced section at the point of fracture must not exceed 80 per cent. of the original section.

Small specimens of one foot in length, of even section of one square inch, or less, should reach in tensile strength 75,000 pounds per square inch, with a modulus and limit of elasticity, and reduction of area before mentioned, and an ultimate stretch of 15 per cent.

All round or flat bars, or flat pieces cut from the web of any shaped bars, must be capable of being bent cold for 180° to a curve whose diameter is no greater than the thickness of the bar, and that without cracking.

The rivets must be made of very ductile steel particularly adapted for that use.

The rods from which the rivets are made must, when tested, have a tensile strength of not less than 70,000 pounds per square inch, and elongate at least 20 per cent. in a length of one foot, and shall reduce at the point of fracture 30 per cent.

If the minimum is reached in any one of these requirements the others must be exceeded by at least 10 per cent. The rod must be capable of being bent cold under a hammer 180°, and the inner surfaces brought in contact without producing any fracture.

. . . .

Cold straightening must be avoided, and when resorted to, the piece so straightened must be annealed afterwards, and all pieces, of which any portion for any cause is reheated, the whole must be annealed and very slowly cooled ; and all pieces in which, from test or otherwise, a want of uniformity is suspected, must be annealed if required by the engineer.

All rivet holes must be drilled, unless some system of punching and reaming approved by the engineer be followed, whereby all of the compressed section around the punched hole be cut away.

The spacing must be accurately done, as no gauging or drifting will be allowed."

CHAPTER XIII.

THE FATIGUE OF METALS.

Art. 96.—Woehler's Law.

IN all the preceding pages, that force or stress, which, by a single or gradual application, will cause the failure or rupture of a piece of material, has been called its "ultimate resistance." It has long been known, however, that a stress less than the ultimate resistance *may* cause rupture if its application be repeated (without shock) a sufficient number of times. Preceding 1859 no experiments had been made for the purpose of establishing any law connecting the number of applications with the stress requisite for rupture, or, with the variation between the greatest and least values of the applied stress.

During the interval between 1859 and 1870, A. Wöhler, under the auspices of the Prussian Government, undertook the execution of some experiments, at the completion of which he had established the following law :

Rupture may be caused not only by a force which exceeds the ultimate resistance, but by the repeated action of forces alternately rising and falling between certain limits, the greater of which is less than the ultimate resistance ; the number of repetitions requisite for rupture being an inverse function both of this variation of the applied force and its upper limit.

This phenomenon of the decrease in value of the breaking load with an increase of repetitions, is known as "*the fatigue of materials.*"

Although the experimental work requisite to give Wöhler's law complete quantitative expression in the various conditions of engineering constructions can scarcely be considered more than begun, yet enough has been done by Wöhler and Spangenberg to establish the *fact* of metallic fatigue, and a few simple formulæ, provisional though they may be. The importance of the subject in its relation to the durability of all iron and steel structures is of such a high character that a synopsis of some of the experimental results of Wöhler and Spangenberg will be given in the next Article.

Art. 97.—Experimental Results.

The experiments of Wöhler are given in "Zeitschrift für Bauwesen," Vols. X., XIII., XVI. and XX., and those of Spangenberg may be consulted in "Fatigue of Metals," translated from the German of Prof. Ludwig Spangenberg, 1876.

These results show in a very marked manner the effect of repeated vibrations on the intensity of stress required to produce rupture.

Spangenberg states that "the experiments show that vibrations may take place between the following limits with equal security against rupture by tearing or crushing:

Wrought iron............	+ 17,600	and	− 17,600	lbs. per sq. in.
	+ 33,000	and	0	" " " "
	+ 48,400	and	+ 26,400	" " " "
Axle cast steel...........	+ 30,800	and	− 30,800	" " " "
	+ 52,800	and	0	" " " "
	+ 88,000	and	+ 38,500	" " " "
Spring steel not hardened..	+ 55,000	and	0	" " " "
	+ 77,000	and	+ 27,500	" " " "
	+ 88,000	and	+ 44,000	" " " "
	+ 99,000	and	+ 66,000	" " " "

And for axle cast steel in shearing:

+ 24,200	and	− 24,200	lbs. per sq. in.
+ 41,800	and	0	" " " "

Phœnix Iron in Tension.

POUNDS STRESS PER SQUARE INCH.	NUMBER OF REPETITIONS.	POUNDS STRESS PER SQUARE INCH.	NUMBER OF REPETITIONS.
From 0 to 52,800	800 rupture.	From 0 to 39,600	480,852 rupture.
From 0 to 48,400	106,910 rupture.	From 0 to 35,200	10,141,645 rupture.
From 0 to 44,000	340,853 rupture.	From 22,000 to 48,400	2.373,424 rupture.
From 0 to 39,600	409,481 rupture.	From 26,400 to 48.400	4,000,000 not broken.

Westphalia Iron in Tension.

From 0 to 52,800	4,700 rupture.	From 0 to 39,600	180,800 rupture.
From 0 to 48,400	83,199 rupture.	From 0 to 39,600	596,089 rupture.
From 0 to 48,400	33,230 rupture.	From 0 to 39,600	433,572 rupture.
From 0 to 44,000	136,700 rupture.	From 0 to 35,200	280,121 rupture.
From 0 to 44,000	159.639 rupture.	From 0 to 35,200	566,344 rupture.

Firth & Sons' Steel in Tension.

From 0 to 66,000	83,319 rupture.	From 0 to 55,000	103,540 rupture.
From 0 to 60,500	168,396 rupture.	From 0 to 53,900	12,200,000 not broken.
From 0 to 55,000	133,910 rupture.	From 0 to 53,900	229,230 rupture.
From 0 to 55,000	185,680 rupture.	From 0 to 52,800	692,543 rupture.
From 0 to 55,000	360.235 rupture.	From 0 to 52,800	12,200,000 not broken.
From 0 to 55,000	186,005 rupture.	From 0 to 50,600	——

Krupp's Axle Steel in Tension.

From 0 to 88,000	18,741 rupture.	From 0 to 55,000	473.766 rupture.
From 0 to 77,000	46,286 rupture.	From 0 to 52,800	13,600,000 not broken.
From 0 to 66,000	170,000 rupture.	From 0 to 50,600	12,200,000 not broken.
From 0 to 60,500	123,770 rupture.	——	——

Phosphor Bronze (unworkea) in Tension.

POUNDS STRESS PER SQUARE INCH.	NUMBER OF REPETITIONS.	POUNDS STRESS PER SQUARE INCH.	NUMBER OF REPETITIONS.
From 0 to 27,500	147,850 rupture.	From 0 to 13,750	1,548,920 rupture.
From 0 to 22,000	408,350 rupture.	From 0 to 13,750	2,340,000 rupture.
From 0 to 16,500	2,731,161 rupture.	——	——

Phosphor Bronze (wrought) in Tension.

From 0 to 22,000	53,900 rupture.	From 0 to 13,750	1,621,300 rupture.
From 0 to 16,500	2,600,000 not broken.	——	——

Common Bronze in Tension.

From 0 to 22,000	4,200 rupture.	From 0 to 11.000	5,447,600 rupture.
From 0 to 16,500	6,300 rupture.	——	——

Phœnix Iron in Flexure (one direction only).

From 0 to 60,500	169,750 rupture.	From 0 to 39,600	4,035,400 ruptnre.
From 0 to 55,000	420,000 rupture.	From 0 to 35,200	3,420,000 rupture.
From 0 to 49,500	481,975 rupture.	From 0 to 33,000	4,820,000 not broken.
From 0 to 44,000	1,320,000 rupture.	——	——

Westphalia Iron in Flexure (one direction only).

From 0 to 52,250	612,065 rupture.	From 0 to 44,000	1,493,511 rupture.
From 0 to 49,500	457,229 rupture.	From 0 to 39,600	3,587,509 rupture.
From 0 to 46,750	799,543 rupture.	——	——

Homogeneous Iron in Flexure (one direction only).

POUNDS STRESS PER SQUARE INCH.	NUMBER OF REPETITIONS.	POUNDS STRESS PER SQUARE INCH.	NUMBER OF REPETITIONS.
From 0 to 60,500	169,750 rupture.	From 0 to 39,600	4,035,400 rupture.
From 0 to 55,000	420,000 rupture.	From 0 to 35,200	3,420,000 not broken.
From 0 to 49,500	481,975 rupture.	From 0 to 33,000	48,200,000 not broken.
From 0 to 44,000	1,320,000 rupture.	——	——

Firth & Sons' Steel in Flexure (one direction only).

POUNDS STRESS PER SQUARE INCH.	NUMBER OF REPETITIONS.	POUNDS STRESS PER SQUARE INCH.	NUMBER OF REPETITIONS.
From 0 to 63,250	281,856 rupture.	From 0 to 52,250	578,323 rupture.
From 0 to 60,500	266,556 rupture.	From 0 to 49,500	5,640,596* rupture.
From 0 to 55,000	1,479,908 rupture.	From 0 to 49,500	13,700,000 not broken.

* Accidental.

Krupp's Axle Steel in Flexure (one direction only).

POUNDS STRESS PER SQUARE INCH.	NUMBER OF REPETITIONS.	POUNDS STRESS PER SQUARE INCH.	NUMBER OF REPETITIONS.
From 0 to 77,000	104,300 rupture.	From 0 to 55,000	729,400 rupture.
From 0 to 66,000	317,275 rupture.	From 0 to 55,000	1,499,600 rupture.
From 0 to 60,500	612,500 rupture.	From 0 to 49,500	43,000,000 not broken.

Krupp's Spring Steel in Flexure (one direction only).

From 0 to 110,000	39,950 rupture.	From 72,600 to 110,000	19,673,300 not broken.
From 0 to 88,000	117,000 rupture.	From 66,000 to 99,000	33,600,000 not broken.
From 0 to 66,000	468,200 rupture.	From 44,000 to 88,000	35,800,000 not broken.
From 0 to 55,000	40,600,000 not broken.	From 44,000 to 88,000	38,000,000 not broken.
From 0 to 49,500	32,942,000 not broken.	From 61,600 to 88,000	36,000,000 not broken.
From 88,000 to 132,000	35,600,000 not broken.	From 27,500 to 77,000	36,600,000 not broken.
From 99,000 to 132,000	33,478,700 not broken.	From 33,000 to 77,000	31,152,000 not broken.

Phosphor Bronze in Flexure (one direction only).

POUNDS STRESS PER SQUARE INCH.	NUMBER OF REPETITIONS.	POUNDS STRESS PER SQUARE INCH.	NUMBER OF REPETITIONS.
From 0 to 22,000	862,980 rupture.	From 0 to 16,500	5,075,169 rupture.
From 0 to 19,800	8,151,811 rupture.	From 0 to 13,200	10,000,000 not broken.

Common Bronze in Flexure (one direction only).

From 0 to 22,000	102,659 rupture.	From 0 to 16,500	837,760 rupture.
From 0 to 19,800	151,310 rupture.	From 0 to 13,200	10,400,000 not broken.

Phœnix Iron in Torsion (both directions).

− 35,200 to + 35,200	56,430 rupture.	− 24,200 to + 24,200	3,632,588 rupture.
− 33,000 to + 33,000	99,000 rupture.	− 22,000 to + 22,000	4,917,992 rupture.
− 28,600 to + 28,600	479,490 rupture.	− 19,800 to + 19,800	19,186,791 rupture.
− 26,400 to + 26,400	909,810 rupture	− 17,600 to + 17,600	132,250,000 not broken.

English Spindle Iron in Torsion (both directions).

− 37,400 to + 37,400	204,400 rupture.	− 30,800 to + 30,800	979,100 rupture.
− 37,400 to + 37,400	147,800 rupture.	− 28,600 to + 28,600	1,142,600 rupture.
− 35,200 to + 35,200	911,100 rupture.	− 28,600 to + 28,600	595,910 rupture.
− 35,200 to + 35,200	402,900 rupture.	− 26,400 to + 26,400	3,823,200 rupture.
− 33,000 to + 33,000	1,064,700 rupture.	− 26,400 to + 26,400	6,100,000 not broken.
− 33,000 to + 33,000	384,800 rupture.	− 22,000 to + 22,000	8,800,000 not broken.
− 30,800 to + 30,800	1,337,700 rupture.	− 22,000 to + 22,000	4,000,000 not broken.

Krupp's Axle Steel in Torsion (both directions).

POUNDS STRESS PER SQUARE INCH.	NUMBER. OF REPETITIONS.	POUNDS STRESS PER SQUARE INCH.	NUMBER OF REPETITIONS.
− 44,000 to + 44,000	367,400 rupture.	− 46,200 to + 46,200	55,100 rupture.
− 39,600 to + 39,600	925,800 rupture.	− 37,400 to + 37,200	797,525 rupture.
− 37,400 to + 37,400	4,900,000 not broken.	− 35,200 to + 35,200	1,665,580 rupture.
− 35,200 to + 35,200	4,800,000 not broken.	− 33,000 to + 33,000	4,163,375 rupture.
− 33,000 to + 33,000	5,000,000 not broken.	− 33,000 to + 33,000	45,050,640 rupture.

In Art. 33 will be found some experiments by the late Capt. Rodman, U. S. A., on the fatigue of cast iron, but they are sufficient in number and character to show the general effect only, and give no quantitative results.

The specimens used in all the preceding experiments were small.

During 1860, '61 and '62, Sir Wm. Fairbairn constructed a built beam of plates and angles with a depth of 16 inches, clear span of 20 feet, and estimated centre breaking load of 26,880 pounds.

This beam was subjected to the action of a centre load of 6,643 pounds, alternately applied and relieved eight times per minute; 596,790 continuous applications produced no visible alterations.

The load was then increased from one-fourth to two-sevenths the breaking weight, and 403,210 more applications were made without apparent injury.

The load was next increased to two-fifths the breaking weight, or to 10,486 pounds; 5,175 changes then broke the beam in the tension flange near the centre.

The total number of applications was thus 1,005,175.

The beam was then repaired and loaded with 10,500 pounds at centre 158 times; then with 8,025 pounds 25,900 times, and finally with 6,643 pounds enough times to make a total of 3,150,000.

In these experiments the load was completely removed each time.

It is thus seen that vibrations (without shock) with one-fourth the calculated breaking centre load produced no apparent effect on the resistance of the beam, but that two-fifths of that load caused failure after a comparatively small number of repetitions.

It is probable that the breaking centre load was calculated too high, in which case the ratios $\frac{1}{4}$ and $\frac{2}{5}$ should be somewhat increased.

Art. 98.—Formulæ of Launhardt and Weyrauch.

Let R represent the intensity (stress per square unit of section) of ultimate resistance for any material in tension, compression, shearing, torsion or bending; R will cause rupture at a single, gradual application. But the material may also be ruptured if it is subjected a sufficient number of times, and alternately, to the intensities P and Q, Q being less than P and both less than R, *while all are of the same kind.* When $Q = 0$ let $P = W$, and let $D = P - Q$. W is called the "primitive safe resistance," since the bar returns to its primitive unstressed condition at each application. In the general case P is called the "working ultimate resistance."

By the notation adopted:

$$P = Q + D \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But by Wöhler's law, P is a function of D; or,

$$P = f(D) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

A sufficient number of experiments have not yet been made in order to completely determine the form of the function $f(D)$.

It is known, however, that:

$$\text{For } Q = 0; \quad P = D = W;$$

and for

$$D = 0; \quad P = Q = R.$$

Provisionally, Launhardt satisfies these two extreme conditions by taking:

$$P = \frac{R - W}{R - P} D = \frac{R - W}{R - P} (P - Q) \quad . \quad . \quad . \quad (3)$$

Even at these limits this is not thoroughly satisfactory, for when $D = 0$, $P = \frac{0}{0} (R - W)$, or, indeterminate.

By solving Eq. (3):

$$P = W \left(1 + \frac{R - W}{W} \cdot \frac{Q}{P} \right) \quad . \quad . \quad . \quad . \quad (4)$$

But if the least value of the total stress to which any member of a structure is subjected is represented by *min B*, and its greatest value by *max B*, there will result $\frac{min\ B}{max\ B} = \frac{Q}{P}$.

Hence:

$$P = W \left(1 + \frac{R - W}{W} \frac{min\ B}{max\ B} \right) \quad . \quad . \quad . \quad . \quad (5)$$

which is Launhardt's formula. In the preceding Article some values of W are shown. In applying Eq. (5) it is only necessary to take the primitive safe resistance, W, for the total number of times which the structure will be subjected to loads.

Since bridges are expected to possess an indefinite duration of life, in such structures that number should be indefinitely large.

Eq. (5), it is to be borne in mind, is to be applied when the piece is *always subjected to stress of one kind, or in one direction only.* It agrees well with some experiments by Wöhler on Krupp's untempered cast spring steel.

If the stress in any piece varies from one kind to another, as from tension to compression, or *vice versa;* or from one direction to another, as in torsion on each side of a state of no stress, Weyrauch has established the following formula by a course of reasoning similar to that used by Launhardt.

If the opposite stresses, which will cause rupture by a certain number of applications, are equal in intensity, and if that intensity is represented by S, then will S be called the "vibration resistance"; this was established by Wöhler for some cases, and some of its values are given in the preceding Article.

Let $+P$ and $-P'$ represent two intensities of opposite kinds or in opposite directions, of which P *is numerically the greater.* Then if $D = P + P'$:

$$P = D - P'.$$

The two following limiting conditions will hold:

$$\text{For } P' = 0; \quad P = D = W;$$

$$\text{For } P' = S; \quad P = S = \tfrac{1}{2}D.$$

But by Wöhler's law, $P = f(D)$, and the two limiting conditions just given will be found to be satisfied by the provisional formula:

$$P = \frac{W - S}{2W - S - P} D = \frac{W - S}{2W - S - P} (P + P') \quad . \quad (6)$$

By the solution of Eq. (6):

$$P = W\left(1 - \frac{W-S}{W} \cdot \frac{P'}{P}\right) \quad . \quad . \quad . \quad . \quad (7)$$

If, without regard to kind or direction, *max B* is numerically the greatest total stress which the piece has to carry, while *max B'* is the greatest total stress of the other kind or direction, then will $\frac{P'}{P} = \frac{max\ B'}{max\ B}$. Hence, there will result the following, which is the formula of Weyrauch:

$$P = W\left(1 - \frac{W-S}{W}\ \frac{max\ B'}{max\ B}\right) \quad . \quad . \quad . \quad . \quad (8)$$

Eqs. (5) and (8) give values of the intensity P which are to be used in determining the cross section of pieces designed to carry given amounts of stress. If n is the safety factor and F the total stress to be carried, the area of section desired will be:

$$A = \frac{nF}{P};$$

in which $\frac{P}{n}$ is the greatest working stress permitted.

If for wrought iron in tension $W = 30{,}000$ and $R = 50{,}000$, Eq. (5) gives:

$$P = 30{,}000\left(1 + \frac{2}{3}\ \frac{min\ B}{max\ B}\right).$$

Hence, if the total stress due to fixed and moving loads in the web member of a truss is $max\ B = 80{,}000$ pounds, while that due to the fixed load alone is $min\ B = 40{,}000$, there will result:

$$P = 30{,}000\left(1 + \frac{2}{3} \cdot \frac{40{,}000}{80{,}000}\right) = 40{,}000.$$

In such a case the greatest permissible working stress with a safety factor of 3 would be about 13,300 pounds.

For steel in tension, if $W = 50{,}000$ and $R = 75{,}000$:

$$P = 50{,}000 \left(1 + \frac{1}{2} \frac{min\ B}{max\ B}\right).$$

For wrought iron in torsion, if $S = 18{,}000$ and $W = 24{,}000$, Eq. (8) will give:

$$P = 24{,}000 \left(1 - \frac{1}{4} \frac{max\ B'}{max\ B}\right).$$

Other methods based on Wöhler's experiments have been deduced by Müller, Gerber and Schäffer, of which synopses may be found in Du Bois' translation of Weyrauch's "Structures of Iron and Steel."

Art. 99.—Influence of Time on Strains.

In the section "*elevation of ultimate resistance and elastic limit*," in Art. 32, the effect of prolonged tensile stress and subsequent rest between the elastic limit and ultimate resistance, was shown to be the elevation of both those quantities. It is a matter of common observation, however, that if a piece of wrought iron be subjected to a tensile stress nearly equal to its ultimate resistance, and held in that condition, that the stretch will increase as the time elapses.

Experiments are still lacking which may show that a piece of metal can be ruptured by a tensile stress much below its ultimate resistance. It may be indirectly inferred, however, from experiments on flexure, that such failure may be produced, as the following by Prof. Thurston will show.

A bar 10 parts tin and 90 parts copper, 1 × 1 × 22 inches and supported at each end, sustained about 65 per cent. of its breaking load at the centre for five minutes. During that time its deflection increased 0.021 inch. The same bar sustained 1,485 pounds at centre for 13 minutes and then failed.

A second bar of the same size, but 90 parts tin and 10 parts copper, was loaded at the centre with 160 pounds, causing a deflection of 1.294 inches. After 10 minutes the deflection had increased 0.025 inch; after one day, 1.00 inch; after two days, 2.00 inches; and after three days, 3.00 inches, when the bar failed under the load of 160 pounds.

Another bar of the same size showed remarkable results; it was composed of 90 parts zinc and 10 parts copper. It gave the same general increase of deflection with time, but eventually broke under a centre load which ran down from 1,233 to 911 pounds, after holding the latter about three minutes.

A bar of the same size and 96 parts copper with 4 parts tin, after it had carried 700 pounds at centre for sixty minutes was loaded with 1,000 pounds, with the following results:

AFTER	DEFLECTION.
0 minute	3.118 inches.
5 minutes	3.540 "
15 minutes	3.660 "
45 minutes	4.102 "
75 minutes	7.634 "
Broke under 1,000 pounds.	

A wrought-iron bar of the same size gave, under a centre load of 1,600 pounds:

AFTER	DEFLECTION.
0 minute	0.489 inch.
3 minutes	0.632 "
6 minutes	0.650 "
16 minutes	0.660 "
344 minutes	0.660 "

It subsequently carried 2,589 pounds with a deflection of 4.67 inches.

During 1875 and 1876 Prof. Thurston made a number of other similar experiments with the same general results.

Metals like tin and many of its alloys showed an increasing rate of deflection and final failure, far below the so-called "ultimate resistance." The wrought-iron bars, however, showed a decreasing increment of deflection, which finally became zero, leaving the deflection constant.

Whether there may be a point for every metal, beyond which, with a given load, the increment of deflection may retain its value or go on increasing until failure, and below which this increment decreases as the time elapses, and finally becomes zero, is yet undetermined, but seems probable.

It does not follow, therefore, that the principle enunciated in the section named at the beginning of this Article, is to be taken without qualification. If "rest" under stress, too near the ultimate resistance, be sufficiently prolonged, it has been seen that it is possible that failure may take place.

In verifying some experimental results by Herman Haupt, determined over forty years ago, Prof. Thurston tested three seasoned pine beams about 1⅛ inches square and 40 inches length of span, and found that 60 per cent. of the ordinary "breaking load" caused failure at the end of 8, 12 and 15 months. In these cases the deflection slowly and steadily increased during the periods named.

Two other sets of three pine beams each, broke under 80 and 95 per cent. of the usual "breaking load," after much shorter intervals of time.

In all these instances it is evident that the molecules under the greatest stress "flow" over each other to a greater or less extent. In the cases of decreasing increments of strain, the new positions afford capacity of increased resistance; in the others, those movements are so great that the distances between some of the molecules exceed the reach of molecular action, and failure follows.

In many cases strained portions of material recover partially or wholly from permanent set. In such cases a portion of the material has been subjected to intensities of stress high enough to produce true "flow" of the molecules, while the remaining portion has not. The internal elastic stresses in the latter portion, after the removal of the external forces, produce in time a reverse flow in consequence of the elastic endeavor to resume the original shape.

It is altogether probable that the phenomena of fatigue and flow of metals are very intimately associated. Some of the prominent characteristics of the latter will be given in the next chapter.

CHAPTER XIV.

THE FLOW OF SOLIDS.

Art. 100.—General Statements.

ALTHOUGH there is no reason to suppose that true solids may not retain a definite shape for an indefinite length of time if subjected to no external force other than gravity,* many phenomena resulting both from direct experiment for the purpose, and incidentally from other experiments involving the application of external stress of considerable intensity, show that a proper intensity of internal stress (in many cases comparatively low) will cause the molecules of a solid to flow, at ordinary temperatures, like those of a liquid. And this flow, moreover, is entirely different from, and independent of, the elastic properties of the material; for it arises from a permanent and considerable relative displacement of the molecules. Nor is it to be confounded with that internal "friction" which, if an elastic body is subjected to oscillations, causes the amplitudes to gradually decrease and finally disappear, even in vacuo. This latter motion is typically elastic and the retarding cause may be considered a kind of elastic friction.

It is evident that if a mass of material be enclosed on all its faces, or outer surfaces, but one or a portion of one, and if external pressure be brought to bear on those faces, the material will be forced to move to and through the free surface; in

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* This, perhaps, may be considered a definition of a true solid.

other words, *the flow of the material will take place in the direction of least resistance.*

Fig. 1

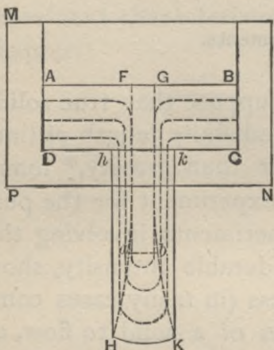


Fig. 2

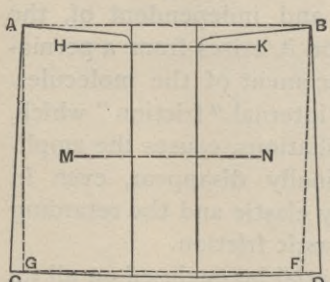


Fig. 3

The theory of the flow of solids to be given is that developed by Mons. H. Tresca in his "Mémoire sur l'Écoulement des Corps Solides," 1865. He made a large number of experiments on hard and soft metals, ceramic pastes, sand and shot.

These different materials all manifested the same characteristics of flow, which are well shown in Fig. 2. *ABCD*, Fig. 1, is supposed to be a cylindrical mass of lead with circular horizontal section, confined in a circular cylinder, *MN*, closed at one end with the exception of the orifice *O*.

This cylinder is supported on the base *PN*, while the face *AB* of the lead receives external pressure from a close-fitting piston. When the pressure is sufficiently increased, the face *AB* in Fig. 1 sinks to *AB* in Fig. 2, while the column *hkHK*, in the latter figure, is forced to flow through the orifice *O*.

In Tresca's experiments with lead, the diameter *AB* was about 3.9 inches; the diameter *HK* of the orifice, from 0.75 in. to 1.5 ins., while the length of the column or jet *hK* varied from 0.4 in. to about 24 ins. The total pressure on the face *AB* varied from 119,000 to 198,000 pounds. The initial thickness *AD* varied from 0.24 inch to 2.4 inches.

Some experiments exhibiting in a remarkably clear manner the flow of metals in cold punching were made by David Townsend in 1878, and the results were given by him in the "Journal of the Franklin Institute" for March of that year. If the dotted rectangle *ABFG*, Fig. 3, shows the original outline of the middle section of a nut before punching, he found that the final outline of the same section would be represented by the full lines. The top and bottom faces were depressed by the punching, as shown; the upper width *AB* remained about the same, but the lower, *GF*, was increased to *CD*. Although the depth of the nut, *AC*, was 1.75 inches, the length of the core punched out was only 1.063 inches. The density of this core was then examined and found to be the same as that of the original nut. Hence a portion of the core equal in length to $1.75 - 1.063 = 0.687$ inch was forced, or flowed, back into the body of the nut. Subsequent experiments showed that this flow did not take place at the immediate upper surface *AB*, nor very much in the lower half of the nut, but that it was chiefly confined to a zone equal in depth to about half that of the nut, the upper surface of which lies a very short distance below the upper face of the nut. The location of this zone is shown by the lines *HK* and *MN* in Fig. 3.

Tresca's experiments on punching showed essentially the same result.

Art. 101.—Tresca's Hypotheses.

The central cylinder *FGKH*, Fig. 1 of Art. 100 was called by Tresca the "primitive central cylinder." As the metal flows, this cylinder will be drawn out into the volume of revolution, whose axis is that of the orifice and whose meridian section is *FGkKHh*, Fig. 2, the diameter *FG* being gradually decreased.

It was found by experiment that if the original mass *AC*,

Fig. 1, was composed of horizontal layers of uniform thickness, the reduced mass in Fig. 2 was also composed of the same number of layers of uniform thickness, except in the immediate vicinity of the central cylinder.

Tresca then assumed these three hypotheses :

1°.—*The density of the material remains the same whether in the cylinder or in the jet;* in other words, the volume of the material in the jet and in the cylinder remains constant.

Let R = radius of the cylinder.
Let R_1 = radius of the orifice.
Let y = variable length of the jet (*i. e.*, hH).
Let D = original depth of material ($BC = AD$, Fig. 1) in the cylinder.
Let d = variable depth of material ($BC = AD$, Fig. 2) in the cylinder.

Then by the hypothesis just stated :

$$R^2d = R^2D - R_1^2y \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

2°. *The rate of compression along any and all lines parallel to the axis of the primitive central cylinder, and taken outside of that limit, is constant.*

If, then, the material lying outside of the central cylinder be divided into horizontal layers of equal thickness, a very small decrease in the variable depth equal to $d(a)$ will cause the same amount of material to move or flow from each of these layers into the space originally occupied by the central cylinder, thus causing a portion of the material previously resting over the orifice to flow through the latter. If $d(d)$ is the indefinitely small change of depth, and dR_1 the indefinitely small change in the radius of the cylindrical portion resting over the orifice, then the equality of volumes expressing this hypothesis is the following :

$$\pi(R^2 - R_1^2) \;.\; d(d) = 2\pi R_1 d \;.\; dR_1 ;$$

or :

$$\frac{d(d)}{d} = \frac{2R_1\, dR_1}{R^2 - R_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

3°.—*The rate of decrease of the radius of the primitive central cylinder is constant throughout its length at any given instant during flow.*

Let r be any radius less than R_1, then if the latter is decreased by the very small amount dR_1, the former will be shortened by the amount dr; and by the last hypothesis there must result :

$$\frac{dR_1}{R_1} = \frac{dr}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This is a perfectly general equation, in which r may or may not be the variable value of the radius of that portion of the primitive central cylinder remaining above the orifice at any instant during flow.

These are the three hypotheses on which Tresca based his theory of the flow of solids. It is thus seen to be put upon a purely geometrical basis, entirely independent of the elastic or other properties of the material.

Art. 102.—The Variable Meridian Section of the Primitive Central Cylinder.

The meridian curve *haH*, or *hbK*, Fig. 2 of Art. 100, may now easily be determined.

Eq. (1) of Art. 101 may take the first of the following forms, while its differential, considering d and y variable, may take the second :

$$d = D - \frac{R_1^2}{R^2} y.$$

$$d(d) = - \frac{R_1^2}{R_2} dy.$$

Dividing the second by the first :

$$\frac{d(d)}{d} = \frac{dy}{y - \frac{R^2}{R_1^2} D} = \frac{2R_1 \, dR_1}{R^2 - R_1^2}.$$

The last member of this equation is simply Eq. (2) of Art. 101 ; and if the value of dR_1, in Eq. (3) of the same Article, be inserted in the third member of this equation, there will result :

$$\frac{2R_1^2}{R^2 - R_1^2} \cdot \frac{dr}{r} = \frac{dy}{y - \frac{R^2}{R_1^2} D}.$$

Integrating between the limits of r and R_1, and remembering that r will be restricted to the representation of the radius of that portion of the primitive central cylinder which remains, at any instant, over the orifice, by taking $y = 0$ for $r = R_1$:

$$\frac{2R_1^2}{R^2 - R_1^2} \log \frac{r}{R_1} = \log \left(\frac{y - \frac{R^2}{R_1^2} D}{- \frac{R^2}{R_1^2} D} \right);$$

"*log*" indicates a Napierian logarithm.

Passing from logarithms to the quantities themselves, and reducing :

$$y = \frac{R^2 D}{R_1^2}\left[1 - \left(\frac{r}{R_1}\right)^{\frac{2R_1^2}{R^2 - R_1^2}}\right] \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is the desired equation of the line, in which r is measured normal to the axis of the cylinder or jet, while y is measured along that axis from the extremity of the jet. When the material is wholly expelled :

$$y = \frac{R^2}{R_1^2} D, \text{ and } \quad r = 0.$$

Eq. (2) is applicable to the jet only. For the line hF or Gk, resort will had to the equation :

$$\frac{d(d)}{d} = \frac{2R_1^2}{R^2 - R_1^2}\,\frac{dr}{r}.$$

Again integrating between the limits d and D, or r and R_1, and reducing :

$$r = R_1\left(\frac{d}{D}\right)^{\frac{R^2 - R_1^2}{2R_1^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This value of r is the radius of that portion of the primitive central cylinder which remains over the orifice when D is reduced to d.

Art. 103.—Positions in the Jet of Horizontal Sections of the Primitive Central Cylinder.

That portion of the primitive central cylinder below ab in Fig. 1 of Art. 100, will be changed to $abKH$ in Fig. 2 of the same Article.

If, in the latter Fig., y' is the distance from HK to ab, measured along the axis, then the volume of *HKab* will have the value

$$\int_0^{y'} \pi r^2\, dy.$$

If d' is the distance $aF = bG$, in Fig. 1, the equality of volumes will give :

$$\int_0^{y'} r^2\, dy = R_1^2(D - d').$$

Eq. (1) of Art. 102 gives :

$$r^2 = R_1^2 \left(1 - \frac{R_1^2 y}{R^2 D}\right)^{\frac{R^2 - R_1^2}{R_1^2}}.$$

$$\therefore \quad \int_0 r^2\, dy = R_1^2 D - R_1^2 D\left(1 - \frac{R_1^2 y'}{R^2 D}\right)^{\frac{R^2}{R_1^2}} = R_1^2(D - d').$$

$$\therefore \quad y' = \frac{R^2}{R_1^2}\left[1 - \left(\frac{d'}{D}\right)^{\frac{R_1^2}{R^2}}\right]D \quad . \quad . \quad . \quad . \quad (1)$$

If N is the number of horizontal layers required to compose the total thickness D, and n the number in the depth d' :

$$\frac{d'}{D} = \frac{n}{N}.$$

Hence :

$$y' = \frac{R^2}{R_1^2}\left[1 - \left(\frac{n}{N}\right)^{\frac{R_1^2}{R^2}}\right]D \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Tresca computed values of y' for some of his experiments, and compared the results with actual measurements. The agreement, though not exact, was very satisfactory. Within limits not extreme, the longer the jet the more satisfactory was the agreement.

Art. 104.—Final Radius of a Horizontal Section of the Primitive Central Cylinder.

Let it be required to determine what radius the section situated at the distance d' from the upper surface of the primitive central cylinder will possess in the jet.

It will only be necessary to put for y in Eq. (1) of Art. 102, the value of y' taken from Eq. (1) of Art. 103. This operation gives :

$$\left(\frac{d'}{D}\right)^{\frac{R_1^2}{R^2}} = \left(\frac{r'}{R_1}\right)^{\frac{2R_1^2}{R^2 - R_1^2}}.$$

Hence :

$$r' = R_1 \left(\frac{d'}{D}\right)^{\frac{R^2 - R_1^2}{2R^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If R_1 is small, as compared with R, there will result approximately :

$$r' = R_1 \left(\frac{d'}{D}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Art. 105.—Path of any Molecule.

The hypotheses on which the theory of flow is based enable the hypothetical path of any molecule to be easily established.

In consequence of the nature of the motion there will be three portions of the path, each of which will be represented by its characteristic equation, as follows :

First : *let the molecule lie outside of the primitive central cylinder.*

Let R' and H be the original co-ordinates of the molecule considered, measured normal to and along the axis of the cylinder, respectively, from the centre of the orifice HK (Fig. 1 Art. 100) as an origin, while r and h are the variable co-ordinates.

The first hypothesis, by which the density remains constant, then gives the following equation :

$$\pi(R^2 - R'^2)H = \pi(R^2 - r^2)h ;$$

or :

$$hR^2 - hr^2 = (R^2 - R'^2)H \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is the equation to the path of the molecule, in which r must always exceed R_1.

As this equation is of the third degree, the curve cannot be one of the conic sections.

Second : *let the molecule move in the space originally occupied by the central cylinder.*

While h and r now vary, the volume $\pi r^2(D - h)$ must remain constant. When $r = R_1$ let $h = h_1$. Hence :

$$r^2(D - h) = R_1^2(D - h_1) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But if $h = h_1$ and $r = R_1$ in Eq. (1) :

$$h_1 = \left(\frac{R^2 - R'^2}{R^2 - R_1^2}\right)H.$$

Placing this value in Eq. (2) :

$$r^2(D - h) = R_1^2\left(D - H\,\frac{R^2 - R'^2}{R^2 - R_1^2}\right) . \quad . \quad . \quad . \quad (3)$$

Third : *let the molecule move in the jet.*

After the molecule passes the orifice, its path will evidently be a straight line parallel to the axis of the jet. Its distance r_1 from that axis will be found by putting $h = 0$ in Eq. (3). Hence :

$$r_1 = R_1 \left(1 - \frac{H}{D} \frac{R^2 - R'^2}{R^2 - R_1^2}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

ADDENDA.

Addendum to Art. 34.

Both Tables in this Addendum show tests of steel used in the St. John Cantilever covered by the specifications of Art. 90. It will be noticed that this steel is of a very mild character, but very ductile, uniform and reliable. Table I. shows tests of specimens from a great variety of bars and shapes, while Table II. gives the results of tests of full size eye bars. With one or two exceptions, the tests of the latter were suspended before failure took place, which accounts for the incomplete record. Both tables are from London "Engineering," of Oct., 1886.

TABLE I.

Test Specimens of Steel.

ORIGINAL FORM OF MATERIAL.	SECTION OF TEST-PIECE.	STRENGTH PER SQUARE INCH.		PER CENT. OF ELONGATION IN 8 INS.	PER CENT. OF REDUCTION AT FRACTURE.
		ELASTIC.	ULTIMATE.		
In.	*Sq.In.*	*Pounds.*	*Pounds.*		
$1\frac{1}{16}$ square steel bar.	1.106	33,092	55,913	27.5	65.3
$1\frac{7}{16}$ " "	2.014	32,373	58,212	32.4	60.3
$2\frac{5}{16}$ " "	1.833	30,000	63,448	17.1	62.0
1 round "	0.773	34,800	56,494	29.6	67.0
$1\frac{1}{4}$ " "	1.206	34,992	56,981	21.0	64.4
$2\frac{1}{4} \times \frac{5}{16}$ flat "	0.683	36,896	55,637	28.1	61.9
$6 \times \frac{3}{8}$ " "	0.888	35,135	58,559	31.9	51.4
$9 \times 1\frac{1}{4}$ " "	1.445	33,010	58,761	35.9	59.6
$9 \times 1\frac{1}{4}$ " "	1.353	32,225	58,558	31.9	58.6
$1\frac{7}{16} \times$ square "	1.990	36,482	58,492	....	55.8
$\frac{27}{32}$ rivet "	0.541	29,944	61,737	26.9	66.5
$\frac{23}{32}$ " "	0.405	37,284	66,222	26.9	62.5
$\frac{7}{8}$ " "	0.558	42,652	67,348	29.6	49.3
$\frac{7}{8}$ " "	0.544	40,441	66,838	26.7	54.8

TABLE I.—*Continued.*

ORIGINAL FORM OF MATERIAL.	SECTION OF TEST-PIECE.	STRENGTH PER SQUARE INCH.		PER CENT. OF ELONGATION IN 8 INS.	PER CENT. OF REDUCTION AT FRACTURE.
		ELASTIC.	ULTIMATE.		
$42 \times \frac{5}{16}$ steel plate.	0.568	....	65,446	18.7	88.4
$42 \times \frac{5}{16}$ "	0.603	34,826	62,189	25.0	48.6
$11\frac{1}{2} \times \frac{3}{8}$ "	0.788	39,344	60,285	29.5	47.6
$11\frac{1}{2} \times \frac{3}{8}$ "	0.980	35,220	60,526	28.8	21.4
$21 \times \frac{1}{2}$ "	1.215	33,498	59,745	33.0	52.3
$21 \times \frac{1}{2}$ "	1.227	39,120	60,456	23.9	33.5
$20 \times \frac{3}{8}$ "	0.908	46,806	65,462	27.1	48.7
$5\frac{1}{2} \times 3\frac{1}{2} \times \frac{5}{8}$ steel angle.	1.232	36,556	61,841	14.0	50.0
$5\frac{1}{2} \times 3\frac{1}{2} \times \frac{1}{2}$ "	1.050	43,801	63,473	16.9	46.6
$5\frac{1}{2} \times 3\frac{1}{2} \times \frac{13}{32}$ "	0.833	46,458	69,900	28.1	51.3
$5\frac{1}{2} \times 3\frac{1}{2} \times \frac{3}{8}$ "	0.990	....	56,303	30.7	52.7
$5\frac{3}{4} \times 2\frac{3}{8} \times \frac{3}{8}$ steel channels.	0.583	39,623	60,703	28.6	55.1
$5\frac{3}{4} \times 2\frac{3}{8} \times \frac{3}{8}$ "	0.754	39,125	61,857	29.8	55.3
$9 \times 2\frac{1}{2} \times \frac{3}{8}$ "	0.866	40,185	64,457	22.4	44.0
$11\frac{7}{8} \times 3 \times \frac{3}{8}$ "	1.072	33,862	54,011	33.0	60.2
$11\frac{7}{8} \times 3 \times \frac{3}{8}$ "	0.994	36,720	58 330	22.6	34.6
$11\frac{7}{8} \times 3 \times \frac{3}{8}$ "	0.973	36,382	57,605	18.3	25.2
$15\frac{3}{4} \times 3 \times \frac{3}{8}$ "	1.140	38,158	65,939	35.5	51.8
$15\frac{3}{4} \times 3 \times \frac{3}{8}$ "	0.572	36,699	59,418	23.9	44.7
$15\frac{3}{4} \times 3 \times \frac{3}{8}$ "	0.800	37,625	59 625	30.0	46.0
$15\frac{3}{4} \times 3 \times \frac{3}{8}$ "	0.629	31,804	60,111	23.7	32.0
Totals		1,215,138	2,135,882	909.0	1.779.4
Average		36,810	61,025	26.7	50.8

TABLE II.

Steel Eye Bar Tests.

DIMENSIONS OF BAR.		SIZE OF BAR HEADS.			STRAINS PER SQ. IN.		GREATEST REDUCTION.
Size.	Length.	Diameter.	Thickness.	Pin-Hole.	Elastic.	Maximum.	
In. *In.*	*Ft.* *In.*	*In.*	*In.*	*In.*			
5.24 × 0.98	16. 3.9	9.00	1.02	3.93	37,350	54,690	
7.10 × 0.94	16 0.1	13.77	1.18	5.46	34,090	56,510	16.2 per cent.
7.14 × 0.94	16 0.1	13.60	1.30	5.44	35,010	56,790	22.2 "
8.13 × 0.98	16 0.1	15.00	1.26	5.92	32,120	55,530	45.4 "
8.06 × 0.96	16 0.1	16.05	1.16	5.92	33,070	56,650	34.2 "
8.05 × 0.99	16 0.1	18.00	1.02	5.94	32,870	53,640	32.5 "
9.79 × 0.98	16 0.1	17.95	1.22	5.92	34,390	54,350	32.5 "
10.01 × 1.03	16 0.1	18.00	1.22	5.94	32,780	53,000	33.5 "
		Total			271,680		
		Average			33,960		

Addendum to Art. 65.

In the autumn of 1883 an extensive series of tests of wrought iron eye beams, subjected to bending by centre loads, was made by G. H. Elmore, C. E., and the writer, at the mechanical laboratory of the Rens. Pol. Inst. The object of these tests was to discover, if possible, the law connecting the value of K for this class of beams with the length of span when the beam is *entirely without lateral support.* The means by which the latter end was accomplished, and a full detailed account of the tests will be found in volume I., No. 1, "Selected Papers of the Rensselaer Society of Engineers." The main results of the tests are given in Table III. All the tests were made on 6 inch eye beams with same area of normal cross section of 4.35

TABLE III.

NO.	SPAN. FEET.	FINAL CENTRE WEIGHT. POUNDS.	$\frac{l}{r}$	K Elas. Lim. Pounds.	K Ultimate. Pounds.	PERMANENT VERTICAL DEFLECTION. INCHES.	PERMANENT LATERAL DEEFLECTION. INCHES.	E POUNDS PER SQ. IN.
1	20	4,060	400	27,726	31,094	0.14		24,170,000
2		4,200	400	29,623	32,885	0.30		26,374,000
3	18	4,390	360	28,264	30.791	0.2	0.5	24,520,000
4		4,570	360	28,264	32,020	0.18	0.4	24,313,000
5	16	4,770	320	26,564	29,579	0.28	1.00	25,771,000
6		5,270	320	29,596	32,632	0.48	1.25	25,003,000
7	14	6,130	280	31,191	33,049	0.30	1.20	26,082,000
8		6,125	280	31,164	33,023	0.30	1.10	23,373,000
9	12	7,161	240	30,221	32,907	0.35	1.08	25,287,000
10		7,350	240	31,314	33.817	0.33	1.09	24,022,000
11	10	9,255	200	33,082	35,358	0.39	1.08	25,115,000
12		9,655	200	33,082	37,064	0.50	1.50	24,218,000
13	8	11,485	160	29,736	35,010	0.30	0.90	21,611,000
14		11,980	160	31,936	36.527	0.29	1.05	21,987,000
15	6	18,300	120	35,497	41,737	0.605	1.53	23,040,000
16		18,145	120	36,617	41,396	0.67	1.88	20,935,000
17	5	22,870	100	34,136	43,434	0.67	1.75	22,023,000
18		23,065	100	34,136	43,813	0.67	1.75	25,272,000
19	4	29,985	80	32,619	45.532	0.96	1.70	24,315,000
20		28,585	80	32,619	44,744	0.60	1.86	21,275,000

square inches. Actual measurement showed the depth d of the beams to be 6.16 inches. The moment of inertia of the beam section about a line through its centre and normal to the web was $I = 24.336$. The radius of gyration of the same section in reference to a line through its centre and *parallel* to the web was $r = 0.6$ inches. l was the length of span in inches.

If M is the bending moment in inch-pounds, W the total centre load (including weight of beam), and K the stress per square inch in extreme fibre, the following formulæ result:

$$K = \frac{Md}{2I} \text{ and } M = \frac{Wl}{4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$\therefore \qquad K = \frac{Wld}{8I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

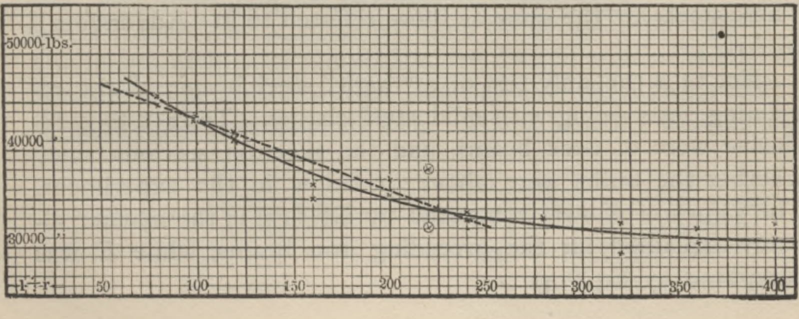
The experimental values of W, l, d and I inserted in the above formula give the values of K shown in the table. The coefficient of elasticity, E, was found by the usual formula:

$$E = \frac{Wl^3}{48wI} \; ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

in which w is the deflection caused by W.

The full line is the graphical representation of the values of K given in Table III. Since K must clearly decrease with the

Plate I.



length of span, and increase with the radius of gyration of the section about an axis through its centre and parallel to the web (the latter, of course, being vertical), K has been plotted in reference to $l \div r$ as shown. No simple formula will closely represent this curve, but the broken line covers all lengths of span used in ordinary engineering practice, and is represented by the formula:

$$K = 51{,}000 - 75 \frac{l}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

For railway structures the greatest allowable stress per sq. in. in the extreme fibres of rolled beams may be taken at:

$$k = 10{,}000 - 15 \frac{l}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Values of k taken from a large scale plate, like Plate I., are, however, far preferable to those given by any formula.

Rolled Steel Beams.

The researches of Mr. James Christie, Supt. of the Pencoyd Iron Co., on the transverse or bending resistance of rolled deck and eye beams, are given in great detail in the "Trans. Am. Soc. C. E." for 1884, from which the results in Table IV. are abstracted. All beams were rolled at the Pencoyd Iron Works. The "mild steel" contained from 0.11 to 0.15 per cent. of carbon, and the "high steel" about 0.36 per cent. of carbon. These steels are the same as those referred to in "Addendum" to Art. 51.

No. 14 is the only test of a "high" steel beam; all the remaining tests being with mild steel shapes. Tests 3 to 9 inclusive were of deck beams, and the skeleton sections show whether the bulbs were above or below in the experiment. The values of K were computed by Eq. (13) in the manner already explained for the preceding iron beams, while E was computed by Eq. (14).

TABLE IV.

Transverse Tests of Steel Beams.

NO.	KIND OF BEAM.	SPAN IN INCHES.	$\frac{l}{r}$	MOMENT OF INERTIA.	FINAL CENTRE LOAD IN POUNDS.	K IN POUNDS PER SQUARE INCH AT		COEFFICIENT OF ELASTICITY E, IN POUNDS PER SQ. IN.
						Elastic.	Ultimate.	
1	Mild 3″ I	59	100	2.76	5,500	41,100	45,200	30,890,000
2	" 3″ "	39	66	2.76	8,300	40,800	45,100	25,011,000
3	" 5″ ⊥	108	200	12	8,800	50,000	55,000	27,718,000
4	" 5″ ⊤	108	200	12	8,400	46.900	52,500	25.489,000
5	" 6″ ⊥	96	152	22	14,860	51,200	54,300	23,692,000
6	" 7″ "	69	97	37.6	34,000	47,100	59,300	18.765,000
7	" 7″ ⊤	69	97	37.6	34,000	47,100	59,300	23,040,000
8	" 9″ ⊥	240	290	84.8	14,500	46,000	51,300	29,923,000
9	" 9″ ⊤	240	290	82.9	13,500	39,800	48,800	30,209,000
10	" 8″ I	240	273	70.2	13.000	37,600	44,400	28.889,000
11	" 8″ "	240	273	70.3	12,930	37,500	44,100	29,055,000
12	" 8″ "	144	164	70.2	19,480	32,800	39,900	31,313,000
13	" 8″ "	96	109	70.2	31,300	40,300	42,800	23,689,000
14	High 3″ I	39		2.74	11,500	54,300		27,515,000
15	Mild 10″ "	156	164	150.5	22,500	35,000		28,414,000
16	" 10″ "	168	177	150.5	21,000	35.200		27,182,000
17	" 10″ "	180	189	150.5	19,500	35,000		29,160,000
18	" 10″ "	192	202	150.5	18,000	34,400		29,727,000
19	" 12″ "	240	238	264.7	24,500	33,400		30,749,000
20	" 12″ "	240	238	267.6	24,200	32,500		29,568,000
21	" 12″ "	228	226	273.8	22,000	27,500		29,164,000
22	" 12″ "	216	214	263.7	29,000	35,600		30,219,000
23	" 12″ "	204	202	256.7	27,000	32,100		30,030,000
24	" 12″ "	192	190	257.8	34,000	38,000		29,709,000
25	" 12″ "	192	190	262.6	34,000	37,300		28,234,000
26	" 12″ "	180	178	262.4	36,700	37,700		27,717,000
27	" 12″ "	168	166	264.0	38,000	36,300		28,784,000
28	" 12″ "	156	154	261.7	43,000	38,400		27,818,000

Beams 3 and 4 were rolled from the same ingot; as were also 6 and 7; as were also 10, 12 and 13, and, as were also 16, 17, 18 and 19. All beams were broken by a centre load, and they were unsupported laterally in either flange. The moments of inertia were computed from the actual beam sections. The length of span is represented by l while r is the radius of gyration of each beam section about an axis through its centre of gravity and parallel to its web. The values of r were as follows:

5 inch ⊥ .... $r = 0.54$ inch.	3 inch I .... $r = 0.59$ inch.
6 " " .... $r = 0.63$ "	8 " " .... $r = 0.88$ "
7 " " .... $r = 0.71$ "	10 " " .... $r = 0.95$ "
9 " " .... $r = 0.83$ "	12 " " .... $r = 1.01$ "

The values of K, both for the elastic limit and ultimate, are so erratic in relation to those of $l \div r$ that no law such as is revealed by Table III. and Plate I. for iron beams, can be discovered. The most marked feature of the Table IV. is the very considerable excess of ultimate and elastic K for deck beams over the same quantities for eye beams. These tests, however, show that for mild structural steel, containing 0.18 or 0.20 per cent. carbon, the working stress in the extreme fibres of eye beams may be taken at 10,000 pounds per sq. in. in railway structures when the length of span does not exceed $150r$, and when the resistance is computed by the exact formula with the moment of inertia.

Addendum to Art. 70.

In "The American Engineer" for March 14th and 21st, 1884, is given a detailed account of some valuable tests of wrought iron and steel built beams. These tests were made under the supervision of Mr. C. L. Strobel, C. E., who gave the data cited. All these girders had the same span of 12 feet, and they were tested in a vertical or natural position. All web plates were nominally 14 inches deep and $\frac{1}{4}$ inch thick, and the flange angles were all 3 inches by 3 inches by nominally $\frac{5}{16}$ inch thick. All these beams were broken by a load applied at the centre, at which point vertical stiffening angles were riveted to the web. Five beams were thus tested to destruction with the results given in the following Table.

TESTS OF PLATE GIRDERS BY C. L. STROBEL, C. E.

12 ft. span, 14 inches depth, 14″ × ¼″ web. 3″ × 3″ × 5/16″ flange angles.

BEAM.	CENTRE LOAD.	FLANGE STRESS PER SQ. IN. AT FAILURE.		MODE OF FAILURE.
		Upper.	Lower.	
Wrought Iron.	63,440 lbs. Punched rivet holes.	48,400 lbs.	53,000 lbs.	Tearing lower flange.
Hard Steel 0.34 C.	110,500 lbs. Punched rivet holes.	79,500 "	87,100 "	" " "
" "	115,700 lbs. Punched and reamed.	80,900 "	88,700 "	Buckling upper "
Soft Steel 0.11 C.	76,700 lbs. Punched and reamed.	56,600 "	62,000 "	" " "
" "	81,900 lbs. Punched rivet holes.	59,300 "	65,000 "	Tearing lower "

The flange stress per sq. in. at failure given above was computed in the extreme fibres by the exact formula from the moment of inertia of the entire section less the rivet holes taken out in the tension flange. The chemical analyses of the two steels were as follows:

	C.	Mn.	Si.	S.	P.
Soft steel........................	.11	.396	.019	.046	.088
Hard "	.34	.954	.063	.113	.175

Test specimens from the hard steel gave ultimate resistances from 94,300 to 107,700 lbs. per sq. in.; elastic limit from 59,300 to 63,300 lbs. per sq. in.; stretch from 14 to 26 per cent.; contraction, 17 to 38 per cent. The corresponding quantities for the soft steel were—ultimate from 60,400 to 62,700 lbs. per sq. in.; elastic limit, from 40,200 to 49,700 lbs. per sq. in.; stretch, from 21 to 28 per cent.; contraction, from 55 to 60 per cent.

These tests show that, for ordinary railway practice, with such mild structural steel as is generally used for plates and angles, the working stresses in plate girder stringers and floor beams may be taken at 10,000 lbs. per sq. in. gross section of compression flange or net section of tension flange. For the same members in wrought iron, the corresponding value would be 8,000 pounds per sq. in.

Addendum to Art. 73.

A butt joint with a set of single or double cover plates or butt straps may be formed in such a manner that the rivets and cover plates will take very nearly or exactly their proper proportional loads. Each set of cover plates is composed of a series uniformly decreasing in length, the longest of the series lying adjacent to the main plates or members joined. One row of rivets parallel to the joint is then put through each end of each cover plate, and, of course, also through those lying underneath. In this manner the number of rivets from the end of the longest or lowest cover plate to any section parallel to the joint is proportional to the sectional area of the covers against which they pull; the joint is consequently of nearly uniform resistance.

The number of butt straps or cover plates in a set depends upon the size of the members joined.

In most cases the rivets cannot take exactly their proportional loads, for the reason that those portions of the members joined which lie within the limits of the joint are not of uniform resistance, as the system of covers is.

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