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A TEXT-BOOK

ON THE

MECHANICS OF MATERIALS

AND OF

BEAMS, COLUMNS, AND SHAFTS.

BY

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PREFACE.

The following pages contain an elementary course of study in the resistance of materials and the mechanics of beams, columns and shafts, designed for the use of classes in technical schools and colleges. It should be preceded by a good training in mathematics and theoretical mechanics, and be followed by a special study of the properties of different qualities of materials, and by detailed exercises in construction and design.

As the plan of the book is to deal mainly with the mechanics of the subject, extended tables of the results of tests on different kinds and qualities of materials are not given. The attempt, however, has been made to state average values of the quantities which express the strength and elasticity of what may be called the six principal materials. On account of the great variation of these values in different grades of the same material the wisdom of this attempt may perhaps be questioned, but the experience of the author in teaching the subject during the past eleven years has indicated that the best results are attained by forming at first a definite nucleus in the mind of the student, around which may be later grouped the multitude of facts necessary in his own particular department of study and work.

As the aim of all education should be to develop the powers of the mind rather than impart mere information, the author has endeavored not only to logically set forth the principles and theory of the subject, but to so arrange the matter that students will be encouraged and required to think for themselves. The problems which follow each article will be found useful for this purpose. Without the solution of many numerical problems it is indeed scarcely possible for the student to become well grounded in the theory. The attempt has been made to give examples, exercises, and problems of a practical nature, and also of such a character as to clearly illustrate the principles of the theory and the methods of investigation.

MANSFIELD MERRIMAN.

SOUTH BETHLEHEM, PA., December, 1889.

NOTE TO THE SEVENTH EDITION.

This edition, like the sixth, contains twice as much matter as the fifth edition. All known errors have been removed, a number of new problems introduced, and several articles rewritten. The most important changes will be found in the Chapter on Columns and in Articles 149–151, RITTER'S rational formula and EULER'S modified formula being presented, while the subject of eccentric loads is more fully treated than before. Articles 117 and 118, on flexure combined with compression or tension, have been also rewritten, and in discussing the second case it has been thought best to briefly introduce the modern idea of hyperbolic functions.

M. M.

LEHIGH UNIVERSITY, May, 1897.

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Evolvi varia problemata. In scientiis enim ediscendis prosunt exempla magis quam præcepta. Qua de causa in his fusius expatiatus sum.—NEWTON.

Nous avons pour but, non de donner un traité complet, mais de montrer, par des exemples simples et variés, l'utilité et l'importance de la théorie mathématique de l'élasticité.—LAMÉ.

MECHANICS OF MATERIALS.

CHAPTER I.

THE RESISTANCE AND ELASTICITY OF MATERIALS.

ARTICLE 1. AVERAGE WEIGHTS.

The principal materials used in engineering constructions are timber, brick, stone, cast iron, wrought iron, and steel. The following table gives their average unit-weights and average specific gravities.

Material.	Average Weight.		Average Specific Gravity.
	Pounds per Cubic Foot.	Kilos per Cubic Meter.	
Timber	40	600	0.6
Brick	125	2 000	2.0
Stone	160	2 560	2.6
Cast Iron	450	7 200	7.2
Wrought Iron	480	7 700	7.7
Steel	490	7 800	7.8

These weights, being mean or average values, should be carefully memorized by the student as a basis for more precise knowledge, but it must be noted that they are subject to more or less variation according to the quality of the material. Brick, for instance, may weigh as low as 100, or as high as 150 pounds per cubic foot, according as it is soft or hard pressed.

Unless otherwise stated the above average values will be used in the examples and problems of this book. In all engineering reference books are given tables showing the unit-weights for different qualities of the above six principal materials, and also for copper, lead, glass, cements, and other materials used in construction.

For computing the weights of bars, beams, and pieces of uniform cross-section, the following approximate simple rules will often be found convenient.

> A wrought iron bar one square inch in section and one yard long weighs ten pounds.
> Steel is about two per cent heavier than wrought iron.
> Cast iron is about six per cent lighter than wrought iron.
> Stone is about one-third the weight of wrought iron.
> Brick is about one-fourth the weight of wrought iron.
> Timber is about one-twelfth the weight of wrought iron.

For example, consider a bar of wrought iron $1\frac{1}{2} \times 3$ inches and 12 feet long; its cross-section is 4.5 square inches, hence its weight is $45 \times 4 = 180$ pounds. A steel bar of the same dimensions will weigh $180 + 0.02 \times 180 =$ about 184 pounds, and a cast iron bar will weigh $180 - 0.06 \times 180 =$ about 169 pounds.

By reversing the above rules the cross-sections of bars are readily computed from their weights per yard. Thus, if a stick of timber 15 feet long weigh 120 pounds, its weight per yard is 24 pounds, and its cross-section is $12 \times 2.4 =$ about 28.8 square inches.

Problem 1. How many square inches in the cross-section of a wrought iron railroad rail weighing 24 pounds per linear foot? In a steel rail? In a wooden beam?

Prob. 2. Find the weights of a wooden beam 6×8 inches in section and 13 feet long, of a steel bar one inch in diameter and 13 feet long, and of a common brick 2×4 inches and 8 inches long.

ART. 2. STRESSES AND DEFORMATIONS.

A 'stress' is a force which acts in the interior of a body and resists the external forces which tend to change its shape. If a weight of 400 pounds be suspended by a rope, the stress in the rope is 400 pounds. This stress is accompanied by an elongation of the rope, which increases until the internal molecular stresses or resistances are in equilibrium with the exterior weight. Stresses are measured in pounds, tons, or kilograms. A 'unit-stress' is the amount of stress on a unit of area; this is expressed either in pounds per square inch, or in kilograms per square centimeter. Thus, if a rope of two square inches cross-section sustains a stress of 400 pounds, the unit-stress is 200 pounds per square inch, for the total stress must be regarded as distributed over the two square inches of cross-section.

A 'deformation' is the amount of change of shape of a body caused by the external forces. If a load be put on a column its length is shortened, and the amount of shortening is a deformation. So in the case of the rope, the amount of elongation is a deformation. Deformations are generally measured in inches, or centimeters.

The word 'strain' is often used in technical literature as synonymous with stress, and sometimes it is also used to designate the deformation, or change of shape. On account of this ambiguity the word will not be employed in this book.

Three kinds of simple stress are produced by forces which tend to change the shape of a body. They are,

Tensile, tending to pull apart, as in a rope.
Compressive, tending to push together, as in a column.
Shearing, tending to cut across, as in punching a plate.

The nouns corresponding to these three adjectives are Tension, Compression, and Shear. The stresses which occur in beams, columns, and shafts are of a complex character, but they may always be resolved into the three kinds of simple stress. The first effect of an applied force is to cause a deformation. This deformation receives a special name according to the kind of stress which accompanies it. Thus,

Tension produces an elongation.
Compression produces a shortening.
Shear produces a detrusion.

This change of shape is resisted by the stresses between the molecules of the body, and as soon as these internal resistances balance the exterior forces the change of shape ceases and the body is in equilibrium. But if the external forces be increased far enough the molecular resistances are finally overcome and the body breaks or ruptures.

In any case of simple stress in a body in equilibrium the total internal stresses or resistances must equal the external applied force. Thus, in the above instance of a rope from which a weight of 400 pounds is suspended, let it be imagined to be cut at any section; then equilibrium can only be maintained by applying at that section an upward force of 400 pounds; hence the stresses in that section must also equal 400 pounds. In general, if a steady force P produce either tension, compression, or shear, the total stress produced is also P, for if not equilibrium does not obtain. In such cases, then, the word 'stress' may be used to designate the external force as well as the internal resistances.

Tension and Compression are similar in character but differ in regard to direction. A tensile stress in a bar occurs when two forces of equal intensity act upon its ends, each in a direction away from the other. In compression the direction of the forces is reversed and each acts toward the bar. Evidently a simple tensile or compressive stress in a bar is to be regarded as evenly distributed over the area of its cross-section, so that if P be the total stress in pounds and A the area of the cross-section in inches, the unit-stress is $\frac{P}{A}$ in pounds per square inch.

Shear requires the action of two forces exerted in parallel planes and very near together, like the forces in a pair of shears, from which analogy the name is derived. Here also the total shearing stress P is to be regarded as distributed uniformly over the area A, so that the unit-stress is $\frac{P}{A}$. And conversely if S represent the uniform unit-stress the total stress P is AS.

In any case of simple stress acting on a body let P be the total stress, A the area over which it is uniformly distributed, and S the unit-stress. Then,

$$(1) \qquad P = AS.$$

Also let λ be the total linear deformation produced by the stress, l the length of the bar, and s the deformation per unit of length. Then this deformation is to be regarded as uniformly distributed over the distance l, so that also,

$$(1)' \qquad \lambda = ls.$$

The laws implied in the statement of these two formulas are confirmed by experiment, if the stress be not too great.

Unit-stress in general will be denoted by S, whether it be tension, compression, or shear. S_t will denote tensile unit-stress, S_c compressive unit-stress, and S_s shearing unit-stress, when it is necessary to distinguish between them.

Prob. 3. A wrought iron rod $1\frac{1}{4}$ inches in diameter breaks under a tension of 67 500 pounds. Find the breaking unit-stress.

Prob. 4. If a wooden bar 1×3 inches breaks under a tensile stress of 33 000 pounds, what stress will break a bar $1\frac{1}{4} \times 2$ inches?

ART. 3. EXPERIMENTAL LAWS.

Numerous tests or experiments have been made to ascertain the strength of materials and the laws that govern stresses and deformations. The resistance of a rope, for instance, may be investigated by suspending it from one end and applying weights to the other. As the weights are added the rope will be seen to stretch or elongate, and the amount of this deformation may be measured. When the load is made great enough the rope will break, and thus its ultimate tensile stress is known. For stone, iron, or steel, special machines, known as testing machines, have been constructed by which the effect of different stresses on different qualities and forms of materials may be accurately measured.

All experiments, and all experience, agree in establishing the five following laws for cases of simple tension and compression, which may be regarded as the fundamental principles of the science of the strength of materials.

(*A*)—When a small stress is caused in a body a small deformation is produced, and on the removal of the stress the body springs back to its original form. For small stresses, then, materials may be regarded as perfectly elastic.

(*B*)—Under small stresses the deformations are approximately proportional to the forces, or stresses, which produce them, and also approximately proportional to the length of the bar or body.

(*C*)—When the stress is great enough a deformation is produced which is partly permanent, that is, the body does not spring back entirely to its original form on removal of the stress. This permanent part is termed a set. In such cases the deformations are not proportional to the stresses.

(*D*)—When the stress is greater still the deformation rapidly increases and the body finally ruptures.

(*E*)—A sudden stress, or shock, is more injurious than a steady stress or than a stress gradually applied.

The words small and great, used in stating these laws, have, as will be seen later, very different values and limits for different kinds of materials and stresses.

The 'ultimate strength' of a material under tension, compression, or shear, is the greatest unit-stress to which it can be subjected. This occurs at or shortly before rupture, and its value is very different for different materials. Thus if a bar whose cross-section is A breaks under a tensile stress P, the ultimate tensile strength of the material is $P \div A$.

Prob. 5. If the ultimate strength of wrought iron is 55 000 pounds per square inch, what tension will rupture a bar 6 feet long which weighs 60 pounds?

Prob. 6. If a bar 1 inch in diameter and 8 feet long elongates 0.05 inch under a stress of 15 000 pounds, how much, according to law (*B*), will a bar of the same size and material elongate whose length is 12 feet and stress 30 000 pounds?

ART. 4. ELASTIC LIMIT AND COEFFICIENT OF ELASTICITY.

The 'elastic limit' is that unit-stress at which the permanent set is first visible and within which the stress is directly proportional to the deformation. For stresses less than the elastic limit bodies are perfectly elastic, resuming their original form on removal of the stress. Beyond the elastic limit a permanent alteration of shape occurs, or, in other words, the elasticity of the material has been impaired. It is a fundamental rule in all engineering constructions that materials can not safely be strained beyond their elastic limit.

The 'coefficient of elasticity' of a bar for tension, compression, or shearing, is the ratio of the unit-stress to the unit-deformation, provided the elastic limit of the material be not exceeded. Let S be the unit-stress, s the unit-deformation, and E the coefficient of elasticity. Then by the definition,

$$(2) \qquad E = \frac{S}{s} \qquad \text{and} \qquad S = Es.$$

By law (B) the quantity E is a constant for each material, until S reaches the elastic limit. Beyond this limit s increases more rapidly than S and the ratio is no longer constant. Equation (2) is a fundamental one in the science of the strength of materials. Since E varies inversely with s, the coefficient of elasticity may be regarded as a measure of the stiffness of the material. The stiffer the material the less is the change in length under a given stress and the greater is E. The values of E for materials have been determined by experiments with testing machines and their average values will be given in the following articles. E is necessarily expressed in the same unit as the unit-stress S. Some authors give the name 'modulus of elasticity' to the quantity E.

Another definition of the coefficient of elasticity for the case of tension is that it is the unit-stress which would elongate a bar to double its original length, provided that this could be done without exceeding the elastic limit. That this definition is in agreement with (2) may be shown by regarding a bar of length l which elongates the amount λ under the unit-stress $\frac{P}{A}$. Here the unit-elongation is $\frac{\lambda}{l}$ and (2) becomes,

$$(2') \qquad E = \frac{P}{A} \div \frac{\lambda}{l} = \frac{Pl}{A\lambda},$$

and if λ be equal to l, E is the same as the unit-stress $\frac{P}{A}$.

Prob. 7. Find the coefficient of elasticity of a bar of wrought iron $1\frac{1}{4}$ inches in diameter and 16 feet long which elongates $\frac{1}{8}$ inch under a tensile stress of 21 000 pounds.

Prob. 8. If the coefficient of elasticity of cast iron is 15 000 000 pounds per square inch, how much will a bar 2 × 3 inches and 6 feet long stretch under a tension of 5 000 pounds?

ART. 5. TENSION.

The phenomena of tension observed when a gradually increasing stress is applied to a bar, are briefly as follows: When the unit-stress S is less than the elastic limit S_e, the unit-elongation s is small and proportional to S. Within this limit the ratio of S to s is the coefficient of elasticity of the material. After passing the elastic limit the bar rapidly elongates and this is accompanied by a reduction in area of its cross-section. Finally when S reaches the ultimate tensile strength S_t, the bar tears apart. Usually S_t is the maximum unit-stress on the bar, but in some cases the unit-stress reaches a maximum shortly before rupture occurs.

The constants of tension for timber, cast iron, wrought iron and steel are given in the following table. The values are average ones and are liable to great variations for different grades and qualities of materials. Brick and stone are not here mentioned, as they are rarely or never used in tension.

Material.	Coefficient of Elasticity, E.	Elastic Limit, S_e.	Ultimate Tensile Strength, S_t.	Ultimate Elongation.
	Pounds per square inch.	Pounds per square inch.	Pounds per square inch.	Per cent.
Timber,	1 500 000	3 000	10 000	1.5
Cast Iron,	15 000 000	6 000	20 000	0.5
Wrought Iron,	25 000 000	25 000	55 000	20
Steel,	30 000 000	50 000	100 000	10

The values of the coefficients of elasticity, elastic limits, and breaking or ultimate strengths are given in pounds per square inch of the original cross-section of the bar. The ultimate elongations are in fractional parts of the original length, or they are the elongations per linear unit; these should be regarded as very rough averages, since they are subject to great variations depending on the shape, size, and quality of the specimen.

The ultimate elongation, together with the reduction in area of the cross-section, furnishes the means of judging of the ductility of the material. The reduction of area in cast iron and in many varieties of steel is scarcely perceptible, while in other varieties of steel and in wrought iron it may be as high as 0.4 of the original section.

A graphical illustration of the principal phenomena of tension is given in Fig. 1. The unit-stresses are taken as ordinates and

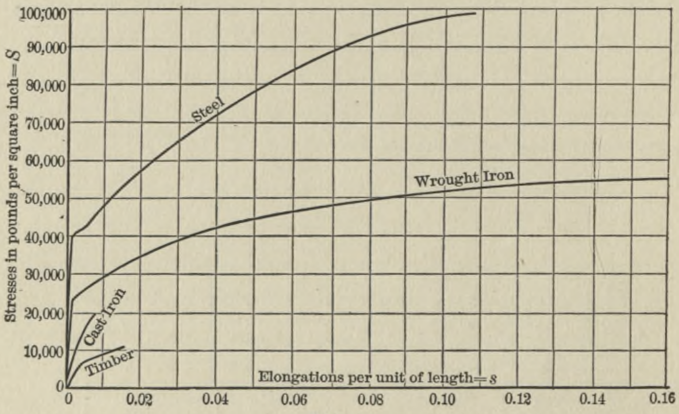

Fig. 1.

the unit-elongations as abscissas. For each unit-stress the corresponding unit-elongation as found by experiment is laid off, and curves drawn through the points thus determined. The curve for each of the materials is a straight line from the origin until the elastic limit is reached, as should be the case according to the law (*B*). The tangent of the angle which this line makes with the axis of abscissas is equal to $S \div s$, which is the same in value as the coefficient of elasticity of the material. At the elastic limit a sudden change in the curve is noticed and the elongation rapidly increases. The termination of the curve indicates the point of rupture. These curves show more plainly to the eye than the values in the table can do the differences in the properties of the materials. It will be seen that the elastic limit is not a well defined point, but that its value is more or less uncertain, particularly for cast iron and timber. It should be also clearly understood that individual curves for special cases would often show marked variations from their mean forms as represented in the diagram.

As a particular example a tensile test of a wrought iron bar $\frac{3}{4}$ inches in diameter and 12 inches long made at the Pencoyd Iron Works will be considered. In the first column of the following table are given the total stresses which were successively applied, in the second the stresses per square inch, in the third the total elongations, and in the fourth the elongations or sets after removal of the stress. The unit-elongations are found by dividing those in the table by 12 inches, the length of the specimen. Then the coefficient of elasticity can be computed for different values of S and s. Thus for the fourth and seventh cases,

Total Stress in Pounds.	Stress per Square Inch.	Elongation.	
		Load on.	Load off.
2 245	5 000	.001	.000
4 490	10 000	.004	.000
6 735	15 000	.005	.000
8 980	20 000	.008	.000
9 878	22 000	.009	.000
10 776	24 000	.010	.000
11 674	26 000	.0105	.000
12 572	28 000	.011	.000
13 470	30 000	.013	.000
14 368	32 000	.014	.000
15 266	34 000	.015	.002
16 164	36 000	.022	.007
17 062	38 000	.416	.3995
17 960	40 000	.5445	.523
25 450	50 000	1.740	1.707
23 175	51 600	2.468	......

Specimen broke with 51 600 pounds per square inch.
Stretch in 12 inches, 2.468 inches.
Stretch in 8 inches, 1.812 inches.
Stretch in 8 inches, 22.65 per cent.
Fractured area, 0.297 square inches.

$$\text{for } S = 20\,000, \quad s = \frac{0.008}{12} \quad \text{and} \quad E = 30\,000\,000;$$

$$\text{for } S = 26\,000, \quad s = \frac{0.0105}{12} \quad \text{and} \quad E = 29\,700\,000.$$

The elastic limit was reached at about 33 000 pounds per square inch, indicated by the beginning of the set and the rapid increase of the elongations. The ultimate tensile strength of the specimen was 51 600 pounds per square inch. The ultimate unit-elongation in 8 inches of the length was 0.226 inches per linear inch. It hence appears that this bar of wrought iron was higher than the average as regards stiffness, elastic limit and ductility, and lower than the average in ultimate strength.

The 'working stress' for a material is that unit-stress to which it is, or is to be, subjected. This should not be greater than the elastic limit of the material, since if that limit be exceeded there is a permanent set which impairs the elasticity. In order to secure an ample margin of safety it is customary to take the working stresses at from one-third to two-thirds the elastic limit S_e. The reasons which govern the selection of proper values of the working stresses will be set forth in the following articles.

To investigate the security of a piece subjected to a tension P, it is necessary first to divide P by the area of the cross-section and thus determine the working stress. Then a comparison of this value with the value of S_e for the given material will indicate whether the applied stress is too great or whether the piece has a margin of safety. For example, if a tensile stress of 4 500 pounds be applied to a wrought iron bar of $\frac{3}{4}$ inches diameter the working unit-stress is,

$$S=\frac{P}{A}=\frac{4\,500}{0.442}=10\,000 \text{ pounds per square inch, nearly.}$$

As this is less than one-half the elastic limit of wrought iron the bar has a good margin of security.

To design a piece to carry a given tension P it is necessary to assume the kind of material to be used and its allowable working stress S. Then $\frac{P}{S}$ is the area of the cross-section of the piece, which may be made of such shape as the circumstances of the case require. For example, if it be required to design a wooden bar to carry a tensile load of 4 500 pounds, the working stress may be assumed at 1 000 pounds per square inch and the required area is 4.5 square inches, so that the bar may be made 2 × 2¼ inches in section.

The elongation of a bar within the elastic limit may be computed by the help of formula (2). For instance, let it be required to find the elongation of a wooden bar 3 × 3 inches and 12 feet long under a tensile stress of 9 000 pounds. From the formulas (2) and (1),

$$E = \frac{S}{s} = \frac{P}{A} \div \frac{\lambda}{l}; \quad \therefore \lambda = \frac{Pl}{AE}.$$

Substituting in this the values $E = 1\,500\,000$, $A = 9$, $l = 144$, and $P = 9\,000$, the probable value of the elongation λ is found to be 0.096 inches.

Prob. 9. Find the size of a round wrought iron rod to safely carry a tensile stress of 100 000 pounds.

Prob. 10. Compute the elongation of a wooden and of a cast iron bar, each being 2 × 3 inches and 16 feet long, under a tensile stress of 6 000 pounds.

ART. 6. COMPRESSION.

The phenomena of compression are similar to those of tension, provided that the length of the specimen does not exceed about five times its least diameter. The piece at first shortens proportionally to the applied stress, but after the elastic limit is passed the shortening increases more rapidly, and is accompanied by a slight enlargement of the cross-section. When the stress reaches the ultimate strength of the material the specimen cracks and ruptures. If the length of the piece exceeds about ten times its least diameter, a sidewise bending or flexure of the specimen occurs, so that it fails under different circumstances than those of direct compression. All the values given in this article refer to specimens whose lengths do not exceed about five times their least diameter. Longer pieces will be discussed in Chapter V under the head of 'columns.' Owing to the difficulty of making experiments on short specimens, the phenomena of compression are not usually so regular as those of tension.

The constants of compression for short specimens are given in the following table, the values, like those for tension, being rough average values liable to much variation in particular cases.

Material.	Coefficient of Elasticity, E.	Elastic Limit, S_e.	Ultimate Compressive Strength, S_c.
	Lbs. per sq. in.	Lbs. per sq. in.	Lbs. per sq. in.
Timber,	1 500 000	3 000	8 000
Brick,			2 500
Stone,	6 000 000		6 000
Cast Iron,	15 000 000	20 000	90 000
Wrought Iron,	25 000 000	25 000	55 000
Steel,	30 000 000	50 000	150 000

The values of the coefficient of elasticity and the elastic limit for timber, wrought iron, and steel here stated are the same as those for tension, but the same reliance cannot be placed upon them, owing to the irregularity of experiments thus far made. There is reason to believe that both the elastic limit and the coefficient of elasticity for compression are somewhat greater than for tension.

The investigation of a piece subjected to compression, or the design of a short piece to be subjected to compression, is effected by exactly the same methods as for tension. Indeed it is customary to employ these methods for cases where the length of the piece is as great as ten times its least diameter.

Prob. 11. Find the height of a brick tower which crushes under its own weight. Also the height of a stone tower.

Prob. 12. Compute the amount of shortening in a wrought iron specimen 1 inch in diameter and 5 inches long under a load of 6 000 pounds.

ART. 7. SHEAR.

Shearing stresses and strains occur whenever two forces, acting like a pair of shears, tend to cut a body between them. When a plate is punched the ultimate shearing strength of the material must be overcome over the surface punched. When a bolt is in tension the applied stress tends to shear off the head and also to strip or shear the threads in the nut and screw. When a rivet connects two plates which transmit tension the plates tend to shear the rivet across.

The ultimate shearing strength of materials is easily determined by causing rupture under a stress P, and then dividing P by the area A of the shorn surface. The value of this for timber is found to be very much smaller along the grain than across the grain; for the first direction it is sometimes called longitudinal shearing strength and for the second transverse shearing strength. The same distinction is sometimes made in rolled wrought iron plates and bars where the process of manufacture induces a more or less fibrous structure. The elastic limit and the amount of detrusion for shearing are dif-

Material.	Coefficient of Elasticity, E.	Ultimate Shearing Strength, S_s.
Timber, Longitudinal,	400 000	600
Timber, Transverse,		3 000
Cast Iron,	6 000 000	20 000
Wrought Iron,	10 000 000	50 000
Steel,	11 000 000	70 000

ficult to determine experimentally. The coefficient of elasticity, however, has been deduced by means of certain calculations and experiments on the twisting of shafts, explained in Chapter VI under the head of torsion.

The investigation and design of a piece to withstand shearing stress is made by means of the equation $P = AS$, in the same manner as for tension and compression. As an instance of investigation, consider the cylindrical wooden specimen shown in Fig. 2, which has the following dimensions: length $ab = 6$ inches, diameter of ends $= 4$ inches, diameter of central part $= 2$ inches. Let this specimen be subjected to a tensile stress in the direction of its length. This not only tends to tear it apart by tension, but also to shear off the ends on a surface whose length is ab and whose diameter is that of the central cylinder. The force P required to cause this longitudinal shearing is,

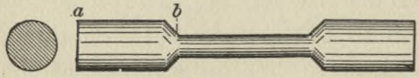

Fig. 2.

$$P = AS_s = 3.14 \times 2 \times 6 \times 600 = 22\,600 \text{ pounds},$$

while the force required to rupture the specimen by tension is,

$$P = AS_t = 3.14 \times 1^2 \times 10\,000 = 31\,400 \text{ pounds}.$$

As the former resistance is only about two-thirds that of the latter the specimen will evidently fail by the shearing off of the ends.

When a bar is subject either to tension or to compression a shear occurs in any section except those perpendicular and parallel to the axis of the bar. Let Fig. 3 represent a bar of cross-section A subject to the tensile stress P which produces in every section perpendicular to the bar the unit-stress $\frac{P}{A}$. Let mn be a plane making an

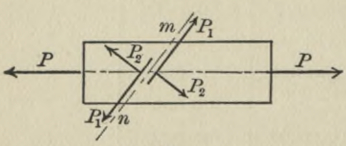

Fig. 3.

angle θ with the axis, and cutting from the bar a section whose area is A_1. On the left of the plane the stress P may be resolved into the components P_1 and P_2, respectively parallel and normal to the plane, and the same may be done on the right. Thus it is seen that the effect of the tensile stress P on the plane mn is to produce a tension P_2 normal to it, and a shear P_1 along it, for the two forces P_1 and P_1 act in parallel planes and in opposite directions. The shearing stress P_1 has the value $P\cos\theta$, which is distributed over the area A_1 whose value is $A \div \sin\theta$. Hence the shearing unit-stress in the given section is,

$$S_1 = \frac{P_1}{A_1} = \frac{P}{A}\sin\theta\cos\theta.$$

When $\theta = 0°$, or $\theta = 90°$, the value of S_1 is zero. The maximum value of S_1 occurs when $\theta = 45°$, and then $S_1 = \frac{1}{2}\frac{P}{A}$, or a tensile unit-stress S on a bar produces a shearing unit-stress of $\frac{1}{2}S$ along every section inclined 45 degrees to the axis of the bar. The above investigation applies also to compression if the direction of P be reversed, and it is sometimes observed in experiments on the compression of short specimens that rupture occurs by shearing along oblique sections.

Prob. 13. A hole $\frac{3}{4}$ inches in diameter is punched in a wrought iron plate $\frac{5}{8}$ inches thick by a pressure on the punch of 78 000 pounds. What is the ultimate shearing strength of the iron?

Prob. 14. A wrought iron bolt $1\frac{1}{2}$ inches in diameter has a head $\frac{3}{4}$ inches long. Find the unit-stress tending to shear off the head when a tension of 3 000 pounds is applied to the bolt.

ART. 8. FACTORS OF SAFETY AND WORKING STRESSES.

The factor of safety for a body under stress is the ratio of its ultimate strength to the actual existing unit-stress. The factor of safety for a piece to be designed is the ratio of the ultimate strength to the proper allowable working stress. Thus if S_t be the ultimate, S the working stress, and f the factor of safety, then

$$f = \frac{S_t}{S}, \quad \text{and} \quad S_t = fS.$$

The factor of safety is hence always an abstract number, which indicates the number of times the working stress may be multiplied before the rupture of the body.

The law (E) in Art. 3 indicates that working stresses should be lower for shocks and sudden stresses than for steady loads and slowly varying stresses. In a building the stresses on the walls are steady, so that the working strength may be taken high and hence the factor of safety low. In a bridge the stresses in the several members are more or less varying in character which requires a lower working strength and hence a higher factor of safety. In a machine subject to shocks the working strength should be lower still and the factor of safety very high. The law (E) from which these conclusions are derived is not merely the result of experience, but can be confirmed by theoretical discussion (Art. 103).

The following are average values of the allowable factors of safety commonly employed in American practice. These values

Material.	For Steady Stress. (Buildings.)	For Varying Stress. (Bridges.)	For Shocks. (Machines.)
Timber,	8	10	15
Brick and Stone,	15	25	30
Cast Iron,	6	15	20
Wrought Iron,	4	6	10
Steel,	5	7	15

are subject to considerable variation in particular instances, not only on account of the different qualities and grades of the material, but also on account of the varying judgment of designers. They will also vary with the range of varying stress, so that different parts of a bridge may have very different factors of safety.

The proper allowable working stress for any material in tension, compression, or shearing, may be at once found by dividing the ultimate strength by the proper factor of safety. Regard should also be paid to the elastic limit in selecting the working stresses, particularly for materials whose elastic limit is well defined. For wrought iron and steel the working strength should be well within the elastic limit, as already indicated in previous articles. For cast iron, stone, brick, and timber it is often difficult to determine the elastic limit, and experience alone can guide the proper selection of the working strength. The above factors of safety indicate indeed the conclusions of experiment and experience extending over the past hundred years.

The student should clearly understand that the exact values given in this and the preceding articles would not be arbitrarily used in any particular case of design. For instance, if a given lot of wrought iron is to be used in an engineering structure, specimens of it should be tested to determine its coefficient of elasticity, elastic limit, ultimate strength, and percentage of elongation. Then the engineer will decide upon the proper working stresses, being governed by its qualities as shown by the tests, the character of the stresses that come upon it, and the cost of workmanship.

The two fundamental principles of engineering design are stability and economy, or in other words:

First, the structure must safely withstand all the stresses which are to be applied to it.

Second, the structure must be built and maintained at the lowest possible cost.

The second of these fundamental principles requires that all parts of the structure should be of equal strength, like the celebrated 'one-hoss shay' of the poet. For, if one part is stronger than another, it has an excess of material which might have been spared. Of course this rule is to be violated if the cost of the labor required to save the material be greater than that of the material itself. Thus it often happens that some parts of a structure have higher factors of safety than others, but the lowest factors should not, as a rule, be less than the values given above. For the design of important structures specifications are prepared which state the lowest allowable unit-stresses that can be used.

The factors of safety stated above are supposed to be so arranged that, if different materials be united, the stability of all parts of the structure will be the same, so that if rupture occurs, everything would break at once. Or, in other words, timber with a factor of safety 8 has about the same reliability as wrought iron with a factor of 4 or stone with a factor of 15, provided the stresses are due to steady loads.

The assignment of working stresses with regard to the elastic limits of materials is more rational than that by means of the factors of safety, and in time it may become the more important and valuable method. But at present the ultimate strengths are so much better known and so much more definitely determinable than the elastic limits that the empirical method of factors of safety seems the more important for the use of students, due regard being paid to considerations of stiffness, elastic limit, and ductility.

As an example, let it be required to find the proper size of a wrought iron rod to carry a steady tensile stress of 90 000 pounds. In the absence of knowledge regarding the quality of the wrought iron, the ultimate strength S_t is to be taken as the average value, 55 000 pounds per square inch. Then, for a factor of safety of 4, the working stress is,

$$S = \frac{55\,000}{4} = 13\,750 \text{ pounds per square inch.}$$

The area of cross-section required is hence,

$$A = \frac{90\,000}{13\,750} = 6.6 \text{ square inches,}$$

which may be supplied by a rod of $2\frac{15}{16}$ inches diameter.

Prob. 15. Determine the size of a short steel piston rod when the piston is 20 inches in diameter and the steam pressure upon it is 67.5 pounds per square inch.

Prob. 16. A wooden frame ABC forming an equilateral triangle consists of short pieces 2×2 inches jointed at A, B, and C. It is placed in a vertical plane and supported at B and C so that BC is horizontal. Find the unit-stress and factor of safety in each of the three pieces when a load of 5 890 pounds is applied at A.

CHAPTER II.

PIPES, CYLINDERS, AND RIVETED JOINTS.

ART. 9. WATER AND STEAM PIPES.

The pressure of water or steam in a pipe is exerted in every direction, and tends to tear the pipe apart longitudinally. This is resisted by the internal tensile stresses of the material. If p be the pressure per square inch of the water or steam, d the diameter of the pipe and l its length, the force P which tends to cause longitudinal rupture is $p \cdot ld$. This is evident from the fundamental principle of hydrostatics that the pressure of water in any direction is equal to the pressure on a plane perpendicular to that direction, or may be seen by imagining the pipe to be filled with a solid substance on one side of the diameter, which would receive the pressure p on each square inch of the area ld and transmit it into the pipe. If t be the thickness of the pipe and S the tensile stress which is uniformly distributed over it, as will be the case when t is not large compared with d, the resistance on each side is $tl \cdot S$. As the resistance must equal the pressure,

$$pld = 2tlS, \qquad \text{or} \qquad pd = 2tS,$$

which is the formula for discussing pipes under internal pressure.

The unit-pressure p for water may be computed from a given head h by finding the weight of a column of water one inch square and h inches high. Or if h be given in feet, the pressure in pounds per square inch may be computed from $p = 0.434h$.

Water pipes may be made of cast or wrought iron, the former being more common, while for steam the latter is preferable.

Wrought iron pipes are sometimes made of plates riveted together, but the discussion of these is reserved for another article. A water pipe subjected to the shock of water ram needs a high factor of safety, and in a steam pipe the factors should also be high, owing to shocks liable to occur from condensation and expansion of the steam. The formula above deduced shows that the thickness of a pipe must increase directly as its diameter, the internal pressure being constant.

For example, let it be required to find the factor of safety for a cast iron water pipe of 12 inches diameter and $\frac{5}{8}$ inches thickness under a head of 300 feet. Here p, the pressure per square inch, equals 130.2 pounds. Then from the formula the unit-stress is,

$$S = \frac{pd}{2t} = \frac{130.2 \times 12}{2 \times \frac{5}{8}} = 1\,250 \text{ pounds per square inch,}$$

and hence the factor of safety is,

$$f = \frac{20\,000}{1\,250} = \text{about } 16,$$

which indicates ample security under ordinary conditions.

Again let it be required to find the proper thickness for a wrought iron steam pipe of 18 inches diameter to resist a pressure of 120 pounds per square inch. With a factor of safety of 10 the working strength S is about 5 500 pounds per square inch. Then from the formula,

$$t = \frac{pd}{2S} = \frac{120 \times 18}{2 \times 5\,500} = 0.2 \text{ inches.}$$

In order to safely resist the stresses and shocks liable to occur in handling the pipes, the thickness is often made somewhat greater than the formula requires.

Prob. 17. What should be the thickness of a cast iron pipe of 18 inches diameter under a head of 300 feet?

Prob. 18. A wrought iron pipe is 3 inches in internal diameter and weighs 24 pounds per linear yard. What steam pressure can it carry with a factor of safety of 8?

ART. 10. THIN CYLINDERS AND SPHERES.

A cylinder subject to the interior pressure of water or steam tends to fail longitudinally exactly like a pipe. The head of the cylinder however undergoes a pressure which tends to separate it from the walls. If d be the diameter of the cylinder and p the internal pressure per square unit, the total pressure on the head is $\frac{1}{4}\pi d^2 . p$. If S be the working unit-stress and t the thickness of the cylinder, the resistance to the pressure is approximately πdtS, if t be so small that S is uniformly distributed. Since the resistance must equal the pressure,

$$\tfrac{1}{4}\pi d^2 . p = \pi dt . S, \qquad \text{or} \qquad pd = 4tS.$$

By comparing this with the formula of the last article it is seen that the resistance of a pipe to transverse rupture is double the resistance to longitudinal rupture.

A thin sphere subject to interior pressure tends to rupture around a great circle, and it is easy to see that the conditions are exactly the same as for the transverse rupture of a cylinder, or that $pd = 4tS$. For thick spheres and cylinders the formulas of this and the last article are only approximate.

A cylinder under exterior pressure is theoretically in a similar condition to one under interior pressure as long as it remains a true circle in cross-section. A uniform interior pressure tends to preserve and maintain the circular form of the cylindrical annulus, but an exterior pressure tends at once to increase the slightest variation from the circle and render it elliptical. The distortion when once begun rapidly increases, and failure occurs by the collapsing of the tube rather than by the crushing of the material. The flues of a steam boiler are the most common instance of cylinders subjected to exterior pressure. In the absence of a rational method of investigating such cases recourse has been had to experiment. Tubes of various diameters, lengths, and thicknesses have been subjected to exterior pressure until they collapse and the results have been compared and discussed. The following for instance are the results of three experiments by FAIRBAIRN on wrought iron tubes.

Length in Inches.	Diameter. in Inches.	Thickness in Inches.	Pressure per Sq. Inch.
37	9	0.14	378
60	14½	0.125	125
61	18¾	0.25	420

From these and other similar experiments it has been concluded that the collapsing pressure varies directly as some power of the thickness, and inversely as the length and diameter of the tube. For wrought iron tubes WOOD gives the empirical formula for the collapsing pressure per square inch,

$$p = 9\,600\,000 \frac{t^{2.18}}{ld}.$$

The values of p computed from this formula for the above three experiments are 397, 120, and 409, which agree well with the observed values.

The proper thickness of a wrought iron tube to resist exterior pressure may be readily found from this formula after assuming a suitable factor of safety. For example, let it be required to find t when $p = 120$ pounds per square inch, $l = 72$ inches, $d = 4$ inches and the factor of safety $= 10$. Then

$$t^{2.18} = \frac{10 \times 120 \times 72 \times 4}{9\,600\,000} = 0.036,$$

from which with the help of logarithms the value of t is found to be 0.22 inches.

Prob. 19. What interior pressure per square inch will burst a cast iron sphere of 24 inches diameter and $\frac{3}{4}$ inches thickness.

Prob. 20. What exterior pressure per square inch will collapse a wrought iron tube 72 inches long, 4 inches diameter and 0.25 inches thickness?

ART. 11. THICK CYLINDERS.

When the walls of a cylinder are thick compared with its interior diameter it cannot be supposed, as in the preceding articles, that the stress is uniformly distributed over the thickness t. Let Fig. 4 represent one-half of a section of a thick cylinder subject to interior pressure over the length l, tending to produce longitudinal rupture. Let r and r_1 be the interior and exterior radii, then $r_1 - r = t$ the thickness. Let S and S_1 be the tensile unit-stresses at the inner and outer edges of the annulus. Before the application of the pressure the volume of the annulus is $\pi(r_1^2 - r^2)l$, after the pressure is applied the radius r_1 is increased to $r_1 + y_1$ and r to $r + y$, so that its volume is $\pi(r_1 + y_1)^2 l - \pi(r + y)^2 l$. The annulus is however really changed only in form, so that the two expressions for the volume are equal, and equating them gives,

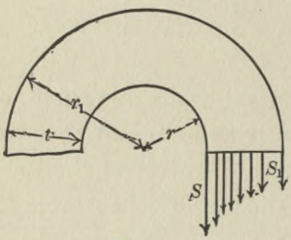

Fig. 4.

$$2ry + y^2 = 2r_1y_1 + y_1^2,$$

or, since y and y_1 are small compared with r and r_1 their squares may be neglected, and hence

$$ry = r_1y_1, \qquad \text{or} \qquad \frac{y}{y_1} = \frac{r_1}{r}.$$

Now if the material is not stressed beyond the elastic limit the unit-stresses S and S_1 are proportional to the corresponding unit-elongations. The elongation of the inner circumference is $2\pi y$ and that of the outer circumference is $2\pi y_1$, and dividing these by $2\pi r$ and $2\pi r_1$ respectively the unit-elongations are found; then,

$$\frac{S}{S_1} = \frac{y}{r} \div \frac{y_1}{r_1} = \frac{r_1}{r} \cdot \frac{y}{y_1}.$$

Substituting in this the value of the ratio $\frac{y}{y_1}$ as above found, gives

$$\frac{S}{S_1} = \frac{r_1^2}{r^2},$$

that is, the unit-stresses in the walls of the cylinder vary inversely as the squares of their distances from the center.

The total stress acting over the area $2t \, . \, l$ is now to be found by summing up the unit-stresses. Let S_x be any unit-stress at a distance x from the center, and S, as before, be that at the inner circumference, which is the greatest of all the unit-stresses. Then by the law of variation,

$$S_x = S\frac{r^2}{x^2}.$$

The stress acting over the area $l \, . \, dx$ is then

$$S_x \, ldx = Sr^2 l \frac{dx}{x^2},$$

and the total stress over the area $2t \, . \, l$ is

$$2Sr^2 l \int_r^{r_1} \frac{dx}{x^2} = 2Sr^2 l \left(\frac{1}{r} - \frac{1}{r_1}\right) = 2Sl\frac{rt}{r+t}.$$

This is the value of the internal resisting stress in the walls of the pipe; if t be neglected in comparison with r it reduces to $2Slt$ which is the same as previously found for thin cylinders; if $t = r$ it becomes Slt or only one-half the resistance of a thin cylinder.

The total interior pressure which tends to rupture the cylinder longitudinally is $2rl \, . \, p$, if p be the unit-pressure (Art. 9).

Equating this to the total internal resisting stress gives

$$p = \frac{St}{r + t},$$

from which one of the quantities S, p, r, or t can be computed when the other three are given.

The above formula was deduced by BARLOW. Although more accurate for thick cylinders than the formula of Art. 10, it is not in practice considered so reliable as the formula of LAMÉ which is deduced in Chapter XIV. If $r = 6$ inches, $t = 1$ inch, and $p = 400$ pounds per square inch, BARLOW'S formula gives $S = 2800$, while LAMÉ'S formula (page 314) gives $S = 2620$ pounds per square inch.

Prob. 21. Prove when the thickness of a pipe equals its interior radius that the exterior circumference elongates one-half as much as the interior circumference.

Prob. 22. If a gun of 3 inches bore is subject to an interior pressure of 1 800 pounds per square inch, what should be its thickness so that the greatest stress on the material may not exceed 3 000 pounds per square inch?

ART. 12. INVESTIGATION OF RIVETED JOINTS.

When two plates which are under tension are joined together by rivets, these must transfer that tension from one plate to another. A shearing stress is thus brought upon each rivet which tends to cut it off. A compressive stress is also brought sidewise upon each rivet which tends to crush it; this particular kind of compression is often called "bearing stress." The exact manner in which it acts upon the cylindrical surface of the rivet is not known, but it is usually supposed to be equivalent to a stress uniformly distributed over the projection of the surface on a plane through the axis of the rivet.

Case I. Lap Joint with single riveting.—Let P be the tensile stress which is transmitted from one plate to the other by means of a single rivet, t the thickness of the plates, d the diameter of a rivet, and a the pitch of the rivets. Let S_t, S_s, and S_c be the unit-stresses in tension, shear, and compression produced by P upon the plates and rivets. Then for the tension on the plate,

$$P = t(a - d)S_t,$$

for the shear on the rivet,

$$P = \tfrac{1}{4}\pi d^2 . S_s,$$

and for compression on the rivet,

$$P = td . S_c.$$

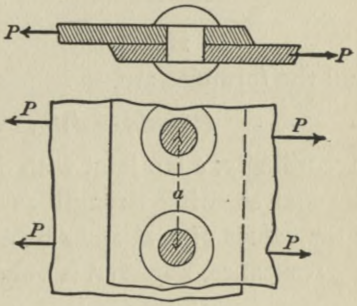

Fig. 5.

From these equations the unit-stresses may be computed, when the other quantities are known, and by comparing them with the proper working unit-stresses the degree of security of the joint is estimated.

Case II. Lap Joint with double riveting.—In this arrangement the plates have a wider lap, and there are two rows of rivets. Let a be the pitch of the rivets in one row, then the tensile stress P is distributed over two rivets, and the three formulas are,

$$P = t(a - d)S_t,$$
$$P = 2 . \tfrac{1}{4}\pi d^2 S_s,$$
$$P = 2 . tdS_c,$$

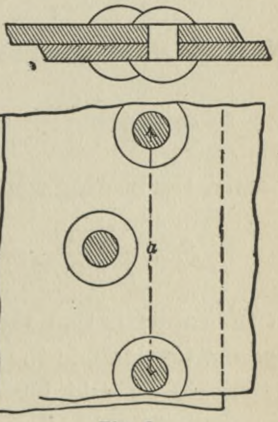

Fig. 6.

from which the unit-stresses may be computed and the strength of the joint be investigated. The loss of strength is here generally less than in the previous case since a can be made larger with respect to d.

Case III. Butt Joint with single riveting.—For this arrangement the shear on each of the rivets comes on two cross-sections, which is said to be a case of double shear, and the formulas are,

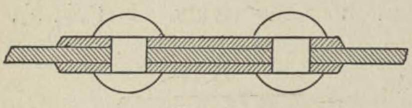

Fig. 7.

$$P = t(a-d)S_t, = 2 \cdot \tfrac{1}{4}\pi d^2 S_s = tdS_c.$$

Accordingly, a lap joint with double riveting has the same tensile and shearing strength as a butt joint with single riveting, if the values of a, d, and t be equal in the two cases; the bearing resistance, however, is only one half as large.

For example, let it be required to investigate a single riveted butt joint consisting of plates 0.75 inches thick with covers 0.375 inches thick, and rivets of 1 inch diameter and 4 inches pitch, when a tension of 8 000 pounds is transmitted through one rivet. First, the working tensile unit-stress on the plate is found to have the value,

$$S_t = \frac{8\,000}{3 \times 0.75} = 3\,560 \text{ pounds per square inch.}$$

Next the shearing unit-stress on the rivets is,

$$S_s = \frac{8\,000}{2 \times 0.785} = 5\,100 \text{ pounds per square inch.}$$

Lastly, the bearing compressive unit-stress on the rivets is,

$$S_c = \frac{8\,000}{1 \times 0.75} = 10\,700 \text{ pounds per square inch.}$$

It thus appears that the joint has the greatest factor of safety against tension and the least against compression of the rivets. It should be said, however, that for wrought iron plates and rivets the highest allowable working stresses for tension, shear, and bearing are generally considered to be about 9 000, 7 500, and 12 000 pounds per square inch respectively; hence the joint has proper security under the given conditions although the degree of security is quite different for the different stresses.

The 'efficiency' of a joint is defined to be the ratio of its highest allowable stress to the highest allowable stress of the unriveted plate. The highest allowable stresses in tension, shear, and compression are the three expressions for P, using, for wrought iron, the values above mentioned; and the highest allowable stress of the unriveted plate is atS_t. Thus result the following values of the efficiency,

For tension, $$e = \frac{a - d}{a},$$

For shear, $$e = \frac{n \,.\, \frac{1}{4}\pi d^2 S_s}{atS_t},$$

For compression, $$e = \frac{m \,.\, dS_c}{aS_t},$$

in which m denotes the number of rivets in the width a which transmit the tension P and n denotes the number of rivet-sections in the same space over which the shear is distributed. The smallest of these values of e is to be taken as the efficiency of the joint. Thus for the above numerical example the three values are 0.75, 0.44, and 0.33; the working strength of the joint is, hence, only 33 per cent of that of the unriveted plate.

If in the above formulas, S_t, S_s, and S_c be taken as the ultimate strengths, the resulting values of e will be the efficiencies at the moment of rupture of the joint. For the same numerical example the three ultimate efficiencies are 0.75, 0.48, and 0.25.

Prob. 23. A boiler is to be formed of wrought iron plates $\frac{3}{8}$ inches thick, united by single lap joints with rivets $\frac{3}{4}$ inches diameter and $1\frac{3}{4}$ inches pitch. Find the efficiency of the joint. Find the factor of safety of the boiler if it is 30 inches in diameter and carries a steam pressure of 100 pounds.

ART. 13. DESIGN OF RIVETED JOINTS.

A theoretically perfect joint with regard to strength is one so arranged that all parts (like the one-hoss shay) have the same degree of security. Thus the resistance of the plate to tension must equal the resistance of the rivets to shearing, and each of these must equal the resistance of the rivets to compression. The three expressions for P of the last Article should hence be equal, or, what amounts to the same thing, the three efficiencies should be equal. Equating then the second to the third and solving for d, gives

$$d = \frac{4mS_c}{\pi nS_s}t,$$

from which d can be computed when t is assumed. Again, equating the first and third and eliminating d gives,

$$a = \frac{4mS_c}{\pi nS_s}\left(1 + \frac{mS_c}{S_t}\right)t,$$

from which the pitch of the rivets can be obtained. Inserting these values of d and a in either of the expressions for e furnishes the formula,

$$e = \frac{1}{1 + \dfrac{S_t}{mS_c}},$$

from which the efficiency can be ascertained. The best joint will be that which has the least loss of strength due to the riveting, or that which has a value of e as near unity as possible.

Using for wrought iron plates and rivets the working unit-stresses $S_t = 9\,000$, $S_s = 7\,500$, and $S_c = 12\,000$ pounds per square inch, the above formulas for a lap joint with single row of rivets where $m = 1$ and $n = 1$, reduce to,

$$d = 2.04t, \qquad a = 4.75t, \qquad e = 0.57,$$

so that, if the thickness of the plate be given and the diameter and pitch of the rivet be made according to these rules, the joint has about 57 per cent of the strength of the unholed plate. For a lap joint with double riveting where $m = 2$ and $n = 2$, they become

$$d = 2.04t, \qquad a = 7.48t, \qquad e = 0.73.$$

This example shows clearly the advantage of double over single riveting, and by adding a third row the efficiency will be raised to about 80 per cent.

The application of the above formulas to butt joints makes the diameter of the rivet equal to the thickness of the plate and makes the pitch much smaller than the above values for lap joints. These proportions are difficult to apply in practice on account of the danger of injuring the metal in punching the holes. For this reason joints are often made in which the strengths of the different parts are not equal. Many other reasons, such as cost of material and facility of workmanship, influence also the design of a joint so that the formulas above deduced are to be regarded only as a rough guide. The old rules which are still often used for determining the pitch in butt joints, are

$$a = d + \frac{\pi d^2}{2t}, \qquad a = d + \frac{\pi d^2}{t},$$

the first being for single and the second for double riveting. These are deduced by making the strength of the joint equal in tension and shear, and taking $S_s = S_t$.

It may be required to arrange a joint so as to secure either strength or tightness. For a bridge, strength is mainly needed; for a gasholder, tightness is the principal requisite; while for a boiler both these qualities are desirable. In general a tight joint is secured by using small rivets with a small pitch. The lap of the plates, and the distance between the rows of rivets, is determined by practical considerations rather than by theoretic formulas.

Prob. 24. A lap joint with double riveting is to be formed of plates $\frac{1}{2}$ inches thick with rivets $\frac{7}{8}$ inches diameter. Find the pitch so that the strength of the plate shall equal the shearing strength of the rivets, and compute the efficiency of the joint.

ART. 14. MISCELLANEOUS EXERCISES.

It will be profitable to the student to thoroughly perform the following exercises and problems and to write upon each a detailed report, which should contain all the sketches and computations necessary to clearly explain the data, the reasoning, the computations, and the conclusions. Problem 26 is intended for students proficient in the use of calculus.

Exercise 1. Visit an establishment where tensile tests are made. Ascertain the kind of machine employed, its capacity, the method of applying the stresses, the method of measuring the stresses, the method of measuring the elongations. Ascertain the kind of material tested, the reason for testing it, and the conclusions derived from the tests. Give full data for the tests on four different specimens, compute the values of coefficient of elasticity, ultimate strength and ultimate elongation for them, and state your conclusions.

Exercise 2. Procure a wrought iron bolt and nut. Measure the diameter of bolt, length of head, and length of nut. State the equation of condition that the head of the bolt may shear off at the same time the bolt ruptures under tension. Compute the length of head for a given diameter. Explain why the length of the head is made greater than theory apparently requires. Compile a table giving dimensions of bolts and nuts of different diameters.

Exercise 3. Go to a boiler shop and witness operations upon a boiler in process of construction. Ascertain length and diameter of boiler, thickness, pitch and diameter of rivets, method of forming holes, method of doing the riveting. Compute the loss of strength caused by the riveting. Compute the steam pressure which would cause longitudinal rupture of the plate along a line of rivets. Ascertain whether the joint is proportioned in accordance with theory.

Prob. 25. A wrought iron pipe $\frac{3}{8}$ inches thick and 20 inches in diameter is to be subjected to a head of water of 345 feet. Compute the probable increase in diameter due to the interior pressure, regarding the pipe as thin.

Prob. 26. Let a pier whose top width is b and length l support a uniformly distributed load P. Let the width of the pier at a distance y below the top be x, its constant length being l at all horizontal sections. Let w be the weight of the masonry per cubic foot. Prove that, in order to make the compressive unit-stress the same for all horizontal sections, the profile of the pier must be such as to satisfy the equation $y = \frac{S}{w} \log_e \frac{x}{b}$ in which $S = \frac{P}{bl}$.

CHAPTER III.

CANTILEVER BEAMS AND SIMPLE BEAMS.

ART. 15. DEFINITIONS.

Transverse stress, or flexure, occurs when a bar is in a horizontal position upon one or more supports. The weight of the bar and the loads upon it cause it to bend and induce in it stresses and strains of a complex nature which, as will be seen later, may be resolved into those of tension, compression, and shear. Such a bar is called a beam.

A simple beam is a bar resting upon supports at its ends. A cantilever beam is a bar on one support in its middle, or the portion of any beam projecting out of a wall or beyond a support may be called a cantilever beam. A continuous beam is a bar resting upon more than two supports. In this book the word beam, when used without qualification, includes all kinds, whatever be the number of the supports, or whether the ends be free, supported, or fixed.

The elastic curve is the curve formed by a beam as it deflects downward under the action of its own weight and of the loads upon it. Experience teaches that the amount of this deflection and curvature is very small. A beam is said to be fixed at one end when it is so arranged that the tangent to the elastic curve at that end always remains horizontal. This may be done in practice by firmly building one end into a wall. A beam fixed at one end and unsupported at the other is a cantilever beam.

The loads on beams are either uniform or concentrated. A uniform load embraces the weight of the beam itself and any load evenly spread over it. Uniform loads are estimated by their intensity per unit of length of the beam, and usually in pounds per linear foot. The uniform load per linear unit is designated by w, then wx will represent the load over any distance x. If l be the length of the beam, the total uniform load is wl which may be represented by W. A concentrated load is a weight applied at a definite point and is designated by P.

In this chapter cantilever and simple beams will be principally discussed, although all the fundamental principles and methods hold good for restrained and continuous beams as well. Unless otherwise stated the beams will be regarded as of uniform cross-section throughout, and in computing their weights the rules of Art. 1 will be found of service.

Prob. 27. Find the diameter of a round steel bar which weighs 48 pounds, its length being 4 feet.

ART. 16. REACTIONS OF THE SUPPORTS.

The points upon which a beam is supported react upward against the beam an amount equal to the pressure of the beam upon them. The beam, being at rest, is a body in equilibrium under the action of a system of forces which consist of the downward loads and the upward reactions. The loads are usually given in intensity and position, and it is required to find the reactions. This is effected by the application of the fundamental conditions of static equilibrium, which for a system of vertical forces, are,

$$\Sigma \text{ of all vertical forces} = 0,$$
$$\Sigma \text{ of moments of all forces} = 0.$$

The first of these conditions says that the sum of all the loads on the beam is equal to the sum of the reactions. Hence if there be but one support, this condition gives at once the reaction.

For two supports the second condition must be used in connection with the first. The center of moments may be taken anywhere in the plane, but it is more convenient to take it at one of the supports. For example, consider a single concentrated

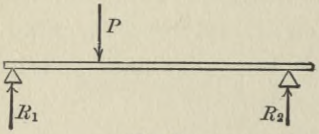

Fig. 8.

load P situated at 4 feet from the left end of a simple beam whose span is 13 feet. The equation of moments, with the center at the left support, is $13R_2 - 4P = 0$, from which $R_2 = \frac{4}{13}P$. Again the equation of moments, with the center at the right support, is $13R_1 - 9P = 0$, from which $R_1 = \frac{9}{13}P$. As a check it may be observed that $R_1 + R_2 = P$.

For a uniform load over a simple beam it is evident, without applying the conditions of equilibrium, that each reaction is one-half the load.

The reactions due to both uniform and concentrated loads on a simple beam may be obtained by adding together the reactions due to the uniform load and each concentrated load, or they may be computed in one operation. As an example of the latter method let Fig. 9 represent a simple beam 12 feet in length between the supports and weighing 35 pounds per

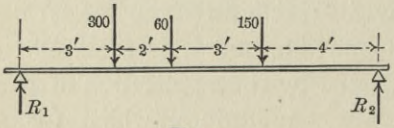

Fig. 9.

linear foot, its total weight being 420 pounds. Let there be three concentrated loads of 300, 60, and 150 pounds placed at 3, 5, and 8 feet respectively from the left support. To find the right reaction R_2 the center of moments is taken at the left support, and the weight of the beam regarded as concentrated at its middle; then the equation of moments is,

$$R_2 \times 12 = 420 \times 6 + 300 \times 3 + 60 \times 5 + 150 \times 8$$

from which $R_2 = 410$ pounds. In like manner to find R_1 the center of moments is taken at the right support, and

$$R_1 \times 12 = 420 \times 6 + 300 \times 9 + 60 \times 7 + 150 \times 4$$

from which $R_1 = 520$ pounds. As a check the sum of R_1 and R_2 is seen to be 930 pounds which is the same as the weight of the beam and the three loads.

When there are more than two supports the problem of finding the reactions from the principles of statics becomes indeterminate, since two conditions of equilibrium are only sufficient to determine two unknown quantities. By introducing, however, the elastic properties of the material, the reactions of continuous beams may be deduced, as will be explained in Chapter IV.

Prob. 28. A simple beam 12 feet long weighs 20 pounds per linear foot and carries a load of 500 pounds. Where should this load be put so that one reaction may be double the other?

Prob. 29. A simple beam weighing 30 pounds per linear foot is 18 feet long. A weight of 700 pounds is placed 5 feet from the left end and one of 500 pounds at 8 feet from the right end. Find the reactions due to the total load.

ART. 17. THE VERTICAL SHEAR.

When a beam is short it sometimes fails by shearing in a vertical section as shown in Fig. 10. The external force which produces this shearing on any section is the resultant of all the vertical forces on one side of that section. Thus, in the second diagram the resultant of all these external forces is the loads and the weight of the part of the beam on the left of the section; in the third diagram the resultant is the loads and the weight on the right, or it is reaction at the wall minus the weight of the beam between the wall and the section.

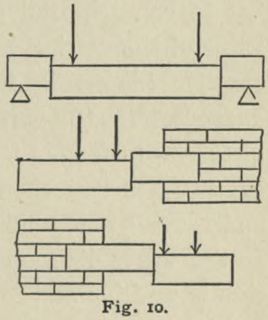

Fig. 10.

'Vertical Shear' is the name given to the algebraic sum of all external forces on the left of the section considered. Let it be denoted by V, then for any section of a simple or cantilever beam,

$$V = \text{Left reaction minus all loads on left of section.}$$

Here upward forces are regarded as positive and downward forces as negative. V is hence positive or negative according as the left reaction exceeds or is less than the loads on the left of the section. To illustrate, consider a simple beam loaded in any manner and cut at any section by a vertical plane *mn*. Let R_1 be the left and R_2 the right reaction. Let ΣP_1 denote the sum of all the loads on the left of the section and ΣP_2 the sum of those on the right. Then, from the definition,

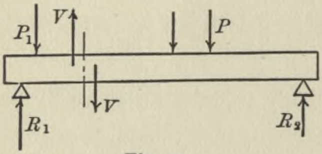

Fig. 11.

$$V = R_1 - \Sigma P_1 .$$

Since $R_1 + R_2 = \Sigma P_1 + \Sigma P_2$ it is clear if $R_1 - \Sigma P_1 = + V$ that $R_2 - \Sigma P_2 = - V$, or that the resultant of all the external forces on one side of the section is equal and opposite to the resultant of those on the other side. They form, in short, a pair of shears acting on opposite sides of the section and tending to cause a sliding or detrusion along the section. The value of the vertical shear for any section of a simple beam or cantilever is readily found by the above equation. When R_1 exceeds ΣP_1, the vertical shear V is positive, and the left part of the beam tends to slide upward relative to the right part. When R_1 is less than ΣP_1, the vertical shear V is negative, and the left part tends to slide downward relative to the other. In the upper diagram of Fig. 10 the shear in the left hand section is positive and that in the right hand section is negative.

The vertical shear varies greatly in value at different sections of a beam. Consider first a simple beam l feet long and weighing w pounds per linear foot. Each reaction is then $\frac{1}{2}wl$. Pass a plane at any distance x from the left support, then from the definition the vertical shear for that section is $V = \frac{1}{2}wl - wx$. Here it is seen that V has its greatest value $\frac{1}{2}wl$ when $x = 0$, that V decreases as x increases, and that V becomes 0 when $x = \frac{1}{2}l$. When x is greater than $\frac{1}{2}l$, V is negative and becomes $-\frac{1}{2}wl$ when $x = l$. The equation $V = \frac{1}{2}wl - wx$ is indeed the equation of a straight line, the origin being at the left support, and may be plotted so that the ordinate at any point will represent the vertical shear for the corresponding section of the beam, as shown in Fig. 12.

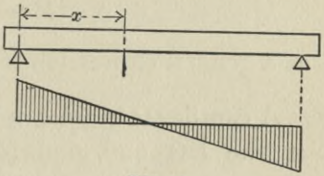

Fig. 12.

Consider again a simple beam as in Fig. 13 whose span is 12 feet and having three loads of 240, 90, and 120 pounds, situated 3, 4, and 8 feet respectively from the left support. By Art. 16 the left reaction is found to be 280 and the right reaction 170 pounds. Then for any section between the left support and the first load the vertical shear is $V = +280$ pounds, for a section between the first and second loads it is $V = 280 - 240 = +40$ pounds, for a section between the second and third loads $V = 280 - 240 - 90 = -50$ pounds, and for a section between the third load and the right support $V = 280 - 240 - 90 - 120 = -170$ pounds, which has the same numerical value as the right reaction. By laying off ordinates upon a horizontal line a graphical representation of the distribution of vertical shears throughout the beam is obtained.

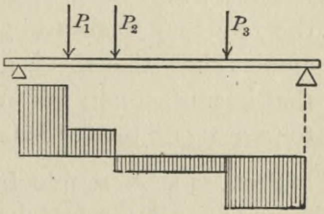

Fig. 13.

For any section of a simple beam distant x from the left support, let R_1 denote the left reaction, w the weight of the uniform load per linear unit, and ΣP_1 the sum of all the concentrated loads between the section and that support. Then the definition gives,

$$V = R_1 - wx - \Sigma P_1$$

as a general expression for the vertical shear at that section.

A cantilever beam can be so drawn that there is no reaction at the left end, and for any section $V = -wx - \Sigma P_1$. Thus, in Fig. 14, the vertical shear for a section in the space a is $V = -wx$, and for a section in the space b it is $V = -wx - P$, and the graphical representation is as shown below the beam.

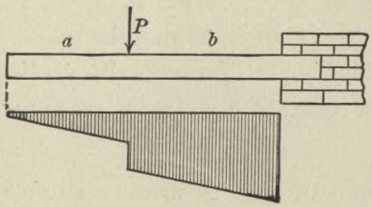

Fig. 14.

The vertical shear for any section of a beam is a measure of the tendency to shearing along that section. The above examples show that this is greatest near the supports. It is rare that beams actually fail in this manner, but it is often necessary to investigate the tendency to such failure.

Prob. 30. A simple beam 12 feet long and weighing 20 pounds per linear foot has loads of 600 and 300 pounds at 2 and 4 feet respectively from the left end. Find the vertical shears at several sections throughout the beam, and draw a diagram to show their distribution.

ART. 18. THE BENDING MOMENT.

The usual method of failure of beams is by cross-breaking or transverse rupture. This is caused by the external forces producing rotation around some point in the section of failure. Thus, in Fig. 14, let a be the distance between the end and the load P, and b be the distance between P and the wall. Then the tendency of P to cause rotation around a point in the section at the wall is measured by its moment Pb; if, however, the load were at the end its tendency to produce rotation around the same point would be measured by the moment $P(a + b)$.

'Bending moment' is the name given to the algebraic sum of the moments of the external forces on the left of the section with reference to a point in that section. Let it be denoted by M. Then, for a cantilever or simple beam,

$M =$ moment of reaction minus sum of moments of loads.

Here the moment of upward forces is taken as positive and that of downward forces as negative. M may hence be positive or negative according as the first or second term is the greater.

For a simple beam of length l, uniformly loaded, each reaction is $\frac{1}{2}wl$. For any section distant x from the left support the moment is $M = \frac{1}{2}wl \,.\, x - wx \,.\, \frac{1}{2}x$, x being the lever arm of the reaction $\frac{1}{2}wl$, and $\frac{1}{2}x$ the lever arm of the load wx. Here $M = 0$ when $x = 0$ and also when $x = l$, and M is a maximum when $x = \frac{1}{2}l$.

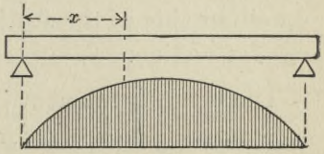

Fig. 15.

The equation, in short, is that of a parabola whose maximum ordinate is $\frac{1}{8}wl^2$ and whose graphical representation is as given in Fig. 15, each ordinate showing the value of M for the corresponding value of the abscissa x.

Consider next a simple beam loaded with only three weights P_1, P_2, and P_3. Here for any section between the left support and the first load $M = Rx$, and for any section between the first and second loads $M = Rx - P_1\,(x-a)$. Each of these expressions is the equation of a straight line, x being the abscissa and M the ordinate, and the graphical representation of bending moments is as shown in Fig. 16. It is seen that for a simple beam all the bending moments are positive.

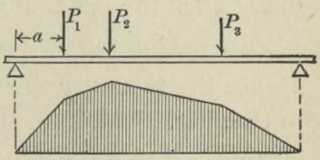

Fig. 16.

For a cantilever there is no reaction at the left end and all the bending moments are negative, the tendency to rotation thus being opposite in direction to that in a simple beam. For instance, for a cantilever beam uniformly loaded and having a load at the end the bending moment is $M = -Px - \frac{1}{2}wx^2$. Here the variation of moments may be represented by a parabola, M being 0 at the free end and a maximum at the wall.

For any given case the bending moment at any section may be found by using the definition given above. The external forces on the left of the section are taken merely for convenience, for those upon the right have also the same bending moment with reference to the section. The bending moment in all cases is a measure of the tendency of the external forces on either side of the section to turn the beam around a point in that section.

The bending moment is a compound quantity resulting from the multiplication of a force by a distance. Usually the forces are expressed in pounds and the distances in feet or inches; then the bending moments are pound-feet or pound-inches. Thus if a load of 500 pounds be at the middle of a simple beam of 8 feet span, the bending moment under the load is,

$$M = 250 \times 4 = 1\,000 \text{ pound-feet} = 12\,000 \text{ pound-inches.}$$

Again let a simple beam of 8 feet span be uniformly loaded with 500 pounds and have a weight of 200 pounds at the middle. Then the bending moment at the middle is,

$$M = 350 \times 4 - 250 \times 2 = 900 \text{ pound-feet.}$$

Hence the tendency to rupture is less in the second case than in the first.

Prob. 31. A beam 6 feet long and weighing 20 pounds per foot is placed upon a single support at its middle. Compute the bending moments for sections distant 1, 2, 3, 4, and 5 feet from the left end, and draw a curve to show the distribution of moments throughout the beam.

Prob. 32. A simple beam of 6 feet span weighs 20 pounds per linear foot and has a load of 270 pounds at 2 feet from the left end. Find the vertical shears for sections one foot apart throughout the beam, and draw the diagram of shears. Find the bending moments for the same sections and draw the diagram to represent them.

ART. 19. INTERNAL STRESSES AND EXTERNAL FORCES.

The external loads and reactions on a beam maintain their equilibrium by means of internal stresses which are generated in it. It is required to determine the relations between the external forces and the internal stresses; or, since the effect of the external forces upon any section is represented by the vertical shear (Art. 17) and by the bending moment (Art. 18), the problem is to find the relation between these quantities and the internal stresses in that section.

Consider a beam of any kind which is loaded in any manner. Imagine a vertical plane *mn* cutting the beam at any cross-section. In that section there are acting unknown stresses of various intensities and directions. Let the beam be imagined to be separated into two parts by the cutting plane and let forces *X*, *Y*, *Z*, etc., equivalent to the internal stresses, be applied to the section as shown in Fig. 17. Then the equilibrium of each part of the beam will be undisturbed, for each part will be acted upon by a system of forces in equilibrium. Hence the following fundamental principle is established.

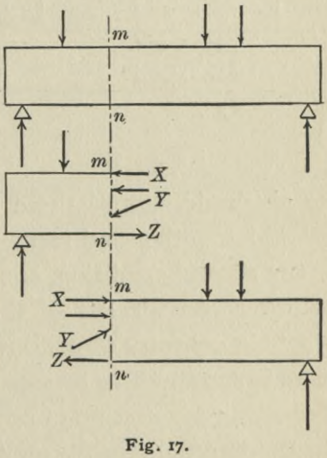

Fig. 17.

The internal stresses in any cross-section of a beam hold in equilibrium the external forces on each side of that section.

This is the most important principle in the theory of flexure. It applies to all beams, whether the cross-section be uniform or variable and whatever be the number of the spans or the nature of the loading.

Thus in the above figure the internal stresses X, Y, Z, etc., hold in equilibrium the loads and reactions on the left of the section, and also those on the right. Considering one part only a system of forces in equilibrium is seen, to which the three necessary and sufficient conditions of statics apply, namely,

$$\Sigma \text{ of all horizontal components} = 0,$$
$$\Sigma \text{ of all vertical components} = 0,$$
$$\Sigma \text{ of moments of all forces} = 0.$$

From these conditions can be deduced three laws concerning the unknown stresses in any section. Whatever be the intensity and direction of these stresses, let each be resolved into its horizontal and vertical components. The horizontal components will be applied at different points in the cross-section, some acting in one direction and some in the other, or in other words, some of the horizontal stresses are tensile and some compressive; by the first condition the algebraic sum of these is zero. The vertical components will add together and form a resultant vertical force V which, by the second condition, equals the algebraic sum of the external forces on the left of the section. As this internal force V acts in contrary directions upon the two parts into which the beam is supposed to be separated, it is of the nature of a shear. Hence for any section of any beam the following laws concerning the internal stresses may be stated.

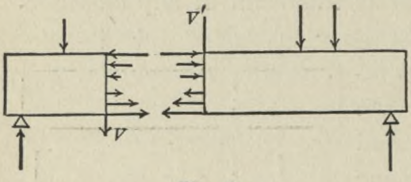

Fig. 18.

1st. The algebraic sum of the horizontal stresses is zero; or the sum of the horizontal tensile stresses is equal to the sum of the horizontal compressive stresses.

2nd. The algebraic sum of the vertical stresses forms a resultant shear which is equal to the algebraic sum of the external vertical forces on either side of the section.

3rd. The algebraic sum of the moments of the internal stresses is equal to the algebraic sum of the moments of the external forces on either side of the section.

These three theoretical laws are the foundation of the theory of the flexure of beams. Their expression may be abbreviated by introducing the following definitions.

'Resisting shear' is the name given to the algebraic sum of the internal vertical stresses in any section, and 'vertical shear' is the name for the algebraic sum of the external vertical forces on the left of the section. 'Resisting moment' is the name given to the algebraic sum of the moments of the internal horizontal stresses with reference to a point in the section, and 'bending moment' is the name for the algebraic sum of the moments of the external forces on either side of the section with reference to the same point. Then the three laws may be thus expressed for any section of any beam,

Sum of tensile stresses = Sum of compressive stresses.
Resisting shear = Vertical shear.
Resisting moment = Bending moment.

The second and third of these equations furnish the fundamental laws for investigating beams. They state the relations between the internal stresses in any section and the external forces on either side of that section. For the sake of uniformity the external forces on the left hand side of the section will generally be used, as was done in Arts. 17 and 18.

Prob. 33. A beam of weight W which is 6 feet long is sustained at one end by a force of 280 pounds acting at an angle of 60 degrees with the vertical, and at the other end by a vertical force Y and a horizontal force X. Find the values of X and Y, and the weight of the beam.

ART. 20. EXPERIMENTAL AND THEORETICAL LAWS.

From the three necessary conditions of static equilibrium, as stated in Art. 19, three important theoretical laws regarding internal stresses were deduced. These alone, however, are not sufficient for the full investigation of the subject, but recourse must be had to experience and experiment. Experience teaches that when a beam deflects one side becomes concave and the other convex, and it is reasonable to suppose that the horizontal tensile stresses are on the convex side and the compressive stresses on the concave. By experiments on beams this is confirmed and the following laws deduced.

(*F*)—The horizontal fibers on the convex side are elongated and those on the concave are shortened, while near the center is a 'neutral surface' which is unchanged in length.

(*G*)—The amount of elongation or compression of any fiber is directly proportional to its distance from the neutral surface. Hence by law (*B*) the horizontal stresses are also directly proportional to their distances from the neutral surface, provided the elastic limit of the material be not exceeded. (See Art. 50.)

From these laws there will now be deduced the following important theorem regarding the position of the neutral surface:

The neutral surface passes through the centers of gravity of the cross-sections.

To prove this let a be the area of any elementary fiber and z its distance from the neutral surface. Let S be the unit-stress on the fiber most remote from the neutral surface at the distance c. Then by law (*G*),

$$\frac{S}{c} = \text{unit-stress at the distance unity,}$$

$$\frac{S}{c}z = \text{unit-stress at the distance } z,$$

therefore $\frac{S}{c}az =$ the total stress on any fiber of area a,

and $\Sigma\frac{Saz}{c} =$ algebraic sum of all horizontal stresses.

But by the first law of Art. 19 this algebraic sum is zero, and since S and c are constants the quantity Σaz must be zero. This, however, is the condition which makes the line of reference pass through the center of gravity as is plain from the definition of the term 'center of gravity.' Therefore, the neutral surface of beams passes through the centers of gravity of the cross-sections.

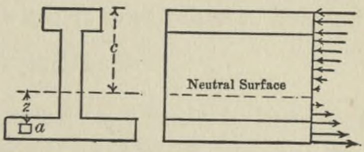

Fig. 19.

The 'neutral axis' of a cross-section is the line in which the neutral surface intersects the plane of the cross-section. On the left of Fig. 19 is shown the neutral axis of a cross-section and on the right a trace of the neutral surface.

Prob. 34. A beam 3 inches wide and 6 inches deep is loaded so that the unit-stress at the remotest fiber of a certain cross-section is 600 pounds per square inch. Find the sum of all the tensile stresses on the cross-section.

Prob. 35. A wooden beam 6 × 12 inches and five feet long is supported at one end and kept level by two horizontal forces X and Z acting at the other end in the median line of the cross-section, the former at 2 inches from the top and the latter at 2 inches from the base. Find the values of X and Z.

ART. 21. THE TWO FUNDAMENTAL FORMULAS.

Consider again any beam loaded in any manner and cut at any section by a vertical plane. The internal stresses in that section hold in equilibrium the external forces on the left of the section, and as shown in Art. 19, the following fundamental laws obtain,

Resisting shear = Vertical shear,
Resisting moment = Bending moment.

The principles established in the preceding pages can now be applied to the algebraic expression of these four quantities.

The resisting shear is the algebraic sum of all the vertical components of the internal stresses at any section of the beam. If A be the area of that section and S_s the shearing unit-stress, regarded as uniform over the area, then from formula (1),

$$\text{Resisting shear} = AS_s.$$

The vertical shear for the same section of the beam being V (Art. 17), the first of the above fundamental laws becomes,

$$AS_s = V, \tag{3}$$

which is the first fundamental formula for the discussion of beams.

The resisting moment is the algebraic sum of the moments of the internal horizontal stresses at any section with reference to a point in that section. To find an expression for its value let S be the horizontal unit-stress, tensile or compressive as the case may be, upon the fiber most remote from the neutral axis and let c be the shortest distance from that fiber to said axis. Also let z be the distance from the neutral axis to any fiber having the elementary area a. Then by law (G) and Fig. 19,

$$\frac{S}{c} = \text{unit-stress at a distance unity,}$$

$$\frac{S}{c}z = \text{unit-stress at distance } z,$$

hence $$\frac{aSz}{c} = \text{total stress on any fiber of area } a,$$

and $\frac{aSz^2}{c}$ = moment of this stress about neutral axis.

Hence $\Sigma\frac{aSz^2}{c}$ = resisting moment of horizontal stresses.

Since S and c are constants this expression may be written $\frac{S}{c}\Sigma az^2$. But Σaz^2, being the sum of the products formed by multiplying each elementary area by the square of its distance from the neutral axis, is the moment of inertia of the cross-section with reference to that axis and may be denoted by I. Therefore,

$$\text{Resisting moment} = \frac{SI}{c}.$$

The bending moment for the same section of the beam being M (Art. 18), the second of the above fundamental laws becomes,

$$\frac{SI}{c} = M, \qquad (4)$$

which is the second fundamental formula for the discussion of beams.

Experience and experiment teach that simple beams of uniform section break near the middle by the tearing or crushing of the fibers and very rarely at the supports by shearing. Hence it is formula (4) that is mainly needed in the practical investigation of beams. The following example and problem relate to formula (3) only, while formula (4) will receive detailed discussion in the subsequent articles.

As an example, consider a wrought iron I beam 15 feet long and weighing 200 pounds per yard, over which roll two locomotive wheels 6 feet apart and each bearing 12 000 pounds. The maximum vertical shear at the left support will evidently occur when one wheel is at the support (Art. 16). The reaction will then be $500 + 12\,000 + \frac{9}{15} \times 12\,000 = 19\,700$ pounds. Accordingly the greatest value of V in the beam is 19 700 pounds. As the area of the cross-section is 20 square inches the average shearing unit-stress by formula (3) is 985 pounds, so that the factor of safety is about 50.

Prob. 36. A wooden beam 6 × 9 inches and 12 feet in span carries a uniform load of 20 pounds per foot besides its own weight and also two wheels 6 feet apart, one weighing 4 000 pounds and the other 3 000 pounds. Find the factor of safety against shearing.

ART. 22. CENTER OF GRAVITY OF CROSS-SECTIONS.

The fundamental formula (4) contains c, the shortest distance from the remotest part of the cross-section to a horizontal axis passing through the center of gravity of that cross-section. The methods of finding c are explained in books on theoretical mechanics and will not here be repeated. Its values for some of the simplest cases are however recorded for reference.

For a rectangle whose height is d, $c = \frac{1}{2}d.$

For a circle whose diameter is d, $c = \frac{1}{2}d.$

For a triangle whose altitude is d, $c = \frac{2}{3}d.$

For a square with side d having one diagonal vertical, $c = d\sqrt{\frac{1}{2}}$

For a I whose depth is d, $c = \frac{1}{2}d.$

For a ⊥ whose depth is d, thickness of flange t, width of flange b, and thickness of web t',

$$c = \frac{\frac{1}{2}t'd^2 + t(b - t')(d - \frac{1}{2}t)}{t'd + t(b - t')}$$

For a trapezoid whose depth is d, upper base b, and lower base b',

$$c = \frac{b + 2b'}{b + b'} \cdot \frac{d}{3}.$$

The student should be prepared to readily apply the principle of moments to the deduction of the numerical value of c for any given cross-section. In nearly all cases the given area may be divided into rectangles, triangles, and circular areas, whose centers of gravity are known, so that the statement of the equation for finding c is very simple.

Prob. 37. Find the value of c for a rail headed beam whose section is made up of a rectangular flange $\frac{3}{4} \times 4$ inches, a rectangular web $\frac{1}{2} \times 5$ inches, and an elliptical head $\frac{3}{4}$ inches deep and $1\frac{1}{2}$ inches wide.

ART. 23. MOMENT OF INERTIA OF CROSS-SECTIONS.

The fundamental formula (4) contains I, the moment of inertia of the cross-section of the beam with reference to a horizontal axis passing through the center of gravity of that cross-section. Methods of determining I are explained in works on elementary mechanics and will not here be repeated, but the values of some of the most important cases are recorded for reference.

For a rectangle of base b and depth d, $I = \frac{bd^3}{12}$.

For a circle of diameter d, $I = \frac{\pi d^4}{64}$.

For an ellipse with axes a and b, the latter vertical, $I = \frac{\pi ab^3}{64}$.

For a triangle of base b and depth d, $I = \frac{bd^3}{36}$.

For a square with side d, having one diagonal vertical, $I = \frac{d^4}{12}$.

For a I with base b, depth d, thickness of flanges t and thickness of web t',

$$I = \frac{bd^3 - (b - t')(d - 2t)^3}{12}.$$

For a ⊥ with base b, depth d, thickness of flange t, thickness of web t' and area A, $I = \frac{bd^3 - (b - t')(d - t)^3}{3} - Ac^2$.

The value of I for any given section may always be computed by dividing the figure into parts whose moments of inertia are known and transferring these to the neutral axis by means of the familiar rule $I_1 = I_0 + Ah^2$, where I_0 is the primitive value for an axis through the center of gravity, I_1 the value for any parallel axis, A the area of the figure and h the distance between the two axes.

Prob. 38. Compute the least moment of inertia of a trapezoid whose altitude is 3 inches, upper base 2 inches, and lower base 5 inches.

Prob. 39. Find the moment of inertia of a triangle with reference to its base, and also with reference to a parallel axis passing through its vertex.

ART. 24. THE MAXIMUM BENDING MOMENT.

The fundamental equation (4), namely $\frac{SI}{c} = M$, is true for any section of any beam, I being the moment of inertia of that section about its neutral axis, c the vertical distance from that axis to the remotest fiber, S the tensile or compressive unit-stress on that fiber, and M the bending moment of all the external forces on one side of the section. For a beam of constant cross-section S varies directly as M, and the greatest S will be found where M is a maximum. The place where M has its maximum value may hence be called the 'dangerous section,' it being the section where the horizontal fibers are most highly strained.

For a simple beam uniformly loaded with w pounds per linear unit, the dangerous section is evidently at the middle, and, as shown in Art. 18, the maximum M is $\frac{wl^2}{8}$.

For a simple beam loaded with a single weight P at the distance p from the left support, the left reaction is $R = P\frac{l-p}{l}$, and the maximum moment is $\frac{P(l-p)p}{l}$. If P be movable the distance p will be variable, and when the load is at the middle the maximum M is $\frac{1}{4}Pl$.

For a beam loaded with given weights, either uniform or concentrated, it may be shown that the dangerous section is at the point where the vertical shear passes through zero. To prove this let P_1 be any concentrated load on the left of the section and p its distance from the left support, and w the uniform load per linear unit. Then, for any section distant x from the left support,

$$M = R_1x - wx \cdot \frac{x}{2} - \Sigma P_1(x-p).$$

To find the value of x which renders this a maximum, the first derivative must be put equal to zero; thus,

$$\frac{dM}{dx} = R_1 - wx - \Sigma P_1 = 0.$$

But $R_1 - wx - \Sigma P_1$ is the vertical shear V for the section x (see Art. 17). Therefore the maximum moment occurs at the section where the vertical shear passes through zero.

To find the dangerous section for any given case the reactions are first to be computed by Art. 16, and then the vertical shears are to be investigated by Art. 17. For a cantilever, however it be loaded, it is seen that the dangerous section is at the wall. For a simple beam with concentrated loads the point where the vertical shear passes through zero must usually be ascertained by trial. Thus, referring to Fig. 9 and the example in Art. 16, the vertical shear just at the left of the first load is $V = 520 - 3 \times 35 = +415$ pounds, and just at the right of the first load it is $V = 520 - 3 \times 35 - 300 = +115$ pounds. Again for the second load the vertical shear just at the left is $V = 520 - 5 \times 35 - 300 = +45$ pounds, and just at the right it is $V = 520 - 5 \times 35 - 360 = -15$ pounds. Hence in this case the vertical shear changes sign, or passes through zero, under the second load, and accordingly this is the position of the dangerous section.

When the dangerous section has been found the bending moment for that section is to be computed by the definition of Art. 18, and this will be the maximum bending moment for the beam. Thus, for the numerical example of the last paragraph, the maximum bending moment is,

$$M = 520 \times 5 - 175 \times 2\tfrac{1}{2} - 300 \times 2 = + 1562.5 \text{ pound-feet.}$$

Again, let a cantilever beam 8 feet long be loaded with 40 pounds per linear foot and carry a weight of 150 pounds at the free end; then the maximum bending moment is,

$$M = - 320 \times 4 - 150 \times 8 = - 2\,480 \text{ pound-feet.}$$

The bending moment for simple beams is seen to be always positive and for cantilever beams always negative. That is to say, in the former case the exterior forces on the left of the section cause compression in the upper and tension in the lower fibers of the beam, while in the latter case this is reversed; or the upper side of a deflected simple beam is concave and the upper side of a deflected cantilever beam is convex.

Prob. 40. A simple beam 12 feet long carries a load of 150 pounds at 5 feet from the left end and a load of 150 pounds at 5 feet from the right end. Find the dangerous section, and the maximum bending moment.

Prob. 41. A simple beam 12 feet long weighs 20 pounds per foot and carries a load of 100 pounds at 4 feet from the left end and a load of 50 pounds at 7 feet from the left end. Find the dangerous section, and the maximum bending moment.

ART. 25. THE INVESTIGATION OF BEAMS.

The investigation of a beam consists in deducing the greatest horizontal unit-stress S in the beam from the fundamental formula (4). This may be written,

$$S = \frac{Mc}{I}.$$

First, from the given dimensions find, by Art. 22, the value of c and by Art. 23 the value of I. Then by Art. 24 determine the value of maximum M. From (4) the value of S is now known. Usually c and I are taken in inches, and M in pound-inches; then the value of S will be in pounds per square inch.

The value of S will be tension or compression according as the remotest fiber lies on the concave or convex side of the beam. If S' be the unit-stress on the opposite side of the beam and c' the distance from the neutral axis, then from law (G),

$$\frac{S}{c} = \frac{S'}{c'} \qquad \text{and} \qquad S' = S\frac{c'}{c}.$$

If S be tension, S' will be compression, and *vice versa*. Sometimes it is necessary to compute S' as well as S in order to thoroughly investigate the stability of the beam. By comparing the values of S and S' with the proper working unit-stresses for the given materials (Art. 8), the degree of security of the beam may be inferred.

As an example consider a wrought iron **I** beam whose depth is 12 inches, width of flange 4.5 inches, thickness of flange 1 inch and thickness of web 0.78 inches. It is supported at its ends forming a span of 12 feet, and carries two loads each weighing 10 000 pounds, one being at the middle and the other at one foot from the right end.

By Art. 1, $w = 56$ pounds per linear foot.
By Art. 16, $R = 6169$ pounds.
By Art. 22, $c = 6$ inches.
By Art. 23, $I = 338$ inches^4.
By Art. 24, $x = 6$ feet for dangerous section.
By Art. 24, max. $M = 36\,006 \times 12$ pound-inches.

Then from formula (4) the unit-stress at the dangerous section is,

$$S = \frac{36\,000 \times 12 \times 6}{338} = 7\,700 \text{ pounds per square inch.}$$

This is the compressive unit-stress on the upper fiber and also the tensile unit-stress on the lower fiber, and being only about one-third of the elastic limit for wrought iron and about one-seventh of the ultimate strength it appears that the beam is entirely safe for steady loads (Art. 8). It will usually be best in solving problems to insert all the numerical values at first in the formula and thus obtain the benefit of cancellation.

A short beam heavily loaded should also be investigated for the shearing stress at the supports in the manner mentioned in Art. 21, but in ordinary cases there is little danger from this cause. Thus for the above example the maximum vertical shear occurs at the right end and is 14 500 pounds; as the area of the cross-section is 16.8 square inches, the mean shearing unit-stress at the right end is from (3),

$$S_c = \frac{14\,500}{16.8} = 863 \text{ pounds per square inch,}$$

so that the factor of safety against shearing is nearly 60.

Prob. 42. A piece of scantling 2 inches square and 10 feet long is hung horizontally by a rope at each end and three painters stand upon it. Is it safe?

Prob. 43. A wrought iron bar one inch in diameter and two feet long is supported at its middle and a load of 500 pounds hung upon each end of it. Find its factor of safety.

ART. 26. SAFE LOADS FOR BEAMS.

The proper load for a beam should not make the value of S at the dangerous section greater than the allowable unit-stress. This allowable unit-stress or working strength is to be assumed according to the circumstances of the case by first selecting a suitable factor of safety from Art. 8 and dividing the ultimate strength of the material by it, the least ultimate strength whether tensile or compressive being taken. For any given beam the quantities I and c are known. Then, by the general formula (4), the bending moment M may be expressed in terms of the unknown loads on the beam, and thus those loads be found. The sign of the bending moment should not be used in (4), since that sign merely denotes whether the upper fiber of the beam is in tension or compression, or indicates the direction in which the external forces tend to bend it.

As an example, consider a cantilever beam whose length is 6 feet, breadth 2 inches, depth 3 inches and which is loaded uniformly with w pounds per linear foot. It is required to find the value of w so that S may be 800 pounds per square inch. Here $c = 1\frac{1}{2}$ inches, $I = \frac{54}{12}$, and $M = 36 \times 6w$. Then from formula (4),

$$216w = \frac{800 \times 54}{1\frac{1}{2} \times 12}, \qquad \text{whence} \qquad w = 11 \text{ pounds.}$$

Since a wooden beam 2×3 inches weighs about 2 pounds per linear foot, the safe load in this case will be about 9 pounds per foot.

Prob. 44. A wooden beam 8×9 inches and of 14 feet span carries a load, including its own weight, of w pounds per linear foot. Find the value of w for a factor of safety of 10.

Prob. 45. A steel railroad rail of 2 feet span carries a load P at the middle. If its weight per yard is 56 pounds, $I = 12.9$ inches^4 and $c = 2.16$ inches, find P so that the greatest horizontal unit-stress at the dangerous section shall be 6 000 pounds per square inch.

ART. 27. DESIGNING OF BEAMS.

When a beam is to be designed the loads to which it is to be subjected are known, as also is its length. Thus the maximum bending moment may be found. The allowable working strength S is assumed in accordance with engineering practice. Then formula (4) may be written,

$$\frac{I}{c} = \frac{M}{S},$$

and the numerical value of the second member be found. The dimensions to be chosen for the beam must give a value of $\frac{I}{c}$ equal to this numerical value, and these in general are determined tentatively, certain proportions being first assumed. The selection of the proper proportions and shapes of beams for different cases requires much judgment and experience. But whatever forms be selected they must in each case be such as to satisfy the above equation.

For instance, a wrought iron beam of 4 feet span is required to carry a rolling load of 500 pounds. Here, by Art. 24, the value of maximum M due to the load of 500 pounds is 6 000 pound-inches. From Art. 8 the value of S for a variable load is about 10 000 pounds per square inch. Then,

$$\frac{I}{c} = \frac{6\,000}{10\,000} = 0.6 \text{ inches}^3.$$

An infinite number of cross-sections may be selected with this value of $\frac{I}{c}$. If the beam is to be round and of diameter d, it is known that $c = \frac{1}{2}d$ and $I = \frac{\pi d^4}{64}$. Hence,

$$\frac{\pi d^3}{32} = 0.6, \qquad \text{whence} \qquad d = 1.83 \text{ inches}.$$

If the cross-section is to be rectangular, the dimensions 1×2 inches would give the value of $\frac{I}{c}$ as $\frac{2}{3}$ which would be a little too large, but it would be well to use it because the weight of the beam itself has not been considered in the discussion. If thought necessary these dimensions may now be investigated by Art. 25 in order to determine how closely the actual unit-stress agrees with the value assumed. Thus the rectangular section 1×2 inches weighs $6\frac{2}{3}$ pounds per foot; the maximum bending moment is then 6 160 pound-inches, and the unit-stress is found to be 9 240 pounds per square inch.

Prob. 46. Design a hollow circular wrought iron beam for a span of 12 feet to carry a load of 320 pounds per linear foot.

Prob. 47. A rectangular wooden beam of 14 feet span carries a load of 1 000 pounds at its middle. If its width is 4 inches find its depth for a factor of safety of 10.

ART. 28. THE MODULUS OF RUPTURE.

The fundamental formula (4) is only true for stresses within the elastic limit, since beyond that limit the law (G) does not hold, and the horizontal unit-stresses are no longer proportional to their distances from the neutral axis, but increase in a less rapid ratio. The sketch shows a case where the fiber stresses above m and below n have surpassed the elastic limit. It is however very customary in practical computations to apply (4) to the rupture of beams.

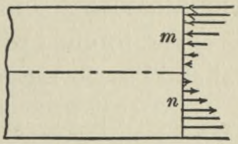

Fig. 20.

The 'modulus of rupture' is the value of S deduced from formula (4) when the beam is loaded up to the breaking point. It is always found by experiment that the modulus of rupture does not agree with either the ultimate tensile or compressive strength of the material but is intermediate between them. If formula (4) were valid beyond the elastic limit, the value of S for rupture would agree with the least ultimate strength, with tension in the case of cast iron and with compression in the case of timber. The modulus of rupture is denoted by S_r.

The average values of the modulus of rupture are given in the following table, which also contains the average ultimate tensile and compressive strengths, previously stated in Arts. 5 and 6, all in pounds per square inch.

Material.	Tensile Strength, S_t.	Modulus of Rupture, S_r.	Compressive Strength, S_c.
Timber	10 000	9 000	8 000
Brick		800	2 500
Stone		2 000	6 000
Cast Iron	20 000	35 000	90 000
Wrought Iron	55 000		55 000
Steel	100 000		150 000

By the use of the experimental values of the modulus of rupture it is easy with the help of formula (4) to determine what load will cause the rupture of a given beam, or what must be its length or size in order that it may rupture under assigned loads. The formula when used in this manner is entirely empirical and has no rational basis.

Prob. 48. What must be the size of a square wooden beam of 8 feet span in order to break under its own weight?

Prob. 49. A cast iron cantilever beam 2 inches square and 6 feet long carries a load P at the end. Find the value of P to cause rupture.

ART. 29. COMPARATIVE STRENGTHS.

The strength of a beam is measured by the load that it can carry. Let it be required to determine the relative strength of the four following cases,

1st, A cantilever loaded at the end with W,
2nd, A cantilever uniformly loaded with W,
3rd, A simple beam loaded at the middle with W,
4th, A simple beam loaded uniformly with W.

Let l be the length in each case. Then, from Art. 24 and formula (4),

For 1st, $M = Wl$ and hence $W = \frac{SI}{lc}$.

For 2nd, $M = \frac{Wl}{2}$ and hence $W = 2\frac{SI}{lc}$.

For 3rd, $M = \frac{Wl}{4}$ and hence $W = 4\frac{SI}{lc}$.

For 4th, $M = \frac{Wl}{8}$ and hence $W = 8\frac{SI}{lc}$.

Therefore the comparative strengths of the four cases are as the numbers 1, 2, 4, 8. That is, if four such beams be of equal size and length and of the same material, the 2nd is twice as strong as the 1st, the 3rd four times as strong, and the 4th eight times as strong. From these equations also result the following important laws.

The strength of a beam varies directly as S, directly as I, inversely as c, and inversely as the length l.

A load uniformly distributed produces only one-half as much stress as the same load when concentrated.

These apply to all cantilever and simple beams whatever be the shape of the cross-section.

When the cross-section is rectangular, let b be the breadth and d the depth, then (Art. 23) the above equations become,

$$W = n\frac{Sbd^2}{6l},$$

where n is either 1, 2, 4, or 8, as the case may be. Therefore,

The strength of a rectangular beam varies directly as the breadth and directly as the square of the depth.

The reason why rectangular beams are put with the greatest dimensions vertical is now apparent.

To find the strongest rectangular beam that can be cut from a circular log of given diameter D, it is necessary to make bd^2 a maximum. Or the value of b is to be found which makes $b(D^2 - b^2)$ a maximum. By placing the first derivative equal to zero this value of b is readily found. Thus,

$$b = D\sqrt{\tfrac{1}{3}} \qquad \text{and} \qquad d = D\sqrt{\tfrac{2}{3}}.$$

Hence very nearly, $b:d::5:7$. From this it is evident that the way to lay off the strongest beam on the end of a circular log is to divide the diameter into three equal parts, from the points of division draw perpendiculars to the circumference, and then join the points of intersection with the ends of the diameter, as shown in the figure.

Fig. 21.

The beam thus cut out is, of course, not as strong as the log, and the ratio of the strength of the beam to that of the log is that of their values of $\frac{I}{c}$, which will be found to be about 0.65.

Prob. 50. Compare the strength of a rectangular beam 2 inches wide and 4 inches deep with that of a circular beam 3 inches in diameter.

Prob. 51. Compare the strength of a wooden beam 4×6 inches and 10 feet span with that of a wrought iron beam 1×2 inches and 7 feet span.

ART. 30. IRON AND STEEL I BEAMS.

Wrought iron **I** beams are rolled at present in about thirteen different depths or sizes; of each there is a light and a heavy weight, and weights intermediate in value may also be obtained. They are extensively used in engineering and architecture. The following table gives mean sizes, weights, and moments of inertia of wrought iron beams most commonly found in the market. The sizes of different manufacturers agree as to depth, but vary slightly with regard to proportions of cross-section, weights per foot, and moments of inertia. Fig. 22 shows the proportions of the light and heavy 6 inch beams. The cross-section of any beam in the table is obtained from its weight per

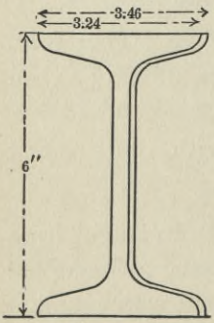

Fig. 22.

foot by multiplying by 3 and dividing by 10, in accordance with the rule of Art. 1.

The moments of inertia in the fourth column of the table are taken about an axis perpendicular to the web at the center, this being the neutral axis of the cross-section when used as a beam. The values of I' are with reference to an axis coinciding with the center line of the web and are for use in Chapter V.

Size. Depth. Inches.	Width of Flange. Inches.	Weight per foot. Pounds.	I. Inches[4].	$\frac{I}{c}$. Inches[3].	I'. Inches[4].
Heavy 15	5.81	80	750	100.	29.9
Light 15	5.55	67	677	90.3	25.4
H 15	5.33	65	614	81.9	20.0
L 15	5.03	50	530	70.6	16.3
H 12	5.09	60	340	56.7	15.5
L 12	4.64	42	275	45.9	11.0
H 10½	4.92	45	201	38.3	10.7
L 10½	4.54	31½	165	31.4	8.01
H 10	4.77	45	187	37.5	11.3
L 10	4.32	30	150	30.0	7.94
H 9	4.33	33	117	26.0	7.14
L 9	4.01	23½	97.5	21.7	5.48
H 8	4.29	35	90.4	22.6	6.96
L 8	3.81	22	69.6	17.5	4.57
H 7	3.91	25	54.3	15.5	4.87
L 7	3.61	18	45.8	13.1	3.72
H 6	3.46	18	28.4	9.48	2.51
L 6	3.24	13½	24.5	8.16	2.00
H 5	2.91	13	14.2	5.69	1.34
L 5	2.73	10	12.3	4.94	1.08
H 4	2.63	10	6.99	3.50	0.87
L 4	2.48	8	6.19	3.10	0.71
H 3	2.52	9	3.54	2.36	0.84
L 3	2.32	7	3.09	2.06	0.55

In investigating the strength of a given I beam the value of $\frac{I}{c}$ is taken from the table and S is computed from formula (4). In designing an I beam for a given span and loads the value of $\frac{I}{c}$ is found by (4) from the data and then from the table that I is selected which has the nearest or next highest corresponding value. Intermediate weights between those given in the table can also usually be obtained; thus, if the computed value of $\frac{I}{c}$ should be 34.0 a 10-inch beam weighing about 38 pounds per foot might be chosen.

For example, let it be required to determine which I should be selected for a floor loaded with 150 pounds per square foot, the beams to be of 20 feet span and spaced 12 feet apart between centers, and the maximum unit-stress S to be 12 000 pounds per square inch. Here the uniform load on the beam is $12 \times 20 \times 150 = 36\,000$ pounds $= W$. From formula (4),

$$\frac{I}{c} = \frac{M}{S} = \frac{36\,000 \times 20 \times 12}{8 \times 12\,000} = 90.$$

and hence from the table, the light 15 inch I should be selected.

I beams of mild steel, about 10 per cent stronger than wrought iron, are now much used, as also other shapes.

Prob. 52. A heavy 15 inch I beam of 12 feet span sustains a uniformly distributed load of 41 net tons. Find its factor of safety. Also the factor of safety for a 24 feet span under the same load.

Prob. 53. A floor, which is to sustain a uniform load of 175 pounds per square foot, is to be supported by heavy 10 inch I beams of 15 feet span. Find their proper distance apart from center to center so that the maximum fiber stress may be 12 000 pounds per square inch.

ART. 31. IRON AND STEEL DECK BEAMS.

Deck beams are used in the construction of buildings, and are of a section such as shown in Fig. 23. The heads are formed with arcs of circles but may be taken as elliptical in computing the values of c and I. The following table gives dimensions of a few wrought iron sections found in the market.

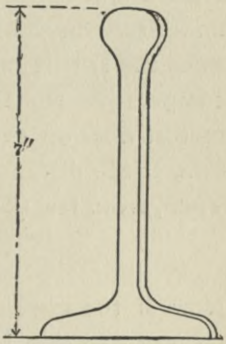
Fig. 23.

By means of formula (4) a given deck beam may be investigated or safe loads be determined for it, or one may be selected for a given load and span. Sometimes T irons are used instead of deck beams; the values of c and I for these are given in the handbooks issued by the manufacturers, or they may be computed with an accuracy usually sufficient by regarding the web and flange as rectangular (Arts. 22 and 23).

Size. Depth. Inches.	Width of Flange. Inches.	Thickness of Web. Inches.	Weight per ft. Pounds.	c. Inches.	I. Inches^4.	$\frac{I}{c}$. Inches^3.
Heavy 9	3.97	0.625	30	4.59	91.9	20.0
Light 9	3.75	0.406	$23\frac{1}{2}$	4.60	78.6	17.1
H 8	4.00	0.750	28	4.49	63.3	14.1
L 8	3.75	0.500	$21\frac{1}{2}$	4.58	52.1	11.6
H 7	3.75	0.625	23	3.98	43.0	10.8
L 7	3.50	0.375	17	4.00	34.4	8.6

Prob. 54. A heavy 7 inch deck beam is loaded uniformly with 50 000 pounds. Find its factor of safety for a span of 22 feet. Also for a span of 11 feet.

Prob. 55. What uniform load should be placed upon a heavy 7 inch deck beam of 22 feet span so that the greatest unit-stress at the dangerous section may be 12 000 pounds per square inch?

ART. 32. CAST IRON BEAMS.

Wrought iron beams are usually made with equal flanges since the resistance of wrought iron is about the same for both tension and compression. For cast iron, however, the flange under tension should be larger than that under compression, since the tensile resistance of the material is much less than its compressive resistance. Let S' be the unit-stress on the remotest fiber on the tensile side and S that on the compressive side, at the distances c' and c respectively from the neutral axis. Then, from law (G),

$$\frac{c}{c'} = \frac{S}{S'}.$$

Now if the working values of S and S' can be selected the ratio of c to c' is known and a cross-section can be designed, but it is difficult to assign these proper values on account of our lack of knowledge regarding the elastic limits of cast iron.

According to HODGKINSON'S investigations the following are dimensions for a cast iron beam of equal ultimate strength.

Thickness of web	=	t,
Depth of beam	=	$13.5t$,
Width of tensile flange	=	$12t$,
Thickness of tensile flange	=	$2t$,
Width of compressive flange	=	$5t$,
Thickness of compressive flange	=	$1\frac{1}{3}t$,
Value of c	=	$9t$,
Value of I	=	$923t^4$.

Here the unit-stress in the tensile flange is one-half that in the compressive flange. Although these proportions may be such as to allow the simultaneous rupture of the flanges, yet it does not necessarily follow that they are the best proportions for ordinary working stresses, since the factors of safety in the flanges as computed by the use of formula (4) would be quite different. The proper relative proportions of the flanges of cast iron beams for safe working stresses have never been definitely established, and on account of the extensive use of wrought iron the question is not now so important as formerly.

As an illustration of the application of formula (4) let it be required to determine the total uniform load W for a cast iron ⊥ beam of 14 feet span, so that the factor of safety may be 6, the depth of the beam being 18 inches, the width of the flange 12 inches, the thickness of the stem 1 inch, and the thickness of the flange $1\frac{1}{4}$ inches. First, from Art. 22 the value of c is found to be 12.63 inches, and that of c' to be 5.37 inches. From Art. 23 the value of I is computed to be 1 031 inches^4. From Art. 24 the maximum bending moment is,

$$M = \frac{wl^2}{8} = 21\,W \text{ pound-inches.}$$

Now with a factor of safety of 6 the working strength S on the remotest fiber of the stem of the dangerous section is to be $\frac{90\,000}{6}$ pounds per square inch. Hence from formula (4),

$$21\,W = \frac{90\,000 \times 1\,031}{6 \times 12.63}, \quad \text{whence} \quad W = 58\,300 \text{ pounds.}$$

Again with a factor of safety of 6 the working strength S' on the remotest fiber of the flange at the dangerous section is to be $\frac{20\,000}{6}$ pounds per square inch. Hence from the formula,

$$21\,W = \frac{20\,000 \times 1\,031}{6 \times 5.37}, \quad \text{whence} \quad W = 30\,400 \text{ pounds.}$$

The total uniform load on the beam should hence not exceed 30 400 pounds. Under this load the factor of safety on the tensile side is 6, while on the compressive side it is nearly 12.

Prob. 56. A cast iron beam in the form of a channel, or hollow half rectangle, is often used in buildings. Suppose the thickness to be uniformly one inch, the base 8 inches, the height

6 inches and the span 12 feet. Find the values of S and S' at the dangerous section under a uniform load of 5 000 pounds.

ART. 33. GENERAL EQUATION OF THE ELASTIC CURVE.

When a beam bends under the action of exterior forces the curve assumed by its neutral surface is called the elastic curve. It is required to deduce a general expression for its equation.

Let *pp* in the figure be any normal section in any beam. Let *mn* be any short length *dl*, measured on the neutral surface, and let *qmq* be drawn parallel to the normal section through *n*. Previous to the bending the sections *pp* and *tt* were parallel; now they intersect at *o* the center of curvature. Previous to the bending *pt* was equal to *dl*, now it has elongated or shortened the amount *pq*. The distance *pq* will be called λ and the distance *mp* is the quantity c (Art. 22). The elongation λ is produced by the unit-stress S, and from (2) its value is,

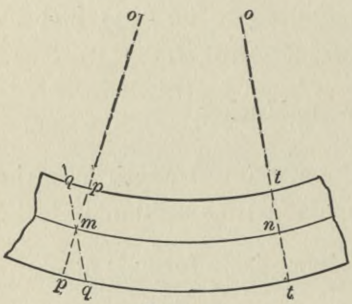

Fig. 24.

$$\lambda = \frac{Sdl}{E},$$

where E is the coefficient of elasticity of the material of the beam. From the similar figures *omn* and *mpq*,

$$\frac{om}{mn} = \frac{mp}{pq}, \qquad \text{or} \qquad \frac{R}{dl} = \frac{c}{\lambda},$$

where R is the radius of curvature *om*. Inserting in this the above value of λ, it becomes,

$$\frac{S}{c} = \frac{E}{R}.$$

But the fundamental formula (4) may be written in the form

$$\frac{S}{c}=\frac{M}{I}.$$

and hence, by comparison,

$$M=\frac{EI}{R}.$$

This is the formula which gives the relation between the bending moment of the exterior forces and the radius of curvature at any section. Where M is 0 the radius R is ∞; where M is a maximum R has its least value.

Now, in works on the differential calculus, the following value is deduced for the radius of curvature of any plane curve whose abscissa is x, ordinate y, and length l, namely,

$$R=\frac{dl^3}{dx\,.\,d^2y}.$$

Hence the most general equation of the elastic curve is,

$$\frac{dl^3}{dx\,.\,d^2y}=\frac{EI}{M},$$

which applies to the flexure of all bodies governed by the laws of Arts. 3 and 20.

In discussing a beam the axis of x is taken as horizontal and that of y as vertical. Experience teaches us that the length of a small part of a bent beam does not materially differ from that of its horizontal projection. Hence dl may be placed equal to dx for all beams, and the above equation reduces to the form,

(5) $$\frac{d^2y}{dx^2}=\frac{M}{EI}.$$

This is the general equation of the elastic curve, applicable to all beams whatever be their shapes, loads or number of spans. M is the bending moment of the external forces for any section whose abscissa is x, and whose moment of inertia with respect to the neutral axis is I. Unless otherwise stated I will be regarded as constant, that is, the cross-section of the beam is constant throughout its length.

To obtain the particular equation of the elastic curve for any special case, it is first necessary to express M as a function of x and then integrate the general equation twice. The ordinate y will then be known for any value of x. It should, however, be borne in mind that formula (5), like formula (4), is only true when the unit-stress S is less than the elastic limit of the material.

Prob. 57. A wooden beam $\frac{1}{2}$ inch wide, $\frac{3}{4}$ inch deep, and 3 feet span carries a load of 14 pounds at the middle. Find the radius of curvature for the middle, quarter points, and ends.

ART. 34. DEFLECTION OF CANTILEVER BEAMS.

Case I. A load at the free end.—Take the origin of co-ordinates at the free end, and as in Fig. 25, let m be any point of the elastic curve whose abscissa is x and ordinate y.

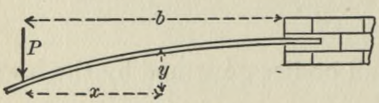

Fig. 25.

For this point the bending moment M is $-Px$ and the general formula (5) becomes,

$$EI\frac{d^2y}{dx^2} = -Px.$$

By integration the first derivative is found to be

$$EI\frac{dy}{dx} = -\frac{Px^2}{2} + C.$$

But $\frac{dy}{dx}$ is the tangent of the angle which the tangent to the elastic curve at m makes with the axis of x, and as the beam is fixed at the wall the value of $\frac{dy}{dx}$ is 0 when x equals l. Hence $C = \frac{1}{2}Pl^2$, and the first differential equation is,

$$EI\frac{dy}{dx} = \frac{Pl^2}{2} - \frac{Px^2}{2}.$$

The second integration now gives,

$$EIy = \frac{Pl^2x}{2} - \frac{Px^3}{6} + C'.$$

But $y = 0$, when $x = 0$. Hence $C' = 0$, and

$$6EIy = P(3l^2x - x^3),$$

which is the equation of the elastic curve for a cantilever of length l with a load P at the free end. If $x = l$ the value of y will be the maximum deflection, which may be represented by Δ. Then,

$$\Delta = \frac{Pl^3}{3EI},$$

and for any point of the beam the deflection is $\Delta - y$.

Case II. A uniform load.—Let the origin be taken at the free end as before, and x and y be the co-ordinates of any point of the elastic curve. Let the load per linear unit be w. Then for any section $M = -\frac{1}{2}wx^2$ and formula (5) becomes,

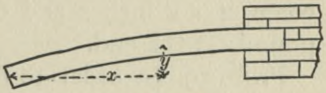

Fig. 26.

$$EI\frac{d^2y}{dx^2} = -\frac{wx^2}{2}.$$

Integrate this, determine the constant of integration by the consideration that $\frac{dy}{dx} = 0$ when $x = l$, and then,

$$6EI\frac{dy}{dx} = wl^3 - wx^3.$$

Integrate again, and after determining the constant, the equation of the elastic curve is,

$$24EIy = w(4l^3x - x^4),$$

which is a biquadratic parabola. For $x = l, y = \Delta$ the maximum deflection, whose value is,

$$\Delta = \frac{wl^4}{8EI} = \frac{Wl^3}{8EI},$$

where W is the total uniform load on the cantilever.

Case III. A load at the free end and also a uniform load.—Here it is easy to show that the maximum deflection is

$$\Delta = \frac{8Pl^3 + 3Wl^3}{24EI},$$

which is the sum of the deflections due to the two loads. Hence it appears that, as in cases of stress, each load produces its effect independently of the other.

In order that the formulas for deflection may be true, the maximum unit-stress S produced by all the loads must not exceed the elastic limit of the material.

Prob. 58. Compute the deflection of a cast iron cantilever beam, 2 × 2 inches and 6 feet span, caused by a load of 100 pounds at the end.

Prob. 59. In order to find the coefficient of elasticity of a cast iron bar 2 inches wide, 4 inches deep, and 6 feet long, it was balanced upon a support and a weight of 4 000 pounds hung at each end, causing a deflection of 0.401 inches. Compute the value of E.

ART. 35. DEFLECTION OF SIMPLE BEAMS.

The deflection of a simple beam due to a load at the middle, or to a uniform load, is readily obtained from the expressions just deduced for cantilever beams. Thus, for a simple beam of span l with a load P at the middle, if Fig. 27 be inverted it will be seen to be equivalent to two cantilever beams of length $\frac{1}{2}l$ with a load $\frac{1}{2}P$ at each end. The formula for the maximum deflection of a cantilever beam hence applies to this figure, if l be replaced by $\frac{1}{2}l$ and P by $\frac{1}{2}P$, which gives $\Delta = \frac{Pl^3}{48EI}$ for the deflection at the middle of the simple beam. It will be well, however, to use the general formula (5) and treat each case independently.

Case I. A single load P at the middle.—Let the origin be taken at the left support. For any section between the left support and the middle the bending moment M is $\frac{1}{2}Px$. Then the general formula (5) becomes,

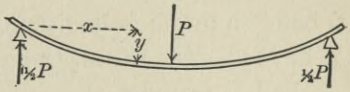

Fig. 27.

$$EI\frac{d^2y}{dx^2} = \frac{Px}{2}.$$

Integrate this and find the constant by the fact that $\frac{dy}{dx} = 0$ when $x = \frac{1}{2}l$. Then integrate again and find the constant by the fact that $y = 0$ when $x = 0$. Thus,

$$48EIy = P(4x^3 - 3l^2x),$$

is the equation of elastic curve between the left hand support and the load. For the greatest deflection make $x = \frac{1}{2}l$, then,

$$\Delta = \frac{Pl^3}{48EI}.$$

Case II. A uniform load.—Let w be the load per linear unit, then the formula (5) becomes,

$$EI\frac{d^2y}{dx^2} = \frac{wlx}{2} - \frac{wx^2}{2}.$$

Integrate this twice, find the constants as in the preceding paragraph, and the equation of the elastic curve is,

$$24EIy = w(-x^4 + 2lx^3 - l^3x),$$

from which the maximum deflection is found to be,

$$\Delta = \frac{5wl^4}{384EI} = \frac{5Wl^3}{384EI}.$$

Case III. A load P at any point.—Here it is necessary first to consider that there are two elastic curves, one on each side of the load, which have distinct equations, but which have a common tangent and ordinate under the load. As in Fig. 28,

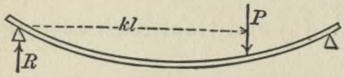

Fig. 28.

let the load be placed at a distance kl from the left support, k being a number less than unity. Then the left reaction is $R = P(1 - k)$. From the general formula (5), with the origin at the left support, the equations are,

On the left of the load,

$$(a) \qquad EI\frac{d^2y}{dx^2} = Rx,$$

$$(b) \qquad EI\frac{dy}{dx} = \tfrac{1}{2}Rx^2 + C_1,$$

$$(c) \qquad EIy = \tfrac{1}{6}Rx^3 + C_1x + C_2.$$

On the right of the load,

$$(a)' \qquad EI\frac{d^2y}{dx^2} = Rx - P(x - kl),$$

$$(b)' \qquad EI\frac{dy}{dx} = \tfrac{1}{2}Rx^2 - \tfrac{1}{2}Px^2 + Pklx + C_3,$$

$$(c)' \qquad EIy = \tfrac{1}{6}Rx^3 - \tfrac{1}{6}Px^3 + \tfrac{1}{2}Pklx^2 + C_3x + C_4.$$

To determine the constants consider in (c) that $y = 0$ when $x = 0$, and hence that $C_2 = 0$. Also in $(c)'$, $y = 0$ when $x = l$; again since the curves have a common tangent under the load, $(b) = (b)'$ when $x = kl$, and since they have a common ordinate at that point $(c) = (c)'$ when $x = kl$. Or,

$$0 = \tfrac{1}{6}Rl^3 - \tfrac{1}{6}Pl^3 + \tfrac{1}{2}Pkl^3 + C_3l + C_4,$$

$$\tfrac{1}{2}Rk^2l^2 + C_1 = \tfrac{1}{2}Rk^2l^2 + \tfrac{1}{2}Pk^2l^2 + C_3,$$

$$\tfrac{1}{6}Rk^3l^3 + C_1kl = \tfrac{1}{6}Rk^3l^3 + \tfrac{1}{3}Pk^3l^3 + C_3kl + C_4.$$

From these three equations the values of C_1, C_3, and C_4 are found. Then the equation of the elastic curve on the left of the load is,

$$6EIy = P(1 - k)x^3 - P(2k - 3k^2 + k^3)l^2x.$$

To find the maximum deflection, the value of x which renders y a maximum is to be obtained by equating the first derivative to zero. If k be greater than $\frac{1}{2}$, this value of x inserted in the above equation gives the maximum deflection; if k be less than $\frac{1}{2}$, the maximum deflection is on the other side of the load. For instance, if $k = \frac{3}{4}$, the equation of the elastic curve on the left of the load is,

$$384EIy = 16Px^3 - 15Pl^2x.$$

This is a maximum when $x = 0.56l$, which is the point of greatest deflection.

Prob. 60. Prove, when k is greater than $\frac{1}{2}$ in Fig. 28, that the maximum deflection is $\Delta = \frac{Pl^3}{3EI}(1 - k)(\frac{2}{3}k - \frac{1}{3}k^2)^{\frac{3}{2}}$.

Prob. 61. In order to find the coefficient of elasticity of Quercus alba a bar 4 centimeters square and one meter long was supported at the ends and loaded in the middle with weights of 50 and 100 kilograms when the deflections were found to be 6.6 and 13.0 millimeters. Show that the mean value of E was 74 500 kilos per square centimeter.

ART. 36. COMPARATIVE DEFLECTION AND STIFFNESS.

From the two preceding articles the following values of the maximum deflections may now be written and their comparison will show the relative stiffness of the different cases.

For a cantilever loaded at the end with W, $\quad \Delta = \frac{1}{3} \cdot \frac{Wl^3}{EI}$.

For a cantilever uniformly loaded with W, $\Delta = \frac{1}{8} \cdot \frac{Wl^3}{EI}$.

For a simple beam loaded at middle with W, $\Delta = \frac{1}{48} \cdot \frac{Wl^3}{EI}$.

For a simple beam uniformly loaded with W, $\Delta = \frac{5}{384} \cdot \frac{Wl^3}{EI}$.

The relative deflections of these four cases are hence as the numbers 1, $\frac{3}{8}$, $\frac{1}{16}$, and $\frac{5}{128}$.

These equations also show that the deflections vary directly as the load, directly as the cube of the length, and inversely as E and I. For a rectangular beam $I = \frac{bd^3}{12}$, and hence the deflection of a rectangular beam is inversely as its breadth and inversely as the cube of its depth.

The stiffness of a beam is indicated by the load that it can carry with a given deflection. From the above it is seen that the value of the load is,

$$W = \frac{mEI\Delta}{l^3},$$

where m has the value 3, 8, 48, or $\frac{384}{5}$ as the case may be. Therefore, the stiffness of a beam varies directly as E, directly as I, and inversely as the cube of its length, and the relative stiffness of the above four cases is as the numbers 1, $2\frac{2}{3}$, 16, and $25\frac{3}{5}$. From this it appears that the laws of stiffness are very different from those of strength. (Art. 29.)

Prob. 62. Compare the strength and stiffness of a joist 3 × 8 inches when laid with flat side vertical and when laid with narrow side vertical.

Prob. 63. Find the thickness of a white pine plank of 8 feet span required not to bend more than $\frac{1}{480}$th of its length under a head of water of 20 feet.

ART. 37. RELATION BETWEEN DEFLECTION AND STRESS.

Let the four cases discussed in Arts. 29 and 36 be again considered. For the strength,

$$W = n\frac{SI}{lc}, \quad \text{where } n = 1, 2, 4, \text{ or } 8.$$

For the stiffness,

$$W = m\frac{EI\Delta}{l^3}, \quad \text{where } m = 3, 8, 48, \text{ or } 76\tfrac{4}{5}.$$

By equating these values of W the relation between Δ and S is obtained, thus,

$$S = \frac{mEc\Delta}{nl^2}, \quad \text{or} \quad \Delta = \frac{nl^2S}{mcE}.$$

These equations, like the general formula (4) and (5), are only valid when S is less than the elastic limit of the materials.

This also shows that the maximum deflection Δ varies as $\frac{l^2}{c}$ for beams of the same material under the same unit-stress S.

From the preceding articles the following table may also be compiled which exhibits the most important results relating to both absolute and relative strength and stiffness.

Case.	Max. Vertical Shear.	Max. Bending Moment.	Max. Stress S.	Max. Deflection.	Relative Strength.	Relative Stiffness.
Cantilever loaded at end,	W	Wl	$\frac{Wlc}{I}$	$\frac{1}{3}\frac{Wl^3}{EI}$	1	1
Cantilever loaded uniformly,	W	$\frac{1}{2}Wl$	$\frac{Wlc}{2I}$	$\frac{1}{8}\frac{Wl^3}{EI}$	2	$2\frac{2}{3}$
Simple beam loaded at middle,	$\frac{1}{2}W$	$\frac{1}{4}Wl$	$\frac{Wlc}{4I}$	$\frac{1}{48}\frac{Wl^3}{EI}$	4	16
Simple beam loaded uniformly,	$\frac{1}{2}W$	$\frac{1}{8}Wl$	$\frac{Wlc}{8I}$	$\frac{5}{384}\frac{Wl^3}{EI}$	8	$25\frac{3}{5}$

Here the signs of the maximum shears and moments are omitted as only their absolute values are needed in computations. Evidently the moments are negative for the first and second cases, and positive for the third and fourth, the direction of the curvature being different.

Prob. 64. Find the deflection of a wrought iron I heavy 10 inch beam of 9 feet span when strained by a uniform load up to the elastic limit.

Prob. 65. A wooden beam of breadth b, depth d, and span x is loaded with P at the middle. Find the value of x so that rupture may occur under the load. Find also the value of x so that rupture may occur by shearing at the supports.

ART. 38. CANTILEVER BEAMS OF UNIFORM STRENGTH.

All cases thus far discussed have been of constant cross-section throughout their entire length. But in the general formula (4) the unit-stress S is proportional to the bending moment M, and hence varies throughout the beam in the same way as the moments vary. Hence some parts of the beam are but slightly strained in comparison with the dangerous section.

A beam of uniform strength is one so shaped that the unit-stress S is the same in all fibers at the upper and lower surfaces. Hence to ascertain the form of such a beam the unit-stress S in (4) must be taken as constant and $\frac{I}{c}$ be made to vary with M. The discussion will be given only for the most important practical cases, namely, those where the sections are rectangular. For these $\frac{I}{c}$ equals $\frac{bd^2}{6}$, and formula (4) becomes,

$$\frac{Sbd^2}{6} = M.$$

In this bd^2 must vary with M for forms of uniform strength.

For a cantilever beam with a load P at the end, $M = Px$ and the equation becomes $\frac{1}{6}Sbd^2 = Px$, in which P and S are constant. If the breadth be taken as constant, d^2 varies with x and the profile is that of a parabola whose vertex is at the load, as shown in Fig. 29. The equation of the parabola is $d^2 = \frac{6P}{Sb}x$ from which d may be found for given values of x. The walking beam of an engine is often made approximately of this shape. If the depth of the cantilever beam be constant then b varies directly as x and hence the plan should be a triangle as shown in Fig. 30. The value of b for given values of P, S, d, and x may be found from the expression $b = \frac{6Px}{Sd^2}$.

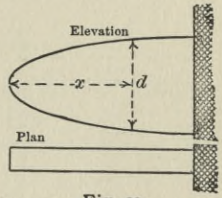

Fig. 29.

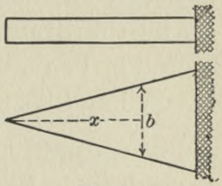

Fig. 30.

For a cantilever beam uniformly loaded with w per linear unit $M = \frac{1}{2}wx^2$, and the equation becomes $\frac{1}{6}Sbd^2 = \frac{1}{2}wx^2$, in which w and S are known. If the breadth be taken as constant then d varies as x and the elevation is a triangle, as in Fig. 31, whose depth at any point is $d = x\sqrt{\frac{3w}{Sb}}$. If however the depth be taken constant, then $b = \frac{3w}{Sd^2}x^2$ which is the equation of a parabola whose vertex is at the free end of the cantilever and whose axis is perpendicular to it. Or the equation may be satisfied by two parabolas drawn upon opposite sides of the center line as shown in Fig. 32.

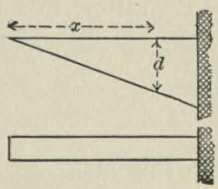

Fig. 31.

The vertical shear modifies in practice the shape of these forms near their ends. For instance, a cantilever beam loaded at the end with P requires a cross-section at the end equal to $\frac{P}{S_c}$ where S_c is the working shearing strength. This cross-section must be preserved until a value of x is reached, where the same value of the cross-section is found from the moment.

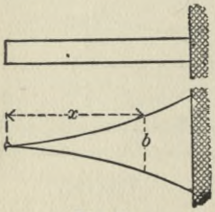

Fig. 32.

The deflection of a cantilever beam of uniform strength is evidently greater than that of one of constant cross-section, since the unit-stress S is greater throughout. In any case it may be determined from the general formula (5) by substituting for M and I their values in terms of x, integrating twice, determining the constants, and then making x equal to l for the maximum value of y.

For a cantilever beam loaded at the end and of constant breadth, as in Fig. 29, formula (5) becomes,

$$\frac{d^2y}{dx^2} = \frac{12Px}{Ebd^3} = \frac{2}{E}\sqrt{\frac{S^3b}{6Px}}.$$

Integrating this twice and determining the constants, as in Art. 34, the equation of the elastic curve is found to be,

$$y = \frac{2}{E}\sqrt{\frac{S^3b}{6P}}\left(\tfrac{4}{3}x^{\frac{3}{2}} - 2l^{\frac{1}{2}}x\right).$$

In this make $x = l$, and substitute for S its value $\frac{6Pl}{bd_1^2}$, where d_1 is the depth at the wall. Then,

$$\Delta = \frac{8Pl^3}{Ebd_1^3},$$

which is double that of a cantilever beam of uniform depth d.

For a cantilever beam loaded at the end and of constant depth, formula (5) becomes,

$$\frac{d^2y}{dx^2} = \frac{12Px}{Ebd^3} = \frac{2S}{Ed}.$$

By integrating this twice and determining the constants as before, the equation of the elastic curve is found, from which the deflection is,

$$\Delta = \frac{6Pl^3}{Ebd^3},$$

which is fifty per cent greater than for one of uniform section.

Prob. 66. A cast iron cantilever beam of uniform strength is to be 4 feet long, 3 inches in breadth and to carry a load of 15 000 pounds at the end. Find the proper depths for every foot in length, using 3 000 pounds per square inch for the horizontal unit-stress, and 4 000 pounds per square inch for the shearing unit-stress.

ART. 39. SIMPLE BEAMS OF UNIFORM STRENGTH.

In the same manner it is easy to deduce the forms of uniform strength for simple beams of rectangular cross-section.

For a load at the middle and breadth constant, $M = \frac{1}{2}Px$, and hence, $\frac{1}{6}Sbd^2 = \frac{1}{2}Px$. Hence $d^2 = \frac{3P}{Sb}x$, from which values of d may be found for assumed values of x. Here the profile of the beam will be parabolic, the vertex being at the support, and the maximum depth under the load.

For a load at the middle and depth constant, $M = \frac{1}{2}Px$ as before. Hence $b = \frac{3P}{Sd^2}x$, and the plan must be triangular or lozenge shaped, the width uniformly increasing from the support to the load.

For a uniform load and constant breadth, $M = \frac{1}{2}wlx - \frac{1}{2}wx^2$, and hence, $d^2 = \frac{3w}{Sb}(lx - x^2)$, and the profile of the beam must be elliptical, or preferably a half-ellipse.

For a uniform load and constant depth, $b = \frac{3w}{Sd^2}(lx - x^2)$ and hence the plan should be formed of two parabolas having their vertices at the middle of the span.

The figures for these four cases are purposely omitted, in order that the student may draw them for himself; if any difficulty be found in doing this let numerical values be assigned to the constant quantities in each equation, and the variable breadth or depth be computed for different values of x.

In the same manner as in the last article, it can be shown that the deflection of a simple beam of uniform strength loaded at the middle is double that of one of constant cross-section if the breadth is constant, and is one and one-half times as much if the depth is constant.

Prob. 67. Draw the profile for a cast iron beam of uniform strength, the span being 8 feet, breadth 3 inches, load at the middle 30,000 pounds, using the same working unit-stresses as in Prob. 66.

Prob. 68. Find the deflection of a steel spring of constant depth and uniform strength which is 6 inches wide at the middle, 52 inches long, and loaded at the middle with 600 pounds, the depth being such that the maximum fiber stress is 20 000 pounds per square inch.

CHAPTER IV.

RESTRAINED BEAMS AND CONTINUOUS BEAMS.

ART. 40. BEAMS OVERHANGING ONE SUPPORT.

A cantilever beam has its upper fibers in tension and the lower in compression, while a simple beam has its upper fibers in compression and the lower in tension. Evidently a beam overhanging one support, as in Fig. 31, has its overhanging part in the condition of a cantilever, and the part near the other end in the condition of a simple beam. Hence there must be a point *i* where the stresses change from tension to compression, and where the curvature changes from positive to negative. This point *i* is called the inflection point; it is the point where the bending moment is zero. An overhanging beam is said to be subject to a restraint at the support beyond which the beam projects, or, in other words, there is a stress in the horizontal fibers over that support.

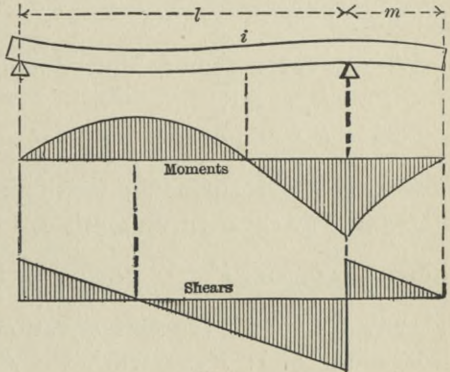

Fig. 33.

Since the beam has but two supports, its reactions may be found by using the principle of moments as in Art. 16. Thus, if the distance between the supports be l, the length of the overhanging part be m, and the uniform load per linear foot be w, the two reactions are,

$$R_1 = \frac{wl}{2} - \frac{wm^2}{2l}, \qquad R_2 = \frac{wl}{2} + wm + \frac{wm^2}{2l},$$

whose sum is equal to the total load $wl + wm$. Here, as in all cases of uniform load, the lever arms are taken to the centers of gravity of the portions considered.

When the reactions have been found, the vertical shear at any section can be computed by Art. 17, and the bending moment by Art. 18, bearing in mind that for a section beyond the right support the reaction R_2 must be considered as a force acting upward. Thus, for any section distant x from the left support,

When x is less than l,	When x is greater than l,
$V = R_1 - wx$,	$V = R_1 + R_2 - wx$,
$M = R_1x - \frac{1}{2}wx^2$.	$M = R_1x + R_2(x - l) - \frac{1}{2}wx^2$.

The curves corresponding to these equations are shown on Fig. 33. The shear curve consists of two straight lines; $V = R_1$ when $x = 0$, and $V = 0$ when $x = \frac{R_1}{w}$; at the right support $V = R_1 - wl$ from the first equation, and $V = R_1 + R_2 - wl$ from the second; $V = 0$ when $x = l + m$. The moment curve consists of two parts of parabolas; $M = 0$ when $x = 0$, M is a maximum when $x = \frac{R_1}{w}$, $M = 0$ at the inflection point where $x = \frac{2R_1}{w}$, M has its negative maximum when $x = l$, and $M = 0$ when $x = l + m$. The diagrams show clearly the distribution of shears and moments throughout the beam.

For example, if $l = 20$ feet, $m = 10$ feet, and $w = 40$ pounds per linear foot, the reactions are $R_1 = 300$ and $R_2 = 900$ pounds.

Then the point of zero shear or maximum moment is at $x = 7.5$ feet, the inflection point at $x = 15$ feet, the maximum shears are $+ 300$, $- 500$, and $+ 400$ pounds, and the maximum bending moments are $+ 1125$ and $- 2\,000$ pound-feet. Here the negative bending moment at the right support is numerically greater than the maximum positive moment. The relative values of the two maximum moments depend on the ratio of m to l; if $m = 0$ there is no overhanging part and the beam is a simple one; if $m = \frac{1}{2}l$ the case is that just discussed; if $m = l$ the reaction R_1 is zero, and each part is a cantilever beam.

After having thus found the maximum values of V and M the beam may be investigated by the application of formulas (3) and (4) in the same manner as a cantilever or simple beam. By the use of formula (5) the equation of the elastic curve between the two supports is found to be,

$$24EIy = 4R_1(x^3 - l^2x) - w(x^4 - l^3x).$$

From this the maximum deflection for any particular case may be determined by putting $\frac{dy}{dx}$ equal to zero, solving for x, and then finding the corresponding value of y.

If concentrated loads be placed at given positions on the beam the reactions are found by the principle of moments, and then the entire investigation can be made by the methods above described.

Prob. 69. Three men carry a stick of timber, one taking hold at one end and the other two at a common point. Where should this point be so that each may bear one third the weight? Draw the diagrams of shears and moments.

Prob. 70. A beam 20 feet long has one support at the right end and one support at 5 feet from the left end. At the left end is a load of 180 pounds, and at 6 feet from the right end is a load of 125 pounds. Find the reactions, the inflection point, and draw the shear and moment diagrams.

ART. 41. BEAMS FIXED AT ONE END AND SUPPORTED AT THE OTHER.

A beam is said to be fixed at the end when it is so restrained in a wall that the tangent to the elastic curve at the wall is horizontal. Thus, in Fig. 33, if the part *m* is of such a length that the tangent over the right support is horizontal, the part *l* is in the same condition as a beam fixed at one end and supported at the other. Fig. 34 shows the practical arrangement of such a beam, the left support being upon the same level as the lower side of the beam at the wall. The reactions of such a beam cannot be determined by the principles of statics alone, but the assistance of the equation of the elastic curve must be invoked.

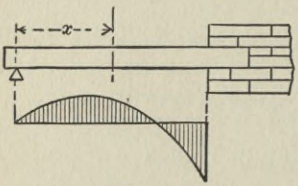

Fig. 34.

Case I. For a uniform load over the whole beam, as in Fig. 34, let R be the reaction at the left end. Then for any section the bending moment is $Rx - \frac{1}{2}wx^2$. Hence the differential equation of the elastic curve is,

$$EI\frac{d^2y}{dx^2} = Rx - \tfrac{1}{2}wx^2.$$

Integrate this once and determine the constant from the necessary condition that $\frac{dy}{dx} = 0$ when $x = l$. Integrate again and find the constant from the fact that $y = 0$ when $x = 0$. Then,

$$24EIy = 4R(x^3 - 3l^2x) - w(x^4 - 4l^3x).$$

Here also $y = 0$ when $x = l$, and therefore $R = \frac{3}{8}wl$.

The moment at any point now is $M = \frac{3}{8}wlx - \frac{1}{2}wx^2$, and by placing this equal to zero it is seen that the point of inflection is at $x = \frac{3}{4}l$. By the method of Art. 24 it is found that the maximum moments are $+\frac{9}{128}wl^2$ and $-\frac{1}{8}wl^2$, and that the distribution of moments is as represented in Fig. 34.

The point of maximum deflection is found by placing $\frac{dy}{dx}$ equal to zero and solving for x. Thus $8x^3 - 9lx^2 + l^3 = 0$, one root of which is $x = +0.4215l$, and this inserted in the value of y gives,

$$\Delta = 0.0054\frac{wl^4}{EI},$$

for the value of the maximum deflection.

Case II. For a load at the middle it is first necessary to consider that there are two elastic curves having a common ordinate and a common tangent under the load, since the expressions for the moment are different on opposite sides of the load. Thus, taking the origin as usual at the supported end,

On the left of the load,

$$(a) \qquad EI\frac{d^2y}{dx^2} = Rx,$$

$$(b) \qquad EI\frac{dy}{dx} = \tfrac{1}{2}Rx^2 + C_1,$$

$$(c) \qquad EIy = \tfrac{1}{6}Rx^3 + C_1x + C_3.$$

On the right of the load the similar equations are,

$$(a)' \qquad EI\frac{d^2y}{dx^2} = Rx - P(x - \tfrac{1}{2}l),$$

$$(b)' \qquad EI\frac{dy}{dx} = \tfrac{1}{2}Rx^2 - \tfrac{1}{2}Px^2 + \tfrac{1}{2}Plx + C_2,$$

$$(c)' \qquad EIy = \tfrac{1}{6}Rx^3 - \tfrac{1}{6}Px^3 + \tfrac{1}{4}Plx^2 + C_2x + C_4.$$

To determine the constants consider in (c) that $y = 0$ when $x = 0$ and hence that $C_3 = 0$. In $(b)'$ the tangent $\frac{dy}{dx} = 0$ when $x = l$ and hence $C_2 = -\frac{1}{2}Rl^2$. Since the curves have a common tangent under the load $(b) = (b)'$ for $x = \frac{1}{2}l$, and thus the value of C_1 is found. Since the curves have a common ordinate under the load $(c) = (c)'$ when $x = \frac{1}{2}l$, and thus C_4 is found. Then,

$$(c) \qquad EIy = \frac{Rx^3}{6} + \frac{Pl^2x}{8} - \frac{Rl^2x}{2},$$

$$(c)' \qquad EIy = \frac{Rx^3}{6} - \frac{Px^3}{6} + \frac{Plx^2}{4} - \frac{Rl^2x}{2} + \frac{Pl^3}{48}.$$

From the second of these the value of the reaction is $R = \frac{5}{16}P$.

The moment on the left of the load is now $M = \frac{5}{16}Px$, and that on the right $M = -\frac{11}{16}Px + \frac{1}{2}Pl$. The maximum positive moment obtains at the load and its value is $\frac{5}{32}Pl$. The maximum negative moment occurs at the wall, and its value is $\frac{3}{16}Pl$. The inflection point is at $x = \frac{8}{11}l$. The deflection under the load is readily found from (c) by making $x = \frac{1}{2}l$. The maximum deflection occurs at a less value of x, which may be found by equating the first derivative to zero.

Case III. For a load at any point whose distance from the left support is kl, the following results may be deduced by a method exactly similar to that of the last case.

Reaction at supported end	=	$\frac{1}{2}P(2 - 3k + k^3)$.
Reaction at fixed end	=	$\frac{1}{2}P(3k - k^3)$.
Maximum positive moment	=	$\frac{1}{2}Plk(2 - 3k + k^3)$.
Maximum negative moment	=	$\frac{1}{2}Pl(k - k^3)$.

The absolute maximum deflection occurs under the load when $x = 0.414l$.

Prob. 71. Draw the diagrams of shears and moments for a load at the middle, taking $P = 600$ pounds and $l = 12$ feet.

Prob. 72. Find the position of load P which gives the maximum positive moment. Find also the position which gives the maximum negative moment.

ART. 42. BEAMS OVERHANGING BOTH SUPPORTS.

When a beam overhangs both supports the bending moments for sections beyond the supports are negative, and in general between the supports there will be two inflection points. If the lengths m and n be equal the reactions will be equal under uniform load, each being one half of the total load. In any case, whatever be the nature of the load, the reactions may be found by the principle of moments (Art. 16), and then the vertical shears and bending moments may be deduced for all sections, after which the formulas (3) and (4) can be used for any special problem.

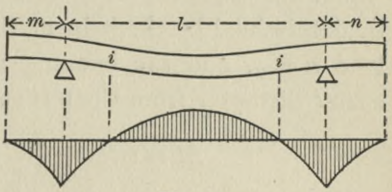

Fig. 35.

Under a uniformly distributed load, and $m = n$, which is the most important practical case, each reaction is $wm + \frac{1}{2}wl$, the maximum shears at the supports are wm and $\frac{1}{2}wl$, the maximum moment at the middle is $+ w(\frac{1}{8}l^2 - \frac{1}{2}m^2)$, the maximum moment at each support is $- \frac{1}{2}wm^2$, and the inflection points are distant $\frac{1}{2}\sqrt{l^2 - 4m^2}$ from the middle of the beam. Fig. 35 shows the distribution of moments for this case. If $m = 0$, the beam is a simple one; if $l = 0$, it consists of two cantilever beams.

Prob. 73. If $m = n$ in Fig. 35, find the ratio of l to m in order that the maximum positive moment may numerically equal the maximum negative moment.

Prob. 74. A beam 30 feet long has one support at 5 feet from the left end, and the other support at 10 feet from the right end. At each end there is a load of 156 pounds and half-way between the supports there is a load of 344 pounds. Construct shear and moment diagrams.

ART. 43. BEAMS FIXED AT BOTH ENDS.

If, in Fig. 35, the distances m and n be such that the elastic curve over the supports is horizontal the central span l is said to be a beam fixed at both ends. The lengths m and n which will cause the curve to be horizontal at the support can be determined by the help of the elastic curve. For uniform load $n = m$ and the bending moment at any section in the span l distant x from the left support is,

$$M = (wm + \tfrac{1}{2}wl)x - \tfrac{1}{2}w(m + x)^2,$$

which reduces to the simpler form,

$$M = M' + \tfrac{1}{2}wlx - \tfrac{1}{2}wx^2,$$

in which M' represents the unknown bending moment $-\tfrac{1}{2}wm^2$ at the left support.

Again, for a single load P at the middle of l in Fig. 35 the elastic curve can be regarded as kept horizontal at the left support by a load Q at the end of the distance m. Then the bending moment at any section distant x from the left support, and between that support and the middle, is,

$$M = (Q + \tfrac{1}{2}P)x - Q(m + x),$$

which reduces to,

$$M = M' + \tfrac{1}{2}Px,$$

in which M' denotes the unknown moment $-Qm$ at the left support. The problem of finding the bending moment at any section hence reduces to that of determining M' the moment at the support.

Case I. For a uniform load the general equation of the elastic curve now is,

$$EI\frac{d^2y}{dx^2} = M' + \tfrac{1}{2}wlx - \tfrac{1}{2}wx^2.$$

Integrating this twice, making $\frac{dy}{dx} = 0$ when $x = 0$ and also when $x = l$, the value of M' is found to be $-\frac{wl^2}{12}$, and the equation of the elastic curve becomes,

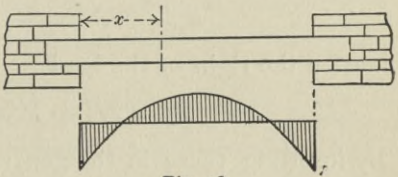

Fig. 36.

$$24EIy = w(-l^2x^2 + 2lx^3 - x^4).$$

From this the maximum deflection is found to be,

$$\Delta = \frac{wl^4}{384EI}.$$

The inflection points are located by making $M = 0$, which gives $x = \frac{1}{2}l \pm l\sqrt{\frac{1}{12}}$. The maximum positive moment is at the middle and its value is $\frac{wl^2}{24}$; accordingly the horizontal stress upon the fibers at the middle of the beam is one half that at the ends. The vertical shear at the left end is $\frac{1}{2}wl$, at the middle 0, and at the right end $-\frac{1}{2}wl$.

Case II. For a load at the middle the general equation of the elastic curve between the left end and the load is,

$$EI\frac{d^2y}{dx^2} = M' + \tfrac{1}{2}Px,$$

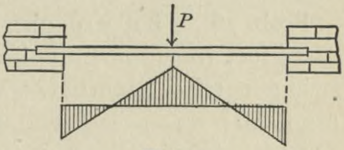

Fig. 37.

and in a similar manner to that of the last case it is easy to find that the maximum negative moments are $\frac{1}{8}Pl$, that the maximum positive moment is $\frac{1}{8}Pl$, that the inflection points are half-way between the supports and the load, and that the maximum deflection is $\frac{Pl^3}{192EI}$.

Case III. For load P at a distance kl from the left end let

M' and V' denote the unknown bending moment and vertical shear at that end. Then on the left of the load,

$$M = M' + V'x,$$

and on the right of the load

$$M = M' + V'x - P(x - kl).$$

By inserting these in the general formula (5), integrating each twice and establishing sufficient conditions to determine the unknown M' and V' and also the constants of integration, the following results may be deduced,

Shear at left end $= P(1 - 3k^2 + 2k^3)$,
Shear at right end $= Pk^2(3 - 2k)$.
Moment at left end $= - Plk(1 - 2k + k^2)$,
Moment at right end $= - Plk^2(1 - k)$,
Moment under load $= + Plk^2(2 - 4k + 2k^2)$.

If $k = \frac{1}{2}$ the load is at the middle and these results reduce to the values found in Case II.

Prob. 75. Show from the results above given for Case III that the inflection points are at the distances $\frac{kl}{1 + 2k}$ and $\frac{(2 - k)l}{3 - 2k}$ from the left end.

Prob. 76. What wrought iron I beam is required for a span of 24 feet to support a uniform load of 40 000 pounds, the ends being merely supported? What one is needed when the ends are fixed?

ART. 44. COMPARISON OF RESTRAINED AND SIMPLE BEAMS.

As the maximum moments for restrained beams are less than for simple beams their strength is relatively greater. This was to be expected, since the restraint produces a negative bending moment and lessens the deflection which would otherwise occur. The comparative strength and stiffness of cantilevers and simple beams is given in Art. 37. To these may now be added four cases from Arts. 41 and 43, and the following table be formed, in which W represents the total load, whether uniform or concentrated.

Beams of Uniform Cross-section.	Maximum Moment.	Maximum Deflection.	Relative Strength.	Relative Stiffness.
Cantilever, load at end	Wl	$\frac{1}{3}\frac{Wl^3}{EI}$	1	1
Cantilever, uniform load	$\frac{1}{2}Wl$	$\frac{1}{8}\frac{Wl^3}{EI}$	2	$2\frac{2}{3}$
Simple beam, load at middle	$\frac{1}{4}Wl$	$\frac{1}{48}\frac{Wl^3}{EI}$	4	16
Simple beam uniformly loaded	$\frac{1}{8}Wl$	$\frac{5}{384}\frac{Wl^3}{EI}$	·8	$25\frac{3}{5}$
Beam fixed at one end, supported at other, load near middle	$0.192\,Wl$	$0.0182\frac{Wl^3}{EI}$	5.2	18.3
Beam fixed at one end, supported at other, uniform load	$\frac{1}{8}Wl$	$0.0054\frac{Wl^3}{EI}$	8	62
Beam fixed at both ends, load at middle	$\frac{1}{8}Wl$	$\frac{1}{192}\frac{Wl^3}{EI}$	8	64
Beam fixed at both ends, uniform load	$\frac{1}{12}Wl$	$\frac{1}{384}\frac{Wl^3}{EI}$	12	128

This table shows that a beam fixed at both ends and uniformly loaded is one and one-half times as strong and five times as stiff as a simple beam under the same load. The advantage of fixing the ends is hence very great.

Prob. 77. Prove, for a uniformly loaded beam with equal overhanging ends, that the deflection at the middle is given by the formula $\frac{wl^2}{384EI}(5l^2 - 24m^2)$.

Prob. 78. Find the deflection of a 9-inch I beam of 6 feet span and fixed ends when loaded at the middle so that the tensile and compressive stresses at the dangerous section are 14 000 pounds per square inch.

ART. 45. GENERAL PRINCIPLES OF CONTINUITY.

A continuous beam is one supported upon several points in the same horizontal plane. A simple beam may be regarded as a particular case of a continuous beam where the number of supports is two. The ends of a continuous beam are said to be free when they overhang, supported when they merely rest on abutments, and restrained when they are horizontally fixed in walls.

The general principles of the preceding chapter hold good for all kinds of beams. If a plane be imagined to cut any beam at any point the laws of Arts. 19 and 20 apply to the stresses in that section. The resisting shear and the resisting moment for that section have the values deduced in Art. 21 and the two fundamental formulas for investigation are,

$$S_sA = V, \tag{3}$$

$$\frac{SI}{c} = M \tag{4}$$

Here S_s is the vertical shearing unit-stress in the section, and S is the horizontal tensile or compressive unit-stress on the fiber most remote from the neutral axis; c is the shortest distance from that fiber to that axis; I the moment of inertia, and A the area of the cross-section. V is the vertical shear of the external forces on the left of the section, and M is the bending moment of those forces with reference to a point in the section. For any given beam evidently S_s and S may be found for any section as soon as V and M are known.

The general equation of the elastic line, deduced in Art. 33, is also valid for all kinds of beams. It is,

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \tag{5}$$

where x is the abscissa and y the ordinate of any point of the elastic curve, M being the bending moment for that section, and E the coefficient of elasticity of the material.

The vertical shear V is the algebraic sum of the external forces on the left of the section, or, as in Art. 17,

$V =$ Reactions on left of section minus loads on left of section.

For simple beams and cantilevers the determination of V for any special case was easy, as the left reaction could be readily found for any given loads. For continuous beams, however, it is not, in general, easy to find the reactions, and hence a different method of determining V is necessary. Let Fig. 38 represent one span of a continuous beam. Let V be the vertical shear for any section at the distance x from the left support, and V' the vertical shear at a section infinitely near to the left support. Also let ΣP_1 denote the sum of all the concentrated loads on the distance x, and wx the uniform load. Then because V' is the algebraic sum of all the vertical forces on its left, the definition of vertical shear gives,

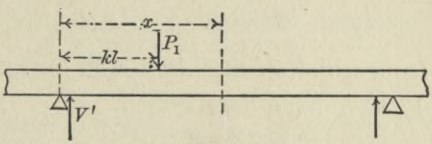

Fig. 38.

$$V = V' - wx - \Sigma P_1. \qquad (6)$$

Hence V can be determined as soon as V' is known.

The bending moment M is the algebraic sum of the moments of the external forces on the left of the section with reference to a point in that section, or, as in Art. 18,

$M =$ moments of reactions minus moments of loads.

For the reason just mentioned it is in general necessary to determine M for continuous and restrained beams by a different method. Let M' denote the bending moment at the left support of any span as in Fig. 38, and M'' that at the right support, while M is the bending moment for any section distant x from the left support. Let P_1 be any concentrated load upon the space x at a distance kl from the left support, k being a fraction less than unity, and let w be the uniform load per linear unit. Let V' be the resultant of all the vertical forces on the left of a section in the given span infinitely near to the left support, and let m be the distance of the point of application of that resultant from that support. Then the definition of bending moment gives,

$$M = V'(m + x) - wx \cdot \tfrac{1}{2}x - \Sigma \cdot P_1(x - kl).$$

But $V'm$ is the unknown bending moment M' at the left support. Hence

$$(7) \qquad M = M' + V'x - \tfrac{1}{2}wx^2 - \Sigma \cdot P_1(x - kl),$$

from which M may be found for any section as soon as M' and V' have been determined.

The vertical shear V' at the support may be easily found if the bending moments M' and M'' be known. Thus in equation (7) make $x = l$, then M becomes M'', and hence,

$$(8) \qquad V' = \frac{M'' - M'}{l} + \frac{wl}{2} + \Sigma \cdot P_1(1 - k).$$

The whole problem of the discussion of restrained and continuous beams hence consists in the determination of the bending moments at the supports. When these are known the values of M and V may be determined for every section, and the general formulas (3), (4), and (5) be applied as in Chapter III, to the investigation of questions of strength and deflection. The formulas (6), (7), and (8) apply to cantilever and simple beams also. For a simple beam $M' = M'' = 0$, and $V' = R$. For a cantilever beam $M' = 0$ for the free end, and M'' is the moment at the wall.

The relation between the bending moment and the vertical shear at any section is interesting and important. At the section x the moment is M and the shear is V. At the next consecutive section $x + dx$ the moment is $M + dM$, which may also be expressed by $M + Vdx$. Hence,

$$V = \frac{dM}{dx}.$$

This may be proved otherwise by differentiating (7) and comparing with (6). From this it is seen that the maximum moments occur at the sections where the shear passes through zero.

Prob. 79. A bar of length $2l$ and weighing w per linear unit is supported at the middle. Apply formulas (6) and (7) to the statement of general expressions for the moment and shear at any section on the left of the support, and also at any section on the right of the support.

ART. 46. PROPERTIES OF CONTINUOUS BEAMS.

The theory of continuous beams presented in the following pages includes only those with constant cross-section having the supports on the same level, as only such are used in engineering constructions. Unless otherwise stated, the ends will be supposed to simply rest upon their supports, so that there can be no moments at those points. Then the end spans are somewhat in the condition of a beam with one overhanging end, and the other spans somewhat in the condition of a beam with two overhanging ends. At each intermediate support there is a negative moment, and the distribution of moments throughout the beam will be as represented in Fig. 39.

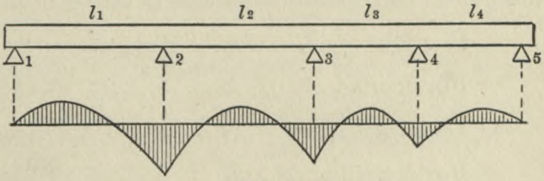

Fig. 39.

As shown in Art. 45, the investigation of a continuous beam depends upon the determination of the bending moments at the supports. In the case of Fig. 39 these moments being those at the supports 2, 3, and 4, may be designated M_2, M_3, and M_4. Let V_1, V_2, V_3, and V_4 denote the vertical shear at the right of each support. The first step is to find the moments M_2, M_3, and M_4. Then from formula (8) the values of V_1, V_2, V_3, and V_4 are found, and thus by formula (7) an expression for the bending moment in each span may be written, from which the maximum positive moments may be determined. Lastly, by formulas (3) and (4) the strength of the beam may be investigated, and by (5) its deflection at any point be deduced.

For example, let the beam in Fig. 39 be regarded as of four equal spans and uniformly loaded with w pounds per linear unit. By a method to be explained in the following articles it may be shown that the bending moments at the supports are,

$$M_2 = -\tfrac{3}{28}wl^2, \qquad M_3 = -\tfrac{2}{28}wl^2, \qquad M_4 = -\tfrac{3}{28}wl^2.$$

From formula (8) the vertical shears at the right of the several supports are,

$$V_1 = \tfrac{11}{28}wl, \qquad V_2 = \tfrac{15}{28}wl, \qquad V_3 = \tfrac{13}{28}wl, \qquad \text{etc.}$$

And from (6) those on the left of the supports 2, 3, 4, etc., are found to be, $-\tfrac{17}{28}wl$, $-\tfrac{13}{28}wl$, $-\tfrac{15}{28}wl$, etc. From formula (7) the general expressions for the bending moments now are,

For first span, $M = +\tfrac{11}{28}wlx - \tfrac{1}{2}wx^2$,

For second span, $M = -\tfrac{3}{28}wl^2 + \tfrac{15}{28}wlx - \tfrac{1}{2}wx^2$,

For third span, $M = -\tfrac{2}{28}wl^2 + \tfrac{13}{28}wlx - \tfrac{1}{2}wx^2$,

For fourth span, $M = -\tfrac{3}{28}wl^2 + \tfrac{17}{28}wlx - \tfrac{1}{2}wx^2$.

From each of these equations the inflection points may be found by putting $M = 0$, and the point of maximum positive moment by putting $\frac{dM}{dx} = 0$. The maximum positive moments are found to have the following values,

$\frac{121}{1568}wl^2$, $\frac{57}{1568}wl^2$, $\frac{57}{1568}wl^2$, and $\frac{121}{1568}wl^2$.

For any particular case the beam may now be investigated by formulas (3) and (4).

The reactions at the supports are not usually needed in the discussion of continuous beams, but if required they may easily be found from the adjacent shears. Thus for the above case,

$$R_1 = \quad 0 + \tfrac{11}{28}wl = \tfrac{11}{28}wl,$$
$$R_2 = \tfrac{17}{28}wl + \tfrac{15}{28}wl = \tfrac{32}{28}wl,$$
$$R_3 = \tfrac{13}{28}wl + \tfrac{13}{28}wl = \tfrac{26}{28}wl, \text{ etc.,}$$

and the sum of these is equal to the total load $4wl$.

The equation of the elastic curve in any span is deduced by inserting in (5) for M its value and integrating twice. When $x = 0$, the tangent $\frac{dy}{dx}$ is the tangent of the inclination at the left support, and when $x = l$ it is the tangent of the inclination at the right support. When $x = 0$, and also when $x = l$, the ordinate $y = 0$, and from these conditions the two unknown tangents may be found. In general the maximum deflection in any span of a continuous beam will be found intermediate in value between those of a simple beam and a restrained beam.

In the following pages continuous beams will only be investigated for the case of uniform load. The lengths of the spans however may be equal or unequal, and the load per linear foot may vary in the different spans.

Prob. 80. In a continuous beam of three equal spans the negative bending moments at the supports are $\frac{1}{10}wl^2$. Find the inflection points, the maximum positive moments and the reactions of the supports.

ART. 47. THE THEOREM OF THREE MOMENTS.

Let the figure represent any two adjacent spans of a continuous beam whose lengths are l' and l'' and whose uniform loads per linear foot are w' and w'' respectively. Let M', M'', and M''' represent the three unknown moments at the supports. Let V' and V'' be the vertical shears at the right of the first and second supports. Then, for any section distant x from the left support in the first span, the moment is,

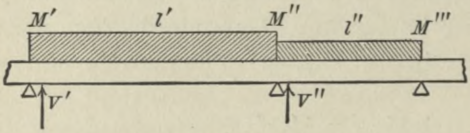

Fig. 40.

$$M = M' + V'x - \tfrac{1}{2}wx^2.$$

If this be inserted in the general formula (5) and integrated twice and the constants determined by the condition that $y = 0$ when $x = 0$ and also when $x = l$, the value of the tangent of the angle which the tangent to the elastic curve at any section in the first span makes with the horizontal is found to be,

$$\frac{dy}{dx} = \frac{12M'(2x - l') + 4V'(3x^2 - l'^2) - w'(4x^3 - l'^3)}{24EI}.$$

Similarly if the origin be taken at the next support the value of the tangent of inclination at any point in the second span is,

$$\frac{dy}{dx} = \frac{12M''(2x - l'') + 4V''(3x^2 - l''^2) - w''(4x^3 - l''^3)}{24EI}.$$

Evidently the two curves must have a common tangent at the support. Hence make $x = l'$ in the first of these and $x = 0$ in the second and equate the results, giving,

$$12M'l' + 8V'l'^2 - 3w'l'^3 = -12M''l'' - 4V''l''^2 + w''l''^3.$$

Let the values of V' and V'' be expressed by (8) in terms of M', M'', and M''', and the equation reduces to,

$$(9)\quad M'l' + 2M''(l' + l'') + M'''l'' = -\frac{w'l'^3}{4} - \frac{w''l''^3}{4},$$

which is the theorem of three moments for continuous beams uniformly loaded.

If the spans are all equal and the load uniform throughout, this reduces to the simpler form,

$$M' + 4M'' + M''' = -\frac{wl^2}{2}.$$

In any continuous beam of s spans there are $s + 1$ supports and $s - 1$ unknown bending moments at the supports. For each of these supports an equation of the form of (9) may be written containing three unknown moments. Thus there will be stated $s - 1$ equations whose solution will furnish the values of the $s - 1$ unknown quantities.

Prob. 81. A simple wooden beam one inch square and 15 inches long is uniformly loaded with 100 pounds. Find the angle of inclination of the elastic curve at the supports.

ART. 48. CONTINUOUS BEAMS WITH EQUAL SPANS.

Consider a continuous beam of five equal spans uniformly loaded. Let the supports beginning on the left be numbered 1, 2, 3, 4, 5, and 6. From the theorem of three moments an equation may be written for each of the supports 2, 3, 4, and 5; thus,

$$M_1 + 4M_2 + M_3 = -\tfrac{1}{2}wl^2,$$
$$M_2 + 4M_3 + M_4 = -\tfrac{1}{2}wl^2,$$
$$M_3 + 4M_4 + M_5 = -\tfrac{1}{2}wl^2,$$
$$M_4 + 4M_5 + M_6 = -\tfrac{1}{2}wl^2.$$

Since the ends of the beam rest on abutments without restraint $M_1 = M_6 = 0$. Hence the four equations furnish the means of finding the four moments M_2, M_3, M_4, M_5. The solution may be abridged by the fact that $M_2 = M_5$, and $M_3 = M_4$, which is evident from the symmetry of the beam. Hence,

$$M_2 = M_5 = -\tfrac{4}{38}wl^2, \qquad M_3 = M_4 = -\tfrac{3}{38}wl^2.$$

From formula (8) the shears at the right of the supports are,

$$V_1 = \tfrac{15}{38}wl, \qquad V_2 = \tfrac{20}{38}wl, \qquad V_3 = \tfrac{19}{38}wl, \quad \text{etc.}$$

From (7) the bending moment at any point in any span may now be found as in Art. 46, and by (3), (4), and (5) the complete investigation of any special case may be effected.

In this way the bending moments at the supports for any number of equal spans can be deduced. The following triangular table shows their values for spans as high as seven in number. In each horizontal line the supports are represented by squares in which are placed the coefficients of $-wl^2$. For example, in a beam of 3 spans there are four supports and the bending moments at those supports are 0, $-\frac{1}{10}wl^2$, $-\frac{1}{10}wl^2$, and 0.

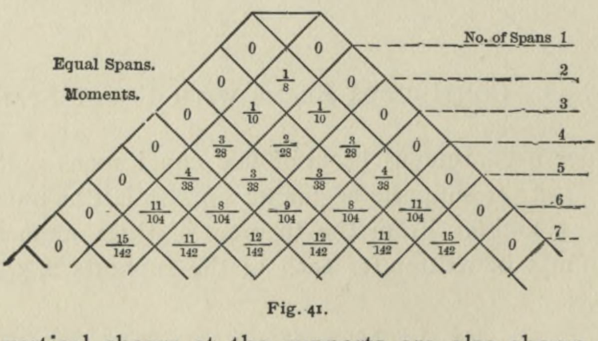

Fig. 41.

The vertical shears at the supports are also shown in the following table for any number of spans up to 5. The space representing a support shows in its left-hand division the shear on the left of that support and in its right-hand division the shear on the right. The sum of the two shears for any support is, of course, the reaction of that support. For example, in a beam of five equal spans the reaction at the second support is $\frac{43}{38}wl$.

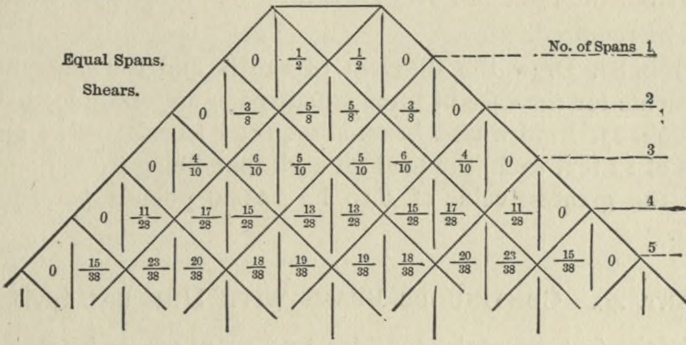

Fig. 42.

It will be seen on examination that the numbers in any oblique column of these tables follow a certain law of increase by which it is possible to extend them, if desired, to a greater number of spans than are here given.

As an example, let it be required to select a **I** beam to span four openings of 8 feet each, the load per span being 14 000 pounds and the greatest horizontal stress in any fiber to be 12 000 pounds per square inch. The required beam must satisfy formula (4), or,

$$\frac{I}{c} = \frac{M}{12\ 000},$$

where M is the maximum moment. From the table it is seen that the greatest negative moment is that at the second support, or $\frac{3}{28}wl^2$. The maximum positive moments are,

For first span, $\quad \max M = \frac{V^2}{2w} = \frac{121}{1568}wl^2,$

For second span, $\quad \max M = M_2 + \frac{V^2}{2w} = \frac{57}{1568}wl^2.$

The greatest value of M is hence at the second support. Then,

$$\frac{I}{c} = \frac{3 \times 14\,000 \times 8 \times 12}{28 \times 12\,000} = 12,$$

and from the table in Art. 30 it is seen that a light 7-inch beam will be required.

Prob. 82. Draw the diagram of shears and the diagram of moments for the case of three equal spans uniformly loaded.

Prob. 83. Find what I beam is required to span three openings of 12 feet each, the load on each span being 6 000 pounds, and the greatest value of S to be 12 000 pounds per square inch.

ART. 49. CONTINUOUS BEAMS WITH UNEQUAL SPANS.

As the first example, consider two spans whose lengths are l_1, l_2, and whose loads per linear unit are w_1 and w_2. The theorem of three moments in (9) then reduces to,

$$2M_2(l_1 + l_2) = -\tfrac{1}{4}w_1l_1^3 - \tfrac{1}{4}w_2l_2^3,$$

and hence the bending moment at the middle support is,

$$M_2 = -\frac{w_1l_1^3 + w_2l_2^3}{8(l_1 + l_2)}.$$

From this the reaction at the left support may be found by (8) and the bending moment at any point by (7).

Next consider three spans whose lengths are l_1, l_2, and l_3, loaded uniformly with w_1, w_2, w_3. The bending moments at the second and third supports are M_2 and M_3. Then from (9),

$$2M_2(l_1 + l_2) + M_3l_2 = -\tfrac{1}{4}w_1l_1^3 - \tfrac{1}{4}w_2l_2^3,$$
$$M_2l_2 + 2M_3(l_2 + l_3) = -\tfrac{1}{4}w_2l_2^3 - \tfrac{1}{4}w_3l_3^3,$$

and the solution of these gives the values of M_2 and M_3. A very common case is that for which $l_2 = l$, $l_1 = l_3 = nl$, and $w_1 = w_2 = w_3 = w$. For this case the solution gives,

$$M_2 = M_3 = -\frac{1 + n^3}{3 + 2n} \cdot \frac{wl^2}{4}.$$

Here if $n = 1$, these two moments become $-\frac{1}{10}wl^2$, as also shown in the last article.

Whatever be the lengths of the spans or the intensity of the loads, the theorem of three moments furnishes the means of finding the bending moments at the supports. Then from (8), (7), and (6) the vertical shears and bending moments at every section may be computed. Finally, if the material be not strained beyond its elastic limit, formula (5) may be used to determine the deflection, while (4) investigates the strength of the beam.

Prob. 84. A continuous beam of three equal spans is loaded only in the middle span. Find the reactions of the end supports due to this load.

Prob. 85. A heavy 12-inch I beam of 36 feet length covers four openings, the two end ones being each 8 feet and the others each 10 feet in span. Find the maximum moment in the beam. Then determine the load per linear foot so that the greatest horizontal unit-stress may be 12 000 pounds per square inch.

ART. 50. REMARKS ON THE THEORY OF FLEXURE.

The theory of flexure presented in this and the preceding chapter is called the common theory, and is the one universally adopted for the practical investigation of beams. It should not be forgotten, however, that the axioms and laws upon which it is founded are only approximate and not of an exact nature like those of mathematics. Laws (*A*) and (*B*) for instance are true as approximate laws of experiment, but probably not as exact laws of science. Law (*G*) has been established by the observed fact that a vertical line, drawn upon the side of the beam before flexure, remains a straight line after flexure, even when the elastic limit of the material is exceeded.

When experiments on beams are carried to the point of rupture and the longitudinal unit-stress S computed from formula (4) a disagreement of that value with those found by direct experiments on tension or compression is observed. This is often regarded as an objection to the common theory of flexure, but it is in reality no objection, since law (G) and formula (4) are only true provided the elastic limit of the material be not exceeded. Experiments on the deflection of beams furnish on the other hand the most satisfactory confirmation of the theory. When E is known by tensile or compressive tests the formulas for deflection are found to give values closely agreeing with those observed. Indeed so reliable are these formulas that it is not uncommon to use them for the purpose of computing E from experiments on beams. If however the elastic limit of the material be exceeded, the computed and observed deflections fail to agree.

On the whole it may be concluded that the common theory of flexure is entirely satisfactory and sufficient for the investigation of all practical questions relating to the strength and stiffness of beams. The actual distribution of the internal stresses is however a matter of very much interest and this will be discussed at some length in Chapter VIII.

The theory of flexure is here applied to continuous beams only for the case of uniform loads. It should be said however that there is no difficulty in extending it to the case of concentrated loads. By a course of reasoning similar to that of Art. 48 it may be shown that the theorem of three moments for single loads is,

$$M'l' + 2M''(l' + l'') + M'''l'' = -P'l'^2(k - k^3) \\ - P''l''^2(2k - 3k^2 + k^3).$$

Here as in Fig. 37 the moments at three consecutive supports are designated by M', M'', and M''' and the lengths of the two spans by l' and l''. P' is any load on the first span at a distance kl' from the left support and P'' any load on the second span at a distance kl'' from the left support, k being any fraction less than unity and not necessarily the same in the two cases. From this theorem the negative bending moments at the supports for any concentrated loads may be found, and the beam be then investigated by formulas (6) and (4). For example, if a beam of three equal spans be loaded with P at the middle of each span, the negative moments at the supports are each $\frac{3}{20}Pl$.

The Journal of the Franklin Institute for March and April, 1875, contains an article by the author in which the law of increase of the quantities in the tables of Art. 48 is explained and demonstrated. A general abbreviated method of deducing the moments at the supports for both uniform and concentrated loads on restrained and continuous beams is given in the Philosophical Magazine for September, 1875. See also Van Nostrand's Science Series, No. 25.

Exercise 4. Consult BARLOW'S Strength of Materials (London, 1837), and write an essay concerning his experiments to determine the laws of the strength and stiffness of beams. Consult also BALL'S Experimental Mechanics.

Exercise 5. Consult Engineering News, Vol. XVIII, pp. 309, 352, 404, 443; Vol. XIX, pp. 11, 28, 48, 84; and Vol. XXII, p. 121. Write an essay concerning certain erroneous views regarding the theory of flexure which are there discussed. Consult also TODHUNTER'S History of the Elasticity and Strength of Materials.

Exercise 6. Procure six sticks of ash each $\frac{1}{8} \times \frac{3}{8}$ inches and of lengths about 8, 12, and 16 inches. Devise and conduct experiments to test the following laws: First, the strength of a beam varies directly as its breadth and directly as the square of its depth. Second, the stiffness of a beam is directly as its breadth and directly as the cube of its depth. Third, a beam fixed at the ends is twice as strong and four times as stiff as a simple beam when loaded at the middle. Write a report describing and discussing the experiments.

Exercise 7. In order to test the theory of continuous beams discuss the following experiments by FRANCIS and ascertain whether or not the ratio of the two observed deflections agrees with theory. "A frame was erected, giving 4 bearings in the same horizontal plane, 4 feet apart, making 3 equal spans, each bearing being furnished with a knife edge on which the beam was supported. Immediately over the bearings and secured to the same frame was fixed a straight edge, from which the deflections were measured. A bar of common English refined iron, 12 feet $2\frac{3}{4}$ inches long, mean width 1.535 inches, mean depth 0.367 inches, was laid on the 4 bearings, and loaded at the center of each span so as to make the deflections the same, the weight at the middle span being 82.84 pounds and at each of the end spans 52.00 pounds. The deflections with these weights were,

At the center of the middle span 0.281 inches.
At center of end spans, 0.275 and 0.284 inches,
mean 0.280 inches.

A piece 3 feet $11\frac{1}{2}$ inches long was then cut from each end of the bar, leaving a bar 4 feet $4\frac{3}{4}$ inches long, which was replaced in its former position and loaded with the same weight (82.84 pounds) as before, when its deflection was found to be 1.059 inches."

Prob. 86. A beam of three spans, the center one being l and the side ones nl, is loaded with P at the middle of each span. Find the value of n so that the reactions at the end may be one-fourth of the other reactions.

Prob. 87. Let a beam whose cross-section is an isosceles triangle have the base b and the depth d. Prove that if $0.13d$ be cut off from the vertex the remaining trapezoidal beam will be about 9 per cent stronger than the triangular one.

CHAPTER V.

COLUMNS OR STRUTS.

ART. 51. CROSS-SECTIONS OF COLUMNS.

A column is a prism, greater in length than about ten times its least diameter, which is subject to compression. If the prism be only about four or six times as long as its least diameter the case is one of simple compression, the constants for which are given in Art. 6. In a case of simple compression failure occurs by the crushing and splintering of the material, or by shearing in directions oblique to the length. In the case of a column, however, failure is apt to occur by a sidewise bending which causes flexural stresses. The word 'strut' is frequently used as synonymous with column, and sometimes also the word 'pillar.'

Wooden columns are usually square or round and they may be built hollow. Cast iron columns are usually round and they are often cast hollow. Wrought iron columns are made of a great variety of forms. I beams may be used, but most columns are usually made of three or more different shape-irons riveted together. The Phœnix column is made by riveting together flanged circular segments so as to form a closed cylinder. It is clear that a square or round section is preferable to an unsymmetrical one, since then the liability to bending is the same in all directions. For a rectangular section the plane of flexure will evidently be perpendicular to the longer side of the cross-section, and in general the plane of flexure will be perpendicular to that axis of the cross-section for which the moment of inertia is the least. In designing a column it is hence advisable that the cross-section should be so arranged that the moments of inertia about the two principal rectangular axes may be approximately equal.

For instance, let it be required to construct a column with two **I** shapes and two plates as shown in Fig. 43. The **I** beams are to be light 10-inch ones weighing 30 pounds per linear foot, and having the flanges 4.32 inches wide. The plates are to be $\frac{1}{2}$ inch thick, and it is required to find their length x so that the liability to bending about the two axes shown in the figure may be the same. From the table in Art. 30 it is ascertained that the moment of inertia I of the beam about an axis through its center of gravity and perpendicular to the web is 150, while the moment of inertia I' about an axis through the same point and parallel to the web is nearly 8. Hence, for the axes shown in the figure, the moments of inertia are,

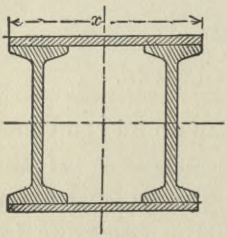

Fig. 43.

For axis perpendicular to plates,

$$2\frac{0.5 \times x^3}{12} + 2 \times 8 + 2 \times 9 \times \left(\frac{x}{2} - 2.16\right)^2.$$

For axis parallel to plates,

$$2\frac{x \times 0.5^3}{12} + 2 \times 0.5x \times 5.25^2 + 2 \times 150.$$

Placing these two expressions equal, the value of x is found to be between 14 and $14\frac{1}{2}$ inches.

Prob. 88. A column is to be formed of two light 12-inch eye-beams connected by a lattice bracing. Find the proper distance between their centers, disregarding the moment of inertia of the latticing.

Prob. 89. Two joists each 2×4 inches are to be placed 6 inches apart between their centers, and connected by two others each 8 inches wide and x inches thick so as to form a closed hollow rectangular column. Find the proper value of x.

ART. 52. GENERAL PRINCIPLES.

If a short prism of cross-section A be loaded with the weight P, the internal stress is to be regarded as uniformly distributed over the cross-section, and hence the compressive unit-stress S_c is $\frac{P}{A}$. But for a long prism, or column, this is not the case; while the average unit-stress is $\frac{P}{A}$, the stress in certain parts of the cross-section may be greater and upon others less than this value on account of the transverse stresses due to the sidewise flexure. Hence in designing a column the load P must be taken as smaller for a long one than for a short one, since evidently the liability to bending increases with the length.

Numerous tests on the rupture of long columns have shown that the load causing the rupture is approximately inversely proportional to the square of the length of the column. That is to say, if there be two columns of the same material and cross-section and one twice as long as the other, the long one will rupture under about one-quarter the load of the short one.

The condition of the ends of columns exerts a great influence upon their strength. Class (a) includes those with 'round ends,' or those in such condition that they are free to turn at the ends. Class (c) includes those whose ends are 'fixed' or in such condition that the tangent to the curve at the ends always remains vertical. Class (b) includes those with one end fixed and the other round. In architecture it is rare that any other than class (c) is used. In bridge construction and in machines, however, columns of classes (b) and (a) are very common. It is evident that class (c)

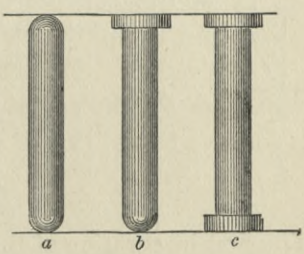

Fig. 44.

is stronger than (*b*), and that (*b*) is stronger than (*a*), and this is confirmed by all experiments. Fig. 44 is intended as a symbolical representation of the three classes of columns, and not as showing how the ends are rendered 'round' and 'fixed' in practical constructions.

The theory of the resistance of columns has not yet been perfected like that of beams, and accordingly the formulas for practical use are largely of an empirical character. The form of the formulas however is generally determined from certain theoretical considerations, and these will be presented in the following articles as a basis for deducing the practical rules.

Prob. 90. A pillar formed of two I beams each weighing 93 pounds per yard is 11 inches square and 3 feet long. What load will it carry with a factor of safety of 5?

ART. 53. EULER'S FORMULA FOR LONG COLUMNS.

Consider a column of cross-section A loaded with a weight P under whose action a certain small sidewise bending occurs. Let the column be round, or free to turn at both ends as in Fig. 45. Take the origin at the upper end, and let x be the vertical and y the horizontal co-ordinate of any point of the elastic curve. The general equation (5) deduced in Art. 33, applies to all bodies subject to flexure provided the bending be slight and the elastic limit of the material be not exceeded. For the column the bending moment is $-Py$, the negative sign being used because the curve is concave to the axis of x; hence,

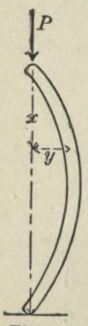

Fig. 45.

$$EI\frac{d^2y}{dx^2} = -Py.$$

The first integration of this gives,

$$EI\frac{dy^2}{dx^2} = -Py^2 + C.$$

But when $y =$ the maximum deflection Δ, the tangent $\frac{dy}{dx} = 0$. Hence $C = P\Delta^2$, and by inversion,

$$dx = \left(\frac{EI}{P}\right)^{\frac{1}{2}} \frac{dy}{\sqrt{\Delta^2 - y^2}}.$$

The second integration now gives,

$$x = \left(\frac{EI}{P}\right)^{\frac{1}{2}} \text{arc sin} \frac{y}{\Delta} + C'.$$

Here C' is 0 because $y = 0$ when $x = 0$. Hence finally the equation of the elastic curve of the column is,

$$y = \Delta \sin x \left(\frac{P}{EI}\right)^{\frac{1}{2}}$$

This equation is that of a sinusoid. But also $y = 0$ when $x = l$. Hence if n be an integer, $l\left(\frac{P}{EI}\right)^{\frac{1}{2}}$ must equal $n\pi$, or,

$$P = EI\frac{n^2\pi^2}{l^2},$$

which is EULER'S formula for the resistance of columns. This reduces the equation of the sinusoid to,

$$y = \Delta \sin n\pi\frac{x}{l}.$$

The three curves for $n = 1$, $n = 2$, and $n = 3$ are shown in Fig. 46. In the first case the curve is entirely on one side of the axis of x, in the second case it crosses that axis at the middle, and in the third case it crosses at $\frac{1}{3}l$ and $\frac{2}{3}l$, the points of crossing being also inflection points where the bending moment is zero. Evidently the greatest deflection will occur for the case where $n = 1$, and

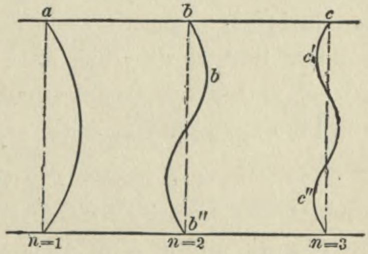

Fig. 46.

this is the most dangerous case. Hence,

$$(a) \qquad P = \frac{\pi^2 EI}{l^2},$$

is EULER'S formula for long columns with round ends.

A column with one end fixed and the other round is closely represented by the portion $b'b''$ of the second case, b' being the fixed end where the tangent to the curve is vertical. Here $n = 2$, and the length $b'b''$ is three-fourths of the entire length, hence,

$$(b) \qquad P = \frac{9}{4}\frac{\pi^2 EI}{l^2}$$

is EULER'S formula for long columns with one end fixed and the other round.

A column with fixed ends is represented by the portion $c'c''$ of the case c. Here $n = 3$, and the length $c'c''$ is two-thirds of the entire length, hence,

$$(c) \qquad P = 4\frac{\pi^2 EI}{l^2}$$

is EULER'S formula for long columns with fixed ends.

From this investigation it appears that the relative resistances of long columns of the classes (a), (b), and (c) are as the numbers 1, $2\frac{1}{4}$, and 4, when the lengths are the same, and this conclusion is approximately verified by experiments. It also appears that, if the resistance of three columns of the classes (a), (b), and (c) are to be equal, their lengths must be as the numbers 1, $1\frac{1}{2}$, and 2.

The moment of inertia I in the above formulas is taken about a neutral axis of the cross-section perpendicular to the plane of the flexure, and in general is the least moment of inertia of that cross-section, since the column will bend in the direction which offers the least resistance. If A be the area of the cross-section and r its least radius of gyration, the value of I is Ar^2, and EULER'S formula may be written

$$\frac{P}{A} = m\pi^2 E \frac{r^2}{l^2},$$

where $m = 1$, $2\frac{1}{4}$, or 4, for the three end conditions.

The value of P in EULER'S formula gives the load which holds the column in equilibrium when it is laterally deflected. If the load be less than this value of P the column, if deflected, will return to its original straight position. If the load be slightly greater than P the bending increases until failure occurs. EULER'S formula is hence the criterion of indifferent equilibrium or the condition for the failure of a column by lateral bending. The maximum deflection Δ is indeterminate, since it cannot be found from the equation of the elastic curve.

EULER'S formula is little used in practice, except in Germany. When so used it takes the form

$$\frac{P}{A} = \frac{m\pi^2 E}{f} \cdot \frac{r^2}{l^2},$$

in which f is an assumed factor of security and P is the load on the column.

Prob. 91. Show that the value of the constant m for a long column which is fixed at one end and entirely free at the other is $m = \frac{1}{4}$.

ART. 54. HODGKINSON'S FORMULAS.

EULER'S formula gives valuable information regarding the laws of flexure of long columns. For cylindrical columns it shows, since $I = \pi d^4/64$, that the load P which causes lateral failure varies directly as the fourth power of the diameter and inversely as the square of the length. HODGKINSON in his experiments observed that this was approximately but not exactly the case. He therefore wrote for each kind of columns the analogous expression,

$$P = \alpha \frac{d^{\beta}}{l^{\delta}},$$

and determined the constants α, β, and δ from the results of his experiments, thus producing empirical formulas.

Let P be the crushing load in gross tons, d the diameter of the column in inches, and l its length in feet. Then HODGKINSON'S empirical formulas are,

For solid cast iron cylindrical columns,

$$P = 14.9 \frac{d^{3.5}}{l^{1.63}} \qquad \text{for round ends,}$$

$$P = 44.2 \frac{d^{3.5}}{l^{1.63}} \qquad \text{for flat ends,}$$

For solid wrought iron cylindrical columns,

$$P = 42 \frac{d^{3.76}}{l^{2}} \qquad \text{for round ends;}$$

$$P = 134 \frac{d^{3.76}}{l^{2}} \qquad \text{for flat ends.}$$

These formulas indicate that the ultimate strength of flat-ended columns is about three times that of round-ended ones. The experiments also showed that the strength of a column with one end flat and the other end round is about twice that of one having both ends round. HODGKINSON'S tests were made upon small columns and his formulas are not so reliable as those which will be given in the following articles. For small cast iron columns however the formulas are still valuable.

By the help of logarithms it is easy to apply these formulas to the discussion of given cases. Usually P will be given and d required, or d be given and P required. By using assumed factors of safety the proper size of cylindrical columns to carry given loads may also be determined. These formulas, it should be remembered, do not apply to columns shorter than about thirty times their least diameters. The word flat used in this Article is to be regarded as equivalent to fixed.

Prob. 92. A cast iron cylindrical column with flat ends is 3 inches diameter and 8 feet long. What load will cause it to fail?

Prob. 93. A cast iron cylindrical column with flat ends is to be 7 feet long and carry a load of 200 000 pounds with a factor of safety of 6. Find the proper diameter.

ART. 55. RANKINE'S FORMULA.

The columns which are generally employed in engineering practice are intermediate in length between short prisms and the long columns to which EULER'S formula applies. They fail under the stresses caused by combined flexure and compression. Fig. 47 shows the flexure very much exaggerated. The load P produces an average unit-stress $\frac{P}{A}$ on any horizontal cross-section whose area is A, but in consequence of the bending this is increased on the concave side and diminished on the convex side by an amount S_1. The value of S_1 depends upon the bending moment $P\Delta$, where Δ is the lateral deflection at the middle of the column. The total unit-stress on the concave side is then $\frac{P}{A} + S_1$, and it is natural to suppose that failure will occur when this is equal to the ultimate strength of the material. Many formulas have been proposed for the investigation of such columns, but all of them are more or less empirical, as certain constants are derived from the results of ex-

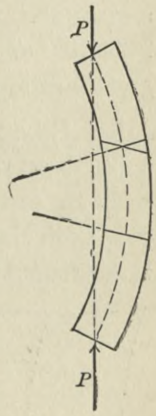

Fig. 47.

periments upon the rupture of columns, or assumptions are made regarding the form of the formula.

The formulas of EULER are unsuitable for practical cases of investigation because they contain no constant indicating the working or ultimate compressive strength of the material, and because they apply only to long columns. HODGKINSON'S formulas are unsatisfactory for similar reasons, and because they do not well agree with later experiments.

The formula which appears to have the best theoretical foundation will now be presented. It is sometimes called GORDON'S formula, and occasionally it is referred to as "GORDON'S formula modified by RANKINE," but the best usage gives to it the name of RANKINE'S formula. (The formula deduced by GORDON differs from (10) in that it applies only to rectangular or circular cross-sections, r being replaced by d, the least side or diameter, and q having different values from those given in the table.)

Let P be the load on the column, l its length, A the area of its cross-section, I the moment of inertia, and r the radius of gyration of that cross-section with reference to a neutral axis perpendicular to the plane of flexure, and c the shortest distance from that axis to the remotest fiber on the concave side.

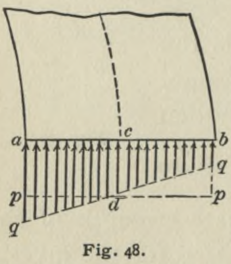

Fig. 48.

The average compressive unit-stress on any cross-section is $\frac{P}{A}$, but in consequence of the flexure this is increased on the concave side, and decreased on the convex side. Thus in Fig. 48 the average unit-stress $\frac{P}{A}$ is represented by cd, but on the concave side this is increased to aq, and on the convex side decreased to bq. The triangles pdq and qdp represent the effect of the flexure exactly as in the case of beams, pq indicating the greatest compressive and qp the greatest tensile unit-stress due to the bending. Let the total maximum unit-stress aq be denoted by S and the part due to the flexure be denoted by S_1. Then,

$$S = \frac{P}{A} + S_1.$$

Now, from the fundamental formula (4) the flexural stress is $\frac{Mc}{I}$, where M is the external bending moment, which for a column has its greatest value when $M = P\Delta$, Δ being the maximum deflection. $I = Ar^2$ is the well-known relation between I and r. Hence the value of S_1 is,

$$S_1 = \frac{P\Delta c}{I} = \frac{P\Delta c}{Ar^2}.$$

By analogy with the theory of beams, as in Art. 37, the value of Δ may be regarded as varying directly as $\frac{l^2}{c}$. Hence if q be a quantity depending upon the kind of material and the condition of the ends, the total unit-stress is,

$$S = \frac{P}{A} + q\frac{Pl^2}{Ar^2}.$$

This may now be written in the usual form,

$$\frac{P}{A} = \frac{S}{1 + q\frac{l^2}{r^2}}, \qquad (10)$$

which is RANKINE'S formula for the investigation of columns.

The above reasoning has been without reference to the arrangement of the ends of the column. By Art. 53 it is known that a column with round ends must be one half the length of one with fixed ends in order to be of equal strength, and that a column with one end fixed and the other round must be three fourths the length of one with fixed ends in order to be of equal strength. Therefore if q be the constant for fixed ends, $(\frac{4}{3})^2 q$ will be the constant for one end fixed and the other round, and $2^2 q$ will be the constant for both ends round.

The values of q to be taken for use in formula (10) for the examples and problems of this chapter may be the following rough values, unless otherwise stated, while the values of the ultimate compressive unit-stress S will be taken from the table in Art. 6.

Material.	Both Ends Fixed.	Fixed and Round.	Both Ends Round.
Timber,	$\frac{1}{3\,000}$	$\frac{1.78}{3\,000}$	$\frac{4}{3\,000}$
Cast Iron,	$\frac{1}{5\,000}$	$\frac{1.78}{5\,000}$	$\frac{4}{5\,000}$
Wrought Iron,	$\frac{1}{36\,000}$	$\frac{1.78}{36\,000}$	$\frac{4}{36\,000}$
Steel,	$\frac{1}{25\,000}$	$\frac{1.78}{25\,000}$	$\frac{4}{25\,000}$

Very wide variation in the values of q is found from experiments on the rupture of different types of columns. Formula (10) when used with such experimental values is an empirical one only. In Art. 61 it is shown how a theoretic value of q may be derived which renders (10) a rational formula.

Prob. 94. Plot the curve represented by formula (10) for cases of wrought-iron columns with fixed and with round ends, taking the values of $\frac{P}{A}$ as ordinates, and the values of $\frac{l}{r}$ as abscissas.

ART. 56. RADIUS OF GYRATION OF CROSS-SECTIONS.

The radius of gyration of a surface with reference to an axis is equal to the square root of the ratio of the moment of inertia of the surface referred to the same axis to the area of the figure. Or if r be radius of gyration, I the moment of inertia, and A the area of the surface, then $I = Ar^2$.

In the investigation of columns by formula (10) the value of r^2 is required, r being the least radius of gyration. These values are readily derived from the expressions for the moment of inertia given in Art. 23, the most common cases being the following,

For a rectangle whose least side is d, $\qquad r^2 = \frac{d^2}{12}$.

For a circle of diameter d, $\qquad r^2 = \frac{d^2}{16}$.

For a triangle whose least altitude is d, $\qquad r^2 = \frac{d^2}{18}$.

For a hollow square section, $\qquad r^2 = \frac{d^2 + d'^2}{12}$.

For a hollow circular section, $\qquad r^2 = \frac{d^2 + d'^2}{16}$.

For I beams and other shapes, r^2 is found by dividing the least moment of inertia of the cross-section by the area of that cross-section. For instance, by the help of the table in Art. 30, the least value of r^2 for a light 12-inch I beam is found to be $\frac{11.0}{12.6} = 0.87$ inches^2.

Prob. 95. Compute the least radius of gyration for a T iron whose width is 4 inches, depth 4 inches, thickness of flange $\frac{1}{2}$ inches, and thickness of stem $\frac{15}{32}$ inches.

ART. 57. INVESTIGATION OF COLUMNS.

The investigation of a column consists in determining the maximum compressive unit-stress S from formula (10). The values of P, A, l, and r will be known from the data of the given case, and q is known from the results of previous experiments. Then,

$$S = \frac{P}{A}\left(1 + q\frac{l^2}{r^2}\right),$$

and, by comparing the computed value of S with the ultimate strength and elastic limit of the material, the factor of safety and the degree of stability of the column may be inferred.

For example, consider a hollow wooden column of rectangular section, the outside dimensions being 4×5 inches and the inside dimensions 3×4 inches. Let the length be 18 feet, the ends fixed, and the load be 5 400 pounds. Here $P =$ 5 400, $A = 8$ square inches, $l = 216$ inches. From the table $q = \frac{1}{3\,000}$. From Art. 57,

$$r^2 = \frac{5 \times 4^3 - 4 \times 3^3}{12 \times 8} = 2.21.$$

Then the substitution of these values gives,

$$S = \frac{5\,400}{8}\left(1 + \frac{216 \times 216}{3\,000 \times 2.21}\right) = 5430 \text{ pounds per square inch.}$$

Here the average unit-stress is 675 pounds per square inch, but the flexure has increased that stress on the concave side to 5 430 pounds per square inch, so that the factor of safety is only about $1\frac{1}{2}$.

Prob. 96. A cylindrical wrought iron column with fixed ends is 12 feet long, 6.36 inches in exterior diameter, 6.02 inches in interior diameter, and carries a load of 98 000 pounds. Find its factor of safety.

Prob. 97. A pine stick 3×3 inches and 12 feet long is used as a column with fixed ends. Find its factor of safety under a load of 3 000 pounds. If the length be only one foot, what is the factor of safety?

ART. 58. SAFE LOADS FOR COLUMNS.

To determine the safe load for a given column it is necessary to first assume the allowable working unit-stress S. Then from formula (10) the safe load is,

$$P = \frac{SA}{1 + q\frac{l^2}{r^2}}.$$

Here A, l, and r are known from the data of the given problem and q is taken from the table in Art. 55.

For example, let it be required to determine the safe load for a fixed-ended timber column, 3 × 3 inches square and 12 feet long, so that the greatest compressive unit-stress may be 800 pounds per square inch. From the formula,

$$P = \frac{800 \times 9}{1 + \dfrac{12^5}{3\,000 \times 3^2}} = \text{about } 700 \text{ pounds.}$$

A short prism 3 × 3 inches should safely carry ten times this load.

Prob. 98. Find the safe load for a heavy wrought iron I of 15 inches depth and 10 feet length when used as a column with fixed ends, the factor of safety being 4.

Prob. 99. Find the safe steady load for a hollow cast iron column with fixed ends, the length being 18 feet, outside dimensions 4 × 5 inches, inside dimensions 3 × 4 inches.

ART. 59. DESIGNING OF COLUMNS.

When a column is to be selected or designed the load to be borne will be known, as also its length and the condition of the ends. A proper allowable unit-stress S is assumed, suitable for the given material under the conditions in which it is used. Then from formula (1) the cross-section of a short column or prism is $\frac{P}{S}$, and it is certain that a greater value of the cross-section than this will be required. Next assume a form and area A, find r^2, and from the formula (10) compute S. If the computed value agrees with the assumed value the correct size has been selected. If not, assume a new area and compute S again, and continue the process until a proper agreement is attained.

For example, a hollow cast iron rectangular column of 18 feet length is to carry a load of 60 000 pounds. Let the working strength S be 15 000 pounds per square inch. Then for a short length the area required would be four square inches. Assume then that about 6 square inches will be needed. Let the section be square, the exterior dimensions 6×6 inches, and the interior dimensions $5\frac{1}{2} \times 5\frac{1}{2}$ inches. Then $A = 5.75$, $l = 18 \times 12$, $P = 60\,000$, $q = \frac{1}{5\,000}$, $r^2 = 5.52$, and from (10),

$$S = \frac{60\,000}{5.75}\left(1 + \frac{18^2 \times 12^2}{5\,000 \times 5.52}\right) = \text{about } 30\,000,$$

which shows that the dimensions are much too small. Again assume the exterior side as 6 inches and the interior as 5 inches. Then $A = 11$, $r^2 = 5.08$, and

$$S = \frac{60\,000}{11}\left(1 + \frac{18^2 \times 12^2}{5\,000 \times 5.08}\right) = \text{about } 15\,700.$$

As this is very near the required working stress, it appears that these dimensions very nearly satisfy the imposed conditions.

In many instances it is possible to assume all the dimensions of the column except one, and then after expressing A and r in terms of this unknown quantity, to introduce them into (10) and solve the problem by finding the root of the equation thus formed. For example, let it be required to find the size of a square wooden column with fixed ends and 24 feet long to sustain a load of 100 000 pounds with a factor of safety of 10. Here let x be the unknown side; then $A = x^2$ and $r^2 = \frac{x^2}{12}$. From (10),

$$800 = \frac{100\,000}{x^2}\left(1 + \frac{24^2 \times 12^3}{3\,000 \times x^2}\right).$$

By reduction this becomes,

$$8x^4 - 1\,000x^2 = 331\,776,$$

the solution of which gives 16.6 inches for the side of the column.

Prob. 100. Find the size of a square wooden column with fixed ends and 12 feet in length to sustain a load of 100 000 pounds with a factor of safety of 10. Find also its size for round ends.

ART. 60. THE STRAIGHT-LINE FORMULA.

In 1886 a straight-line formula for columns, as a substitute for RANKINE'S, was proposed by THOMAS H. JOHNSON, which has since been extensively used on account of its simplicity when expressed in numerical form. The notation being the same as in Art. 55, this formula is,

$$\frac{P}{A} = S - k\frac{l}{r},$$

in which k is a constant whose value is,

$$k = \frac{S}{3}\sqrt{\frac{4S}{3m\pi^2 E}},$$

where m is 1, $2\frac{1}{4}$, or 4, depending on the condition of the ends, as in EULER'S formula (Art. 53).

This formula is not a rational one, the equation of the straight line being assumed merely as a good representation of the results of experiments on the rupture of columns. The value of k is deduced by making the straight line tangent to the curve which represents EULER'S formula. Thus, let the values of $\frac{P}{A}$ and $\frac{l}{r}$ be regarded as the ordinates and abscissas of a curve, and be designated by u and v respectively. Then the equations of EULER'S curve and of the assumed straight line are,

$$u = \frac{m\pi^2 E}{v^2} \qquad \text{and} \qquad u = S - kv.$$

By placing equal the values of u in these two equations and also the values of the first derivatives, the ordinate and abscissa of the point of tangency are found to be,

$$u_1 = \tfrac{1}{3}S \qquad \text{and} \qquad v_1 = \sqrt{\frac{3m\pi^2 E}{S}},$$

and then the value of k, as above given. results. The value of v_1 is the limiting value of $\frac{l}{r}$, within which the straight-line formula is to be used.

The values of S to be used for cases of rupture are such as make the straight-line agree best with experimental results. The values derived by JOHNSON in his discussion are given in the following table, together with the corresponding values of k, and the limiting values of $\frac{l}{r}$.

Kind of Column.	S	k	Limit $\frac{l}{r}$
Wrought iron:			
Flat ends,	42 000	128	218
Hinged ends,	42 000	157	178
Round ends,	42 000	203	138
Mild steel:			
Flat ends,	52 500	179	195
Hinged ends,	52 500	220	159
Round ends,	52 500	284	123
Cast iron:			
Flat ends,	80 000	438	122
Hinged ends,	80 000	537	99
Round ends,	80 000	693	77
Oak:			
Flat ends,	5 400	28	128

It will be noticed that the values of S in the above table are less than the average values of ultimate strength given in Art. 6. For ductile materials, like wrought iron and mild steel, this should be the case in columns, since when the elastic limit is passed a flow of metal begins which causes the lateral deflection to increase, and failure then rapidly follows.

Reference is made to JOHNSON'S paper in Transactions of

American Society of Civil Engineers for July 1886, for a fuller discussion of the straight-line formula. Although much used for computing values of P/A, it is inconvenient for finding S, since this quantity is included in k and a cubic equation in S results.

Prob. 101. Solve Problems 99 and 100 by the straight-line formula, using the values given in the table.

ART. 61. RITTER'S RATIONAL FORMULA.

Several attempts have been made to establish a formula for columns, which shall be theoretically correct, like formula (4) for beams, when the material is not stressed beyond the elastic limit. The most successful attempt is that of RITTER, who in 1873 proposed the formula

$$\frac{P}{A} = \frac{S}{1 + \frac{S_e}{m\pi^2 E} \cdot \frac{l^2}{r^2}},$$

in which P is the load on the column, l its length, A the area and r the least radius of gyration of the cross-section, E the coefficient of elasticity, S_e the unit-stress at the elastic limit, and S the greatest compressive unit-stress on the concave side, while $m = 4$ when both ends are fixed, $m = 2\frac{1}{4}$ when one end is round and the other fixed, $m = 1$ when both ends are round, and $m = \frac{1}{4}$ when one end is fixed and the other free.

The form of this formula is the same as that of RANKINE'S formula, (10) in Art. 55, but it deserves a special name because it completes the deduction of the latter formula by finding for q a value which is closely correct when the stress S does not exceed the elastic limit S_e.

To justify RITTER'S formula let it be noted that EULER'S formula (Art. 53) gives the value of P which causes the failure of a long column by lateral bending. In the actual long column, however, the load must be less than given by

EULER'S formula, since it is required that stable equilibrium shall prevail. Now, if there be written

$$\frac{P}{A} = \frac{S}{S_e} \cdot m\pi^2 E \cdot \frac{r^2}{l^2}$$

for the long column, stable equilibrium prevails when S is less than S_e, while failure occurs when S equals S_e, because then the column will not spring back if laterally deflected by a slight force. S_e/S is hence the factor of security f for a long stable column, as noted at the end of Art. 53. Now RITTER'S formula reduces to the above form for long columns when l/r is so great that unity in the denominator may be neglected in comparison with the following term. It also reduces to the form $P/A = S$ for short blocks, when $l = 0$. The formula hence satisfies the two limiting conditions, and its general form is justified by the reasoning of Art. 55.

Another demonstration may be given as follows: Let P/A be denoted by y and l/r by x. Then the formulas of RANKINE and of EULER for stable equilibrium are

$$y = \frac{S}{1 + qx^2} \quad \text{and} \quad y = \frac{m\pi^2 E}{fx^2}.$$

Now let the curves which these equations represent be tangent to each other. For the point of tangency the two ordinates are equal, as also the two derivatives of y with respect to x. Solving these two equations there results $x_1 = \infty$ for the point of tangency, while

$$q = \frac{Sf}{m\pi^2 E} = \frac{S_e}{m\pi^2 E}$$

is the constant in RANKINE'S formula. Thus,

$$\frac{P}{A} = \frac{S}{1 + \frac{S_e}{m\pi^2 E} \cdot \frac{l^2}{r^2}},$$

which is the rational formula of RITTER.

To find the mean theoretical values of q the mean values of S_e and E, as stated in Art. 6, may be used, whence results the following table:

RATIONAL VALUES OF q FOR FORMULA (10).

Material.	Both Ends Fixed.	Fixed and Round.	Both Ends Round.
Timber...........	$\frac{1}{20\,000}$	$\frac{1.78}{20\,000}$	$\frac{4}{20\,000}$
Cast Iron.........	$\frac{1}{30\,000}$	$\frac{1.78}{30\,000}$	$\frac{4}{30\,000}$
Wrought Iron.....	$\frac{1}{40\,000}$	$\frac{1.78}{40\,000}$	$\frac{4}{40\,000}$
Steel...............	$\frac{1}{24\,000}$	$\frac{1.78}{24\,000}$	$\frac{4}{24\,000}$

These theoretical values of q are smaller than the empirical values given in Art. 55, except that for steel. Values of P computed from RITTER'S formula are hence generally larger than those found from the empirical formulas in common use. In specifications, however, the values of q and S to be used are generally stated, so that the designer is free from the responsibility of selecting them.

It should be noted that this rational formula, having been deduced from the laws which govern the behavior of materials within the elastic limit, is not necessarily true for cases of rupture. In RANKINE'S formula (10) the unit-stress S may be the ultimate strength or any smaller value, but in RITTER'S formula S cannot exceed the elastic limit S_e. This rational formula cannot hence be justified or disproved because it agrees or fails to agree with the results of experiments on the rupture of columns.

Prob. 102. Compute by RITTER'S formula the safe steady load for a hollow cast-iron column with fixed ends, the outside dimensions being 4 × 5 inches, the inside dimensions 3 × 4 inches, and the length 18 feet.

Prob. 103. Find by RITTER'S formula the size of a square wooden strut with fixed ends, and 12 feet long, to carry a load of 100 000 pounds, taking S as 800 pounds per square inch.

ART. 62. ECCENTRIC LOADS.

In all that precedes, the load on the column has been supposed to be applied at the center of gravity of the cross-section. Let now the case be considered where the load P is applied at the horizontal distance a from that center of gravity and in the same vertical plane with the least radius of gyration r. The reacting load at the other end is similarly situated with respect to the cross-section at that end. Let an origin be taken at the upper end of the column and let x be the vertical and y the horizontal co-ordinate of any point in the elastic curve. The column having round ends, the bending moment for the end is Pa, and for any other point $P(a + y)$. Then from (5) the differential equation of the elastic curve is,

$$EI\frac{d^2y}{dx^2} = -P(a + y),$$

where the negative sign is used because the curve is concave to the axis of x. This may be written,

$$\frac{d^2y}{dx^2} = -\beta^2(a + y), \quad \text{where} \quad \beta = \sqrt{\frac{P}{EI}},$$

and by two integrations there is found,

$$y = -a + a\frac{\sin\beta(l - x) + \sin\beta x}{\sin\beta l},$$

as may be proved by differentiating the last equation twice.

The deflection at the middle of the column is found by making $x = \frac{1}{2}l$ and $y = \Delta$, whence

$$\Delta = a(\sec\tfrac{1}{2}\beta l - 1).$$

Here Δ becomes indeterminate when $a = 0$ and $\sec \frac{1}{2}\beta l = \infty$; when $\sec \frac{1}{2}\beta l = \infty$, the arc $\frac{1}{2}\beta l$ equals $\frac{1}{2}\pi$, whence $\beta l = \pi$, or inserting for β its value and squaring,

$$P = \frac{\pi^2 EI}{l^2},$$

which is EULER'S formula for columns with round ends.

Resuming now the reasoning of Art. 55, the maximum compressive unit-stress S on the concave side of the column is given by,

$$S = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{A} + \frac{P}{A}\frac{(a + \Delta)c}{r^2},$$

where c is the distance from the axis of the column to the concave side. Inserting the value of Δ found above,

$$S = \frac{P}{A}\left(1 + \frac{ac}{r^2} \sec \tfrac{1}{2}\beta l\right), \quad \text{where} \quad \beta l = \frac{l}{r}\sqrt{\frac{P}{AE}},$$

and this is the formula for a round-ended column with an eccentric load. Here also, if $a = 0$, $\sec \frac{1}{2}\beta l = \infty$, and the unit-stress S is indeterminate.

For any finite value of the eccentricity a the deflection Δ and the unit-stress S may be computed. For example, let a column with round ends have $A = 16.8$ square inches, $r = 3.01$ inches, $c = 4.45$ inches, $l = 192$ inches, $a = 1$ inch, $P = 1\,680\,000$ pounds, $E = 30\,000\,000$ pounds per square inch. Here $P/A = 10\,000$ pounds per square inch. The value of the arc $\frac{1}{2}\beta l$ is 0.582 and the angle in degrees is $0.582 \times 57°.3 = 33°\ 20'$; then $\sec \frac{1}{2}\beta l = 1.197$, and the above formula for the deflection gives $\Delta = 0.197$ inches. Next $ac/r^2 = 0.489$, and the above formula for the unit-stress on the concave side gives $S = 16\,900$ pounds per square inch. In this case the influence of the eccentric load is to increase the unit-stress 69 per cent.

The value of P/A cannot be deduced from the above formula for S, since it is contained in β. It can be found, however, by successive trials when all the other quantities in the formula are given. For this purpose it is well to write the formula in the form,

$$\frac{P}{A} = \frac{S}{1 + \frac{ac}{r^2} \sec \frac{l}{2r}\sqrt{\frac{P}{AE}}},$$

and to compute the second member for an assumed value of P/A; then insert this computed value and compute again, and so continue until P/A is obtained to the required degree of precision.

Further matter relating to columns and struts under compression will be found in Arts. 108, 117, 149, 150, and 151.

Prob. 104. Show that $\sec \frac{1}{2}\beta l$ in the formulas of this article should be written $\sec \frac{1}{8}\beta l$ for a column with one end round and the other free, and $\sec \frac{1}{4}\beta l$ for a column with both ends fixed.

Prob. 105. A round-ended column has $A = 2.12$ square inches, $r = 1.18$ inches, $c = 1.75$ inches, $a = 0.25$ inches, $l = 180$ inches, and $E = 30\,000\,000$ pounds per square inch. Find the load P which will cause S to be $12\,500$ pounds per square inch.

CHAPTER VI.

TORSION, AND SHAFTS FOR TRANSMITTING POWER.

ART. 63. THE PHENOMENA OF TORSION.

Torsion occurs when applied forces tend to cause a twisting of a body around an axis. Let one end of a horizontal shaft be rigidly fixed and let the free end have a lever p attached at right angles to its axis. A weight P hung at the end of this lever will twist the shaft so that fibers such as *ab*, which were originally horizontal, assume a spiral form *ad* like the strands of a rope. Radial lines such as *cb* will also have moved through a certain angle *bcd*.

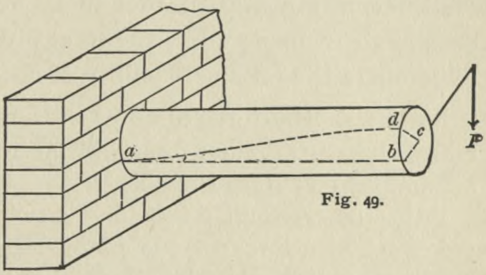

Fig. 49.

Experiments have proved, that if P be not so large as to strain the material beyond its elastic limit, the angles *bcd* and *bad* are proportional to P and that on the removal of the stress the lines *cd* and *ad* return to their original positions *cb* and *ab*. The angle *bcd* is evidently proportional to the length of the shaft, while *bad* is independent of the length. If the elastic limit be exceeded this proportionality does not hold, and if the twisting be great enough the shaft will be ruptured. These laws are but a particular case of the general axioms stated in Art. 3.

The product Pp is the moment of the force P with respect to the axis of the shaft, p being the perpendicular distance from that axis to the line of direction of P, and is called the twisting moment. Whatever be the number of forces acting at the end of the shaft, their resulting twisting moment may always be represented by a single product Pp.

A graphical representation of the phenomena of torsion may be made as in Fig. 1, the angles of torsion being taken as abscissas and the twisting moments as ordinates. The curve is then a straight line from the origin until the elastic limit of the material is reached, when a rapid change occurs and it soon becomes nearly parallel to the axis of abscissas. The total angle of torsion, like the total ultimate elongation, serves to compare the relative ductility of specimens.

Prob. 106. If a force of 80 pounds at 18 inches from the axis twists a shaft 60°, what force will produce the same result when acting at 4 feet from the axis?

Prob. 107. A shaft 2 feet long is twisted through an angle of 7 degrees by a force of 200 pounds acting at a distance of 6 inches from the axis. Through what angle will a shaft 4 feet long be twisted by a force of 500 pounds acting at a distance of 18 inches from the axis?

ART. 64. THE FUNDAMENTAL FORMULA FOR TORSION.

The stresses which occur between any two cross-sections of a bar under torsion are similar to those of shearing, each section tending to shear off from the one adjacent to it. When equilibrium obtains the external twisting moment is exactly balanced by the sum of the moments of these resisting internal stresses, or,

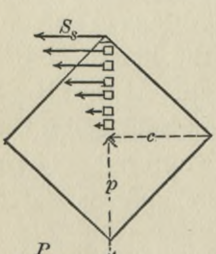

Fig. 50.

Resisting moment = twisting moment.

The law governing the distribution of these internal stresses is to be taken the same as in beams, namely, that they vary directly as the distance from the axis, provided that the elastic limit of the material be not exceeded.

If P be the force acting at a distance p from the axis about which the twisting takes place, the value of the twisting moment is Pp. To find the resisting moment, let c be the distance from the axis to the remotest part of the cross-section where the unit-shear is S_s. Then since the stresses vary as their distances from the axis,

$$\frac{S_s}{c} = \text{unit-stress at a unit's distance from axis,}$$

$$\frac{S_s z}{c} = \text{unit-stress at a distance } z \text{ from axis,}$$

$$\frac{aS_s z}{c} = \text{total stress on an elementary area } a,$$

$$\frac{aS_s z^2}{c} = \text{moment of this stress with respect to axis,}$$

$$\frac{\Sigma aS_s z^2}{c} = \text{internal resisting moment.}$$

This may be written $\frac{S_s}{c}\Sigma az^2$. But Σaz^2 is the polar moment of inertia of the cross-section with respect to the axis, and may be denoted by J. Therefore,

$$\frac{S_s J}{c} = Pp, \tag{11}$$

which is the fundamental formula for torsion.

The analogy of formula (11) with formula (4) for the flexure of beams will be noted. Pp, the twisting moment, is often the resultant of several forces, and might have been expressed by a single letter like the M in (4). By means of (11) a shaft subjected to a given moment may be investigated, or the proper size be determined for a shaft to resist given forces.

Prob. 108. Three forces of 120, 90, and 70 pounds act at distances of 6, 11, and 8 inches from the axis and at different distances from the end of a shaft, the direction of rotation of the second force being opposite to that of the others. Find the three values of the twisting moment Pp.

Prob. 109. A circular shaft is subjected to a maximum shearing unit-stress of 2 000 pounds when twisted by a force of 90 pounds at a distance of 27 inches from the center. What unit-stress will be produced in the same shaft by two forces of 40 pounds, one acting at 21 and the other at 36 inches from the center?

ART. 65. POLAR MOMENTS OF INERTIA.

The polar moment of inertia for simple figures is readily found by the help of the calculus, as explained in works on elementary mechanics. It is also a fundamental principle that,

$$J = I_1 + I_2,$$

where J is the polar moment of inertia, I_1 the least and I_2 the greatest rectangular moment of inertia about two axes passing through the center. The following are values of J for some of the most common cases.

For a circle with a diameter d, $\qquad J = \frac{\pi d^4}{32}$,

For a square whose side is d, $\qquad J = \frac{d^4}{6}$,

For a rectangle with sides b and d, $\qquad J = \frac{bd^3}{12} + \frac{b^3 d}{12}$.

The value of c in all cases is the distance from the axis about which the twisting occurs, usually the center of figure of the cross-section, to the remotest part of the cross-section. Thus,

For a circle with diameter d, $\qquad c = \frac{1}{2}d$,

For a square whose side is d, $\qquad c = d\sqrt{\frac{1}{2}}$,

For a rectangle with sides b and d, $\qquad c = \frac{1}{2}\sqrt{b^2 + d^2}$.

It is rare in practice that formulas for torsion are needed for any cross-sections except squares and circles.

Prob. 110. Find the values of J and c for an equilateral triangle whose side is d.

Prob. 111. Find, from the data in Art. 30, the values of J and c for a light 6 inch I section.

ART. 66. THE CONSTANTS OF TORSION.

The constant S_s computed from experiments on the rupture of shafts by means of formula (11) may be called the modulus of torsion, in analogy with the modulus of rupture as computed from (4). The values thus found agree closely with the ultimate shearing unit-stress given in Art. 7, viz.,

For timber, $S_s = 2\,000$ pounds per square inch,
For cast iron, $S_s = 25\,000$ pounds per square inch,
For wrought iron, $S_s = 50\,000$ pounds per square inch,
For steel, $S_s = 75\,000$ pounds per square inch.

By the use of these average values it is hence easy to compute from (11) the load P acting at the distance p which will cause the rupture of a given shaft.

The coefficient of elasticity for shearing may be computed from experiments on torsion in the following manner. Let a circular shaft whose length is l and diameter d be twisted through an arc θ by the twisting moment Pp. Here a point on the circumference of one end is twisted relative to a corresponding point on the other end through the arc θ or through the distance $\frac{1}{2}\theta d$, so that the detrusion per unit of length is

$$s = \frac{\frac{1}{2}\theta d}{l}.$$

From the fundamental definition of the coefficient of elasticity E as given in (2),

$$E = \frac{S_s}{s} = \frac{2S_s l}{\theta d},$$

and inserting for S_s its value from (11), there results,

$$E = \frac{32Ppl}{\pi\theta d^4},$$

from which E can be computed when all the quantities in the second member have been determined by experiment, provided that the elastic limit of the material be not exceeded. The numerical value of θ must here be expressed in terms of the same unit as π.

Prob. 112. What force P acting at the end of a lever 24 inches long will twist asunder a steel shaft 1.4 inches in diameter?

Prob. 113. An iron shaft 5 feet long and 2 inches in diameter is twisted through an angle of 7 degrees by a force of 5 000 pounds acting at 6 inches from the center, and on the removal of the force springs back to its original position. Find the value of E for shearing.

ART. 67. SHAFTS FOR THE TRANSMISSION OF POWER.

Work is the product of a resistance by the distance through which it acts, and is usually measured in foot-pounds. A horse-power is 33 000 foot-pounds of work done in one minute. It is required to determine the relation between the horse-power H transmitted by a shaft and the greatest internal shearing unit-stress S_s produced in it.

Let a shaft making n revolutions per minute transmit H horse-power .The work may be applied by a belt from the motor to a pulley on the shaft, then, by virtue of the elasticity and resistance of the material of the shaft, it is carried through other pulleys and belts to the working machines. In doing this the shaft is strained and twisted, and evidently S_s increases with H. Let P be the resistance acting at the circumference of the pulley and p the radius of the pulley. In making one revolution the force P acts through the distance $2\pi p$ and performs the work $2\pi pP$, and in n revolutions it performs the work $2\pi pPn$. Then if P be in pounds and p in inches, the imparted horse-power is,

$$H = \frac{2\pi pPn}{33\,000 \times 12}.$$

The twisting moment Pp in this expression may be expressed, as in formula (11), by the resisting moment $\frac{S_s J}{c}$. Hence the equation becomes,

$$(12) \qquad H = \frac{\pi n S_s J}{198\,000 c}.$$

This is the formula for the discussion of shafts for the transmission of power, and in it J and c must be taken in inches and S_s in pounds per square inch, while n is the number of revolutions per minute.

Prob. 114. A wooden shaft 6 inches square breaks when making 40 revolutions per minute. Find the horse-power then probably transmitted.

ART. 68. ROUND SHAFTS.

For round shafts of diameter d, the values of J and c are to be taken from Art. 65 and inserted in the last equation, giving,

$$S_s = 321\,000 \frac{H}{nd^3}, \qquad \text{or} \qquad d = 68.5 \sqrt[3]{\frac{H}{nS_s}}.$$

The first of these may be used for investigating the strength of a given shaft when transmitting a certain number of horse-power with a known velocity. The computed values of S_s, compared with the ultimate values in Art. 67, will indicate the degree of security of the shaft. Here d must be taken in inches and S_s will be in pounds per square inch.

The second equation may be used for determining the diameter of a shaft to transmit a given horse-power with a given number of revolutions per minute. Here a safe allowable value must be assumed for S_s in pounds per square inch, and then d will be found in inches. This equation shows that the diameter of a shaft varies directly as the cube root of the transmitted horse-power and inversely as the cube root of its velocity.

Prob. 115. Find the factor of safety for a wrought iron shaft $2\frac{1}{2}$ inches in diameter when transmitting 25 horse-power while making 100 revolutions per minute.

Prob. 116. Find the diameter of a wrought iron shaft to transmit 90 horse-power with a factor of safety of 8 when making 250 revolutions per minute, and also when making 100 revolutions per minute.

ART. 69. HOLLOW SHAFTS.

Hollow forged steel shafts are now coming into use for ocean steamers, their strength being greater than solid shafts of the same sectional area. If D be the exterior and d the interior diameter, and A the area of the cross-section, the polar moment of inertia is,

$$J = \frac{\pi}{32}(D^4 - d^4) = \frac{A}{8}(D^2 + d^2),$$

and the discussion of any case can be made by formula (12), c being replaced by $\frac{1}{2}D$.

For example, let it be required to determine the interior diameter of a nickel-steel shaft, when $D = 17$ inches, to transmit 16 000 horse power at 50 revolutions per minute, with a stress of 25 000 pounds per square inch on the exterior circumference. Here everything is given except d, and by solution its value is found to be 11 inches nearly.

Shafts are subject to flexural stresses due to their own weight and to applied loads, as well as to torsional stresses. The effect of these will be discussed in Art. 76.

Prob. 117. Find the diameter of a solid shaft for the conditions of the above example, and compare its weight with that of the hollow one.

Prob. 118. Find the horse-power transmitted by a hollow shaft, when $D = 15\frac{3}{4}$ inches, $d = 9\frac{3}{4}$ inches, and $S_s = 12\,500$ pounds per square inch, the number of revolutions per minute being 50.

ART. 70. MISCELLANEOUS EXERCISES.

Exercise 8. Make experiments to verify the phenomena of torsion stated in Art. 63. Show by your experiments that the strength of a round shaft varies directly as the cube of its diameter, and is independent of its length.

Exercise 9. Make a theoretical investigation to ascertain if the strength of a square shaft can be increased by cutting off material from the corners. If such is found to be the case write an essay explaining the reasoning, the computations and the conclusion.

Exercise 10. Go to a testing room and inspect THURSTON'S testing machine for torsion. Ascertain the dimensions and kind of specimens tested thereon. Explain with sketches the construction of the machine, the method of its use, and the torsion diagrams. State how the quality of the specimens is inferred from the torsion diagrams.

Prob. 119. Compare the strength of a square shaft with that of a circular shaft of equal area.

Prob. 120. Compare the strengths of two shafts when stressed to their elastic limits; the first shaft is solid, 24 inches in diameter, and has an elastic limit of 25 000 pounds per square inch; the second shaft is hollow, 18 inches outside and 9 inches inside diameter, and its elastic limit is 45 000 pounds per square inch.

CHAPTER VII.

COMBINED STRESSES.

ART. 71. COMBINED TENSION AND COMPRESSION.

Tensile and compressive forces acting upon a bar in the direction of its length, produce a resultant stress equal to their numerical difference, which may be either tensile or compressive. This case is of frequent occurrence in the members of bridge trusses.

A tensile force acting upon a bar produces a tensile unit-stress S and unit-elongation s. It is found by experiment that the lateral unit-contraction of the bar, when S is within the elastic limit, is about $\frac{1}{3}s$, and hence the internal compressive unit-stress normal to the length of the bar is about $\frac{1}{3}S$. Thus internal stresses may exist in a body in directions which do not correspond with any of the applied exterior forces. In general, if s be any unit-deformation, the corresponding internal unit-stress is sE (Art. 4).

If three tensile forces P_1, P_2, and P_3 act normally upon the sides of a rectangular prism whose areas are A_1, A_2, and A_3, the unit-stresses apparently produced upon those sides are $S_1 = P_1 \div A_1$, $S_2 = P_2 \div A_2$, and $S_3 = P_3 \div A_3$; but the real effective internal unit-stresses are much smaller, their values, by the principle of the last paragraph, being $T_1 = S_1 - \frac{1}{3}S_2 - \frac{1}{3}S_3$, $T_2 = S_2 - \frac{1}{3}S_1 - \frac{1}{3}S_3$, and $T_3 = S_3 - \frac{1}{3}S_1 - \frac{1}{3}S_2$. These formulas apply when some or all of the external forces are compressive as well as tensile, if the apparent unit-stresses be taken positive when tension and negative when compression. For example, let a cube whose edge is unity be subject to a compression of 60 pounds upon two opposite sides and to 45 pounds tension upon two other opposite sides, the third pair of sides having no forces applied to them. Then the effective internal unit-stress normal to the first pair of sides is 75 pounds compression, that for the second pair is 65 pounds tension, and that for the third pair is 5 pounds tension.

Prob. 121. A common brick $2 \times 4 \times 8$ inches is subject to compression of 3 200 pounds upon its top and bottom faces, 500 pounds upon its sides, and 60 pounds upon its ends. Find the effective internal unit-stresses in the three directions.

ART. 72. STRESSES DUE TO TEMPERATURE.

If a bar be unstrained it expands when the temperature rises and contracts when the temperature falls. But if the bar be under stress, so that the change of length cannot occur, an additional unit-stress must be produced which will be equivalent to the unit-stress that would cause the same change of length in the unstrained bar. Thus if a rise of temperature elongates a bar of length unity the amount s when free from stress, it will cause the unit-stress $S = sE$ (see Art. 4) when the bar is prevented from expanding by external forces.

Let l be the length of the bar, α its coefficient of linear expansion for a change of one degree, and λ the change of length due to the rise or fall of t degrees. Then,

$$\lambda = \alpha t l.$$

and the unit-deformation s is,

$$s = \frac{\lambda}{l} = \alpha t.$$

The unit-stress produced by the change in temperature hence is,

$$S = \alpha t E$$

which is seen to be independent of the length of the bar. The total stress on the bar is then AS.

The following are average values of the coefficients of linear expansion for a change in temperature of one degree Fahrenheit.

For brick and stone,	$\alpha = 0.000\,00\,50$,
For cast iron,	$\alpha = 0.000\,00\,62$,
For wrought iron,	$\alpha = 0.000\,00\,67$,
For steel,	$\alpha = 0.000\,00\,65$.

As an example consider a wrought iron tie rod 20 feet in length and 2 inches in diameter which is screwed up to a tension of 9 000 pounds in order to tie together two walls of a building. Let it be required to find the stress in the rod when the temperature falls 10° F. Here,

$$S = 0.000\,00\,67 \times 10 \times 25\,000\,000 = 1\,675 \text{ pounds.}$$

The total tension in the rod now is,

$$9\,000 + 3.14 \times 1\,675 = 14\,000 \text{ pounds.}$$

Should the temperature rise 10° the tension in the rod would become,

$$9\,000 - 3.14 \times 1\,675 = 4\,000 \text{ pounds.}$$

In all cases the stresses caused by temperature are added or subtracted to the tensile or compressive stresses already existing.

Prob. 122. A cast iron bar is confined between two immovable walls. What unit-stress will be produced by a rise of 40° in temperature?

ART. 73. COMBINED TENSION AND FLEXURE.

Consider a beam in which the flexure produces a unit-stress S_1 at the fiber on the tensile side most remote from the neutral axis. Let a tensile stress P be then applied to the ends of the bar uniformly distributed over the cross-section A. The tensile unit-stress at the neutral surface is then $\frac{P}{A}$ and all the longitudinal stresses due to the flexure are increased by this amount. Then $S = \frac{P}{A} + S_1$ is an approximate value of the maximum tensile unit-stress.

In designing a beam under combined tension and flexure the dimensions must be so chosen that $\frac{P}{A} + S_1$ shall not exceed the proper allowable working unit-stress. For instance, let it be required to find the size of a square wooden beam of 12 feet span to hold a load of 300 pounds at the middle while under a longitudinal stress of 2 000 pounds, so that the maximum tensile unit-stress may be about 1 000 pounds per square inch. Let d be the side of the square. From formula (4),

$$S_1 = \frac{6M}{d^3} = \frac{6 \times 150 \times 72}{d^3}.$$

Then from the conditions of the problem,

$$\frac{2\,000}{d^2} + \frac{64\,800}{d^3} = 1\,000,$$

from which results the cubic equation,

$$d^3 - 2d = 64.8,$$

whose solution gives for d the value 4.25 inches.

In investigating a beam under combined tension and flexure the maximum combined unit-stress S is computed, and the factor of safety found by comparing it with the ultimate tensile strength of the material. The method here given is approximate; more accurate methods are in Art. 118.

Prob. 123. A heavy 12-inch I beam of 6 feet span carries a uniform load of 200 pounds per linear foot, besides its own weight, and is subjected to a longitudinal tension of 80 000 pounds. Find the factor of safety of the beam.

Prob. 124. What I beam of 12 feet span is required to carry a uniform load of 200 pounds per linear foot when subjected to a tension of 50 000 pounds, the maximum tensile stress at the dangerous section to be 9 000 pounds per square inch?

ART. 74. COMBINED COMPRESSION AND FLEXURE.

Consider a beam in which the flexure produces a unit-stress S_1 in the fiber on the compressive side most remote from the neutral axis. Let a compressive stress P be applied in the direction of its length uniformly over the cross-section A. Then at the neutral surface the unit-stress is $\frac{P}{A}$ and at the remotest fiber it is $\frac{P}{A} + S_1$. The discussion of this case is hence exactly similar to that of the last article. If the beam is short the total working unit-stress is to be taken as for a short prism; if long it should be derived from RANKINE'S formula for columns.

The method of investigation explained in this and the preceding article is the one ordinarily used in practice on account of the complexity of the formulas which result from the strict mathematical determination of the moments of the applied forces. Although not exact the method closely approximates to the truth, giving values of the stresses a little too large for the case of tension and a little too small for the case of compression. (See Arts. 117 and 118.)

A rafter of a roof is a case of combined compression and flexure. Let b be its width, d its depth, l the length, w the load per linear unit, and ϕ the angle of inclination. To find the horizontal reaction H the center of moments is to be taken at the lower end, and

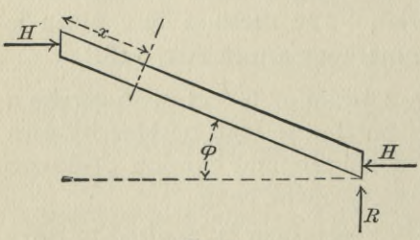

Fig. 51.

$$H \,.\, l \sin\phi = wl \,.\, \frac{l\cos\phi}{2}, \quad \text{whence} \quad H = \frac{wl}{2}\cot\phi.$$

For any section whose distance from the upper end is x, the flexural unit-stress now is from (4),

$$S_1 = \frac{6M}{bd^2} = \frac{6(Hx \sin\phi - \frac{1}{2}wx^2 \cos\phi)}{bd^2}.$$

and the uniform compressive unit-stress is,

$$S_2 = \frac{H \cos\phi + wx \sin\phi}{bd}.$$

The total compressive unit-stress on the upper fiber hence is,

$$S = S_1 + S_2 = \frac{3w \cos\phi}{bd^2}(lx - x^2) + \frac{wl \cot\phi \cos\phi}{2bd} + \frac{wx \sin\phi}{bd}.$$

This can be shown to be a maximum when

$$x = \tfrac{1}{2}l + \tfrac{1}{6}d \tan\phi,$$

and substituting this, the maximum unit-stress is,

$$S = \frac{3wl^2 \cos\phi}{4bd^2} + \frac{wl \operatorname{cosec}\phi}{2bd} + \frac{w \sin\phi \tan\phi}{12b}$$

which formula may be used to investigate or to design rafters subject to uniform loads.

In any inclined rafter let P denote all the load above a section distant x from the upper end. Then reasoning as before the greatest unit-stress for that section is found to be,

$$S_x = \frac{Mc}{I} + \frac{P \sin\phi}{A} + \frac{H \cos\phi}{A},$$

from which S_x may be computed for any given case.

Prob. 125. A roof with two equal rafters is 40 feet in span and 15 feet in height. The wooden rafters are 4 inches wide and each carries a load of 450 pounds at the center. Find the depth of the rafter so that S may be 700 pounds per square inch.

Prob. 126. A wooden beam 10 inches wide and 8 feet long carries a uniform load of 500 pounds per linear foot and is subjected to a longitudinal compression of 40 000 pounds. Find the depth of the beam so that the maximum working unit-stress may be about 800 pounds per square inch.

ART. 75. SHEAR COMBINED WITH TENSION OR COMPRESSION.

Let a bar whose cross-section is A be subjected to the longitudinal tension or compression P and at the same time to a shear V at right angles to its length. The longitudinal unit-stress is $\frac{P}{A}$ which may be denoted by p, and the shearing unit-stress is $\frac{V}{A}$ which may be denoted by v. It is required to find the maximum unit-stresses produced by the combination of p and v. In the following demonstration P will be regarded as a tensile force, although the reasoning and conclusions apply equally well when it is compressive.

Consider an elementary cubic particle with edges one unit in length acted upon by the horizontal tensile force p and p, and by the vertical shear v and v, as shown in Fig. 52. These forces are not in equilibrium unless a horizontal couple be applied as in the figure, each of whose forces is equal to v. Therefore at every point of a body under vertical shear there exists a horizontal shear, and the horizontal shearing unit-stress is equal to the vertical shearing unit-stress.

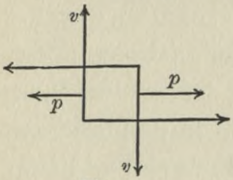

Fig. 52.

Let a parallelopipedal element have the length dm, the height dn, and a width of unity. The tensile force $p \, . \, dn$ tends to pull it apart longitudinally. The vertical shear vdn tends to cause rotation and this is resisted, as shown above, by the horizontal shear vdm. These forces may be resolved into rectangular components parallel and perpendicular to the diagonal dz, as shown in Fig. 53. The components parallel to the diagonal form a shearing force sdz, and

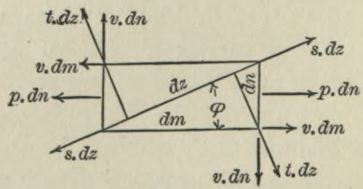

Fig. 53.

those perpendicular to it a tensile force tdz, s being the shearing and t the tensile unit-stresses. Let ϕ be the angle between dz and dm. The problem is first to state expressions for sdz and tdz in terms of ϕ, and then to determine the value of ϕ, or the ratio of dm to dn, which gives the maximum values of s and t.

By simple resolution of forces,

$$sdz = pdn \cos \phi + vdm \cos \phi - vdn \sin \phi,$$
$$tdz = pdn \sin \phi + vdm \sin \phi + vdn \cos \phi.$$

Divide each of these by dz, for $\frac{dn}{dz}$ put its value $\sin \phi$ and for $\frac{dm}{dz}$ its value $\cos \phi$. Then the equations take the form,

$$s = p \sin \phi \cos \phi + v(\cos^2 \phi - \sin^2 \phi),$$
$$t = p \sin^2 \phi + 2v \sin \phi \cos \phi.$$

These may be written,

$$s = \tfrac{1}{2}p \sin 2\phi + v \cos 2\phi,$$
$$t = \tfrac{1}{2}p(1 - \cos 2\phi) + v \sin 2\phi.$$

By placing the first derivative of each of these equal to zero it is found that,

$$s \text{ is a maximum when } \tan 2\phi = \frac{p}{2v},$$

$$t \text{ is a maximum when } \tan 2\phi = -\frac{2v}{p}.$$

Expressing $\sin 2\phi$ and $\cos 2\phi$ in terms of $\tan 2\phi$ and inserting them in the above the following values result:

$$(13) \qquad \begin{cases} \text{max. } s = \pm \sqrt{v^2 + \tfrac{1}{4}p^2}, \\ \text{max. } t = \tfrac{1}{2}p + \sqrt{v^2 + \tfrac{1}{4}p^2}. \end{cases}$$

These formulas apply to the discussion of the internal stresses in beams, as well as to combined longitudinal stress and vertical shear directly applied by external forces. If p is tension t is tension, if p is compression t is also compression. If when p is tension the negative sign be used before the radical, the resultant value of t is the maximum compressive unit-stress.

Prob. 127. A bolt $\frac{3}{4}$-inch in diameter is subjected to a tension of 2 000 pounds and at the same time to a cross shear of 3 000 pounds. Find the maximum tensile and shearing unit-stresses and the directions they make with the axis of the bolt.

ART. 76. COMBINED FLEXURE AND TORSION.

This case occurs when a shaft for the transmission of power is loaded with weights. Let S be the greatest flexural unit-stress computed from (4) and S_s the torsional shearing unit-stress computed from (12) or by the special equations of Arts. 67 and 68. Then, according to the last article, the resultant maximum unit-stresses are,

$$\text{max. ten. or comp.}\quad t = \tfrac{1}{2}S + \sqrt{S_s^2 + \tfrac{1}{4}S^2}$$
$$\text{max. shear}\quad s = \pm\sqrt{S_s^2 + \tfrac{1}{4}S^2}.$$

For wrought iron or steel it is usually necessary to regard only the first of these unit-stresses, but for timber the second should also be kept in view.

For example, let it be required to find the factor of safety of a wrought iron shaft 3 inches in diameter and 12 feet between bearings, which transmits 40 horse-power while making 120 revolutions per minute, and upon which a load of 800 pounds is brought by a belt and pulley at the middle. Taking the shaft as fixed over the bearings the flexural unit-stress is,

$$S = \frac{4Pl}{\pi d^3} = 5\,400 \text{ pounds per square inch.}$$

From Art. 68 the torsional unit-stress is,

$$S_s = 321\,000\frac{H}{nd^3} = 4\,000 \text{ pounds per square inch.}$$

The maximum tensile and compressive unit-stress now is,

$$t = 2\,700 + \sqrt{4\,000^2 + 2\,700^2} = 7\,600 \text{ pounds per square in.}$$

and the factor of safety is hence over 7.

As a second example, let it be required to find the size of a square wooden shaft for a water-wheel weighing 3 000 pounds which transmits 8 horse-power while making 20 revolutions per minute. The length of the shaft is 16 feet, and one-third of the weight is concentrated at the center and the remainder is equally divided between two points, each 6 feet from the center. Here the greatest flexural unit-stress is,

$$S = \frac{6(1\,500 \times 96 - 1\,000 \times 72)}{d^3} = \frac{432\,000}{d^3},$$

and from Art. 69 the torsional unit-stress is,

$$S_s = \frac{267\,500 \times 8}{20d^3} = \frac{107\,000}{d^3}.$$

From the formula of the last Article the combined tensile or compressive unit-stress is,

$$t = \frac{470\,400}{d^3}.$$

Now if the working value of t be taken at 600 pounds per square inch the value of d will be about 9 inches. From formula (13) also

$$s = \frac{254\,400}{d^3},$$

and if the working value of s be taken at 150, the value of d is found to be about 12 inches. The latter value should hence be chosen for the size of the shaft.

By similar reasoning it may be proved that the formula for finding the diameter of a round iron shaft is,

$$d^3 = \frac{16M}{\pi t} + \frac{16}{t}\sqrt{\frac{M^2}{\pi^2} + \frac{402\,500\,000H^2}{n^2}},$$

where M is the maximum bending moment of the transverse forces in pound-inches, H the number of transmitted horse-power, n the number of revolutions per minute, and t the safe allowable tensile or compressive working strength of the material.

Prob. 128. Find the factor of safety for the data of Prob. 115 when the shaft is in bearings 12 feet apart and carries a load of 200 pounds at the middle.

ART. 77. COMBINED COMPRESSION AND TORSION.

In the case of a vertical shaft the torsional unit-stress S_s combines with the direct compressive stress due to the weights upon the shaft, and produces a resultant compression t and shear s. From formulas (13) the combined unit-stresses are,

$$t = \tfrac{1}{2}S_c + \sqrt{S_s^2 + \tfrac{1}{4}S_c^2},$$
$$s = \sqrt{S_s^2 + \tfrac{1}{4}S_c^2}.$$

The use of these is the same as those of the last Article, S_s being found from the formulas of Chapter VI, while S_c is computed from formula (1) if the length of the shaft be less than ten times its diameter and from (10) for greater lengths.

In order to prevent vibration and flexure it is usual to place bearings at frequent intervals on a vertical shaft so that probably the use of formula (10) will rarely be required, particularly if t be taken at a low value. For a round shaft the expression for t becomes,

$$t = \frac{4P}{\pi d^2} + \sqrt{321\,000^2 \frac{H^2}{n^2 d^6} + \frac{16P^2}{\pi^2 d^4}},$$

in which P is the load. From this the diameter d may be found when t and the other data are given.

Prob. 129. A vertical shaft, weighing with its loads 6000 pounds, is subjected to a twisting moment by a force of 300 pounds acting at a distance of 4 feet from its center. If the shaft is wrought iron, 4 feet long and 2 inches in diameter, find its factor of safety.

Prob. 130. Find the diameter of a short vertical steel shaft to carry loads amounting to 6 000 pounds when twisted by a force of 300 pounds acting at a distance of 4 feet from the center, taking the unit-stress against compression as 10 000 and against shearing as 7 000 pounds per square inch.

ART. 78. HORIZONTAL SHEAR IN BEAMS.

The common theory of flexure as presented in Chapters III and IV considers that the internal stresses at any section are resolved into their horizontal and vertical components, the former producing longitudinal tension and compression and the latter a transverse shear, and that these act independently of each other. Formula (3) supposes further that the vertical shear is uniformly distributed over the cross-section of the beam. A closer analysis will show that a horizontal shear exists also and that this, together with the vertical shear, varies in intensity from the neutral surface to the upper and lower sides of the beam. It is well known that a pile of boards which acts like a beam deflects more than a solid timber of the same depth, and this is largely due to the lack of horizontal resistance between the layers. The common theory of flexure in neglecting the horizontal shear generally errs on the side of safety. In a few experiments however beams have been known to crack along the neutral surface and it is hence desirable to investigate the effect of horizontal shear in tending to cause rupture of that kind. That a horizontal shear exists simultaneously with the vertical shear is evident from the considerations in Art. 75.

Let Fig. 54 represent a portion of a bent beam of uniform section. Let a rectangular notch *nmpq* be imagined to be cut into it, and let forces be applied to it to preserve the equilibrium. Let H be the sum of all the horizontal components of these forces acting on *mn* and H' the sum of those acting on *qp*. Now H' is greater or less than H, hence the difference $H' - H$ must act along *mq* as a horizontal shear. Let the distance *mq* be dx, the thickness *mm'* be b, and the area *mqmm'* be at a distance c' above the neutral surface. Let c be the distance from that neutral surface to the remotest fiber where the unit-stress is S. Let a be the cross-section of any fiber. Let M be the bending moment at the section *mn* and M' that at the section *qp*. Now from the fundamental laws of flexure,

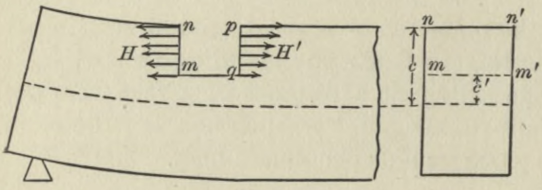

Fig. 54.

$$\frac{S}{c} = \text{unit-stress at a unit's distance from neutral surface,}$$

$$\frac{S}{c}y = \text{unit-stress at distance } y \text{ from neutral surface,}$$

$$\frac{aSy}{c} = \text{total stress on fiber } a \text{ at distance } y,$$

$$\frac{S}{c}\Sigma_{c'}^{c} ay = \text{sum of horizontal stresses between } m \text{ and } n.$$

The value of H hence is, since $\frac{S}{c} = \frac{M}{I}$,

$$H = \frac{M}{I}\Sigma_{c'}^{c} ay,$$

and likewise for the other section,

$$H' = \frac{M'}{I}\Sigma_{c'}^{c} ay.$$

The horizontal shear therefore is expressed by

$$H' - H = \frac{M' - M}{I} \Sigma_{c'}^{c} ay.$$

Now since the distance mq is dx, the value of $M' - M$ is dM. Also if S_h be the horizontal shearing unit-stress upon the area bdx the value of $H' - H$ is $S_h bdx$. Hence,

$$S_h = \frac{dM}{Ibdx} \Sigma_{c'}^{c} ay.$$

Again from Art. 45 it is plain that $\frac{dM}{dx}$ is the vertical shear V at the section under consideration. Therefore,

(14) $$S_h = \frac{V}{Ib} \Sigma_{c'}^{c} ay,$$

is the formula for the horizontal shearing unit-stress at any point of any section of the beam.

This expression shows that the horizontal shearing unit-stress is greatest at the supports, and zero at the dangerous section where V is zero. The summation expression is the statical moment of the area $mm'nn'$ with reference to the neutral axis; it is zero when $y = c$, and a maximum when $y = 0$. Hence the longitudinal unit-shear is zero at the upper and lower sides of the beam and is a maximum at the neutral surface. The formula for the maximum horizontal shearing unit-stress at any section therefore is,

$$S_s = \frac{V}{Ib} \Sigma_{o}^{c} ay.$$

Here I is the moment of inertia of the whole cross-section with reference to the neutral axis (Art. 23), b is the width of the beam along the neutral surface, and $\Sigma_{o}^{c} ay$ is the statical moment of the area of the part of the cross-section on one side of the neutral axis. Let A_1 be the area of the cross-section on one side of the neutral axis and c_1 the distance of its center of gravity from that axis; then $\Sigma_0^c ay = A_1c_1$, and the formula becomes,

$$(14)' \qquad S_s = \frac{VA_1c_1}{Ib},$$

which gives the maximum shearing unit-stress, both horizontal and vertical, at the neutral surface. The mean unit-stress given by (3) is always less than this maximum.

For a rectangular beam of breadth b and depth d, the value of I is $\frac{bd^3}{12}$, and $A_1c_1 = \frac{bd}{2} \cdot \frac{d}{4} = \frac{bd^2}{8}$. Then,

$$S_s = \frac{3V}{2bd}.$$

By inserting in this the value of V for any section the corresponding value of S_s at the neutral surface is found. In this particular instance it is seen that the approximate formula (3) gives values of S_s which are 33 per cent lower than the true maximum value.

Prob. 131. In the Journal of the Franklin Institute for February, 1883, is detailed an experiment on a spruce joist $3\frac{7}{8} \times 12$ inches and 14 feet long, which broke by tension at the middle and afterwards by shearing along the neutral axis at the end when loaded at the middle with 12 545 pounds. Find the tensile and shearing unit-stresses.

ART. 79. MAXIMUM INTERNAL STRESSES IN BEAMS.

From the last Article it is evident that at every point of a beam there exists a horizontal unit-shear of the intensity S_h and also a vertical unit-shear of the same intensity, whose value is given by (14). At every point there also exists a longitudinal tension or compression which may be computed from (4) with the aid of the principle that these stresses vary directly as their distances from the neutral axis. Let v denote the unit-shear thus determined and p the tensile or compressive unit-stress. Then from Art. 75 the maximum unit-shear at that point is,

$$s = \sqrt{v^2 + \tfrac{1}{4}p^2},$$

and it makes an angle ϕ with the neutral surface such that,

$$\tan 2\phi = \frac{p}{2v}.$$

Also the maximum tensile or compressive unit-stress at that point is,

$$t = \tfrac{1}{2}p + \sqrt{v^2 + \tfrac{1}{4}p^2},$$

and it makes an angle θ with the neutral surface such that,

$$\tan 2\theta = -\frac{2v}{p}.$$

From these formulas the lines of direction of the maximum stresses may be traced throughout the beam.

For the maximum shear v is greatest and p is zero at the neutral surface, while v is zero and p is greatest at the upper and lower surfaces. Hence for the neutral surface ϕ is 0, it increases with p, and becomes 45° at the upper and lower surfaces.

For the maximum tension t is greatest and equal to p on the convex side where $v = 0$ and $\theta = 0$. As the neutral surface is approached v increases, p decreases, and θ increases. At the neutral surface v is greatest, p is zero, and $\theta = -45°$. Here the maximum tension and compression are each equal to v.

For the maximum compression in like manner θ is 0° at the concave surface and 45° at the neutral surface. The lines of maximum tension if produced beyond the neutral surface would evidently cut those of maximum compression at right angles and be vertical at the concave surface.

The following figure is an attempt to represent the lines which indicate the directions of the maximum unit-stress in a beam. The full lines above the neutral surface are those of maximum compression, while those below are maximum tension. The broken lines are those of maximum shear. On any line the intensity of stress varies with the inclination, being greatest where the line is horizontal and least where its inclination is 45°. The lines of maximum shear cut those of maximum tension and compression at angles of 45°. The lines of maximum tension above the neutral surface and those of maximum compression below it are not shown; if drawn they would cut the others at right angles and become vertical at the upper and lower edges of the beam.

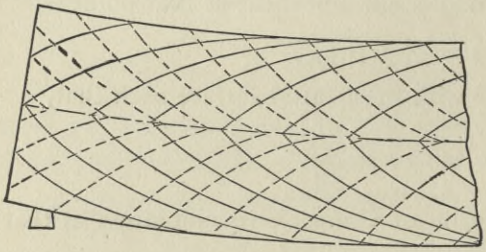

Fig. 55.

It appears from the investigation that the common theory of flexure gives the horizontal unit-stress correctly at the dangerous section of a simple beam where the vertical shear is zero. At other sections the stress S as computed from (4) is correct for the remotest fiber, but for other fibers the unit-stress t is greater. It is hence seen that the main practical value of the theory of internal stress is in showing that the intensity of the shear varies throughout the cross-section of the beam. For a restrained beam, where the vertical shear suddenly changes sign at the dangerous section, the common theory gives the horizontal stress S correctly for the remotest fiber only, and it might be possible in some forms of cross-sections for the maximum stress t to be slightly greater than S for a fiber nearer to the neutral surface. All that has here been deduced justifies the validity of the common theory of flexure as a correct guide in the practical design and investigation of beams.

Prob. 132. A joist fixed at both ends is 3 × 12 inches and 12 feet long, and is strained by a load at the middle, so that the value of S as computed from (4) is 4 000 pounds per square inch. Find the value of t for points over the support distant 3, 4, and 5 inches from the neutral surface.

Prob. 133. Show, for a point between the neutral surface and the convex side, that there exists a maximum compression as well as a maximum tension. Deduce an expression for the value of this maximum compression and its direction. Draw a figure showing the curves over the entire beam for both these stresses.

CHAPTER VIII.

THE STRENGTH OF MATERIALS.

ART. 80. MEAN CONSTANTS.

The term 'strength of materials' is generally understood to refer to stresses caused by slowly applied loads, where the conditions are those of statics only. The ultimate tensile strength of timber, for example, is obtained by placing a specimen, like that shown in Fig. 2 of Art. 8, in a testing machine and pulling it by a force which gradually increases until rupture occurs. When forces are applied suddenly or with shock, the condition is one of work or resilience (see Chapter IX).

The following tables recapitulate the mean values of the constants of the strength of materials which have been given in the preceding pages. It is here again repeated that these values are subject to wide variations dependent on the kind and quality of the material, and for many other reasons. Timber, for instance, varies in strength according to the climate where grown, the soil, the age of the tree, the season of the year when cut, the method and duration of the process of seasoning, the part of the tree used, the knots and wind shakes, the form and size of the test specimen, and the direction of its fibers, so that it is a difficult matter to state definite numerical values concerning its elasticity and strength. The quality of the material causes a yet wider variation, so wide in fact that in some cases testing machines alone could scarcely distinguish between wrought iron and steel; for while the higher grades of steel have much greater strength than the tables give, the mild structural and merchant steels may have values almost as low as the average constants for wrought iron. In general, therefore, the following values should not be used in actual cases of investigation and design except for approximate computations.

Detailed tables giving the results of experiments upon numerous kinds and qualities of materials may be found in the following books.

WOOD'S Resistance of Materials; New York, 1880.
THURSTON'S Materials of Engineering; New York, 1884.
TRAUTWINE'S Engineers' Pocket Book; New York, 1885.
LANZA'S Applied Mechanics; New York, 1885.
UNWIN'S Testing of Materials; London, 1888.
BURR'S Elasticity and Strength of Materials; New York, 1888.

TABLE I.

Material.	Mean Weight.		Coefficient of Linear Expansion.	
	Pounds per cubic foot.	Kilograms per cubic meter.	For 1° Fah.	For 1° Cent.
Timber,	40	600	0.0000020	0.0000036
Brick,	125	2 000	0.0000050	0.0000090
Stone,	160	2 560	0.0000050	0.0000090
Cast Iron,	450	7 200	0.0000062	0.0000112
Wrought Iron,	480	7 700	0.0000067	0.0000121
Steel,	490	7 800	0.0000065	0.0000117

TABLE II.

Material.	Elastic Limit.		Coefficient of Elasticity.	
	Pounds per square inch.	Kilograms per square centimeter.	Pounds per square inch.	Kilograms per square centimeter.
Timber,	3 000	210	1 500 000	105 000
Cast Iron,	6 000	420	15 000 000	1 050 000
Wrought Iron,	25 000	1 750	25 000 000	1 750 000
Steel,	50 000	3 500	30 000 000	2 100 000

TABLE III.

Material.	Ultimate Tensile Strength.		Ultimate Compressive Strength.	
	Pounds per square inch.	Kilograms per square centimeter.	Pounds per square inch.	Kilograms per square centimeter.
Timber,	10 000	700	8 000	560
Brick,	200	14	2 500	175
Stone,			6 000	420
Cast Iron,	20 000	1 400	90 000	6 300
Wrought Iron,	55 000	3 850	55 000	3 850
Steel,	100 000	7 000	150 000	10 500

TABLE IV.

Material.	Ultimate Shearing Strength.		Modulus of Rupture.	
	Pounds per square inch.	Kilograms per square centimeter.	Pounds per square inch.	Kilograms per square centimeter.
Timber,	{ 600 3 000 }	{ 42 210 }	9 000	630
Stone,			2 000	140
Cast Iron,	20 000	1 400	35 000	2 450
Wrought Iron,	50 000	3 500		
Steel,	70 000	4 900		

ART. 81. HISTORICAL.

Some topics in the mechanics of materials were discussed and partially developed in advance of precise knowledge regarding elastic resistance and ultimate strength. The first investigations were those by GALILEI in 1638 on the flexure of beams and on forms of uniform strength. In 1678 HOOKE announced the law "ut tensio sic vis," namely, that the elongation of a spring is proportional to the force which causes it; three years earlier he had performed experiments in the presence of the king of England illustrating the law, which indeed he had published in 1660 concealed in the anagram "c e i i i n o s s s t t u u."

During the eighteenth century but slight advances were made in practical knowledge, although the theory of flexure was improved by BERNOULLI, LEIBNITZ, COULOMB, EULER and others. The few experimental results during this century were on the rupture of timber by flexure and by tension, questions of ultimate strength being only investigated while that of elastic limit was scarcely recognized.

Early in the nineteenth century appeared the 'Lectures on Natural Philosophy' by YOUNG in which is found a clear presentation of many of the laws of flexure both under static forces and under shock. It also introduces for the first time the coefficient or modulus of elasticity, E, but fails to note that it can only be deduced or applied when the elastic limit of the material is not surpassed. A little later BARLOW, TREDGOLD, and HODGKINSON experimented on timber and cast iron, both in the form of beams and of columns; their methods and results, although now seeming rude and defective, are deserving of praise as the first of real practical value.

The complete theory of the flexure of beams and the equations of the elastic curve are due to NAVIER, who from 1820 to 1833 established the modern mathematical theory of elasticity on a solid foundation, which by his followers LAMÉ, ST. VENANT and BOUSSINESQ has been applied to very complex problems, a full account of which is given in TODHUNTER and PIERSON'S History of the Elasticity and Strength of Materials (3 volumes, 1886–1892).

In 1849 was published the 'Report of the Commissioners on the Application of Iron to Railway Structures,' which may be regarded as the landmark of the beginning of modern methods of testing. It contains the records of valuable tests by WILLIS, JAMES, HODGKINSON and GALTON on the strength of cast and wrought iron as well as upon the resistance to impact, investigations of the increase in stress caused by a rolling load on a beam, experiments on the fatigue of metals, and the evidence given by leading British engineers as to their opinions on proper factors of safety under different conditions. The immediate result of this report was the decision by the English board of trade that the factor of safety for cast iron should be twice as great for rolling loads as for steady ones, while throughout both Europe and the United States it excited a marked interest and impetus in the subject of testing materials.

A volume would be required to outline the progress and the results of the experimental work done since 1850. The main conclusions will be noted in subsequent articles, and many others will be found in the books quoted in Art. 80. The following additional references to works of the principal experimenters will enable students to consult original authorities as well. It should be noted, however, that very important contributions have appeared in technical periodicals and in the transactions of engineering societies; for the titles of many of these see 'Descriptive Index of Engineering Literature,' published by the Association of Engineering Societies.

WADE, W., and RODMAN, T. J.: Reports of Experiments on the Strength and other Properties of Metals for Cannon. Two volumes, quarto: Philadelphia, 1856, pp. 482; Boston, 1861, pp. 308.

KIRKALDY, D.: Experiments on Wrought Iron and Steel. Second edition, London, 1861, octavo, pp. 227, plates xvi.

FAIRBAIRN, W.: An Experimental Inquiry into the Strength, Elasticity, Ductility and other Properties of Steel. London, 1869, octavo, pp. 51.

STYFFE, K.: The Elasticity, Extensibility and Tensile Strength of Iron and Steel. London, 1869, octavo, pp. 171, plates ix.

BAUSCHINGER, J.: Mittheilungen aus dem mechanischtechnischen Laboratorium der polytechnischen Schule in München. Munich, 1873–1890, folio; usually published annually, each part dealing with a special subject.

GILMORE, Q. A.: The Compressive Resistance of Freestone, Brick Piers, Hydraulic Cements, Mortars and Concretes. New York, 1888, octavo, pp. 198, plates viii.

Reports of the U. S. Board appointed to Test Iron, Steel, and other Metals. Washington, two volumes, 1878 and 1881, octavo, pp. 592 and 684, with many plates.

German Railroad Union: Die Eigenschaften von Eisen und Stahl. Wiesbaden, 1880, quarto, pp. 230, plates x.

U. S. Tenth Census: Forest Trees of North America. Washington, 1884, quarto; containing results of tests on 412 kinds of timber.

U. S. Ordnance Bureau: Tests of Metals. Washington, annually since 1883, containing records of tests made at the Watertown arsenal.

Prob. 134. Ascertain something about the tests made by GIRARD, PRONY, RENNIE, KENNEDY, HOWARD, and J. B. JOHNSON.

ART. 82. TESTING MACHINES.

Tests on the flexure and rupture of beams were made in the seventeenth century, and machines for this purpose are comparatively simple. The load is usually applied at the middle of a beam supported at its ends and gradually increased until rupture occurs, the deflection being measured also as a test of stiffness. The apparatus for determining the deflection should be attached to the supports so that the compression of these may not affect the observed values. In the simplest case weights are hung upon a ring or stirrup placed upon the middle of the beam, and are added in regular increments of 100 pounds, more or less, depending upon the size of the beam. When the elastic limit is not exceeded the coefficient of elasticity E may be computed from the observed deflection by the formula in Case I of Art. 35. From the breaking load P the modulus of rupture S_r may be computed by the formula (4) of Art. 21; the value of this lies between the ultimate tensile and compressive strengths of the material. As a method of comparison of different qualities of materials this test is an excellent one.

Machines for tensile tests are usually arranged so as to operate on a specimen of the shape shown in Fig. 2 of Art. 8. The heads are either griped in jaws or are provided with threads so that they may be screwed into nuts to which the tension is applied. The power may be furnished by a lever in machines of small capacity, in others by a screw or by hydraulic pressure, the total tension being weighed off on a compound lever. In commercial tests the ultimate elongation is alone measured; this is done by marking inch spaces on the specimen and measuring them after rupture. In scientific tests an extensimeter is attached to the specimen so that the elongation can be read at each increment of weight. The elongation is usually expressed as a percentage of the original length between two marks whose distance apart is more than eight times the diameter of the specimen. In the case of ductile metals a marked diminution in diameter occurs at the point of rupture, and the two parts of the specimen are seen to have a taper on each side of the fracture. On this account the percentage of ultimate elongation will depend upon the distance between the two marks. As no standard proportions have yet been adopted, it is desirable that the actual distance on which the elongation is measured should be always stated.

In ductile materials, like wrought iron and mild steel, the final strength is less than the maximum strength, owing to the rapid flow of metal which is the cause of the taper and contraction. The maximum strength is usually alone recorded, and in this volume the term 'ultimate strength' is to be understood, not as that at the instant of rupture, but as the maximum unit-stress observed shortly before rupture.

The contraction of area, introduced by KIRKALDY as an element to be noted in tests of ductile metals, is now regarded as of equal importance with the ultimate elongation, since it is subject to less variation with the length of the specimen. This is expressed as a percentage of the original area of the cross-section. According to KIRKALDY, the tensile strength and ultimate contraction of area, when considered jointly, furnish the best index for judging the quality of wrought iron and steel.

In all these tensile tests the load is applied gradually, and not suddenly or with impact. A test, however, may be made slowly or quickly, and it is found that the degree of rapidity has a marked influence upon the results. In general, a quick test gives a higher elastic limit and a higher strength than a slow one, while the ultimate elongation is less. Attempts have hence been made to specify the degree of rapidity with which the test should be conducted, and undoubtedly some uniformity in this respect will in time be required in standard specifications. (Art. 88.)

Compressive tests are mainly confined to brick and stone, and are but little used for metals on account of the difficulty of securing a uniform distribution of stress over the surfaces. Rupture in these cases rarely occurs by true crushing, but by a diagonal shearing or splitting, and it is not easy to obtain precise measures of the amount of shortening. Cement, which is always used in compression, is indeed usually tested by tension, and this is found to be the cheaper and more satisfactory method.

The first testing machines in the United States were those built by WADE and RODMAN between 1850 and 1860 for testing gun-metal for the government. About this time the rapid introduction of iron bridges led to experiments by PLYMPTON and by ROEBLING. About 1865 machines were made by FAIRBANKS, which have since been developed into forms applicable to all classes of work, as also have those made by OLSEN and by RIEHLE. A machine devised by THURSTON soon after 1870 has done excellent work on stresses of torsion. The machine built by EMERY for the United States Board of 1878–81, and now in the Watertown arsenal, is the most precise one ever devised, and its work has been of great value in advancing our knowledge of materials. Large machines for testing eye-bars have been built by bridge companies since 1880, and several testing laboratories now exist provided with machines for every kind of work.

The capacity of a testing machine is the tension or pressure which it can exert. A small machine for testing cement need not have a capacity greater than 1000 pounds. Machines of 50 000, 100 000, and 150 000 pounds for testing metals are common. The Watertown machine has a capacity of 1 000 000 pounds and can test a heavy bar 30 feet long and a small hair with equal precision. The eye-bar machine at Athens, Pa., has a capacity of 1 244 000 pounds, and that at Phœnixville, Pa., a capacity of 2 160 000 pounds.

For descriptions of the principal testing machines the student should consult ABBOTT'S papers in Van Nostrand's Magazine, Vol. XXX, reprinted as Van Nostrand's Science Series, No. 74. KENT'S Strength of Materials, No. 41 in that Series, contains also valuable information and discussions of the methods of making and interpreting tests. KENT'S Mechanical Engineers' Pocket Book (New York, 1895) may be referred to as containing an excellent synopsis of the results of tests.

Prob. 135. A steel eye-bar, $10 \times 2\frac{5}{8}$ inches, tested at Phœnixville in 1893, broke under a tension of 1 626 322 pounds, with

20.5 per cent elongation in 47 feet, and a reduction of area of 50.4 per cent. Compute the ultimate tensile strength per square inch of original area, and also per square inch of fractured area.

ART. 83. TIMBER.

Good timber is of uniform color and texture, free from knots, sapwood, wind-shakes, worm-holes or decay; it should also be well seasoned, which is best done by exposing it for two or three years to the weather to dry out the sap. The heaviest timber is usually the strongest; also the darker the color and the closer the annular rings the stronger and better it is, other things being equal. The strength of timber is always greatest in the direction of the grain, the sidewise resistance to tension or compression being scarcely one fourth of the longitudinal.

The following table gives average values of the ultimate strength in pounds per square inch of a few of the common

Kind.	Pounds per Cubic Foot.	Tensile Strength.	Modulus of Rupture.	Compressive Strength.
Hemlock......	25	8 000	6 000	5 000
White Pine....	27	8 000	6 000	5 500
Chestnut......	40	12 000	7 000	5 000
Red Oak......	42	9 000	7 000	6 000
Yellow Pine...	45	15 000	11 000	9 000
White Oak.....	48	12 000	10 000	8 000

kinds of timber as determined from tests of small specimens carefully selected and dried. Large pieces of timber such as are actually used in engineering structures will probably have an ultimate strength of from fifty to eighty per cent of these values. Moreover the figures are liable to a range of 25 per cent on account of variations in quality and condition arising from place of growth, time when cut, and method and duration of seasoning. To cover these variations the factor of safety of 10 is not too high.

The shearing strength of timber is still more variable than the tensile or compressive resistance. White pine across the grain may be put at 2500 pounds per square inch, and along the grain at 500. Chestnut has 1500 and 600 respectively, yellow pine and oak perhaps 4 000 and 600 respectively.

The elastic limit of timber is poorly defined. In precise tests on good specimens it is sometimes observed at about one half the ultimate strength, but under ordinary conditions it is safer to put it at one third. The coefficient of elasticity ranges from 1 000 000 to 2 000 000 pounds per square inch, 1 500 000 being a good mean value to use in general computations. The ultimate elongation is small, usually being between 1 and 2 per cent.

The tests published in the Census Report on the Forest Trees of North America (1884) are very comprehensive as they include 412 species of timber. Of these 16 species have a specific gravity greater than 1.0 and 28 species less than 0.4. Even in the same species a great variation in weight was often found; for instance white oak ranged from 42 pounds to 54 pounds per cubic foot. The heaviest wood weighed 81 pounds per cubic foot and had a compressive strength of 12 000 pounds per square inch; the lightest wood weighed 16 pounds per cubic foot and had a compressive strength of 200 pounds per square inch.

ART. 84. BRICK.

Brick is made of clay which consists mainly of silicate of alumina with compounds of lime, magnesia, and iron. The clay is prepared by cleaning it carefully from pebbles and sand, mixing it with about one half its volume of water, and tempering it by hand stirring or in a pug-mill. It is then moulded in rectangular boxes by hand or by special machines, and the green bricks are placed under open sheds to dry. These are piled in a kiln and heated for nearly two weeks until those nearest to the fuel assume a partially vitrified appearance.

Three qualities of brick are taken from the kiln; 'arch brick' are those from around the arches where the fuel is burned—these are hard and often brittle; 'body brick,' from the interior of the kiln, are of the best quality; 'soft brick,' from the exterior of the pile, are weak and only suitable for filling. Paving brick are burned in special kilns, often by natural gas or by oil, the rate of heating being such as to ensure toughness and hardness.

The common size is $2 \times 4 \times 8\frac{1}{4}$ inches, and the average weight $4\frac{1}{2}$ pounds. A pressed brick, however, may weigh nearly $5\frac{1}{2}$ pounds. Good bricks should be of regular shape, have parallel and plane faces, with sharp angles and edges. They should be of uniform texture and when struck a quick blow should give a sharp metallic ring. The heavier the brick, other things being equal, the stronger and better it is.

Poor brick will absorb when dry from 20 to 30 per cent of its weight of water, ordinary qualities absorb from 10 to 20 per cent, while hard paving brick should not absorb more than 2 or 3 per cent. An absorption test is valuable in measuring the capacity of brick to resist the disintegrating action of frost, and as a rough general rule the greater the amount of water absorbed the less is the strength and durability.

The crushing strength of brick is variable; while a mean value may be 2500 pounds per square inch, soft brick will scarcely stand 500, pressed brick may run to 10 000, and the best qualities of paving brick have given 15 000 pounds per square inch, or even more. Crushing tests are difficult and expensive to make on account of the labor of preparing the specimen cubes so as to secure surfaces truly parallel.

A flexural test for brick is often used. The brick is supported at the middle of the flat side upon a fixed steel support about one half inch in width, and around its ends are placed stirrups to which is hung a vessel for holding weights. The loads being applied until rupture occurs, the modulus of rupture can be computed from the formula

$$S_r = \frac{3Wl}{2bd^2},$$

in which l is the length of the brick between stirrups, b its breadth, d its thickness, and W the breaking load. For example, if $l = 8$ inches, $b = 4$ inches, $d = 2$ inches, and $W = 1000$ pounds, then $S_r = 1500$ pounds per square inch. This modulus of rupture is intermediate between the ultimate tensile and compressive strengths, and although not a physical constant it serves the means of making excellent comparisons of different qualities of brick.

Tensile and shearing tests of bricks are rarely made and but little is known of their behavior under such stresses. The ultimate tensile strength may perhaps range from 50 to 500 pounds per square inch. As for elastic limit and coefficient of elasticity, so little is known that nothing can be said.

ART. 85. CEMENT AND MORTAR.

Common mortar is composed of one part of lime to five parts of sand by measure. When six months old its tensile strength is from 15 to 30, and its compressive strength from 150 to 300 pounds per square inch. Its strength slowly increases with age, and it may be considerably increased by using a smaller proportion of sand.

Hydraulic mortar is composed of hydraulic cement and sand in varying proportions. The less the proportion of sand the greater is its strength. If S be the strength of neat cement, that is, cement with no sand, and S_1 be the strength of mortar having p parts of sand to 1 part of cement, the formula

$$S_1 = \frac{S}{1 + p}$$

gives a rough approximation to the strength of the mortar. A common proportion is 3 parts of sand to 1 of cement, the strength of this being about one-fourth that of the cement itself. The strength of hydraulic mortar also increases with its age.

There are two classes of hydraulic cements, the Rosendale or natural cements, and the Portland or artificial cements. The former are of lighter color, lower weight, and lesser strength than the latter, but they are quicker in setting and cheaper in price. The following table gives average ultimate tensile strengths in pounds per square inch of mortars of both classes of different ages and different proportions of sand.

Proportion of Sand.	Rosendale.				Portland.			
	One Week.	One Month.	Six Months.	One Year.	One Week.	One Month.	Six Months.	One Year.
0	120	200	300	350	300	400	450	500
1 to 1	70	125	200	250	125	200	300	350
2 to 1	30	60	120	180	100	150	250	300
3 to 1	20	30	70	130	80	125	200	250
4 to 1	10	20	60	100	50	70	120	150

These figures are obtained from tests of briquettes carefully made in molds, and are probably higher than mortar made under usual circumstances would give. The briquette is removed from the mold when set, allowed to remain in air for one day, and then put under water for the remainder of the time.

As cements and mortars are only used in compression, the tensile test may seem an inappropriate one. Compressive tests, however, are expensive and unsatisfactory to make under ordinary conditions. It may be taken as a general rule that the compressive strength of a material increases with the tensile strength, and it is certain that the universal adoption of the tensile tests has done much to greatly improve the quality of hydraulic cements.

For neat cement a one-day test is frequently employed, the briquette being put under water as soon as it has set. For Rosendale cement a briquette one square inch in section should give a tensile strength of 75 pounds, while one of Portland cement should give 125 pounds. To secure these results, however, the material should be thoroughly rammed into the moulds, no more water being used than necessary, and the quality of the cement must be good.

The compressive strength of hydraulic cements and mortars is much higher than the tensile strength, and it increases more rapidly with age. After an immersion of six days in water the average compressive strengths are about as follows in pounds per square inch:

	Neat.	1 Sand, 1 Cement.	2 Sand, 1 Cement.	3 Sand, 1 Cement.
Rosendale,	60	40	30	20
Portland,	300	200	150	100

The adhesion of cement to stone or brick is somewhat less than the tensile strength. The shearing strength of cement or mortar is much less than the tensile strength—usually only about one-fourth.

The strength of cement and mortar is influenced by many causes: the quality of the stone or materials from which the cement is made, the method of manufacture, the age of the cement, the kind of sand, the method of mixing, and even the amount of water used. Tensile tests of briquettes, more than all others, are valuable on account of the ease with which they can be made, and the reliable conclusions that can be drawn from the results.

ART. 86. STONE.

Sandstone, as its name implies, is sand, usually quartzite, which has been consolidated under heat and pressure. It varies much in color, strength, and durability, but many varieties form most valuable building material. In general it is easy to cut and dress, but the variety known as Potsdam sandstone is very hard in some localities.

Limestone is formed by consolidated marine shells, and is of diverse quality. Marble is limestone which has been reworked in the laboratory of nature so as to expel the impurities, and leave a nearly pure carbonate of lime ; it takes a high polish, is easily worked, and makes one of the most beautiful building stones.

Granite is a rock of aqueous origin metamorphosed under heat and pressure; its composition is quartz, feldspar, and mica, but in the variety called gneiss the mica is replaced by hornblende. It is fairly easy to work, usually strong and durable, and some varieties will take a high polish.

Trap, or basalt, is an igneous rock without cleavage. It is hard and tough, and less suitable for building constructions than other rocks, as large blocks cannot be readily obtained and cut to size. It has, however, a high strength, and is remarkable for durability.

The average weights and ultimate compressive strengths of these four classes are as follows :

Kind.	Weight per Cubic Foot.	Compressive Strength.	Modulus of Rupture.
	Pounds.	Lbs. per sq. in.	Lbs. per sq. in.
Sandstone	150	5 000	1 500
Limestone	160	7 000	1 500
Granite	165	12 000	2 000
Trap	175	16 000	

These figures, however, refer to small specimens such as can be used in a testing machine, and it is known that the strength of large blocks per square inch is materially less. The rupture of a cube or prism of stone under compression often occurs by splitting, or rather shearing, in planes making an angle of about 45 degrees with the direction of the pressure (see Fig. 3, Art. 8). In order to insure a perfect distribution of pressure over the surfaces, cushions of wood, leather, and lead are often placed upon them, but the advantage of using them is doubtful.

The coefficient of elasticity of stone has been found to range from 5 000 000 to 10 000 000 pounds per square inch. The elastic limit is difficult to observe, if indeed any exists. Little is known of the tensile and shearing strengths of stone, except that they are smaller than the compressive strength in some varieties. Tests to determine the modulus of rupture by loading a stone beam to destruction are easily made, and will probably serve to compare the quality of different specimens better than other tests of strength.

Slate is an argillaceous stone consolidated under very heavy pressure so as to form a marked cleavage at right angles to the direction of that pressure. It is split into plates $\frac{1}{4}$ inch in thickness for use as roofing slate, and in larger blocks is used for pavements and steps. Its weight per cubic foot is about 175 pounds, its compressive strength about 10 000 pounds per square inch, and its modulus of rupture about 7 000 pounds per square inch. Unfortunately it is liable to corrode under the action of the atmosphere, and its marked cleavage and grain render its strength variable in different directions. See Transactions American Society of Civil Engineers, Sept. 1892 and Dec. 1894.

The quality of a building stone cannot be safely inferred from tests of strength, as its durability depends largely upon its capacity to resist the action of the weather. Hence corrosion and freezing tests, impact tests, and observations of the behavior of stone under conditions of actual use are more important than the determination of crushing strength in a compression machine. See BAKER'S Masonry Construction for full information regarding these subjects.

ART. 87. CAST IRON.

Cast iron is a modern product, having been first made in England about the beginning of the fifteenth century. Ores of iron are melted in a blast furnace, producing pig iron. The pig iron is remelted in a cupola furnace and poured into moulds, thus forming castings. Beams, columns, pipes, braces, and blocks of every shape required in engineering structures are thus produced.

Pig iron is divided into two classes, Foundry pig and Forge pig, the former being used for castings and the latter for making wrought iron. Foundry pig has a dark-gray fracture, with large crystals and a metallic luster; forge pig has a light-gray or silver-white fracture, with small crystals. Foundry pig has a specific gravity of from 7.1 to 7.2, and it contains from 6 to 4 per cent of carbon; forge pig has a specific gravity of from 7.1 to 7.4, and it contains from 4 to 2 per cent of carbon. The higher the percentage of carbon the less is the specific gravity, and the easier it is to melt the pig. Besides the carbon there are present from 1 to 5 per cent of other impurities, such as silicon, manganese, and phosphorus.

The properties and strength of castings depend upon the quality of the ores and the method of their manufacture in both the blast and the cupola furnace. Cold-blast pig produces stronger iron than the hot-blast, but it is more expensive. Long-continued fusion improves the quality of the product, as also do repeated meltings. The darkest grades of foundry pig make the smoothest castings, but they are apt to be brittle; the light-gray grades make tough castings, but they are apt to contain blow-holes or imperfections.

The percentage of carbon in cast iron is a controlling factor which governs its strength, particularly that percentage which is chemically combined with the iron. For example, the following are the results of tests by WADE of three classes of cast iron for guns, the tensile strength being expressed in pounds per square inch:

No.	Specific Gravity.	Percentage of Carbon. Graphite.	Percentage of Carbon. Combined.	Ultimate Tensile Strength.
1	7.204	2.06	1.78	28 800
2	7.154	2.30	1.46	24 800
3	7.087	2.83	0.82	20 100

Here it is seen that the total carbon is about the same in the three kinds, but the smaller the percentage of combined carbon the less is the specific gravity and the ultimate strength.

As average values for the ultimate strength of cast iron, 20 000 and 90 000 pounds per square inch in tension and compression respectively are good figures. In any particular case, however, a variation of from 10 to 20 per cent from these values may be expected, owing to the great variation in quality. For first-class gun iron WADE found a tensile strength of over 30 000 and a compressive strength of over 150 000 pounds per square inch. On the other hand medium-quality castings often have a tensile strength less than 16 000 pounds per square inch.

In tensile tests the elongations increase faster than in the simple ratio of the applied stresses, so that the elastic limit is poorly defined, and the coefficient of elasticity may range from 10 000 000 to 20 000 000 pounds per square inch. HODGKINSON deduced formulas for finding the elongation for different stresses, but these are of little practical importance.

The flexural test is a good one for comparing the strength of different bars of cast iron. A bar 2 × 1 inch in cross-section and 27 inches long, laid flatwise on two supports 24 inches apart, should carry a load of 2000 pounds, or more, applied at the middle; that is, the modulus of rupture should be 36 000 pounds per square inch, or more.

The high compressive strength and cheapness of cast iron render it a valuable material for many purposes, but its brittleness, low tensile strength, and ductility forbid its use in structures subject to variations of load or to shocks. Its ultimate elongation being scarcely one per cent, the work required to cause rupture in tension is small compared to that for wrought iron and steel, and hence as a structural material the use of cast iron is very limited. POLE'S Iron as a Material of Construction (London, 1872) is an elementary work which the student may consult with advantage.

ART. 88. WROUGHT IRON.

The ancient peoples of Europe and Asia were acquainted with wrought iron and steel to a limited extent. It is mentioned in Genesis, iv., 22, and in one of the oldest pyramids of Egypt a piece of iron has been found. It was produced, probably, by the action of a hot fire on very pure ore. The ancient Britons built bloomaries on the tops of high hills, a tunnel opening toward the north furnishing a draught for the fire which caused the carbon and other impurities to be expelled from the ore, leaving behind nearly pure metallic iron.

Modern methods of manufacturing wrought iron are mainly by the use of forge pig (Art. 87), the one most extensively used being the puddling process. Here the forge pig is subjected to the oxidizing flame of a blast in a reverberatory furnace, where it is formed into pasty balls by the puddler. A ball taken from the furnace is run through a squeezer to expel the cinder and then rolled into a muck bar. The muck bars are cut, laid in piles, heated, and rolled, forming what is called merchant bar. If this is cut, piled, and rolled again a better product called best iron is produced. A third rolling gives 'best-best' iron, a superior quality, but high in price.

The product of the rolling mill is bar iron, plate iron, shape iron, beams, and rails. Bar iron is round, square, and rectangular in section; plate iron is from $\frac{1}{4}$ to 1 inch thick, and of varying widths and lengths; shape iron includes angles, tees, channels, and other forms used in structural work; beams are **I** shaped, and of the deck or rail form (Art. 31).

Wrought iron is metallic iron containing less than 0.25 per cent of carbon, and which has been manufactured without fusion. Its tensile and compressive strengths are closely equal on the average from 50 000 to 55 000 pounds per square inch. The elastic limit is well defined at about 25 000 pounds per square inch, and within that limit the law of proportionality of stress to deformation is strictly observed. It is tough and ductile, having an ultimate elongation of from 20 to 30 per cent. It is stiffer than cast iron, the coefficient of elasticity being 25 000 000 pounds per square inch. It is malleable, can be forged and welded, and has a high capacity to withstand the action of shocks. It cannot, however, be tempered nor can it be melted at any ordinary temperature.

The cold-bend test for wrought iron is an important one for judging of general quality. A bar perhaps $\frac{3}{4} \times \frac{3}{4}$ inches and 15 inches long is bent when cold either by pressure or by blows of a hammer. Bridge iron should bend through an angle of 90 degrees to a curve whose radius is twice the thickness of the bar, without cracking. Rivet iron should bend through 180 degrees until the sides of the bar are in contact, without showing signs of fracture. Wrought iron that breaks under this test is lacking in both strength and ductility.

The tensile test on a small specimen not less than 12 inches in length and 0.5 square inches in cross-section is mainly employed. The elongation should be measured on a length not less than 8 inches. The following requirements for structural iron in tension are those recommended by a committee of the American Society of Civil Engineers in 1895 :

Kind of Wrought Iron.	Yield Point. Lbs. per sq. in.	Maximum Strength. Lbs. per sq. in.	Ultimate Elongation. Per cent.	Reduction of Area. Per cent.
Bars	26 000	50 000	20	30
Tension Plates and Shapes	26 000	48 000	15	20
Compression Plates and Shapes	26 000	48 000	12	16
Web Plates	26 000	46 000	8	12

The term 'yield point' is here used instead of elastic limit, as the latter is strictly the point at which the elongation ceases to be proportional to the applied load, and it is not easy to determine its exact value. On a delicate machine, however, the sudden increase of elongation soon after the elastic limit is passed cannot fail to be noticed by the drop of the scale beam, and this is called the yield point. It is further recommended that the stress be applied to the specimen at the uniform rate of from 4 000 to 5 000 pounds per square inch per minute below the yield point and 7 000 to 8 000 pounds per square inch per minute above the yield point.

The process of manufacture, as well as the quality of the pig iron, influences the strength of wrought iron. The higher the percentage of carbon the greater is the strength. Best iron is 10 per cent stronger than ordinary merchant iron owing to the influence of the second rolling. Cold rolling causes a marked increase in elastic limit and ultimate strength, but a decrease in ductility or ultimate elongation. Annealing lowers the ultimate strength, but increases the elongation. Boiler-plate iron will generally have an ultimate strength greater than 55 000 pounds per square inch. Iron wire, owing to the process of drawing, has a high tensile strength, sometimes greater than 100 000 pounds per square inch.

Good wrought iron when broken by tension shows a fibrous structure. If, however, it be subject to shocks or to repeated stresses which exceed the elastic limit, the molecular structure becomes changed so that the fracture is more or less crystalline. The effect of a stress slightly exceeding the elastic limit is to cause a small permanent set, but the elastic limit will be found to be higher than before. This is decidedly injurious to the quality of the material on account of the accompanying change in structure, and hence it is a fundamental principle that the working unit-stresses should not exceed the elastic limit. For proper security indeed the allowable unit-stress should seldom be greater than one half the elastic limit.

In a rough general way the quality of wrought iron may be estimated by the product of its tensile strength and ultimate elongation, this product being an appropriate measure of the work required to produce rupture. A more precise measure of this work K is, however, given by the formula

$$(15) \qquad K = \tfrac{1}{3}s(S_e + 2S_t)$$

in which s is the ultimate unit-elongation, S_e is the elastic limit, and S_t the tensile strength. For example, take two wrought-iron specimens, the first having $S_e = 27\,000$, $S = 66\,000$, $s = 13$ per cent $= 0.13$, and the second having $S_e = 24\,000$, $S_t = 51\,000$, $s = 24$ per cent $= 0.24$. Then the values of K are

$$K_1 = 6\,890, \qquad K_2 = 10\,080,$$

which shows that the work required to rupture the second specimen will be much greater than for the first. Thus high tensile strength is not a good quality when accompanied by low elongation. The value of K is the number of inch-pounds of work required for the rupture of a specimen one square inch in section and one inch long.

Wrought iron has no proper modulus of rupture since, owing to the change of molecular condition after the elastic limit is exceeded, bars can be bent to almost any extent without fracture. The same is true of soft and medium steel.

ART. 89. STEEL.

Steel was originally produced directly from pure iron ore by the action of a hot fire, which did not remove the carbon to a sufficient extent to form wrought iron. The modern processes, however, involve the fusion of the ore, and the definition of the United States law is that "steel is iron produced by fusion by any process, and which is malleable." Chemically, steel is a compound of iron and carbon generally intermediate in composition between cast and wrought iron, but having a higher specific gravity than either. The following comparison points out the distinctive differences between the three kinds of iron:

	Per cent of Carbon.	Spec. Grav.	Properties.
Cast iron,	5 to 2	7.2	Fusible, not malleable.
Steel,	1.50 to 0.10	7.8	Fusible and malleable.
Wrought iron,	0.30 to 0.05	7.7	Malleable, not fusible.

It should be observed that the percentage of carbon alone is not sufficient to distinguish steel from wrought iron; also, that the mean values of specific gravity stated are in each case subject to considerable variation.

The three principal methods of manufacture are the crucible process, the open-hearth process, and the Bessemer process. In the crucible process impure wrought iron or blister steel, with carbon and a flux, are fused in a sealed vessel to which air cannot obtain access; the best tool steels are thus made. In the open-hearth process pig iron is melted in a Siemens furnace, wrought-iron scrap being added until the proper degree of carbonization is secured. In the Bessemer process pig iron is completely decarbonized in a converter by an air blast and then recarbonized to the proper degree by the addition of spiegeleisen. The metal from the open-hearth furnace or from the Bessemer converter is cast into ingots, which are rolled in mills to the required forms. The open-hearth process produces steel for guns, armor plates, and for some structural purposes; the Bessemer process produces steel for railroad rails and also for structural shapes.

The physical properties of steel depend both upon the method of manufacture and upon the chemical composition, the carbon having the controlling influence upon strength. Manganese promotes malleability and silicon increases the hardness, while phosphorus and sulphur tend to cause brittleness. The higher the percentage of carbon within reasonable limits the greater is the ultimate strength and the less the elongation. THURSTON proposes the formula

$$S_t = 60\,000 + 70\,000C \qquad (16)$$

for the ultimate tensile strength in pounds per square inch, C being the per cent of carbon. Thus, with 0.40 per cent of carbon the value of S_t is 88 000 pounds per square inch. The ultimate elongation is approximately inversely proportional to the tensile strength, a formula frequently given being:

$$\text{elongation in per cent} = \frac{150}{6 + 7C} = \frac{1\,500\,000}{S_t};$$

thus when $C = 0.70$, the tensile strength is about 109 000 pounds per square inch and the elongation about 14 per cent. These approximate formulas refer only to unannealed steel.

A classification of steel according to the percentage of carbon and its physical properties of tempering and welding is as follows:

Extra hard,	1.00 to 0.60% C.,	takes high temper, but not weldable.
Hard,	0.70 to 0.40% C.,	temperable, but welded with difficulty.
Medium,	0.50 to 0.20% C.,	poor temper, but weldable.
Mild,	0.40 to 0.05% C.,	not temperable, but easily welded.

It is seen that these classes overlap so that there are no distinct lines of demarcation. The extra-hard steels are used for tools, the hard steels for piston rods and other parts of machines, the medium steels for rails, ties, and guns, and the mild or soft steels for beams and structural purposes.

The influence of annealing, or keeping the metal in contact with a light fire for some days, is marked in reducing the ultimate strength and increasing the elongation. As an example the following table gives some of the results of a large series of tests exhibited by the Bethlehem Iron Company at the World's Columbian Exposition of 1893, all being flat bars of

Per cent of Carbon.	Tensile Strength. Pounds per square in.		Ultimate Elongation. Per cent.	
	Unannealed.	Annealed.	Unannealed.	Annealed.
0.08	58 000	56 000	27	31
0.25	84 000	75 000	21	25
0.50	125 000	99 000	11	19
0.67	136 000	112 000	6	16
1.04	153 000	128 000	3½	11

Bessemer steel. The process of annealing is thus seen to greatly improve the capacity of the high-carbon steels to resist work, since the product of tensile strength and elongation is materially increased (Art. 88).

For bridges and buildings the following requirements are recommended by the committee report of the American Society of Civil Engineers in 1895, for ultimate tensile strength:

For low steel, $60\,000 \pm 4\,000$,
For medium steel, $65\,000 \pm 4\,000$,
For high steel, $70\,000 \pm 4\,000$,

all being in pounds per square inch. The yield point is required to be 55 per cent of the ultimate strength, the per cent of elongation to be 1 500 000 divided by the ultimate strength, and the per cent of reduction of area to be 2 800 000 divided by the ultimate strength. Specimens of medium steel cut from bars or plates must stand bending through 180 degrees to an inner radius of one and one half times the thickness of the specimen without sign of fracture, while for the high steel the same must be the case to a radius of twice the thickness of the specimen.

The strength of steel may be greatly increased by compressing it while fluid, by the use of nickel as an alloy, and by the processes of forging. Nickel steel has been made with an elastic limit over 100 000 and with an ultimate tensile strength of 277 000 pounds per square inch. By tempering both strength and hardness are increased, and by annealing its resistance to shock is improved. See paper by R. W. DAVENPORT in *Engineering News* for Nov. 23, 1893.

The compressive strength of steel is always higher than the tensile strength. The maximum value recorded for hardened steel is 392 000 pounds per square inch. The expense of commercial tests of compression is, however, so great that they are seldom made. The shearing strength is about three-fourths of the tensile strength. Soft and structural steels have no modulus of rupture, since bars can be bent through 180 degrees by transverse pressure.

The elastic limit is usually well defined and closely coincident with the yield point. In tension, as a rough rule, it may be taken as one-half of the ultimate strength, in compression at somewhat less than one-half, perhaps one-third, for the hard steels. The coefficient of elasticity is subject to but little variation with the percentage of carbon, and the mean value of 30 000 000 pounds per square inch may be used in ordinary computations both for tensile and compressive stresses that do not exceed the elastic limit. In shearing the coefficient of elasticity may be taken as a mean at two-fifths of that for tension.

Steel castings are extensively used for axle boxes, cross-heads, and joints in structural work. They contain from 0.25 to 0.50 per cent of carbon, ranging in tensile strength from 60 000 to 100 000 pounds per square inch.

Steel has entirely supplanted wrought iron for railroad rails, and largely so for structural purposes. Its price being the same, its strength greater, its structure more homogeneous, the low and medium varieties are coming more and more into use as a satisfactory and reliable material for large classes of engineering constructions.

ART. 90. OTHER MATERIALS.

Concrete, composed of hydraulic mortar and broken stone, is an ancient material, having been extensively used by the Romans. It is mainly employed for foundations and monolithic structures, but in some cases large blocks have been made which are laid together like masonry. Like mortar, its strength increases with age. When six months old its mean compressive strength ranges from 700 to 1 500 pounds per square inch, and when one year old it is probably about fifty per cent greater.

Beton is an artificial stone made of hydraulic cement and sand which has been subject to prolonged trituration. Its strength is about double that of ordinary concrete.

Ropes are made of hemp, of manilla, and of iron or steel wire with a hemp center. A hemp rope one inch in diameter has an ultimate strength of about 6 000 pounds, and its safe working strength is about 800 pounds. A manilla rope is slightly stronger. Iron and steel ropes one inch in diameter have ultimate strengths of about 36 000 and 50 000 pounds respectively, the safe working strengths being 6 000 and 8 000 pounds. As a fair rough rule, the strength of ropes may be said to increase as the squares of their diameters.

Phosphor bronze is an alloy of copper and tin containing from 2 to 6 per cent of phosphorus. It is remarkable for its complete fluidity so that most perfect castings can be made. It has been used for journal bearings, valve seats, and even for cannon. It is hard and tough, and its ultimate tensile strength may range from 40 000 to 100 000 pounds per square inch.

Aluminum is a silver-gray metal which is malleable and ductile and not liable to corrode. Its specific gravity is about 2.65, so that it is light, weighing only 168 pounds per cubic foot. Its ultimate tensile strength is about 25 000 pounds per square inch. It has a low coefficient of elasticity, and its ultimate elongation is also low. Alloys of aluminum and copper have been made with a tensile strength and elongation exceeding those of wrought iron, but have not come into use as structural materials.

Numerous brasses and bronzes composed of copper, tin, and zinc have been made. The strongest was ascertained by THURSTON to be that composed of 55 parts of copper, 43 of zinc, and 2 of tin, its ultimate tensile strength of 68 900 pounds per square inch, with an elongation of 48 per cent and a reduction of area of 70 per cent. See THURSTON'S Materials of Engineering, Vol. III.

Brass, which is composed of copper and zinc, is almost the only alloy which has come into extensive use in the arts and which at the same time is a fully reliable material. In the form of castings it has a tensile strength of about 20 000 pounds per square inch, in the form of rolled sheets or wire it has a much greater strength. Brass water-pipes are now frequently used in houses by those who can afford to pay as high a price as 20 cents per pound.

The strength of lead is only about one-tenth of that of brass, and it attains a permanent set under a small tensile stress.

Glass has a tensile strength of about 5 000 and a compressive strength of about 8 000 pounds per square inch.

ART. 91. THE FATIGUE OF MATERIALS.

The ultimate strength S_u is usually understood to be that steady unit-stress which causes rupture at one application. Experience and experiments, however, teach that if a unit-stress somewhat less than S_u be applied a sufficient number of times to a bar rupture will be caused. The experiments of WÖHLER have been of great value in establishing the laws which govern the rupture of metals under repeated applications of stress. For instance, he found that the rupture of a bar of wrought iron by tension was caused in the following different ways.

By 800 applications of 52 800 pounds per square inch.
By 107 000 applications of 48 400 pounds per square inch.
By 450 000 applications of 39 000 pounds per square inch.
By 10 140 000 applications of 35 000 pounds per square inch.

The range of stress in each of these applications was from 0 to the designated number of pounds per square inch. Here it is seen that the breaking stress decreases as the number of applications increase. In other experiments where the initial stress was not 0, but a permanent value S, the same law was seen to hold good. It was further observed that a bar could be strained from 0 up to a stress near its elastic limit an enormous number of times without rupture. From a discussion of these numerous experiments the following laws may be stated.

1. By repeated applications of stress rupture may be caused by a unit-stress less in value than the ultimate strength of the material.
2. The greater the range of stress the less is the unit-stress required to produce rupture after an enormous number of applications.
3. When the stress ranges from 0 up to a value about equal

to the elastic limit the number of applications required to rupture it is enormous.

4. A range of stress from tension into compression, or *vice versa*, produces rupture with a less number of applications than the same range in stress of one kind only.
5. When the range of stress in tension is equal to that in compression the stress which will produce rupture after an enormous number of applications is a little greater than one-half the elastic limit.

The term 'enormous number' used in stating these laws means about 40 millions, that being roughly the number used by WÖHLER to cause rupture under the conditions stated. For all practical cases of repeated stress, except in fast moving machinery, this great number would seldom be exceeded during the natural life of the piece.

In Art. 8 it was recognized that the working stress should be less for pieces subject to varying stresses than for those carrying steady loads only. For many years indeed it has been the practice of designers to grade the working stress according to the range of stresses to which it might be liable to be subjected. WÖHLER'S laws and experiments afford however a means of grading these values in a more satisfactory manner than mere judgment can do, and formulas for that purpose will be deduced in the next Article. After the working stress S_w is determined the cross-section of the piece is found in the usual way, if in tension by formula (1), and if in compression by formula (1) or (10) as the case may require.

Prob. 136. How many years will probably be required for a tie bar in a bridge truss to receive 40 million repetitions of stress?

ART. 92. REPEATED STRESSES.

Consider a bar in which the unit-stress varies from S' to S, the latter being the greater numerically. Both S' and S may be tension or both may be compression, or one may be tension and the other compression. In the last case the sign of S' is to be taken as minus. Consider the stress to be repeated an enormous number of times and rupture to then occur. By the second law above stated S is some function of the range of stress, or,

$$S = \psi(S - S').$$

This may be expressed in another way, thus,

$$S = \phi\left(1 - \frac{S'}{S}\right),$$

or, in words, the rupturing stress S after an enormous number of repetitions is a function of the ratio of the limiting stresses.

Let u be the utimate strength of the material, tensile if S is tension and compressive if S is compression. Let e be the unit-stress at the elastic limit, and f the unit-stress which produces rupture after an enormous number of repetitions when the range of stress in tension is equal to that in compression. It is required to find the value of S in terms of u, e, f, and the ratio $\frac{S'}{S}$. For this purpose let the values of the ratio be regarded as abscissas and those of S as ordinates, the former ranging from $+1$ to -1 as seen in the figure. Now if this ratio is $+1$ there is no range of stress and $S = u$ as in cases of steady load. Again when the ratio is 0 the third law gives $S = e$; and lastly when the ratio is -1 the fifth law gives $S = f$. The most rational assumption as to the law of variation of S is that it represented by some curve passing through the

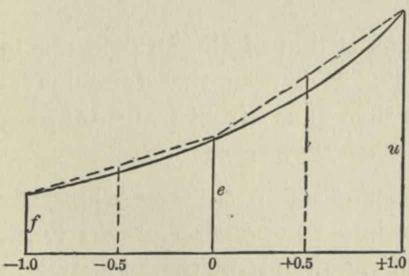

Fig. 56.

three points determined by the ordinates u, e, and f. The simplest curve is a parabola, whose equation is,

$$S = me + n\frac{S'}{S} + p\left(\frac{S'}{S}\right)^2$$

in which m, n, and p are quantities to be determined from the conditions just stated, and doing this there results

$$(17) \qquad S = e + \frac{u-f}{2}\cdot\frac{S'}{S} + \frac{u+f-2e}{2}\left(\frac{S'}{S}\right)'.$$

This formula is not to be regarded as the true law of rupturing strength under repeated stresses, but merely as an empirical statement which agrees with the limiting values determined by experiment, and which will give approximately intermediate values.

The formulas most frequently used for determining the unit-stress which will cause rupture under repeated loads are those of LAUNHARDT and WEYRAUCH, that of the former being applicable when the limiting stresses S' and S are both tension or both compression, and that of the latter when one limiting stress is tension and the other compression. LAUNHARDT supposes that S varies uniformly between the ordinates u and e so that its equation is that of a straight line, or

$$S = e + (u - e)\frac{S'}{S}$$

and the graphical representation is that of the straight line in the right hand part of Fig. 56. It is seen that formula (17) gives values of S slightly less than those from LAUNHARDT'S, except for the ratios 0 and 1 when they agree.

The formula of WEYRAUCH applies to the case where the range of stress is from tension into compression or *vice versa*, and it also supposes the law of variation to be that of a straight line between the limiting ordinates given by experiment, or

$$S = e - (e - f)\frac{S'}{S},$$

in which the numerical value of the ratio $S' : S$ is to be taken as positive. This equation is represented by the straight line in the left hand part of Fig. 56. Here also formula (17) gives less values for S than those obtained by WEYRAUCH'S formula.

In designing a bar which is to be subject to an enormous number of repetitions of stress, ranging from P' to P, the ratio $\frac{P'}{P}$ is the same as $\frac{S'}{S}$, and formula (17) gives the unit-stress S which will cause rupture after an enormous number of repetitions. To be sure of safety a factor of security must be applied; then the working unit-stress is found by dividing S by this factor, which is here usually taken the same as the factor of safety for a steady load where there is no range of stress. For example, consider a kind of wrought iron for which $u = 52\,000$, $e = 26\,000$, and $f = 13\,000$ pounds per square inch, and let the factor of security be 4. Then formula (17) becomes,

$$S_w = 6\,500\left(1 + \frac{3}{4}\frac{S'}{S} + \frac{1}{4}\left(\frac{S'}{S}\right)^2\right),$$

from which the allowable value of the working unit-stress can be computed for assigned values of the ratio $S' : S$.

For example, let it be required to find the proper cross-section of a wrought iron bar which is to be subjected to a repeated tension ranging from 30 000 pounds under dead load to 90 000 pounds under full live load. Here

$$\frac{S'}{S} = \frac{P'}{P} = \frac{30\,000}{90\,000} = \frac{1}{3},$$

and from the formula just deduced,

$$S_w = 6\,500(1 + \tfrac{3}{4}\cdot\tfrac{1}{3} + \tfrac{1}{4}\cdot\tfrac{1}{9}) = 8\,300.$$

Then the cross-section of the bar is,

$$A = \frac{90\,000}{8\,300} = 10.9 \text{ square inches.}$$

But if the bar is to be subjected to repeated stress varying from 30 000 pounds compression to 90 000 pounds tension, then

$$\frac{S'}{S} = \frac{-30\,000}{+90\,000} = -\frac{1}{3},$$

and from the special formula,

$$S_w = 6\,500(1 - \tfrac{3}{4}\cdot\tfrac{1}{3} + \tfrac{1}{4}\cdot\tfrac{1}{9}) = 5\,050,$$

so that the cross-section of the bar should be,

$$A = \frac{90\,000}{5\,050} = 17.8 \text{ square inches,}$$

which is 63 per cent larger than required for the smaller range.

The quantity f which is the unit-stress required to produce rupture after an enormous number of repetitions in alternating stress of equal amplitudes, is called the 'vibration strength' by some writers. Its value for wrought iron is about one-half and for steel a little greater than one-half the elastic limit. For other materials there is as yet no experimental knowledge regarding its value.

Prob. 137. A steel bar one inch in diameter is subject to repeated stress ranging between 15 000 pounds tension and 40 000 pounds tension. Will it break after an enormous number of repetitions?

Prob. 138. Show that, according to the above investigation, the working unit-stress for wrought iron bars subject to repeated applications of equal tension and compression should be about one-fourth of that for a steady stress.

CHAPTER IX.

THE RESILIENCE OF MATERIALS.

ART. 93. SUDDEN LOADS AND IMPACT.

When a tensile load is slowly and uniformly applied to a bar it increases slowly from 0 up to the final value P, and the stress in the bar at any instant is equal to the tensile force existing at that instant; the elongation of the bar increases proportionally to the stress from 0 up to the final limit λ, if the elastic limit is not exceeded. The work done upon the bar by the external force is then equal to its mean intensity $\frac{1}{2}P$ multiplied by the distance λ, or $\frac{1}{2}P\lambda$; the work of the molecular forces is also equal to this same quantity $\frac{1}{2}P\lambda$.

A load P is said to be suddenly applied when its intensity is the same from the beginning to the end of the elongation. The stress in the bar, however, increases from 0 up to a limit Q. Let y be the elongation produced by the sudden load P; then the work of this external force is Py. If the stresses are within the elastic limit so that they increase proportionally to the elongation, the mean stress is $\frac{1}{2}Q$ and the work of the resisting forces is $\frac{1}{2}Qy$. Hence, as these two works must be equal,

$$\tfrac{1}{2}Qy = Py \qquad \text{or} \qquad Q = 2P.$$

Now let λ be the elongation due to the load P when gradually applied, then by law (B),

$$\frac{y}{\lambda} = \frac{Q}{P} \qquad \text{or} \qquad y = 2\lambda.$$

Therefore is established the following important theoretical law,

> A suddenly applied load produces double the stress and double the deformation caused by the same load when applied slowly with uniform increments.

This law is only true when all the stresses are within the elastic limit of the material. The sudden load P thus causes the end of the bar to move from 0 to 2λ, when the stress becomes $2P$ the resultant force tending to move the end is $P - 2P$ or $-P$ and hence the end moves backward, until after a series of oscillations it comes to rest with the elongation λ due to the static stress P. The time of this oscillation, as also the velocity of the end of the bar at any instant, can be computed by the principles of dynamics.

Impact is said to be produced upon the end of a bar when a load P falls from a height h upon it. Here the stress in the bar will increase from 0 up to a certain limit Q and the deformation from 0 up to a certain limit y. If the elastic limit of the material be not exceeded, the stress at any instant will be proportional to the deformation, so that the work of the internal stresses will be $\frac{1}{2}Qy$. The work done by the exterior force P in the same time is $P(h + y)$. Hence

$$\tfrac{1}{2}Qy = P(h + y).$$

But if λ be the deformation due to a static load P, the law of proportionality gives

$$\frac{Q}{P} = \frac{y}{\lambda}.$$

Combining these two equations there is found,

$$Q = P\left(1 + \sqrt{2\frac{h}{\lambda} + 1}\right),$$

$$y = \lambda\left(1 + \sqrt{2\frac{h}{\lambda} + 1}\right).$$

If $h = 0$ these formulas reduce to $Q = 2P$ and $y = 2\lambda$, which is the case of a suddenly applied load; if $h = 4\lambda$, they become $Q = 4P$ and $y = 4\lambda$; if $h = 12\lambda$ they give $Q = 6P$ and $y = 6\lambda$. Since λ is a small quantity for any metallic bar, it follows that a load P dropping from a moderate height may produce great stresses and deformations. Experiments made upon springs show that the theory here presented is correct, provided the elastic limit of the material is not surpassed by the stress Q.

The effect of loads applied with impact is therefore to cause stresses and deformations greatly exceeding those produced by the same static loads, so that the elastic limit may perhaps be often exceeded. Moreover the rapid oscillations and the rapid variations in the stresses cause a change in molecular structure which impairs the elasticity of the material. Generally it will be found that the appearance of a fracture of a bar which has been subject to shocks is of a crystalline nature, whereas the same material, if ruptured under a gradually increasing stress, would exhibit a tough fibrous structure. Shocks which produce stresses above the elastic limit cause the material to become stiff and brittle, and hence it is that the working unit-stresses based upon static loads should be taken very low (Art. 8).

Prob. 139. In an experiment upon a spring a weight of 14.79 ounces produced an elongation of 0.42 inches, but when dropped from a height of 7.72 inches it produced a stress of 102.3 ounces and an elongation of 2.90 inches. Compare theory with experiment.

ART. 94. THE MODULUS OF RESILIENCE.

When an applied stress causes a deformation work is done. Thus if a tensile stress P be applied by increments to a bar, so that the stress gradually increases from 0 to the value P, the work done is the product of the average stress by the total elongation λ. This product is termed the resilience of the bar. If the stress does not exceed the elastic limit of the material the average stress is $\frac{1}{2}P$, and the work or resilience is $\frac{1}{2}P\lambda$. If the cross-section of the bar be A and its length l, the unit-stress is $P \div A = S$ and the unit-elongation is $\lambda \div l = s$, so that the work of the internal resisting stresses performed on each unit of length of the bar per unit of cross-section is $\frac{1}{2}Ss$. From formula (2) the value of s is $\frac{S}{E}$, and accordingly this work may be written,

$$K = \frac{1}{2}\frac{S^2}{E}. \qquad (18)$$

If S be the unit-stress at the elastic limit, the quantity K is called the modulus of resilience of the material.

Resilience is often regarded as a measure of the capacity of a material to withstand impact, for if a shock or sudden blow be produced by a falling body, its intensity depends upon the weight and the height through which it has fallen, that is, upon its kinetic energy or work. Hence the higher the resilience of a material the greater is its capacity to endure work that may be performed upon it. The modulus of resilience is a measure of this capacity within the elastic limit only.

The following are values of the modulus of resilience as computed from (18) by the use of the average constants given in Art. 5.

For timber,	$K = 3.0$ inch-pounds,
For cast iron,	$K = 1.2$ inch-pounds,
For wrought iron,	$K = 12.5$ inch-pounds,
For steel,	$K = 41.7$ inch-pounds.

The ultimate resilience of materials cannot be expressed by a rational formula, because the law of increase of elongation beyond the elastic limit is unknown. In Fig. 1 the ultimate resilience is indicated by the area between any curve and the axis of abscissas, since that area has the same value as the total work performed in producing rupture. For timber and cast iron the ratio of these areas is about the same as that of the values of K, but for wrought iron and steel the areas are nearly equal.

Prob. 140. What horse-power engine is required to strain 125 times per minute a bar of wrought iron 18 feet long and 2 inches in diameter from 0 up to 12 500 pounds per square inch.

ART. 95. EXTERNAL WORK AND RESILIENCE.

When a body is deformed by applied forces the work done by these forces is called the external work. For example, if a bar is subjected to a tensile force which is slowly applied until it reaches the intensity P, an elongation λ is produced, and the external work is $\frac{1}{2}P\lambda$. Again, if a beam be subject to a concentrated load P gradually applied, a deflection Δ occurs under the load and the external work is $\frac{1}{2}P\Delta$. If the load is applied suddenly so that its full intensity is P during the entire time of application, then the external works in the two cases are $P\lambda$ and $P\Delta$ respectively. If P falls from a height h above the top of the bar or beam the external works are $P(h + \lambda)$ and $P(h + \Delta)$ respectively.

If a beam be uniformly loaded with w per linear unit the load on any short length dx is wdx, and if y be the deflection at the point whose abscissa is x, the elementary external work for a gradually applied load is $\frac{1}{2}wy\,.\,dx$. The integration of this over the entire length of the beam will give the total external work of the uniform load.

As external force is resisted by internal stress so external work is resisted by internal work. Each elementary stress multiplied by its displacement gives a corresponding elementary work, and the sum of all these products is the total internal work. By the law of conservation of energy,

$$\text{Internal Work} = \text{External Work},$$

provided that no work is lost in heat by the application of the external forces. Now the word 'resilience' is used to denote the internal work of the stresses, and hence

$$\text{Resilience} = \text{External Work},$$

or the resilience of a body is its capacity to resist the work of external forces.

Elastic resilience is internal work when the body is not stressed beyond the elastic limit. Non-elastic resilience is internal work when the stresses range from the elastic limit to the point of rupture.

In Art. 94 an expression for resilience within the elastic limit was deduced for a bar of unit length and unit cross-section. If a bar of cross-section A and length l be subject to a tensile or compressive force P, the deformation λ is produced. If the load be gradually applied the external work is $\frac{1}{2}P\lambda$. Let S be the unit-stress produced, and E be the coefficient of elasticity of the material. Then, from Arts. 2 and 4,

$$P = AS, \qquad \lambda = \frac{Pl}{AE};$$

and accordingly the internal work or elastic resilience is

$$(18)' \qquad K = \frac{1}{2}\frac{S^2}{E} \cdot Al;$$

that is, the elastic resiliences of bars of the same material under the same unit-stress are proportional to their volumes. If S be the stress at the elastic limit the quantity $S^2/2E$ is the modulus of resilience, and accordingly the maximum elastic resilience of a bar is the product of its modulus of resilience by its volume.

A theoretic expression for non-elastic resilience cannot be deduced, but in Art. 97 it will be shown how this can be estimated when sufficient experimental data are given. Non-elastic resiliences, however, are generally closely proportional to the volumes of bodies, the material and the maximum stress being constant.

Prob. 141. How many foot-pounds of work are required to strain a wrought-iron bar, 4 inches in diameter and 54 inches long, from 6 000 pounds per square inch up to 12 000 pounds per square inch?

ART. 96. ELASTIC RESILIENCE OF BEAMS.

When a beam deflects under the action of a load the fibers on one side of the neutral surface are elongated while those on the other side are shortened. If the elastic limit is not exceeded the stress in any fiber is proportional to its distance from the neutral surface (Art. 20). The internal work or elastic resilience of the beam is the half-sum of the products formed by multiplying the stress upon each elementary area by its corresponding change of length. The half-sum instead of the sum is taken, because the stress uniformly increases from 0 up to its maximum value as the load is applied. Thus, if T be the unit-stress under the elongation e, the unit-stress for an elongation x is $\frac{Tx}{e}$, and the work in the distance dx is $\frac{Tx}{e}dx$; integrating this between the limits 0 and e gives $\frac{1}{2}Te$ as the internal work.

Using the same notation as in Chapter III, the horizontal unit-stress upon the remotest fiber at the dangerous section of the beam is called S, and the distance of that fiber from the neutral surface is called c. Let a single concentrated load W be gradually applied to the beam, and let Δ be the deflection beneath it. The external work of the load is then $\frac{1}{2}W\Delta$, and this equals the elastic resilience if the unit-stress S does not surpass the elastic limit. If l be the length of the beam, and I the moment of inertia of the cross-section, the value of W is, from Art. 29,

$$W = n\frac{SI}{cl},$$

where n is 1 for a cantilever loaded at the end and 4 for a simple beam loaded at the middle. Also from Art. 37,

$$\Delta = \frac{nSl^2}{mcE},$$

where m is 3 for the cantilever and 48 for the simple beam. The internal work of the beam hence is :

$$K = \tfrac{1}{2}W\Delta = \frac{1}{2}\frac{S^2}{E}\cdot\frac{n^2Il}{mc^2}.$$

or, putting for I its value Ar^2 where A is the area and r the least radius of gyration of the cross-section,

$$(19) \qquad K = \frac{n^2}{m}\cdot\frac{1}{2}\frac{S^2}{E}\cdot\frac{r^2}{c^2}\cdot Al$$

which is a general expression for the elastic resilience of a beam under a single concentrated load.

If the beam be strained to the elastic limit the factor $\frac{1}{2}\frac{S^2}{E}$ is the modulus of resilience of the material (Art. 94). For either the cantilever or the simple beam the value of $\frac{n^2}{m}$ is $\frac{1}{3}$. For a rectangular beam $\frac{r^2}{c^2}$ is $\frac{1}{3}$. Thus for a rectangular beam the internal work is

$$(19)' \qquad K = \frac{1}{2}\frac{S^2}{E}\cdot\tfrac{1}{9}Al = \frac{1}{18}\frac{S^2}{E}\cdot Al;$$

that is, the work required to deflect a rectangular beam by a concentrated load is proportional to its volume Al, and the work required to cause the stress S to reach the elastic limit is the product of the modulus of resilience and one-ninth of its volume.

For a cantilever loaded uniformly with w pounds per linear foot the load on any short length dx is wdx, and if y be the deflection at that point the elementary external work is $\frac{1}{2}wydx$. Inserting for y its value from Art. 34, there is found

$$K = \frac{w^2}{48EI}\int_0^l (4l^3x - x^4)dx = \frac{9w^2l^5}{240EI} = \frac{3W^2l^3}{80EI}$$

for the total external work of the uniform load. This must be equal to the internal work. Substituting for W its value in terms of S from Art. 29, the elastic resilience of a rectangular cantilever is then

$$K = \frac{1}{2}\frac{S^2}{E}\cdot\tfrac{1}{10}Al = \frac{1}{20}\frac{S^2}{E}\cdot Al,$$

which is nine-tenths of that found for the concentrated load, and also proportional to the volume and modulus of resilience. A similar result is easily deduced for a simple beam uniformly loaded.

Formula (19) shows that the internal work or resilience developed within the elastic limit is proportional to the product of the volume of the beam and the ratio r^2/c^2. As, however, this ratio always has a numerical value which is the same for similar sections, it may be stated as a general law, that the elastic resiliences of beams of similar cross-section are proportional to their volumes.

As a numerical example let it be required to determine the horse-power necessary to deflect 50 times per second a rectangular wrought-iron beam 6 feet long, 2 inches wide, and 3 inches deep, so that at each deflection the unit-stress S may range from 5 000 to 10 000 pounds per square inch, the beam being a cantilever with the load applied at the end. From formula (19)′ the work in fifty deflections is

$$K = \frac{50 \times 2 \times 3 \times 72}{18 \times 25\,000\,000}(10\,000^2 - 5\,000^2) = 3\,600 \text{ inch-pounds},$$

which is 300 foot-pounds, and hence the power required is $300/550 = 0.52$ horse-powers.

The strength of a rectangular beam increases with the square of its depth and its stiffness with the cube of the depth (Arts. 29 and 36). The elastic resilience, however, increases only with the area of the cross-section; hence for a given unit-stress S it is immaterial whether the short or the long side of a beam be placed vertical when its office is the resistance of external work only.

Prob. 142. Deduce an expression for the elastic resilience K of a beam fixed at both ends and loaded in the middle; also for a beam fixed at both ends and uniformly loaded.

ART. 97. ULTIMATE RESILIENCE.

The ultimate resilience of a body under stress is equal to the total external work required to produce rupture. The elastic resilience is that part of the ultimate resilience in which the stresses do not surpass the elastic limit of the material. In wrought iron and steel the ultimate resilience greatly surpasses the elastic resilience, being sometimes five hundred times as large.

In order to show this fact the particular case of a steel specimen 12 inches long and 0.505 inches in diameter will be taken, which was tested in a tension machine and the elongations observed at certain intervals. In the following table the first

Load. Pounds.	Stress. Pounds per square in.	Elongation. Per cent.	Partial Work. Inch-Lbs.	Total Work Inch-Lbs.
200	1 000	0.00		0
1 000	5 000	0.01	0.3	0
3 000	15 000	0.04	3.0	3
5 000	25 000	0.07	6.0	9
7 000	35 000	0.10	9.0	18
9 000	45 000	0.14	16.0	34
9 600	48 000	0.16	9.3	44
10 000	50 000	0.70	264.6	308
12 000	60 000	1.90	660.0	968
14 000	70 000	3.62	1120.0	2 088
16 000	80 000	8.50	3660.0	5 748
16 800	83 600	15.20	5480.6	11 229
15 000	75 000	24.50	7374.9	18 604

column gives the total applied load, the second the corresponding stress per square inch, and the third the elongation expressed in per cent. The elastic limit was observed at 48 000 pounds per square inch with 0.10 per cent elongation. The elongation then rapidly increased with the unit-stress (as seen in the diagram Fig. 1 of Art. 5). At 83 600 pounds per square inch the maximum tensile strength was reached and the material was elongating very rapidly. The load was then slowly removed, but the elongation continued to increase until rupture occurred at 75 000 pounds per square inch with a total elongation of 24.5 per cent.

The external work per cubic inch of material may be approximately computed for any interval by multiplying the average stress during that interval by the elongation which occurs. The given elongations divided by 100 give the elongations per linear inch. Thus while the load ranged from 1 000 to 3 000 pounds the elongation per inch increased from 0.0001 to 0.0004. Hence the work done upon one cubic inch of the specimen in that interval was,

$$\tfrac{1}{2}(15\,000 + 5\,000)\ (0.0004 - 0.0001) = 3.0 \text{ inch-pounds.}$$

Similarly the external work per cubic inch performed in the last interval is

$$\tfrac{1}{2}(83\,600 + 75\,000)\ (0.245 - 0.152) = 7374.9 \text{ inch pounds,}$$

and thus the quantities in the fourth column of the table are separately computed.

The summation of the fourth column gives the total external work per cubic inch required to stress the bar from 0 up to the point of rupture. In the last column sums are given for each value of the unit-stress, and these are closely equal to the internal works. Thus it is seen in this particular case that for one cubic inch of material

Elastic Resilience = 44 inch-pounds,
Ultimate Resilience = 18604 inch-pounds;

and therefore that the ultimate resilience is more than 400 times as great as the elastic resilience.

The total external work required to rupture the specimen is found by multiplying the above ultimate resilience by the volume of the specimen, which is 2.4 cubic inches. This work is found to be 3721 foot-pounds.

The example here given is a case of static resilience, where the internal work is slowly developed under the action of an external force gradually applied. The more common cases of resilience, however, are those developed by the impact of a falling body, and a general discussion of these will be presented in the following articles.

Prob. 143. Compute the elastic and ultimate static resilience for the specimen of wrought iron whose test is given in Art. 5, page 11.

ART. 98. EARLY HISTORY OF RESILIENCE.

The matter in the remainder of this chapter is taken from an address on 'The Resistance of Materials under Impact,' delivered by the author in August, 1894, before the Section of Mechanical Science and Engineering of the American Association for the Advancement of Science.

In 1807 THOMAS YOUNG announced the fundamental ideas of the resistance of materials under impact. "The action which resists pressure," he said, "is called strength, and that which resists impulse may properly be called resilience." He stated that the resilience of a body is proportional to its strength and extension jointly, and that it is measured by the height through which a given weight must fall to cause rupture. The resilience of beams of the same kind he made proportional to their volumes, as also the resilience of shafts, whether solid or hollow. He further suggested that a very high velocity of the moving weight may rupture a body by impact before its full resilience can be developed.

Resilience is thus the capacity of a body to resist applied work, or it is the internal work which can be developed by the energy of a moving body. External force is resisted by internal stress, and the resulting deformation is a secondary consequence; but impact is resisted by resilience, where the deformation is of equal importance with the stress, for internal work is the product of these two factors.

In YOUNG'S time the elastic limit of materials was but vaguely recognized, and HOOKE'S law of proportionality of stress to elongation was often applied to all the phenomena preceding rupture. YOUNG'S statements are valid in a general way, but we now know that it is necessary to distinguish between the two cases of elastic resistance and non-elastic resistance. Hence there are two divisions of the subject of impact: first, that of elastic resilience, where the molecular forces do not surpass the elastic limit of the material; and second, that of ultimate resilience, where the elastic limit is exceeded and rupture finally occurs.

The problems of elastic resilience are largely theoretical and mathematical. Their discussion begins with YOUNG, and has been continued by a long line of investigators to the present time; it forms, indeed, the most prominent part of the theory of elasticity, which has been so thoroughly set forth in the history of TODHUNTER and PIERSON. Starting with HOOKE'S law of proportionality of stress to deformation, and applying to this the mechanical laws of force, velocity, and work, the stresses, displacements, and resilience of elastic bodies subject to impact have been deduced. The results thus theoretically found have been confirmed by many experiments, as must necessarily be the case, since all the laws of the discussion are those of experiment and experience.

The first problems of elastic resilience were those of bars or rods subject to the longitudinal impact of a moving weight. The vibrations of stretched wires had been discussed by EULER, LAGRANGE, and others; but NAVIER, in 1823, appears to have been the first to investigate the oscillations and maximum stresses in a horizontal bar due to a weight impinging on its end. PONCELET, in 1829, treated the same problem for a vertical bar subject to longitudinal impact by a falling weight. These discussions showed that for the case of horizontal impact the maximum elongation is a mean proportional between twice the height of fall and the extension due to the same weight when applied gradually, while for vertical impact it is the sum of the static extension and the hypothenuse of a right-angled triangle, whose sides are this static extension and the dynamic extension stated for the horizontal bar. It is a corollary from this that under the sudden application of a load, which is the particular case of impact when the height of fall is zero, the maximum elongation, and hence the maximum internal stress, is twice as great as that caused by the same load when applied gradually. (Art. 93.)

While speaking of PONCELET, it is well to pause to remark, that his work on industrial mechanics which contains these investigations deserves especial mention as the book that marks the beginning of technical education; his conceptions and explanations of these problems of external work and internal resilience may indeed still be regarded as models of clear and accurate reasoning.

The solutions of NAVIER and PONCELET for longitudinal impact on a bar were approximate in the sense that its weight was regarded as small compared to that of the striking body. Later investigations by SAINT VENANT, BOUISSINESQ, and others, have completely resolved the problem when the relative masses are taken into account, and have also disclosed all the attendant circumstances of duration and intensity of the forces at the surface of impact.

Another important problem is that of transverse impact on a beam giving rise to flexural resilience. The first investigation of this was by HODGKINSON, 1833, in connection with the discussion of experiments, while the full elastic theory for different ratios of weights of the beam and falling body was developed by COX in 1848, and later most fully extended by the French elasticians in a similar manner to that of longitudinal impact. The law of proportionality of resilience to volume was shown to be true only when the latter is increased an amount proportional to one-half the ratio of the weight of the beam to that of the striking body, and similar modifications are necessary regarding deflections and internal stresses. In many problems the velocity of transmission of stress, which is the same as the velocity of sound in the given material, forms an important element, as indeed YOUNG had suggested in 1807, at the very beginning of the consideration of impact resistances.

While the conclusions of all these investigations of elastic resilience are true and valuable, it should never be forgotten that they are only true when the conditions are observed under which they are deduced, namely, that the elastic limit is not exceeded by the maximum internal stress. Now, while the elastic limit for some materials is as high as one-half the ultimate strength, the elongation up to the elastic limit is small compared with the ultimate elongation; and hence the elastic resilience may be very small—less perhaps than one part in a thousand, compared with the total ultimate resilience. The phenomena of elastic resilience form, indeed, such a small portion of those occurring in practice, that it is difficult to observe them with precision, while those of non-elastic resilience are ever present. Many a misconception has arisen and many a paradox has been founded on the assumption that the theory of the smaller part is applicable also to the larger part of the phenomena (Art. 100).

The investigation of ultimate resilience is necessarily experimental, since beyond the elastic limit no theoretical relation between stress and deformation is known. In 1818 TREDGOLD made experiments on wooden beams subject to the impact of a falling ball, and concluded that YOUNG'S law of proportionality of resilience to volume was not justified. HODGKINSON, in 1835, experimented on cast-iron beams under lateral impact from a pendulum, and found that the deflections were proportional to the velocities of impact, that the same work was required to break the beam whether struck at the middle or at the quarter point, and that the weight of the beam increased its ultimate resilience—all of which is more nearly in accordance with the elastic theory than perhaps might be expected. In 1849 was published the "Report of the Commissioners appointed to inquire into the application of iron to Railway Structures," which contains some of the most comprehensive and valuable experiments ever undertaken. In view of later practice it was perhaps unfortunate that these were mostly made upon cast iron, but the methods of investigation and the conclusions deduced render this report an epoch-making one in the history of the resistance of materials. Here can only be noted those points relating to impact on which experiments were made by HODGKINSON, WILLIS, and GALTON. Transverse impact on beams by oft-repeated blows of a ball swung as a pendulum, and also by pressure applied by a revolving cam, indicated for the first time the laws of fatigue under repeated stresses; transverse impact with a single blow confirmed YOUNG'S theorem of resilience and volume, as the same amount of work was required to rupture a rectangular beam whether struck on its narrow or broad side; and the inertia of the beam was shown to be an important factor in increasing its resilience.

Weights suddenly applied without impact gave deflections nearly double the static deflections, and the same ratio was observed in the breaking loads. Experiments on the effects of a load moving with different velocities over a beam showed that the deflection could amount to more than double the static deflection, and the elaborate theoretical discussions of this case by STOKES, together with the evidence given by experienced engineers, form the basis of many practical rules since used in bridge design. The immediate result indeed was the revision of the factors of safety by the British Board of Trade and the establishment of a rule that for cast iron the factor for live load should be double that for dead load.

ART. 99. MODERN EXPERIMENTS.

The thirty years following the middle of the century may be designated as the period of development of the modern methods of static tests upon tensile specimens. In the United States this development, begun by WADE, RODMAN, and PLYMPTON, later produced the numerous testing machines of FAIRBANKS, OLSEN, and RIEHLÉ, and at last has culminated in the precise apparatus of EMERY and in the powerful machines at Athens and Phœnixville. These static tensile tests seem to have diverted attention from the question of impact and to have caused the neglect of the theory of resilience, for the objects of the method have been mainly to determine the elastic limit, maximum strength, and ultimate elongation, considerations of work having been largely overlooked. Yet the diagram formed by laying off the unit-stresses as ordinates and the corresponding elongation as abscissas shows by its area the internal work which resists rupture, and thus is approximately a measure of the ultimate resilience per cubic unit of the material. The phenomena in a slow test, however, are not in all respects the same as in a rapid one, the elastic limit and maximum strength being generally greater in the latter, while the elongation is less. How the diagram would look under the very rapid development of stress is yet to be determined, and not until this is done by an autographic device can a full knowledge of the total field of resilience be obtained. High values of the ultimate elongation and contraction of area have been regarded as of great importance in judging of the quality of iron and steel, and this has given rise to what may be called an unconscious manipulation of the machine, the power being applied at such a rate as to secure these results consistent with other conditions. It is safe to say that stresses are not applied and that rupture does not occur in this manner in practice ; and here it is that static tests fail to give full satisfaction.

It would be easy to take a great deal of space in criticism of the imperfections of the static method of testing, to show the uncertainties of percentages of elongation, to dwell on the discrepant results found from specimens of different shape, and to state the various views regarding contraction of area and the appearance of the fracture. But this is not the place. It may be said again, however, that the work per cubic unit developed before rupture is an important element, by the study of which much can be learned; for the total resilience depends not merely on maximum strength and ultimate elongation, but upon the rate of variation of these throughout the test. Each material, or specimen, by virtue of its physical and chemical properties and its method of manufacture, has the capacity for resisting a certain amount of applied work, of which the quantities measured in the static test are not complete factors. The full resilience can indeed only be found from an autographic diagram like that used in THURSTON'S torsion machine, which has done good service in this direction.

During all this development of static testing one impact test has survived and everywhere held its own. This is the cold-bend test for wrought iron and steel. In the rolling-mill it is used to judge of the purity and quality of the muck bar; in the steel mill it serves to classify and grade the material almost as well as chemical analysis can do, and in the purchase of shape iron it affords a quick and reliable method of estimating toughness, ductility, strength, and resilience. It is true that numerical values of these qualities are not obtained, but the indications are so valuable, that if all tests except one were to be abandoned, the simple cold-bend test would probably be the one which the majority of engineers would desire to retain.

Returning now to impact experiments proper, the experiments of KIRKALDY, in 1862, deserve notice. He subjected wrought-iron specimens to sudden tensile stress by loads which were applied without velocity, and found that they broke with 82 per cent less load than under static conditions. This has since been quoted as a contradiction of the law that the stress produced by a gradual load is one-half that caused by the same load when suddenly applied; but really no contradiction exists, since the law is only true when the elastic limit is not exceeded. For ultimate resilience the ratio of gradual to sudden load producing the same stress is always greater than one-half, and approaches nearer to unity the more ductile the material. KIRKALDY also introduced the ultimate contraction of area as an element of equal importance with the ultimate elongation; and undoubtedly this is the case, since it is subject to less variation with the length of the specimen. However, little satisfactory evidence exists that contraction of area is a measure of resistance to impact, as some have supposed.

The impact hammer or ram delivering repeated blows, as used by SANDBERG and STYFFE in 1868 on railroad rails and by the United States Board for Testing Metal in 1878 in experiments on chain iron, is perhaps the most common method of conducting tests for resilience. The weight of the ram multiplied by the height of fall measures the work done in one blow, and the number of foot-pounds required to produce rupture furnishes an index of the ultimate resilience of the bar or specimen. Although valuable for comparative purposes, like the cold-bend test, this method fails to give proper numerical results, since the effect of a second blow is not really added to that of the first on account of the dissipation of internal work into heat which follows the release of the stress when this is greater than the elastic limit. That the work of several blows affords little indication of ultimate resilience was well shown by UCHATIUS, in 1874, who experimented on rods of small size under the impact of a small ram, and found that rupture could be caused by one blow of 28 inches fall giving 6 foot-pounds, by four blows of 21 inches fall giving 18 foot-pounds, by 37 blows of $10\frac{1}{2}$ inches fall giving 83 foot-pounds, or finally by 2 050 blows of $3\frac{1}{2}$ inches fall giving 1 536 foot-pounds of work. Indeed, the fact that one strong blow is more efficient than many weak ones is well known to every artisan. The comparative weights of the ram and bar are also of importance in such tests, as will be later noticed.

The rise of the elastic limit after the release of a stress which surpasses it, first shown by THURSTON in 1873 ,and the consequent stiffening of the material under several such applications of stress, is also observed in cases of repeated impact. Thus MARTENS, in 1879, tested a railroad rail by a ram falling alternately on the head and base; the second blow produced only 95 per cent of the deflection caused by the first, and after four or five blows only 90 per cent was obtained. Here, of course, a large part of the work of each blow was dissipated in heat due to the change of molecular structure, so that the total work expended gives no valid numerical measure of the true ultimate resilience of the material. Our knowledge as to what occurs under such repeated impacts is of the most uncertain kind. Most of the stresses are no doubt injurious, as the final result proves, but a few may perhaps be beneficial in improving the quality of the material like the process of cold-hammering and cold-rolling in manufacture. BAUSCHINGER'S discovery in 1885, that the raising of the tensile elastic limit is accompanied by a lowering of the compressive elastic limit, is a most important one, which is likely in connection with future investigations to lead to a clearer knowledge of the laws of stress in materials under repeated impact.

The investigations of WÖHLER between 1860 and 1870, and the later ones of SPANGENBERG, on the resistance of materials under repeated stresses applied with little impact, have been of the greatest value in influencing the rational design of bridges and other structures subject to variations of load. They establish the facts that repeated stresses below the elastic limit do not injure the material, that repeated stresses beyond this limit cause injury in proportion to the range between the maximum and minimum limits, and that the greater the range the less the number of repetitions required to produce rupture. (Art. 91.) The prompt adoption of rules for designing based on these conclusions, in place of the arbitrary methods of adding 20 or 30 per cent to the static stresses, indicated that the engineering profession appreciated the importance of providing for the resistance to impact in a rational way. Yet probably greater impacts occur on bridges than these repeated stresses, for failures are not infrequent, and the life of a bridge under heavy traffic scarcely exceeds a dozen years. The further study of the effect of repeated impact is thus imperative, for what we now know is but little compared to what is to be learned.

MAITLAND, in 1887, showed by many experiments on tensile specimens subject to many blows of a falling ram that the ultimate elongation was greater than in static tests, it being in some cases nearly doubled. This might, perhaps, be expected from what has been said regarding the beneficial influence of some of the stresses. He also exploded powder and gun-cotton between two cylinder heads joined together by rods, and found their ultimate elongation to be more than double the corresponding ones under static stress. This, on the other hand, might not have been expected from the general conclusions regarding the effect of time on stress and elongation, and it seems not to be the case for the impact of a single blow in later experiments. Thus it appears in problems of ultimate resilience, where no theory has yet been developed, that general principles derived from static tests should be applied with caution.

The tests made by E. D. ESTRADA, in 1893, have recently led to much interesting discussion on the subject of impact. Specimens of wrought iron and steel were tested by the usual tension machine, and also under the blow of a ram weighing 100 pounds falling through vertical heights ranging from 5 to 25 feet. Over 40 specimens were broken under repeated blows, they being so arranged that they were brought into tension by the pressure caused by the impact. The number of blows ranged from 2 to 14, the height of fall varying in different cases. These valuable experiments show clearly that the ultimate elongation under repeated impact is greater than for static stresses, the average of all being about one-third greater, while the contraction of area shows only a small increase; but no conclusions regarding the elastic limit or the maximum strength under impact can be derived from them. The impact was not directly applied to the specimen, but through a number of plates and bolts through which it was transmitted, and thus much of the work was spent in acting against their inertia and resilience. This, however, in no way invalidates the important deductions regarding the influence of repeated impact on ultimate elongation and contraction of area, and the conclusion of KIRKALDY regarding the value of the latter as an index of toughness and ductility is thoroughly confirmed.

The elongation before rupture appears to vary approximately as the amount of work expended by the ram, but the work required to produce rupture does not give a satisfactory comparison of the ultimate resilience of different specimens, except in those cases where the height of fall was the same.

The experimental work thus briefly reviewed has been done by special improvised apparatus, and with little or no uniformity of method; but the time may come when machines for impact tests will be put on the market and be in common use. At present almost the only one that can be mentioned is that devised by W. J. KEEP, about 1889, for testing the resilience of cast iron. The blows are delivered by a hammer weighing 25 pounds, falling like a pendulum through heights less than 6 inches, and produce horizontal flexure at the middle of a small bar 1 foot in length. Beginning with a fall of $\frac{1}{8}$ inch, successive blows are applied, each with a fall $\frac{1}{8}$ inch greater than the preceding, until rupture occurs. The deflections and sets of the bars are graphically recorded by the machine itself, and thus excellent comparisons of the ultimate resilience of different grades of metal may be obtained.

Impact tests are most important in the case of railroad rails, car wheels, tires and axles, and other forms liable to shock. Such rail tests have been carried on for many years in Europe, and since 1890 in the United States by P. H. DUDLEY. Already at least three prominent railroads require drop tests of steel rails to be made at the mill. One rail butt $4\frac{1}{2}$ feet long is to be taken from each heat, placed on solid supports with either head or base upwards, and a weight of 2000 pounds is dropped upon it, the height of fall being 16 feet for rails lighter than 70 pounds per yard and 20 feet for heavier ones, while the distance between the supports is 3 feet for the former and 4 feet for the latter. Under this test 90 per cent of the rails must not break, and the elongation of the base or head under the greatest tension must be more than 5 per cent. This requirement of testing by a single blow is in every respect more satisfactory than by several repeated ones, as the deflection and deformation are produced by a known quantity of work and the complex phenomena of stiffening and the loss of work in heat are largely avoided.

Aside from the physical qualities of the metal, impact tests on rails and wheels promise to give important conclusions regarding the influence of temperature, of chemical composition, and of methods of manufacture, and thus to lead to a better, more uniform, and cheaper product. The discovery by GOSS, in 1892, that the driving wheels of a locomotive lift up from the rail during a part of each revolution when moving at high speed shows that impacts are of more common occurrence than generally supposed, and to satisfactorily resist these an increased resilience in wheels, rails, and bridges is required.

At the closing session of the Engineering Congress held in 1893 in Chicago, a resolution was offered by DEBRAY that uniform methods of testing are desirable for purposes of comparison, and it was adopted unanimously. For common static tests of tension and compression the time has certainly arrived when rules to secure uniformity should be framed and followed. The conferences held at Munich in 1884 and at Dresden in 1886 made a good beginning, and the later work done in 1891 by the Committee of the American Society of Mechanical Engineers will undoubtedly bear good fruit. With respect to impact, however, our knowledge is not yet sufficient to frame uniform rules. The recommendation that the weight of the anvil, or supporting blocks which hold the specimen, should be at least ten times that of the ram is an excellent one, but others fully as important will doubtless be developed by further rational investigation and experiment.

The influence of the inertia of the resisting body in modify-ing the effect of the impact should not be forgotten. If the ram be light, local damage in the body struck will be the result rather than the development of its resilience. COTTERILL has suggested for the case of transverse impact that if the total work of the ram be taken as proportional to its weight plus one-half the weight of the beam, then the work spent in developing resilience may be taken as proportional to the weight of the ram and that spent in local damage to one-half the weight of the beam. This is a good rule to keep in mind, for longitudinal impact producing pure tension, one-third being the fraction to be used instead of one-half. The fall of the ram should not be too great, for a high velocity of impact is also apt to cause the work to be spent locally, since time is required for the transmission of internal stress. The cutting of hard steel plates by the impact of particles of sand at high velocities is an example familiar to engineers, and that drops of water will wear away stone is known to all; hence a low fall combined with a heavy ram seems best adapted for conducting impact tests where the most satisfactory numerical results are desired.

It may further be suggested that the blow should be delivered directly to the beam or specimen without the intervention of intermediate blocks or parts which may absorb work. At the same time the surface of contact should be sufficiently large so that the compression of the ram may not, if possible, exceed the elastic limit, and thus loss of work in heat be avoided. In elastic resilience the applied work is not transformed into heat, in resilience accompanied by permanent deformation it is; and hence the tests should be so conducted that the specimen and not the ram may be heated. Lastly, it may be noted that the rebound of the ram should be subtracted from the total fall to give an exact measure of the work actually performed by it upon the body which is tested.

ART. 100. PARADOXES OF RESILIENCE.

Having now passed over in rapid review the state of our knowledge of the resistance of materials under impact, a brief consideration should be given to the contradictions or paradoxes that constantly arise in discussions of the subject. Of these the most common is that the theory of elastic resilience is incorrect, because it is not verified by the phenomena of ultimate resilience. It has already been pointed out that the principles of the two cases are necessarily different. It is a principle of elastic resilience that a force suddenly applied to a rod or beam produces double the elongation and double the stress caused by the same force when gradually applied; but this is not true when the elastic limit is exceeded, and it should not be expected to be true when materials are tested to destruction. HODGKINSON found it closely verified for cast iron, but this is because the ratio between stress and elongation of that material is not subject to wide variation, while an elastic limit scarcely exists. KIRKALDY found it not verified for wrought iron, and this is because of its great elongation beyond the elastic limit. So with the deflection of beams under ultimate resilience: cast iron agrees fairly with the elastic law, while wrought iron does not; and the reason is the same as before. So with the elastic law of the proportionality of resilience to the volume of the bar: cast iron is found to fairly agree under conditions of rupture, while wrought iron and steel probably do not, and in neither case are the elastic laws affected. The old paradox in the theory of beams is of a similar nature; the modulus of rupture derived from the static formula does not agree with the tensile or compressive strength of the material; but this in no wise reflects on the truth of the formula whose theory supposes the elastic limit to be not exceeded.

The numerous efforts to compute the pressure caused by the impact of a ram form a series of paradoxes which originate in defective mechanical conceptions. The frequent assumption that the pressure is equal to the momentum is one of these, which has, of course, no reasonable basis, except in the particular case when the ram is brought to rest by a uniform resisting force in one second of time. The assumption that the pressure varies directly with the fall, or with the work done by the fall of the ram, has no reasonable basis at all. The correct principle here is, as well elucidated by SKINNER in 1877, that the mean value of the pressure multiplied by the distance through which it acts is equal to the total work done in the fall; but this gives no idea of the variation of the pressure or its maximum intensity, except for a few problems in elastic resilience.

In recent discussions of ESTRADA'S tests the paradox of equating foot-pounds of work to pounds of stress has arisen, with the result that it has been claimed that the elastic limit drops almost to zero in cases of impact. It is scarcely necessary to note here that the principles of mechanics lead to no such conclusion. As already stated, elastic resilience forms so small a part of total resilience that it often passes unobserved, but resilience is not stress, and there is no way in which they can be correlated except by combining the latter with the deformation that it produces. KEEP'S tables for impact are nothing more than the statement that the pressure produced by a blow is 5.42 pounds for each foot-pound of work; but it should be said that he fully regards this as arbitrary, and valuable for comparative purposes only.

These unnecessary paradoxes and misunderstandings can be dispelled only by a clear conception and correct application of the fundamental laws of mechanics. The definitions of force and stress, work and energy, and all the words employed, must be clearly understood, and the law of conservation of energy be made the basis of every step of the reasoning. Too great attention cannot be given in our technical schools to the elucidation of these fundamental conceptions in view of the lack of precision which is so frequently apparent. Even the discussions of HERBERT SPENCER in the first volume of his Philosophy appear poorly adapted to inculcate sound principles; for there are no laws of persistence of force and of continuity of motion as his words imply, but the phenomena of both force and motion are merely consequences of the great law of energy, and in any particular case either force or motion may vanish as the energy becomes either kinetic or potential. Language is ofttimes an imperfect instrument for communicating thought, and algebra usually beclouds the reasoning whose tool it is; hence perhaps these fundamental conceptions may be better fixed in the minds of students by practical experiments and numerical computations than by the mere study of books and the solution of algebraic exercises. The laws of mechanics are the laws of experience, and the best knowledge of nature is the solid ground of direct interrogation and investigation.

In conclusion, it may be noted that the pages of engineering literature contain much testimony as to the importance and value of impact tests. As a single example, BAKER, the engineer of the Forth bridge, said in 1884 that for judging the ductility of metal the method of direct tension would apply in one case in a hundred, but in the other ninety-nine the simple cold bending of a bar was generally used, and was just as sound a test. The idea that the internal resisting work of a material is the best measure of all its physical qualities has been advocated in Germany by HARTIG, and the law of proportional resistances announced by KICK, in 1879, indicates a wide field of valuable applications for the future. The theoretical work done in France, the drop tests for rails through all Europe, the experiments of DUDLEY and others in the United States, furnish additional evidence of the growing appreciation of the value of the study of impact. Indeed, it is not too much to hope that methods may be devised by which full numerical information regarding all the physical qualities of materials can be obtained from simple tests of their elastic and ultimate resilience.

Further discussions of the subject of impact and resilience will be found in Arts. 103, 111, 112, 114, 115, 120, 122, and also in the Appendix.

CHAPTER X.

TENSION AND COMPRESSION.

ART. 101. ELONGATION UNDER OWN WEIGHT.

When an unloaded bar is hung vertically by one end there is no stress on the lower end, while at the upper end there is a stress equal to the weight of the bar. In many practical cases this stress is so small that it can be neglected in comparison with that caused by the applied tension.

The elongation of a vertical bar under its own weight is one-half that caused by the same load applied at the end. To show this, let l be the length of the bar, A the area of the cross-section, and W the weight; let x be any distance from the lower end, then $W\frac{x}{l}$ is the weight of this portion. The elementary elongation caused by it on the length dx is, from Art. 5,

$$d\lambda_1 = \frac{Wxdx}{AEl},$$

and the integral of this between the limits 0 and l gives

$$\lambda_1 = \frac{Wl}{2AE},$$

which is one-half of that caused by a load W at the end.

The work performed by gravity in elongating the bar is one-third of that done by the same load applied at the end. For, the elementary work done upon the element dx by the lower portion of the bar is

$$dK = \frac{1}{2}\frac{Wx}{l}\,d\lambda_1 = \frac{W^2x^2dx}{2AEl^2},$$

the integral of which, between the limits 0 and l, gives

$$K = \frac{W^2 l}{6AE} = \tfrac{1}{3} W\lambda_1,$$

while that done by W at the end of the bar is $\frac{1}{2} W\lambda$ or $W\lambda_1$.

The total elongation caused by the weight W and a load P at the end hence is

$$\lambda = (P + \tfrac{1}{2}W)\frac{l}{AE},$$

and the total work of elongation is

$$K = \tfrac{1}{2}(P^2 + PW + \tfrac{1}{3}W^2)\frac{l}{AE}.$$

These formulas, let it be remembered, are only valid when the unit-stress $(P + \frac{1}{2}W)/A$ is less than the elastic limit of the material. They apply also to the case of compression within the elastic limit, λ being the amount of shortening.

Prob. 144. Find the length of a wrought-iron bar, and its elongation, when suspended at the upper end so that the unit-stress at that end may be equal to the elastic limit.

ART. 102. BAR OF UNIFORM STRENGTH.

A suspension bar of constant cross-section is stressed at the lower end by the load P, and at the upper end by P plus the weight of the rod. When the bar is very long and heavy its section should be less at the lower than at the upper end in order to economize material. The bar in such cases is sometimes made in parts, the cross-section of any part being less than that of the one above it.

The theoretic form for a bar of uniform strength may be determined as follows: Let P be the load applied to the lower end, and let S be the allowable working unit-stress. Then the area of the lower end is

$$A_0 = \frac{P}{S}.$$

Let A be the area of any section at a distance y from the lower end, then $A + dA$ will be the area at the distance $y + dy$, and the area dA must provide for the weight in the distance dy. Let w be the weight per cubic unit of the bar, then

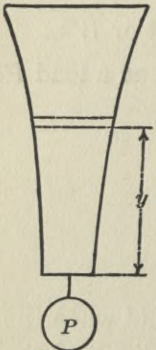

FIG. 57.

$$dA = \frac{wAdy}{S},$$

from which by integration, and observing that $A = A_0$ when $y = 0$, there is found

$$y = \frac{S}{w}(\log_e A - \log_e A_0).$$

Replacing for S its value P/A_0, this may be written

$$\log_e A = \log_e A_0 + \frac{wA_0}{P}y,$$

where the logarithms are in the Naperian system. Passing to common logarithms, it becomes

$$\log A = \log A_0 + 0.43429\,\frac{wA_0}{P}y,$$

which is the form for practical computation.

For example, let $P = 10\,000$ pounds, $S = 5\,000$ pounds per square inch, and $w = 0.25$ pounds per cubic inch. Then $A_0 = 2$ square inches, and the formula becomes

$$\log A = 0.30103 + 0.0000217y.$$

Now for $y = 100$ inches, $\log A = 0.30320$, and $A = 2.01$ square inches; for $y = 1000$ inches, $\log A = 0.32274$ and $A = 2.10$ square inches; while when $y = 10\,000$ inches, $A = 3.30$ square inches. Thus it is only in the case of very long bars that an appreciable increase in cross-section is found.

Prob. 145. Let a pier whose top section is a rectangle of length l and breadth b support the load P, as in Prob. 26. Deduce the value of $\log x$ in terms of b, l, and P.

ART. 103. LONGITUDINAL IMPACT.

By longitudinal impact is understood the impinging of a moving body upon the end of a bar so as to produce in it either tensile or compressive stress. In Art. 93 this subject was discussed and formulas were deduced for the elongation and stress under impact. These formulas, however, take no account of the resisting influence of the inertia of the bar, which may modify the results when the weight of the bar is large compared to that of the moving body.

Let a body of weight P be free to move along a horizontal bar, and let v be its velocity when it reaches the end of the

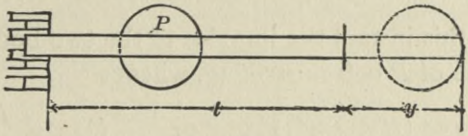

FIG. 58.

bar. The work then stored in it is $P\frac{v^2}{2g}$, or Ph, if h be the height due to the velocity v. This work is expended in overcoming the inertia of the particles of the bar and in elongating it through a distance y. A series of oscillations then results until finally the end of the bar comes to rest at its original position, provided that the elastic limit be not exceeded by the maximum stress produced.

The load P has the velocity v before it strikes the end of the bar. When in complete contact both P and the end of the bar are moving with the velocity V which is less than v. At this instant any element of the bar dW is moving with a velocity u. Thus the work stored up by P and the bar at this instant of complete contact is

$$K = P\frac{V^2}{2g} + \int_0^l dW.\frac{u^2}{2g}.$$

Now $u = 0$ for the fixed end, and $u = V$ for the free end of the bar, and in general u is proportional to the distance x from the fixed end. Thus

$$u = V\frac{x}{l} \qquad \text{and} \qquad dW = W\frac{dx}{l},$$

and introducing these values, the integral in the above expression is found to be $\frac{1}{3}W\frac{V^2}{2g}$, or one third of that which would obtain if the entire bar were in motion. Then

$$K = (P + \tfrac{1}{3}W)\frac{V^2}{2g}$$

is the work stored up by load and bar when the end of the bar and the load are moving with the common velocity V.

Now at this instant the impulse of the two bodies is equal to the impulse of P before striking, whence

$$Pv = (P + \tfrac{1}{3}W)V, \qquad \text{or} \qquad V = \frac{Pv}{P + \frac{1}{3}W},$$

and hence the work K is

$$K = \frac{P^2}{P + \frac{1}{3}W}\frac{v^2}{2g} = \frac{Pv^2}{2g(1 + \frac{1}{3}k)} = \frac{Ph}{1 + \frac{1}{3}k},$$

in which k denotes the ratio of W to P. This work is expended in elongating the bar through the distance y, the internal stress increasing from 0 up to Q, so that the entire internal work is $\frac{1}{2}Qy$. Hence

$$\tfrac{1}{2}Qy = \frac{Ph}{1 + \frac{1}{3}k}$$

is the equation between external and internal work.

If the elastic limit be not exceeded the forces Q and P are proportional to the elongations they can produce. Let λ be the static elongation due to P. Then $Q/P = y/\lambda$, and the equation gives

$$y = \sqrt{\frac{2h\lambda}{1 + \frac{1}{3}k}} \qquad \text{or} \qquad Q = P\sqrt{\frac{2h}{\lambda(1 + \frac{1}{3}k)}},$$

from which the elongation y and the stress Q are determined.

The static elongation λ due to a load P and a cross-section A is, from Art. 5,

$$\lambda = \frac{Pl}{AE},$$

and this may be inserted in the above formulas either in literal or numerical form. The formulas apply equally well to the case of compression where the load P impinges upon the end of the bar, provided that l be not so long that lateral flexure occurs.

If the bar be placed in a vertical position the additional work Py is performed while the load descends through the distance y, and the equation of work is

$$\tfrac{1}{2}Qy = \frac{Ph}{1 + \frac{1}{3}k} + Py,$$

which leads to the formulas

$$(20) \qquad y = \lambda\left(1 + \sqrt{\frac{2h}{\lambda(1 + \frac{1}{3}k)} + 1}\right), \qquad Q = P\left(1 + \sqrt{\frac{2h}{\lambda(1 + \frac{1}{3}k)} + 1}\right),$$

and these are the same as those of Art. 93 if the ratio k be made equal to 0.

The influence of the weight of the bar is hence to diminish the elongation and stress under impact. For instance, in the case of the horizontal bar let S be the unit-stress produced if the weight of the bar be very small; then $\frac{S}{\sqrt{1 + \frac{1}{3}k}}$ is the unit-stress when the weight of the bar is k times that of the load. Thus if $W = P$ the unit-stress is $0.87S$, and if $W = 9P$ the unit-stress is $0.5P$; but these high values of k are unusual.

As a numerical example, let it be required to find the stress produced in a vertical wrought-iron bar, one square inch in section and 18 feet long by the impact of a body weighing 30 pounds falling through a height of one foot. Here $W = 60$ pounds and $k = 2$. The static elongation is

$$\lambda = \frac{Pl}{AE} = 0.000\,2592 \text{ inches.}$$

Then, as $h = 12$ inches, formulas (20) give

$$y = 0.000\,2592 \times 49.1 = 0.01264 \text{ inches,}$$
$$Q = 30 \times 49.1 = 1470 \text{ pounds,}$$

which shows that very high stresses may be produced by the impact of light bodies falling through moderate heights.

Prob. 146. A weight of 60 pounds moving horizontally impinges upon the end of a bar of wrought iron 2 inches in diameter and 12 feet long. Find the velocity v which will stress the bar up to the elastic limit.

ART. 104. CENTRIFUGAL STRESS.

When a body of weight P revolves around an axis with the uniform velocity v, and r is its distance from the axis, a centrifugal force Q is generated whose value, as deduced in theoretical mechanics, is

$$Q = \frac{Pv^2}{gr},$$

and which acts as a stress in the cord or bar that connects the body with the axis. The case shown in Fig. 59 is that of a

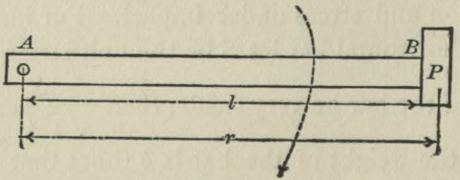

FIG. 59.

bar of uniform cross-section and length l having a weight P

attached to one end while it revolves around an axis A at the other end. It is required to find the centrifugal stress in the bar at A when the speed of n revolutions per second is maintained.

Let x be any distance from the axis; the velocity at this distance is $2\pi xn$, or, if ω be the angular velocity, the velocity at the distance x is $x\omega$, and

$$\omega = 2\pi n, \qquad \text{or} \qquad n = \frac{\omega}{2\pi}.$$

Now let W be the weight of the bar, and dW an element at the distance x. Then the centrifugal stress at A is

$$Q = \frac{Pr}{g}\omega^2 + \int_0^l dW \frac{\omega^2 x^2}{gx}.$$

But $dW = Wdx/l$; inserting this and integrating,

$$Q = \left(Pr + \tfrac{1}{2}Wl\right)\frac{\omega^2}{g},$$

which gives the centrifugal stress at the axis.

As an example let a bar of wrought iron 2 × 2 inches and 6 feet long have a weight of 400 pounds attached at $6\frac{1}{4}$ feet from the axis of revolution. It is required to find the number of revolutions per second in order to produce rupture. Solving the last equation for ω, these results,

$$\omega^2 = \frac{Qg}{Pr + \frac{1}{2}Wl},$$

in which $Q = 2 \times 2 \times 55\,000 = 220\,000$, $P = 400$ pounds, $W = 80$ pounds, $g = 32.16$ feet per second per second, $l = 6$ feet, $r = 6\frac{1}{4}$ feet; then $\omega = 50.1$, and

$$n = \frac{50.1}{2\pi} = 8 \text{ nearly},$$

which is the speed required.

Another case is that of a thin circular rim or hoop of mean radius r and thickness t which revolves uniformly around its center as an axis. Let W be its weight, which is closely equal to $2\pi rtw$, if w be the weight of the material per cubic unit and the length perpendicular to the plane of the drawing be unity. The total radial centrifugal force due to the angular velocity ω is

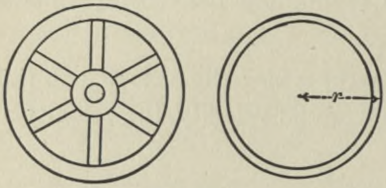

FIG. 60.

$$Q = \frac{Wr\omega^2}{g} = \frac{2\pi r^2 tw\omega^2}{g},$$

and the centrifugal force per square unit is

$$p = \frac{Q}{2\pi r} = \frac{rtw\omega^2}{g},$$

which acts upon the hoop in the same manner as the internal pressure of fluid in a pipe (Art. 9). Let S be the tangential tensile unit-stress caused in the hoop; then for equilibrium

$$2rp = 2tS,$$

and hence the value of S is

$$S = \frac{rp}{t} = \frac{w}{g}r^2\omega^2 = \frac{w}{g}(2\pi rn)^2,$$

which shows that the tensile unit-stress in a thin hoop is r times the centrifugal force of a cubic unit of the revolving material.

For example, let it be required to find the unit-stress in a cast-iron hoop 2 inches thick, 4 inches wide, and 62 inches outer diameter when making 300 revolutions per minute. Here $w = 450/1728$ pounds per cubic inch, $g = 32.16 \times 12$ inches per second per second, $r = 30$ inches, $n = 5$ revolutions per second; then S is found to be 630 pounds per square inch, which is a safe value for cast iron in tension under such conditions.

Prob. 147. A cast-iron bar is 3×2 inches in section, and 9 feet long. Through the middle and normal to the flat side is a hole $\frac{3}{4}$ inches in diameter. If the bar be revolved around an axis through this hole, how many revolutions per second will produce rupture?

ART. 105. SHRINKAGE OF HOOPS.

Hoops and tires are frequently turned with the interior diameter slightly less than that of the wheels or cylinders upon which they are to be placed. They are then expanded by heat and placed in position, and upon cooling are held firmly in position by the radial stress thus produced. The effect of this radial stress is to cause tension in the hoop, and compression throughout the mass that it encircles.

When the hoop is thin compared to the diameter of the cylinder upon which it is to be shrunk, the entire stress due to the shrinkage may be practically regarded as confined to the hoop. The tangential unit-stress in the hoop will then be due only to the increase in length of the circumference, and this will be proportional to the increase in its diameter.

Let D be the diameter of the cylinder upon which the hoop is to be shrunk, and d be the interior diameter to which the hoop is turned. Supposing that D is unchanged by the shrinkage, d will be increased to D, and the relative change in length or the unit-elongation of the hoop will be

$$s = \frac{D - d}{d},$$

and hence the unit-stress produced will be

$$S = sE = \frac{D - d}{d} E,$$

where E is the coefficient of elasticity of the material.

A very common rule for the case of steel hoop shrinkage is to make $D - d$ equal to $\frac{1}{1500}D$, that is, the hoop is turned so that the interior diameter is $\frac{1}{1500}$th less than the diameter of the cylinder. Then

$$s = \frac{D - d}{d} = \frac{1}{1499},$$

or s may be taken also as $\frac{1}{1500}$. Thus the tangential unit-stress S will be 20 000 pounds per square inch. For wrought iron s may be taken as $\frac{1}{2000}$ and S will be 12 500 pounds per square inch.

These values of S are too high, because a part of the effect of shrinkage is expended in producing compression in the cylinder. In Art. 142 the subject will be further discussed, and the final decrease in D as well as the increase in d will be determined.

Prob. 148. Upon a cylinder 18 inches in diameter a wrought-iron hoop 2 inches thick is to be placed. The hoop is turned to an interior diameter of 17.98 inches and shrunk on. Compute the tensile unit-stress in the hoop.

ART. 106. SPHERICAL ROLLERS.

Let a sphere of radius r be placed between two plates of equal thickness r_1 and be subject to compression by a load W.

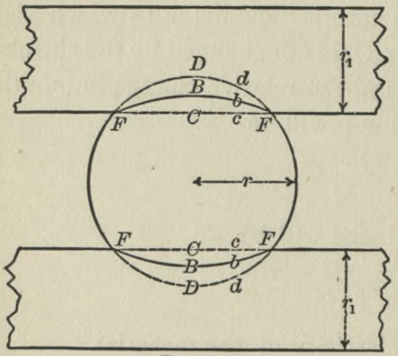

FIG. 61.

The vertical diameter DD is thus shortened to BB, while any vertical line dd is shortened to bb. Let the total shortenings CD be called λ, and the shortenings cd at any point be called y. Let the unit-stresses at B and b be denoted by S and S_y. Then, if the elastic limit is not exceeded,

$$\frac{S}{S_y} = \frac{\lambda}{y}, \quad \text{or} \quad S_y = \frac{S}{\lambda}y.$$

The unit-stress S is evidently the maximum, and it is required to express it in terms of W, r, and r_1. This will now be done approximately, assuming that each vertical element acts independently of the others and that no lateral bulging of the sphere occurs.

The value of λ may be expressed by noting that BD is the shortening of the radius r, and CB that of the thickness r_1; thus

$$\lambda = S\frac{r}{E} + S\frac{r_1}{E_1} = S\left(\frac{r}{E} + \frac{r_1}{E_1}\right),$$

in which E and E_1 are the coefficients of elasticity of the sphere and plates respectively. Also the sum of all the vertical stresses in the spherical segment FDF must equal the total load, or

$$W = \int S_y dA = \frac{S}{\lambda}\int y dA,$$

where dA denotes an elementary area of the circle whose radius is CF. Thus two equations have been found connecting the two unknown quantities S and λ.

To solve these equations consider that y is any ordinate cd of the spherical segment corresponding to an abscissa Cc or x. Thus $\int y dA$ is the volume of the segment, which is very nearly equal to one-half the cylinder having CF as the radius of its base and CD as its altitude, since the arc of contact FBF is very small. Now $CD = \lambda$ and $CF = \sqrt{2r\lambda - \lambda^2} = \sqrt{2r\lambda}$ nearly, and hence

$$W = \frac{S}{\lambda}.\frac{\lambda}{2}.\pi 2r\lambda = \pi S r\lambda.$$

Inserting for λ its value, this becomes

$$W = \pi S^2 r\left(\frac{r}{E} + \frac{r_1}{E_1}\right),$$

which is an approximate formula for the investigation of spherical rollers when the upper and lower plates are of equal thickness.

As a numerical example let a steel sphere 2 inches in diameter be placed between two cast-iron plates, each 3 inches in thickness. It is required to find the load which will cause a unit-stress at the top and bottom of the sphere of 15 000 pounds per square inch. Here $r = 1$ inch, $r_1 = 3$ inches, $S = 15\,000$, $E = 30\,000\,000$, $E_1 = 15\,000\,000$ pounds per square inch. Inserting all values, W is found to be about 165 pounds. If, however, the two plates be 10 inches thick, the value of W is about 495 pounds.

By solving the last equation for S the unit-stress may be computed when the load, the size of the roller, and the thickness of the plates are given; or it may be solved for r when W, S, and r_1 are the given quantities.

Prob. 149. A load of 144 000 pounds is to be supported upon spherical steel rollers included between steel plates whose thickness is equal to the diameter of the rollers. If the working unit-stress is 12 500 pounds per square inch, how many rollers are required?

ART. 107. CYLINDRICAL ROLLERS.

The reasoning of the last article applies also to a cylindrical roller included between two plates of equal thickness. Let Fig. 61 represent a cross-section perpendicular to the axis of the roller whose length is l. Then, as before,

$$\lambda = S\left(\frac{r}{E} + \frac{r_1}{E_1}\right), \qquad W = \frac{S}{\lambda}\int y dA,$$

are the two equations for determining S. The area A is that of a plane with width FF and length l, so that $\int y dA$ is the volume of the cylindrical segment whose cross-section is FDF. Since the area of FDF is very small, it is closely two-thirds of the rectangle whose base is FF and altitude is CD. Now $CD = \lambda$ and $FF = 2\sqrt{2r\lambda}$ very nearly, and hence

$$W = \frac{S}{\lambda}\cdot\tfrac{2}{3}l\lambda\,.\,2\sqrt{2r\lambda} = \tfrac{4}{3}Sl\sqrt{2r\lambda},$$

or inserting for λ its value, this becomes

$$W = \tfrac{4}{3}l\sqrt{2rS^3\left(\frac{r}{E} + \frac{r_1}{E_1}\right)},$$

which is an approximate formula for the investigation of cylindrical rollers.

If $r_1 = r$ and $E = E_1$, that is, if the thickness of the plates is equal to the radius of the rollers, and are made of the same material, and if w be the load per unit of length, the formula becomes

$$w = \tfrac{8}{3}r\sqrt{\frac{S^3}{E}}.$$

Taking $S = 15\,000$, and $E = 30\,000\,000$ for steel or cold rolled iron, this reduces to $w = 890r$, that is, the safe load per linear inch is 890 pounds multiplied by the radius of the roller in inches. In general, the safe load is probably proportional to the radius of the roller.

The above is not a very satisfactory investigation, and indeed it may be said that the common rule for proportioning cylindrical rollers in bridge work is that the safe load in pounds per linear inch should not exceed $500\sqrt{D}$ where D is the diameter in inches. This is equivalent to $707\sqrt{r}$. The practice of making the safe load vary as the square root of the radius or diameter of the roller seems to be based on the authority of GRASHOF, who was the first to deduce the above formula for W, but who appears to have placed a doubtful interpretation upon it. The practical rule, however, is probably on the safe side. See an important paper by CRANDALL and MARSTON in Transactions of American Society of Civil Engineers for August, 1894.

Prob. 150. A load of 90 000 pounds is to be carried on wrought-iron rollers 12 inches long and 3 inches in diameter. How many rollers are required?

ART. 108. ECCENTRIC LOADS.

Let a load P be applied to the end of a short bar at a horizontal distance p from the center of gravity of its cross-section, as shown in Fig. 62. If two forces equal to P and acting in opposite directions be supposed to be applied to the end of the bar, the equilibrium is undisturbed, and it is seen that the downward force produces pure tension, while the couple formed by P and the other force produces flexure. Let A be the cross-section of the bar and S the maximum unit-stress caused by the flexure; then the resultant unit-stresses are

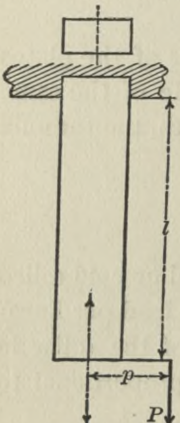

FIG. 62.

$$S_1 = \frac{P}{A} + S, \qquad S_2 = \frac{P}{A} - S,$$

the former being on the side of the bar nearest to P and the latter on the other side.

Let I be the moment of inertia of the cross-section with respect to an axis through the center of gravity and normal to the direction of the arm p. Let r be the radius of gyration of the cross-section so that $I = Ar^2$. Let c_1 and c_2 be the distances from the axis to the sides of the section nearest to and farthest from P. Then from the formula (4) of Art. 21 the two values of S are found, and the above expressions reduce to the practical formulas

$$(21) \qquad S_1 = \frac{P}{A}\left(1 + \frac{pc_1}{r^2}\right), \quad S_2 = \frac{P}{A}\left(1 - \frac{pc_2}{r^2}\right),$$

which give the greatest and least tensile stresses under the eccentric application of the load.

These formulas apply also to compression under an eccentric load, provided that the bar be short so that no lateral flexure can occur. As an example let a short block have a rectangular section, the dimension parallel to the arm p being d, while the one normal to it is b. Then the formulas reduce to

$$S_1 = \frac{P}{A}\left(1 + 6\frac{p}{d}\right), \qquad S_2 = \frac{P}{A}\left(1 - 6\frac{p}{d}\right),$$

and in Fig. 63 are shown the distribution of stresses for several values of p. In the first sketch P is applied at the center of the section so that the unit-stresses are uniform and both S_1 and S_2 are equal to $\frac{P}{A}$. In the second sketch p is taken as $\frac{1}{12}d$,

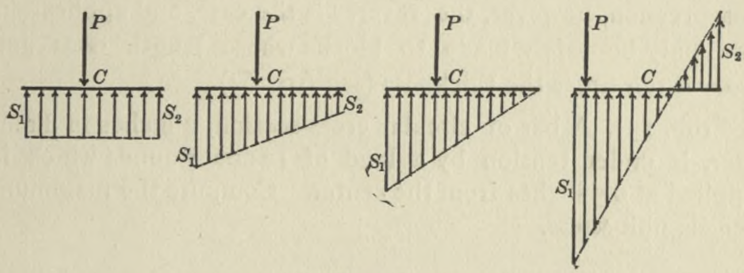

FIG. 63.

which gives $S_1 = \frac{3}{2}\frac{P}{A}$ and $S_2 = \frac{1}{2}\frac{P}{A}$. In the third sketch p is taken as $\frac{1}{6}d$, so that $S_1 = 2\frac{P}{A}$ and $S_2 = 0$. As the load moves further away from the center C, the stress S_1 increases, while S_2 becomes negative, showing that it is tension. Thus in the last sketch where p is equal to $\frac{1}{2}d$ the compressive unit-stress S_1 is $3\frac{P}{A}$, while the tensile unit-stress S_2 is $2\frac{P}{A}$. In all these cases the unit-stress at the center C is the mean value $\frac{P}{A}$.

It is thus seen that the eccentric application of a load may materially increase the direct stress of tension or compression. In the case of a rectangular masonry pier the greatest deviation of P from the center should never be greater than $\frac{1}{6}d$, in order that no tension may be brought upon the joint. In other words the point of application of P should be kept within the 'middle third' of the base.

In both the cases above discussed the body under tension or compression has been supposed to be very short, so that no material lateral movement can result. If in Fig. 62 the bar be long the arm p will be different for different parts of the length, and the greatest bending moment will occur at the lower end, so that the formulas given apply only to that end, while for other sections they give results too large. For compression, however, the reverse is the case, and application can only be safely made to blocks whose length does not exceed ten times the thickness (see Art. 62).

Prob. 151. A bar of circular cross-section, 2 inches in diameter, is under tension by a load of 12 000 pounds which is applied at 0.5 inches from the center. Compute the maximum tensile unit-stress.

CHAPTER XI.

FLEXURE OF BEAMS.

ART. 109. THE WORK OF FLEXURE.

This subject was treated in Art. 96, but it will now be discussed again in order to deduce a more general relation between the work of the external forces and that of the internal stresses.

Let P be a load upon a beam and Δ the deflection under it. The load being gradually applied the work done by P is $\frac{1}{2}P\Delta$, and this must be equal to the internal work or resilience of the molecular stresses.

For a beam loaded uniformly with w per linear unit let y be the deflection at any point whose abscissa is x. Then the load wdx on the short distance dx deflects the amount y, and the elementary external work is $\frac{1}{2}wydx$. Thus the integral of this, if y is known in terms of x, will give the total work of the uniform load, if the integration be extended over the entire length of the beam. This will be equal to the internal work, or resilience.

A general expression for the resilience, or internal work of the horizontal fibers, in terms of the bending moment M of the applied forces will now be deduced; it is applicable to all cases in which the elastic limit of the material is not surpassed by the maximum fiber stress S.

When a beam deflects under the action of a load the horizontal fibers upon one side of the neutral surface are elongated and upon the other side are compressed. The internal work will be found by taking the sum of the products formed by multiplying the stress upon any elementary area by its elongation or compression.

Using the same notation as in Chapter III., the horizontal unit-stress at any distance z from the neutral axis is represented by $\frac{Sz}{c}$. In the distance dx the elongation or compression due to this unit-stress, is by (2) found to be $\frac{Sz\,dx}{cE}$. The elementary work of a fiber of the area a under this gradually applied unit-stress hence is,

$$\frac{1}{2} \cdot \frac{Saz}{c} \cdot \frac{Szdx}{cE}.$$

The work done in the distance dx by all the fibers in the cross-section now is,

$$dK = \frac{S^2 \Sigma az^2}{2c^2E} dx.$$

Here $\Sigma az^2 = I$ and from formula (4), the value of $\frac{S^2}{c^2}$ is $\frac{M^2}{I^2}$

Therefore $$dK = \frac{M^2 dx}{2EI}.$$

This is the formula for the work done in the distance dx. By expressing M as a function of x, and integrating, the total internal work K between assigned limits can be found.

For example, consider a cantilever beam loaded at the end with a weight P. Here $M = -Px$. Inserting this and integrating between the limits 0 and l, gives,

$$K = \frac{P^2 l^3}{6EI},$$

for the total internal work in the beam due to a load which is gradually applied.

The preceding furnishes a new method of deducing the deflection of a beam loaded with a single weight P. Let Δ be the deflection under the weight. Then $\frac{1}{2}P\Delta$ is the external work done by the load P upon the beam, and this must equal the internal work K. Hence the formula,

$$P\Delta = \int \frac{M^2 dx}{EI},$$

from which Δ may be found for particular cases.

For example, consider a cantilever beam loaded at the end with P. Then the internal work is, as shown above, $\frac{P^2 l^3}{6EI}$. Hence the deflection Δ is,

$$\Delta = \frac{Pl^3}{3EI},$$

which is the same as otherwise found in Art. 34.

For a simple beam loaded at the middle the value of M is $\frac{Px}{2}$ and then

$$P\Delta = 2\int_0^{\frac{1}{2}l} \frac{P^2x^2dx}{4EI} = \frac{P^2l^3}{48EI},$$

from which the deflection is,

$$\Delta = \frac{Pl^3}{48EI},$$

which is the same as found in Art. 35 by the use of the elastic curve.

Prob. 152. Prove that the internal work caused by a uniformly distributed load on a cantilever beam is $\frac{3}{20}$ths of that caused by the same load applied at the end.

Prob. 153. Deduce by the method of Art. 35, and also by the use of the principle of internal work, the deflection under a load P which is placed upon a simple beam at a distance $\frac{1}{4}l$ from one end.

ART. 110. STATIC AND SUDDEN DEFLECTIONS.

A static deflection is one produced by the gradual application of the load, so that at each instant the beam is in a condition of static equilibrium and hence no oscillations occur. The formulas for the deflection of beams deduced in Chapters III and IV, as well as the discussion of the last Article, are all static deflections where the elastic limit of the material is not surpassed. In such case the load increases from 0 up to its maximum value P, and simultaneously the deflection increases from 0 up to its maximum value Δ. If Q be the load at any instant and δ the corresponding deflection, then the deflections are proportional to the loads or $\delta/\Delta = Q/P$.

A load is said to be suddenly applied when it acts with uniform intensity during the full period. For instance, let a load be attached to a ring which is placed around the middle of a beam, and let the load be supported so that the ring just touches the upper surface of the beam; then if the support of the load be suddenly withdrawn the force of gravity acts upon the load with uniform intensity during the entire period of deflection. In this case the maximum deflection is greater than for a static load, but as soon as it is reached a series of oscillations occurs until finally the beam comes to rest with a deflection due to the static load. The maximum stress in the beam evidently occurs at the instant of greatest deflection. If S be the unit-stress due to the static load P under the deflection Δ, and if T be the unit-stress due to the sudden load P under the deflection δ, both loads being applied at the same point on the same beam, then from Art. 37 the stresses are proportional to the deflections, or $\delta/\Delta = T/S$.

The deflection under a sudden load is double that under the same static load, and the stress under a sudden load is double that under the same static load. In order to prove this refer to Art. 96 and note that the total internal work or resilience due to the maximum unit-stress S in the beam is

$$K = \frac{1}{2}\frac{S^2}{E} \cdot C \cdot Al,$$

where Al is the volume of the beam, and C is a constant depending upon the shape of its cross-section and the arrangement of its ends. Thus for a given beam the internal work varies as S^2. Now for the gradually applied load P the deflection is Δ and the external work is $\frac{1}{2}P\Delta$. Hence as internal work equals external work,

$$\tfrac{1}{2}P\Delta = \frac{1}{2}\frac{S^2}{E}CAl,$$

which gives the relation between Δ and S. Again for the sudden load the deflection is δ and the maximum unit-stress is T; the external work, however, is $P\delta$, since the load acts with full intensity during the entire period of deflection, while the stress increases gradually from 0 up to T; hence

$$P\delta = \frac{1}{2}\frac{T^2}{E}CAl,$$

which gives the relation between δ and T. By comparing these two equations there is found

$$\frac{\delta}{\frac{1}{2}\Delta} = \frac{T^2}{S^2},$$

and remembering that $T/S = \delta/\Delta$, as shown above, this gives

$$\delta = 2\Delta, \quad \text{and} \quad T = 2S,$$

which proves the proposition as stated.

Another method of establishing the same truth is to compare the sudden load P with the static load W which will produce the same deflection Δ and hence the same unit-stress S. The internal work is thus the same in the two cases, but the external work of the sudden load is $P\Delta$, while that of the static load is $\frac{1}{2}W\Delta$. Therefore $P = \frac{1}{2}W$; that is, the sudden load that produces a given stress is one-half of the static load that produces the same stress.

These propositions are true only when the maximum stresses caused by the deflections are within the elastic limit of the material, since all the reasoning by which they are established supposes the law of proportionality of stress to deformation to be observed. It is hence not to be expected that they can be verified by the rupture of beams. It has, however, been found to be approximately true for cast iron, but for wrought iron and steel the sudden load that produces rupture is greater than one-half of the static load.

The following experiments of HODGKINSON on cast-iron beams illustrate very well the agreement of theory with practice near the elastic limit and the disagreement when the elastic limit is exceeded. Each beam was laid on supports 9 feet apart, and a lever at the middle prevented deflection when the load was applied. This was hung to the beam by two rings, one on each side of the lever and placed as close to it as possible without touching. The lever being instantaneously removed the load was brought suddenly into action, and the deflection was registered upon a vertical sheet of paper by means of a pencil screwed to the side of the beam. Before applying the loads in this sudden manner the beams were tested in the usual way by gradually applied loads. The results here given are the mean of two or three tests upon different beams. It will be found that for each beam the first load gives a unit-stress less than the elastic limit, while the second gives a

Size of Beam. Inches.	Load. Pounds.	Deflection in Inches. Gradual.	Sudden.	Ratio of Deflections, Sudden to Gradual.
1 × 2	112	0.253	0.515	2.04
1 × 2	224	0.580	0.933	1.61
1 × 3	224	0.163	0.303	1.86
1 × 3	560	0.410	0.720	1.76
4 × 1½	448	0.770	1.510	1.96
4 × 1½	784	1.275	2.225	1.73

greater value. Thus for the first beam under 112 pounds at the middle S is 4 890, and for 224 pounds S is 9 770 pounds per square inch.

The breaking loads of these beams were also observed under both gradual and sudden loads with the following results:

Size of Beam. Inches.	Breaking Load in Pounds. Gradual.	Sudden.	Ratio of Loads, Gradual to Sudden.
1 × 2	1 000	569	1.76
1 × 3	2 008	1219	1.65
4 × 1½	1911	1082	1.77

Here the ratio is seen to be about the same as for the heavier loads in the cases of deflection.

KIRKALDY'S experiments on wrought-iron beams show that the ratio of the gradual to the sudden breaking load ranges between 1.2 and 1.3, so that the elastic law does not even approximately apply. There is a very good reason why it should not apply, and for this the student may consult the Proceedings of the Engineers' Society of Western Pennsylvania for May, 1894.

Prob. 154. Compute the static deflection and the unit-stress S for the third experiment on the cast-iron beams given above.

ART. 111. DEFLECTION UNDER IMPACT.

As the deflection under a sudden load is greater than for a static load, so the deflection under impact is greater still. A load falling through a small height upon a beam may easily produce a deflection and a stress that may prove dangerous to its stability. In Arts. 93 and 103 rules were deduced for bars subject to impact, and a similar investigation leading to similar results will now be made for beams.

Let the beam be placed in a horizontal position and let a weight P move horizontally with the velocity v so as to strike it at the middle and cause a lateral deflection. The work stored up by the moving weight is $P\frac{v^2}{2g}$, or Ph if h be the height due to the velocity. Let W be the weight of the beam, δ the maximum deflection produced by the impact, and T the corresponding maximum unit-stress. Let $\varDelta$ be the deflection due to a static load P, and S be the corresponding maximum unit-stress. Then $\delta/\varDelta = T/S$, or the deflections are proportional to the stresses. Also, let Q be a static load which will produce the deflection δ, then likewise $Q/P = \delta/\varDelta$, or the static loads are proportional to the deflections. The internal work or resilience of the beam now is $\frac{1}{2}Q\delta$, and this must equal the external work Ph, or

$$\tfrac{1}{2}Q\delta = \frac{1}{2}\cdot\frac{\delta}{\varDelta}P\cdot\delta = Ph.$$

Also substituting for δ its value in terms of T, S, and $\varDelta$, an equation between S and T results. Thus,

$$(22) \qquad \delta = \sqrt{2h\varDelta}, \qquad T = S\sqrt{\frac{2h}{\varDelta}},$$

are the formulas for finding the maximum deflection and corresponding unit-stress in a simple beam under impact at the middle, $\varDelta$ and S being given by the expressions of Chapter III,

$$\varDelta = \frac{Pl^3}{mEI}, \qquad S = \frac{Pcl}{nI},$$

where l is the length of the beam, m and n numbers depending upon the arrangement of its ends, I the moment of inertia of the cross-section, and c the distance from the neutral axis to the fiber whose stress is S or T.

The above results will be somewhat modified by the resistance of inertia of the beam, as in the case of longitudinal impact discussed in Art. 115. An instant before the contact the weight P has the velocity v and the beam is at rest; when the contact is complete both P and the middle of the beam are moving with a common velocity V. Plainly V is less than v, as a portion of the energy $P\frac{v^2}{2g}$ has been communicated to the beam whose weight is W. When the velocity V is acquired any element dW is moving with the velocity u, and the combined energy of weight and beam then is

$$K = P\frac{V^2}{2g} + \int dW \frac{u^2}{2g} = \left(P + qW\right)\frac{V^2}{2g},$$

and this is the energy which is effective in producing the deflection. But also, as soon as the velocity V is acquired the condition for the impact of elastic bodies must obtain, namely,

$$Pv = (P + qW)V.$$

Eliminating V from these equations gives

$$K = \frac{P^2v^2}{2g(P + qW)} = \frac{Ph}{1 + qk},$$

in which k is the ratio W/P and q is a number whose value is to be determined.

The above value of K being placed equal to $\frac{1}{2}Q\delta$, as before, there are found

$$(22)' \quad \delta = \sqrt{\frac{2h\Delta}{1 + qk}}, \qquad T = S\sqrt{\frac{2h}{\Delta(1 + qk)}},$$

as the modified formulas for deflection and stress due to horizontal impact.

It now remains to determine the value of q, and this will depend upon the arrangement of the ends of the beam. The general value of q is seen to be

$$q = \int \frac{u^2}{V^2} \cdot \frac{dW}{W},$$

and the integration is to be extended over the entire length of the beam. Now $dW/W = dx/l$; also if y be the deflection at any point and y_1 the deflection at the point whose velocity is V, then $u/V = y/y_1$. Thus the integral becomes

$$q = \int \frac{y^2dx}{y_1^2l},$$

which may be applied to any particular case where y/y_1 can be expressed as a function of x.

For a beam supported at the ends and loaded in the middle the ordinate y of the elastic curve in terms of the maximum deflection y_1 and the abscissa x is, from Art. 35,

$$y = y_1\left(4\frac{x^3}{l^3} - 3\frac{x}{l}\right),$$

and for this case the integral becomes

$$q = \frac{2}{l}\int_0^{\frac{1}{2}l}\left(4\frac{x^3}{l^3} - 3\frac{x}{l}\right)^2 dx = \frac{17}{35}.$$

Therefore for a beam supported at the ends and impinged upon in the middle there obtains

$$1 + qk = 1 + \frac{17}{35}\frac{W}{P},$$

which is the vaiue to use in formula (22)′. This result was first established theoretically by COX in 1848, but HODGKINSON had previously found by experiments on cast-iron beams that the value of q was about $\frac{1}{2}$.

As a numerical example let a cast-iron beam on supports 9 feet apart be 1 inch wide and 2 inches deep, and have a static load of 50 pounds at the middle. The deflection and stress at the middle due to this load are

$$\varDelta = 0.131 \text{ inches}, \quad S = 2025 \text{ pounds per square inch.}$$

Now suppose that this load of 50 pounds moves horizontally with a velocity due to a fall of $\frac{1}{2}$ inch. Then from (22)

$$\delta = 0.362 \text{ inches}, \quad T = 5590 \text{ pounds per square inch,}$$

which are the results, supposing that the beam has no weight. Taking this into account, the weight W is 56.4 pounds, whence $k = 1.128$, and by (22)′

$$\delta = 0.290 \text{ inches}, \quad T = 4470 \text{ pounds per square inch,}$$

which are more exact results.

It is seen by this investigation that very small velocities of impact may produce very high stresses in a beam. Thus in (22) the static deflection Δ is always small, and if $h = 2\Delta$, T is $2S$. It is also seen that the influence of the resisting inertia of the beam increases with k, that is, with the ratio of the weight of the beam to the impinging weight.

When the weight falls vertically upon the horizontal beam, h being the height of fall to the top of the beam, the formulas (22) and (22)′ are to be modified slightly, since the external work is $P(h + \delta)$. Thus are found

$$(22)'' \qquad \delta = \Delta\left(1 + \sqrt{\frac{2h}{\Delta(1 + qk)} + 1}\right),$$

$$T = S\left(1 + \sqrt{\frac{2h}{(1 + qk)} + 1}\right),$$

which are seen to be the same in form as those deduced for longitudinal impact in Art. 103. For instance, if the load in the above example drops $\frac{1}{2}$ inch, then $\delta = 0.295$ inches instead of 0.290 inches. (See Appendix.)

Prob. 155. Prove for the case of impact against the middle of a beam fixed at both ends that the value of q is $\frac{13}{35}$.

Prob. 156. A piece of railroad rail 5 feet long is laid upon two supports 4 feet apart and a weight of 2240 pounds dropped upon it from a height of 2 inches. The rail weighs 95 pounds per yard, its height is 5 inches, the neutral axis of the cross-section is 2.4 inches above the base, and the moment of inertia of the cross-section is 31.4 inches. Compare the static deflection and stress with those resulting from the impact.

ART. 112. PRESSURE DUE TO IMPACT.

When a weight P falls from a height h upon a beam, a pressure is produced at the point of contact. This pressure is variable during the period of deflection, being zero at the first instant of contact, increasing rapidly to a maximum, and then decreasing, until at the moment of greatest deflection there exists a pressure sufficient to cause the weight to rebound a short distance. The laws governing the variation of the pressure are not understood, but the mean or average pressure existing during the impact can be satisfactorily ascertained.

Let h be the height of fall above the top of the rail, and δ the deflection produced by P falling through this height. Let R be the mean or average pressure existing between the rail and weight during the impact. The work of the falling ram is $P(h + \delta)$ and the work of the mean pressure R is $R\delta$. Equating these values, there is found

$$R = P\left(\frac{h}{\delta} + 1\right),$$

from which R can be computed when δ has been measured.

This formula is correct both for elastic and non-elastic deflections. If the elastic limit be not exceeded, δ can be computed by (22)″ of the last article. If the elastic limit is exceeded, δ can only be found by actual measurement.

To illustrate, let a ram weighing 2 000 pounds fall from a height of 20 feet upon a railroad rail, this being one method of testing rails at the mill. The rail will be deflected an amount δ, depending upon its size, shape, length, and the quality of the material. For $\delta = 1$ inch R will be 482 000 pounds; for $\delta = 2$ inches R will be 242 000 pounds; for $\delta = 4$ inches R will be 122 000 pounds. Thus the mean pressure decreases approximately inversely as the deflection.

The maximum pressure in such cases cannot be theoretically ascertained, but it probably is not as great as double the mean pressure. The maximum pressure per square inch will depend, of course, upon the area of contact between the ram and the rail, and upon the manner in which the total pressure is distributed over that area.

Prob. 157. Compute the mean pressure on the rail for the data given in Prob. 156.

ART. 113. CENTRIFUGAL STRESS.

The rod that connects the cross-head of an engine with the crank pin is subject to a centrifugal stress owing to the fact that one end revolves in a circle. The horizontal rod, or parallel bar joining two driving wheels of a locomotive is another instance of centrifugal stress; this is simpler than the connecting rod, because all points are revolving with the same velocity, and hence it will be discussed first.

Let V be the velocity of a locomotive in feet per second, and v the velocity of revolution of the end of the parallel rod around the axle of the driver to which it is attached. Let R be the radius of the driver, and r the radius of the circle of revolution of the end of the parallel rod. Then since the

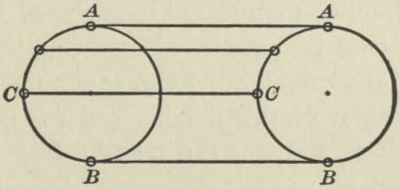

FIG. 64.

velocity of revolution of the circumference of the driver is the same as that of the speed of the train, the value of v is

$$v = \frac{r}{R} V.$$

Now, not only the end but every point in the parallel rod is revolving with the velocity v in a circle whose radius is r. Thus a centrifugal force is generated which produces stresses. When the rod is at its lowest position BB, this centrifugal force acts as a downward uniform load producing flexure; at the highest position AA it acts as an upward uniform load producing flexure; at the position CC, on the same level as the axles, it produces a compressive stress in the direction of the length of the rod.

Let w be the weight of the parallel rod per linear unit; then from mechanics the centrifugal force of this weight is

$$f = \frac{wv^2}{gr} = \frac{wrV^2}{gR^2},$$

which may be called the centrifugal load per unit of length. The rod being a beam supported at its ends, having a length l, a breadth b, and a depth d, the maximum unit-stress due to this uniform load is, from (4),

$$S = \frac{Mc}{I} = \frac{3fl^2}{4bd^2},$$

which is the flexural stress due to centrifugal force when the bar is at its highest or at its lowest position.

In this formula g is the acceleration of gravity, or 32.16 feet per second per second. In applying it numerically, however, all quantities should be expressed in terms of the same linear unit, the inch being preferable. For example, let a locomotive be running at 60 miles per hour, the radius of the drivers being 3 feet and that of the parallel rod 1 foot, this being of steel, 4 inches deep, 2 inches thick, and 8 feet long. Here $V = 88$ feet per second $= 12 \times 88$ inches per second, $g = 32.16 \times 12$ inches per second per second, $R = 3 \times 12$ inches, $r = 1 \times 12$ inches, $l = 8 \times 12$ inches, $b = 2$ inches, $d = 4$ inches, and $w = 2.27$ pounds per linear inch. The centrifugal load per inch then is

$$f = \frac{2.27 \times 12 \times (88 \times 12)^2}{32.16 \times 12 \times (3 \times 12)^2} = 61 \text{ pounds},$$

and then the maximum fiber stress is

$$S = \frac{3 \times 61 \times (8 \times 12)^2}{4 \times 2 \times 16} = 13\,200 \text{ pounds per square inch},$$

which is not probably sufficiently low when it is considered that the parallel rod is subject to vibrations and shocks.

The connecting rod moves in a circle of radius r at the crank pin, while the other end moves only in a straight line. Thus at the end A there is no centrifugal load, while at B the centrifugal load is the same as given by the above expression for f. When the rod is in the position shown in Fig. 65, it is a beam acted upon by a centrifugal load which varies uniformly from 0 at A to f at B. The total load is hence $\frac{1}{2}fl$, the reaction at A is $\frac{1}{6}fl$ and that at B is $\frac{1}{3}fl$. The bending moment for any section distant x from A is

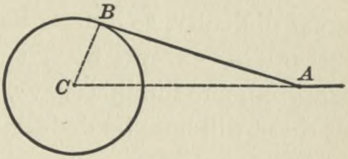

FIG. 65.

$$M = \tfrac{1}{6}fl \cdot x - \tfrac{1}{2}f\frac{x^2}{l} \cdot \tfrac{1}{3}x = \tfrac{1}{6}fl^2\left(\frac{x}{l} - \frac{x^3}{l^3}\right),$$

and the maximum value of M occurs for $x = l/\sqrt{3}$, which gives max. $M = 0.0638fl^2$. Hence from (4),

$$S = 0.383\frac{fl^2}{bd^2},$$

in which f is given by the same expression as above.

By comparing this with the value of S for the parallel rod it is seen that the former is about twice as great, if the length and cross-section be the same in the two cases. The parallel rod needs the greatest cross-section at the middle, while the connecting rod needs the greatest cross-section at about $0.6l$ from the cross-head.

Prob. 158. The connecting rod of an engine is 2 feet long and it is attached to a crank pin at a distance of 6 inches from the axis of a fly-wheel. If the wheel makes 750 revolutions per minute, find a square cross-section for the connecting rod so that the centrifugal unit-stress S may be 4200 pounds per square inch.

ART. 114. LIVE-LOAD VELOCITY.

It is well known that when a live load moves over a beam or bridge the deflections and stresses are greater than those due to the same load at rest. In general the greater the velocity the greater also are the deflections and the stresses. The exact theoretical investigation of this question is one of very great difficulty, as differential equations arise which cannot be integrated except by an unsatisfactory tentative process. Approximate formulas have, however, been established, and one of these will here be deduced.

Let a static load P rest on the middle of a simple beam. From Chapter III the deflection and the maximum unit-stress under the load are

$$\Delta = \frac{Pl^3}{48EI}, \qquad S = \frac{Pcl}{4I},$$

where l is the length of the beam, I the moment of inertia of the cross-section with respect to the neutral axis, c the distance from that axis to the remotest fiber where the unit-stress is S. Now, let the load P be moving horizontally across the beam with the velocity v, and when it reaches the center let the deflection be δ and the maximum unit-stress be T. It is required to find δ and T in terms of Δ and S.

When the load P runs over the beam, the curve in which it moves is found by putting $kl = x$ in the first equation on page 77, or

$$y = \frac{P}{3EIl}(2lx^3 - l^2x^2 - x^4).$$

Differentiating this twice, and then making $x = \frac{1}{2}l$, gives

$$\left(\frac{d^2y}{dx^2}\right)_{x=\frac{1}{2}l} = \frac{1}{R} = \frac{Pl}{3EI},$$

as the reciprocal of the radius of curvature of this curve at the middle of the beam.

It is assumed that when P reaches the middle of the beam it produces the same deflection Δ as if it were at rest, and also that it causes a downward pressure F due to the centrifugal force arising from motion in the curve. This pressure F gives rise to an additional deflection, thus increasing $\varDelta$ to δ, and S to T. Then,

$$\delta = \frac{(P+F)l^3}{48EI} = \varDelta\left(1 + \frac{F}{P}\right),$$

$$T = \frac{(P+F)cl}{4I} = S\left(1 + \frac{F}{P}\right).$$

Now, to find F, the expression for centrifugal force is known from mechanics, and inserting in it the above value of R, there results

$$F = \frac{Pv^2}{g\mathrm{R}} = \frac{P^2lv^2}{3gEI} = \frac{2P^2lh}{3EI},$$

in which $v^2/2g$ has been replaced by h, the height of fall which will produce v. In this last expression P/EI may be replaced by its value in terms of $\varDelta$ or by its value in terms of S from the first equations given. Making these substitutions in the formulas for δ and S, they become

$$\delta = \varDelta\left(1 + \frac{32\varDelta h}{l^2}\right),$$

(23)

$$T = S\left(1 + \frac{8Sh}{3Ec}\right),$$

which give the approximate deflection and maximum unit-stress at the middle of the beam due to a load P moving with the velocity $v = \sqrt{2gh}$.

As an example let a wrought-iron plate girder have a span of 80 feet, a depth of 7 feet 2 inches, a flange cross-section of 38 square inches, and a moment of inertia of about 134 000 inches. Let it be required to find the deflection and maximum unit-stress when a single load of 60 000 pounds crosses the girder at a velocity of 80 miles per hour. Here $P = 60\,000$ pounds, $l = 960$ inches, $I = 134\,000$ inches, $c = 43$ inches, and $E = 25\,000\,000$ pounds per square inch. Then when the load is at rest at the middle of the beam,

$\varDelta = 0.33$ inches, $S = 4\,620$ pounds per square inch.

Now, a speed of 80 miles per hour corresponds to a velocity of 117 feet per second, and the velocity head is

$$h = \frac{v^2}{2g} = \frac{117 \times 117}{2 \times 32.16} = 213 \text{ feet.}$$

Then from (23) are found the increased deflection and unit-stress,

$$\delta = 0.33(1 + 0.029) = 0.34 \text{ inches,}$$

$$T = 4620(1 + 0.029) = 4750 \text{ pounds per square inch,}$$

which show the influence of the velocity to be small.

When a uniform live load is moving over the beam or bridge, a similar investigation may be made, regarding the centrifugal force at each point as a vertical load. Let w be the uniform live load per linear unit; then when this extends over the whole beam,

$$\Delta = \frac{5wl^4}{384EI}, \quad S = \frac{wcl^2}{8I},$$

are the static deflection and maximum fiber stress at the middle. Let δ be the deflection and T the unit-stress when the entire uniform load is moving with the velocity v, and let h be the head due to this velocity. Then by a method similar in principle to the above, it may be shown that

$$(23)' \qquad \delta = \Delta\left(1 + \frac{16\Delta h}{l^2}\right), \qquad T = S\left(1 + \frac{5Sh}{3Ec}\right),$$

which are the approximate deflection and unit-stress at the middle of the beam due to the moving load wl.

As an example take the plate girder whose data are given above and let a uniform load of 1800 pounds per linear foot be moving over it. Then

$$\Delta = 0.63 \text{ inches}, \quad S = 8820 \text{ pounds per square inch,}$$

are the static deflection and unit-stress at the middle. As before, $h = 213$ feet, and then from (23)′

$\delta = 0.65$ inches, $T = 9070$ pounds per square inch,
which are only about 3 per cent greater than the static values.

It should be remarked that the allowance made for velocity of a live load in practice is much greater than the above formulas indicate. It is often customary to add from 10 to 30 per cent to the statical stresses in order to cover the effect of velocity, the greater values being used for the shorter bridges. It should also be said that experiments indicate a greater increase in deflection than these formulas give. The most extensive set of experiments in this direction is that made by JAMES, WILLIS, and GALTON, for the British board of 1848, and in some of these the statical deflection was more than doubled under heavy loads.

It may be further noted that a perfect formula for the effect of velocity of live load would show in the case of very high speeds that there would be no increase in deflection, since there would then not be sufficient time for the load to fall through the distance δ. This was recognized by the board above mentioned, and the fact ascertained in several tests. For instance a wrought-iron beam 9 feet long, 1 inch wide, and 3 inches deep was subjected to a load of 1778 pounds moving at different velocities, with the following results:

Velocity in feet per second,	0	15	29	36	43,
Deflection in inches,	0.29	0.38	0.50	0.62	0.46.

Here it will be seen that the deflection for 43 feet per second is less than that for 36 feet per second. Unfortunately these experiments were made without regard to the elastic limit of the material, and hence the results are of little use as a test of theory; for instance, the static load of 1778 pounds causes a unit-stress of 32 000 pounds per square inch on the middle of the wrought-iron beam, and thus at all velocities the elastic limit was surpassed.

The formulas of this article are confessedly imperfect, as they do not take into account the influence of the inertia of the beam which will tend to modify them materially. This very complex question cannot here be investigated, but the student is referred to Appendix B of the Report of the Commissioners on the Application of Iron to Railway Purposes (London, 1849) for an excellent general discussion. It may also be noted that the above formula (23) agrees with the first two terms of the series given on page 203 of that Report.

Prob. 159. Deduce the values of δ and T given in formula (23)′ for the uniform moving load.

Prob. 160. What velocity v must the load P have so that, in crossing the above plate-girder, there would not be sufficient time for it to fall through the vertical deflection of 0.33 inches?

ART. 115. WORK OF VERTICAL SHEARS.

In the discussion of Art. 109 the entire external work of the load P was supposed to be expended in the work of elongating and compressing the horizontal fibers of the beam. In reality, however, a part of the external work is expended in the slipping or detrusion due to the vertical shears throughout the beam.

As in Chapter III suppose the vertical shear to be uniformly distributed over the cross-section of the beam. Let V be the vertical shear at any distance x from an origin and ϕ be the angle of detrusion in the distance dx. Then taking the shear to increase slowly from 0 up to its value V, the work done by it in the distance dx is, since ϕ is very small,

$$dK = \tfrac{1}{2}V\,.\,dx\,.\tan\phi = \tfrac{1}{2}V\phi dx,$$

in which ϕ in the last value is to be taken in circular measure.

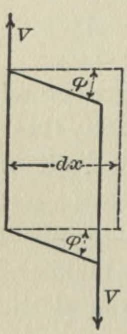

FIG. 66.

Let A be the cross-section of the beam, S_s the shearing unit-stress, and E_s the coefficient of elasticity for shearing. As ϕ is the amount of detrusion per unit of displacement, its value is found by the same process as the unit-deformation s for tension or compression, namely

$$\phi = \frac{S_s}{E_s} = \frac{V}{AE_s}.$$

This being inserted, the expression for dK becomes

$$dK = \frac{V^2 dx}{2E_sA},$$

which is the elementary work of shearing in the distance dx. By expressing V as a function of x and integrating over the entire length of the beam, the total work of shearing is found.

For instance, a simple beam loaded with P at the middle has the shear V constant throughout and equal to $\frac{1}{2}P$. Then

$$K = \frac{P^2 l}{8E_sA},$$

is the internal work or resilience due to all the shearing forces over the span l. In Art. 127 the internal work of the horizontal stress was found to be

$$K' = \frac{P^2 l^3}{96EI},$$

so that the ratio of the former to the latter is

$$\frac{K}{K'} = \frac{12EI}{E_sAl^2} = 12\frac{Er^2}{E_sl^2};$$

in which r is the radius of gyration of the cross-section with respect to the neutral axis.

For example, let a cast-iron beam have a square cross-section of side d; then $r^2 = \frac{1}{12}d^2$, and $E = \frac{5}{2}E_s$ approximately (Art. 121). The ratio of the internal work of shearing to that of the fiber stresses is then $5d^2/2l^2$. If the length is 30 times the depth, or $l = 30d$, then this ratio is $\frac{1}{360}$; if $l = 60d$, the ratio is $\frac{1}{1440}$. For a very short beam, such as $l = 2d$, the ratio is $\frac{5}{8}$, showing that in such cases the work of shearing is greater than that of the horizontal fiber stresses.

Prob. 161. Deduce an expression for the work of shearing in a beam supported at the ends and uniformly loaded.

ART. 116. DEFLECTION DUE TO SHEARING.

The treatment of the deflection of beams in the previous pages has been solely from the standpoint of the horizontal stresses as expressed by the external bending moment. The last Article shows, however, that the resisting work of shearing may be a material amount for short beams, and it hence appears that the former investigations are more or less incomplete.

Let a load P produce the deflection Δ beneath it. The external work, if P has been gradually applied, is $\frac{1}{2}P\Delta$, and this must equal the internal work, or resilience, of the molecular stresses. The work of the horizontal stresses of tension and compression is deduced in Art. 109 and that of the vertical stresses of shearing in Art. 115. Then the external work equated to the sum of these, gives

$$(24) \qquad \Delta = \int \frac{M^2dx}{PEI} + \int \frac{V^2dx}{PE_sA},$$

in which M is the bending moment and V the vertical shear at any section distant x from the origin. To apply this to a particular case, M and V are to be expressed in terms of x and the integration extended over the entire length of the beam.

For example, let a simple beam of span l have a load P at the middle. Then $M = \frac{1}{2}Px$ and $V = \frac{1}{2}P$. Inserting these and bearing in mind that each integral is equal to twice the value between the limits 0 and $\frac{1}{2}l$, there is found,

$$\Delta = \frac{Pl^3}{48EI} + \frac{Pl}{4E_sA},$$

which is the deflection under the load. The second term here gives the deflection due to the vertical shears. By placing

$A = I/r^2$, where r is the radius of gyration, this becomes

$$\Delta = \frac{Pl^3}{48EI}\left(1 + 12\frac{Er^2}{E_s l^2}\right),$$

which is the formula for deflection, taking into account the effect of the vertical shears. For long beams the deflection due to shearing is scarcely appreciable; for short beams, however, it may be larger than that due to the bending moment.

Attention was first called to the inaccuracy of the ordinary formula for deflection in the case of short beams in a paper by NORTON read before the American Association for the Advancement of Science in 1870. A number of experiments on white pine beams of different lengths and sizes were made, and it was shown that the deflections were directly proportional to the loads and inversely proportional to the breadth of the beam, as the common formula requires. The deflections were, however, not proportional to the cubes of the spans nor to the cubes of the depths of the beams, as the formula requires. An examination into the reason of these discrepancies showed that it was due to influence of the vertical shears, and NORTON deduced the formula

$$\Delta = \frac{Pl^3}{4Ebd^3} + C\frac{Pl}{bd}$$

as applicable to beams of breadth b and depth d, where C was a constant whose value he did not theoretically determine. From one series of experiments he found for the white pine beams

$$E = 1\,428\,000, \quad C = 0.0000094,$$

and using these values the deflections for other series computed from the formula agreed very well with those observed.

From the theoretic formula for Δ deduced above, it is seen that NORTON'S value of C is $1/4\,E_s$, or

$$E_s = \frac{1}{4C} = 266\,000 \text{ pounds per square inch,}$$

which should be the coefficient of elasticity for the shearing of white pine across the grain. This is probably not far from the actual value of that coefficient, since THURSTON, by experiments on torsion, found $E_s = 220\,000$ pounds per square inch for white pine. The experiments of NORTON, therefore, confirm the theoretic formula above deduced for the true deflection of a simple beam loaded at the middle.

When a uniform load extends over the beam and it is desired to find the deflection at a particular point, let M and V be the bending moment and vertical shear due to the uniform load, and M' and V' the bending moment and vertical shear due to a load P' placed at the particular point. Then the deflection at that point is given by

$$(24)' \qquad \Delta = \int \frac{M'Mdx}{P'EI} + \int \frac{V'Vdx}{P'E_sA},$$

in which both bending moments and vertical shears are to be expressed in terms of x, and the integration extended over the entire length of the beam. For instance, let a simple beam have the uniform load wl, and let it be required to find the deflection at the middle. Here $M' = \frac{1}{2}P'x$, $M = \frac{1}{2}w(lx - x^2)$, $V' = \frac{1}{2}P'$, and $V = \frac{1}{2}wl - wx$. Inserting and integrating between the limits 0 and l, there results

$$\Delta = \frac{5wl^4}{384EI} + \frac{wl^2}{8E_sA},$$

which is the deflection at the middle, the first term being the deflection due to the horizontal stresses of tension and compression and the second that due to the vertical stresses of shearing. The ratio of the second term to the first is seen to be slightly greater than in the case of a single load at the middle.

Prob. 162. Prove formula (24)′ by considering that the work due to the imaginary load P' is equal to the stress caused by that load multiplied by the deformation caused by the stress due to the total uniform load.

ART. 117. FLEXURE AND COMPRESSION.

Let a beam be subject to flexure by transverse loads and also to a compression in the direction of its length. If the longitudinal compression be not large the combined maximum stress due to flexure and compression may be computed by the approximate method of Art. 74. It is clear, however, that if the compression be large the deflection Δ will be increased, and hence the effective bending moment and maximum fiber stresses will be greater than given by that method. A closer approximation will now be established.

Let P be the longitudinal compressive force and M the bending moment of the flexural forces. Let M_1 be the actual bending moment for the section where the deflection is Δ; this is greater than M, on account of the moment $P\Delta$ of the force P, or $M_1 = M + P\Delta$. Now the maximum fiber unit-stress S_1 which results from this moment M_1 is, from (4),

$$S_1 = \frac{M_1c}{I} = \frac{(M + P\Delta)c}{I},$$

where I is the moment of inertia of the cross-section and c the distance from the neutral axis to the remotest fiber. The value of Δ may be expressed in terms of S_1 regarding Δ to vary with S_1 in the same manner as for a beam subject to no longitudinal compression. Inserting then for Δ its value from Art. 37, and solving for S_1, gives

$$S_1 = \frac{Mc}{I - \frac{nPl^2}{mE}},$$

where n and m are numbers depending upon the arrangement of the ends and the kind of loading. This formula was first deduced by J. B. JOHNSON, who regarded m/n as 10 for all kinds of loading. Art. 37 shows, however, that m/n depends on the arrangement of the ends as well as on the load; for a simple beam uniformly loaded $m/n = 9.6$, and for a load at the middle $m/n = 12$.

The maximum compressive unit-stress on the concave side of the beam is $S = S_1 + P/A$. For example, let a wooden beam 8 feet long, 10 inches wide, and 9 inches deep be under a compression of 40 000 pounds, while at the same time it carries a total uniform load of 4000 pounds. Here $M = \frac{1}{8}Wl = 48\,000$ pound-inches, $c = 4\frac{1}{2}$ inches, $I = \frac{1}{12}bd^3 = 607\frac{1}{2}$ inches^4, $l = 96$ inches, $P = 40\,000$ pounds, $n = 8$, $m = \frac{384}{5}$, and $E = 1\,500\,000$ pounds per square inch. Inserting these values in the formula the flexural stress S_1 is found to be 371 pounds per square inch. The compressive unit-stress due directly to P is $P/A = 40\,000/90 = 444$, so that the total stress $S = 371 + 444 = 815$ pounds per square inch.

While the above method is better than that of Art. 74, it is not exact, and gives in general values of S which are too small. The exact method of dealing with combined flexure and compression is by the help of the elastic curve. The results will now be developed for the most common case.

Let a simple beam of span l be uniformly loaded with w per linear unit, and at the same time be under the longitudinal compression P. The bending moment for any point whose co-ordinates are x and y is,

$$M = \tfrac{1}{2}wlx - \tfrac{1}{2}wx^2 + Py,$$

and the differential equation of the elastic curve is,

$$EI\frac{d^2y}{dx^2} = -\tfrac{1}{2}wlx + \tfrac{1}{2}wx^2 - Py,$$

where the negative sign of the bending moment is taken because the curve is concave to the axis of x. By two integrations results

$$y = -\frac{wlx}{2P} + \frac{wx^2}{2P} + \frac{wEI}{P^2}\left(\frac{\cos \beta(x - \frac{1}{2}l)}{\cos \frac{1}{2}\beta l} - 1\right),$$

in which β, as in Art. 117, is an abbreviation for $(P/EI)^{\frac{1}{2}}$, or

$$\tfrac{1}{2}\beta l = \frac{l}{2}\sqrt{\frac{P}{EI}},$$

is an arc expressed in terms of the radius as unity. In this equation of the elastic curve let $x = \frac{1}{2}l$, then $y = \Delta$, and

$$\Delta = -\frac{wl^2}{8P} + \frac{wEI}{P^2}(\sec \tfrac{1}{2}\beta l - 1).$$

Inserting this in the expression for S_1, there is found

$$S_1 = \frac{wcE}{P}(\sec \tfrac{1}{2}\beta l - 1),$$

which is an exact expression for the maximum compressive flexural unit-stress. Lastly, $S_1 + P/A$ is the total unit-stress S due to the combined flexure and compression.

To illustrate this method let the data of the above numerical example be again used. Here $w = 4000/96$ pounds per linear inch, and the other quantities as before. The arc $\frac{1}{2}\beta l$ is found to be 0.318, and the corresponding angle is 18° 13′, whence $\sec \frac{1}{2}\beta l = 1.0545$. Then from the formula the flexural unit-stress S_1 is 416 pounds per square inch. Lastly, the total unit-stress is $416 + 444 = 860$ pounds per square inch. A comparison for this numerical example shows that the rough method of Art. 74 gives 799, and the method of JOHNSON 815, while the exact method gives 860 pounds per square inch for the maximum unit-stress.

If $w = 0$ in the above formulas the case reduces to that of a column where $\sec \frac{1}{2}\beta l = \infty$, and hence both Δ and S_1 are indeterminate. If on the other hand $P = 0$, the case is that of a simple beam uniformly loaded, and it may be shown that Δ and S_1 will reduce to the expressions for that case.

Prob. 163. Show, by the calculus method of evaluating indeterminate quantities, that the statement in the last sentence is correct.

Prob. 164. Let a simple wooden beam 32 feet long, 9 inches wide, and 10 inches deep, carry a total uniform load of 2000 pounds while at the same time it is under a longitudinal compression of 9000 pounds. Compute the maximum unit-stress S by the three methods.

ART. 118. FLEXURE AND TENSION.

Let a beam be subject to flexure by transverse loads and then to a tension in the direction of its length. The effect of the tension is to decrease the deflection, and thus also the tensile flexural stress. If M be the bending moment of the transverse loads, and M_1 that of the combined flexure and tension, then $M_1 = M - P\Delta$. Let S_1 be the resulting unit-stress on the fiber most remote from the neutral surface; then, JOHNSON'S formula in the last article gives S_1, if the minus sign in the denominator be changed to plus. Finally, $S_1 + P/A$ is the total unit-stress on the convex side of the beam resulting from the combined flexure and tension.

As an example take a steel eye-bar 18 feet long, 1 inch thick, and 8 inches deep, under a longitudinal tension of 80 000 pounds, E being 29 000 000 pounds per square inch. The weight of the bar is 490 pounds, and $M = \frac{1}{8} \times 490 \times 18 \times 12 = 13\,230$ pound-inches. Also $c = 4$ inches, $I = 42.67$ inches^4, $m/n = 9.6$, $P = 80\,000$ pounds, $l = 216$ inches. Then the maximum flexural tensile stress S_1 is 943 pounds per square inch. Finally, the total tensile stress is $S = 943 + 10\,000 = 10\,943$ pounds per square inch.

As in the last article, a more accurate way of dealing with this case is by use of the general equation of the elastic curve. The expression for the bending moment is,

$$M = \tfrac{1}{2}wlx - \tfrac{1}{2}wx^2 - Py,$$

which is the same as before, except in the sign of P. Hence by changing the sign of P in the expressions for Δ and S_1, they apply to the case of combined flexure and tension. In doing this $\frac{1}{2}\beta l$ becomes $\frac{1}{2}\beta l\sqrt{-1}$, and this changes the circular secant to the hyperbolic secant; thus,

$$\Delta = + \frac{wl^2}{8P} + \frac{wEI}{P^2}(\text{sech}\,\tfrac{1}{2}\beta l - 1)$$

is the deflection of the beam, and

$$S_1 = \frac{wcE}{P}(1 - \text{sech}\,\tfrac{1}{2}\beta l)$$

is the unit-stress due to the flexure. Finally $S_1 + P/A$ is the total unit-stress due to the combined flexure and tension.

Since many students are unacquainted with hyperbolic functions, it may be here noted that they are closely analogous with the circular functions. Thus for circular functions $\cos^2 + \sin^2 = 1$, but for hyperbolic functions $\cosh^2 - \sinh^2 = 1$. A table of hyperbolic sines, cosines, and tangents is given in "Higher Mathematics" (Wiley & Sons, 1896). In the absence of a table the hyperbolic cosine and secant can be computed from

$$\cosh\theta = \frac{e^\theta + e^{-\theta}}{2}, \quad \text{sech}\,\theta = \frac{2}{e^\theta + e^{-\theta}},$$

where e is the base of the Naperian system of logarithms.

As an example, for the above eye-bar $w = 2.267$ pounds per linear inch, and $\frac{1}{2}\beta l = 0.868 = \theta$; then $\text{sech}\,\theta = 0.714$, and the flexural stress $S_1 = 940$ pounds per square inch. Lastly, the total unit-stress $S = 940 + 10\,000 = 10\,940$ pounds per square inch, which differs but little from the result found before.

Prob. 165. Compute the deflection Δ for the above eye-bar before and after the tension is applied.

Prob. 166. Show that the deflection of a cantilever beam loaded at the end with W and under the longitudinal tension P is $\Delta = WP^{-1}(l - \beta^{-1}\tanh\beta l)$, where $\beta = (P/EI)^{\frac{1}{2}}$.

CHAPTER XII.

SHEAR AND TORSION.

ART. 119. STRESSES CAUSED BY SHEAR.

It is shown in Art. 8, and also in Art. 75, that forces of tension or compression acting upon a body produce not only internal tensile or compressive stresses, but also internal shearing stresses. Conversely, an external shear acting upon a body produces in it not only internal shearing stresses, but also internal tensile and compressive stresses.

For example, the rectangle *ABCD* in the web of a plate girder, shown in Fig. 67, may be considered. Let V be the shear at the sections AB and CD, which are taken very near together so that the weight in the rectangle itself can be disregarded. This vertical shear or couple must be accompanied by a horizontal shear V_1, which in this case is caused by the resistance of the flange rivets. Let the thickness of the material be one unit; then if S and S_1 be the shearing unit-stresses,

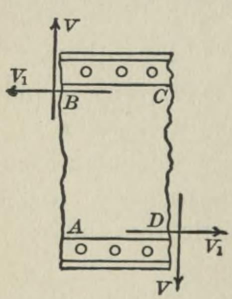

FIG. 67.

$$S = \frac{V}{AB}, \quad S_1 = \frac{V_1}{AD},$$

and it is now to be shown that S and S_1 are equal. Taking either A or D as a center of moments, the equation of moments is,

$$V \times AD = V_1 \times AB,$$

and hence by division

$$\frac{V}{AB} = \frac{V_1}{AD}, \quad \text{or} \quad S = S_1,$$

that is, the shearing unit-stresses on adjacent sides of the rectangle are equal. This is without regard to the weight of the rectangle itself, which will cause a slight modification, because the V on the left will then be greater than the V on the right. But if AD be very small the conclusion is strictly true that the vertical shearing unit-stress is equal to the horizontal shearing unit-stress (Art. 79).

The resultant of V and V_1 acts as a tension on the diagonal BD and as a compression on the diagonal AC, thus tending to deform the rectangle into a rhombus. The maximum value of this resultant will be when $V_1 = V$, that is, when the rectangle is a square. The resultant tension or compression then acts at an angle of 45 degrees with the length of the beam, and its value is $V\sqrt{2}$. The tensile or compressive unit-stress is obtained by dividing $V\sqrt{2}$ by the area normal to its direction, or

$$\frac{V\sqrt{2}}{AB\sqrt{2}} = \frac{V}{AB} = S;$$

that is, the tensile or compressive unit-stress caused by shear is equal to the shearing unit-stress.

This may also be proved from the discussion in Art. 75. Thus in formula (13) let $p = 0$; then max. $t = \pm v$, which is the same result. The action of tension or compression on a bar produces a shearing unit-stress equal to one half the tensile or compressive unit-stress, but the action of a shear produces tensile and compressive unit-stresses equal to the shearing unit-stress itself. This may be regarded as a most fortunate arrangement in view of the fact that the shearing strength of materials is usually less than the tensile or compressive strength.

Prob. 167. A square bar is subject to a tension of 6000 pounds per square inch in the direction of its length and to a lateral compression of 2000 pounds per square inch on two opposite sides. Show that the maximum shearing unit-stress in the bar is 4000 pounds per square inch.

ART. 120. RESILIENCE UNDER SHEAR.

The resilience of a body under the action of shear is governed by similar laws to that of tension and flexure, namely, it is proportional to the square of the maximum unit-stress and to the volume of the body. Thus in Fig. 68 let a shearing unit-stress S_s act upon a parallelopiped of length l and cross-section A, deforming it into a rhombus and causing each right angle on the side to be increased or decreased by the amount ϕ. The external work done by the total shear V, supposing it be gradually applied, is $\frac{1}{2}Vl \tan \phi$, or since ϕ is a very small angle, simply $\frac{1}{2}Vl\phi$, if ϕ be expressed in circular measure. This is equal to the internal work or resilience of all the shearing stresses. But $V = AS_s$, and if E_s be the coefficient of elasticity for shearing, then $E_s = S_s\phi$ (Art. 4). Therefore

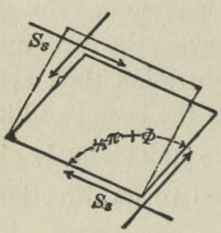

FIG. 68.

$$K = \frac{1}{2}Vl\phi = \frac{1}{2}\frac{S_s^2}{E_s} \cdot Al$$

is an expression for the work of shearing. The first factor $S_s^2/2E_s$ may be called the modulus of resilience for shearing, in analogy with that for tension (Art. 94), and the second factor Al is the volume of the body. This expression, however, is only valid when the stress S_s is within the elastic limit of the material.

Practically a shear cannot act upon a body of any considerable size without causing flexure or torsion, and thus only a part of the external work will be expended in the internal work of shearing. Hence the above rule for resilience under shear is of limited application, unless the effect of the accompanying flexure be considered. In the case of long beams the work of shearing is indeed but a small part of that due to the flexure (Art. 115).

The ultimate resilience of shearing is far more difficult to estimate than that of tension or flexure. It can, however, be experimentally determined by the power required to punch a hole through a plate, although even in this case some of the applied work is lost in heat and friction.

Prob. 168. Estimate the horse-power required to punch 50 holes per minute in a wrought-iron plate $\frac{3}{4}$ inch thick, the diameter of the holes being 2 inches.

ART. 121. THE COEFFICIENTS OF ELASTICITY.

The coefficient of elasticity for shearing has a certain relation to the coefficient of elasticity for tension, which will now be deduced. Let *abcd* represent one side of a cube which under the tensile unit-stress S is elongated into the parallelopiped

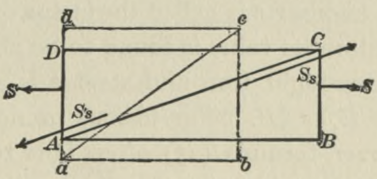

FIG. 69.

ABCD, the length *ab* being increased to *AB*, and the breadth *ad* being decreased to *AD*. The ratio of the lateral decrease to the longitudinal increase is designated by ϵ, a mean value of which for iron and steel is $\frac{1}{3}$, as already mentioned in Art. 71. The distance $AB - ab$ is the unit-elongation s, and the distance $ad - AD$ is the lateral unit-contraction ϵs.

The distortion of the square into the parallelogram may be regarded as caused by the shearing stresses acting along the two diagonals AC and BD. The angle cad, originally $\frac{1}{4}\pi$, is changed into CAD, which is $\frac{1}{4}\pi - \frac{1}{2}\phi$; the total change of angle between the two diagonals of $abcd$ being the distortion ϕ due to shearing. Now

$$\tan(\tfrac{1}{4}\pi - \tfrac{1}{2}\phi) = \frac{BC}{AB} = \frac{1 - \epsilon s}{1 + s},$$

and since ϕ is very small, the value of $\tan(\frac{1}{4}\pi - \frac{1}{2}\phi)$ is $1 - \phi$ very nearly. Hence

$$1 - \phi = \frac{1 - \epsilon s}{1 + s}, \qquad \text{or,} \qquad \phi = (1 + \epsilon)s,$$

since $s\phi$ is a quantity of the second order and can be neglected. Lastly, replacing for ϕ its value S_s/E_s, and for s its value S/E, and remembering that $S_s = \frac{1}{2}S$, as shown in Art. 7, there is found the important formula,

$$(25) \qquad E_s = \frac{E}{2(1 + \epsilon)},$$

which gives the coefficient of elasticity for shearing in terms of the coefficient of elasticity for tension.

The abstract number ϵ is called the factor of lateral contraction. For cast iron its value is found to be about $\frac{1}{4}$, and thus $E_s = \frac{2}{5}E$. For wrought iron and steel ϵ is about $\frac{1}{3}$, and for these materials $E_s = \frac{3}{8}E$. For fibrous or non-homogeneous materials, however, formula (25) often fails to apply, for the reason that E is not the same in all directions as it is in a homogeneous body. Using the mean values of E given in Art. 80, the mean values of the coefficients of elasticity for shearing are,

For cast iron,	$E_s =$ 6 000 000,
For wrought iron,	$E_s =$ 9 400 000,
For steel,	$E_s =$ 11 200 000,

all being in pounds per square inch. By means of experiments on the torsion of shafts (Art. 66) these values have been verified.

Prob. 169. Prove that tan $(\frac{1}{4}\pi - \frac{1}{2}\phi)$ is $1 - \phi$ very nearly, when ϕ is small.

Prob. 170. A cast-iron shaft 60 inches long and 2 inches in diameter is twisted through an angle of 7 degrees by a force of 2500 pounds acting at 12 inches from the center, and on the removal of the force springs back to its original position. Compute the factor of lateral contraction ϵ.

ART. 122. RESILIENCE UNDER TORSION.

When a shaft is twisted by a force P acting with a lever arm p, as in Fig. 49 of Art. 63, each element of the cross-section is subject to a shearing unit-stress S. The stress being slowly developed, the internal work, or resilience, is equal to $\frac{1}{2}S$ multiplied by its displacement ϕ, or if E_s denote the coefficient of elasticity for shearing, the work of any elementary area a and length dx is

$$dK = \tfrac{1}{2}S\phi \,.\, adx = \frac{1}{2}\frac{S^2}{E_s}\, adx.$$

Now let S_s be the shearing unit-stress at the part of the cross-section most remote from the axis, and let c be its distance from that axis; also let z be the distance of S from the axis. Then $S = S_s\frac{z}{c}$, and

$$dK = \frac{1}{2}\frac{S_s^2}{E_s}\cdot\frac{az^2}{c^2}\,.\,dx$$

is an expression for the internal work. To integrate this over the entire volume of the shaft all cross-sections being similar, Σaz^2 is the polar moment of inertia J, and Σdx is the length of the shaft l. Thus

$$(26) \qquad K = \frac{1}{2}\frac{S_s^2}{E_s}\cdot\frac{J}{c^2}l = \frac{1}{2}\frac{S_s^2}{E_s}\cdot\frac{r^2}{c^2}Al,$$

where r is the polar radius of gyration of the cross-section defined by $J = Ar^2$. Now Al is the volume of the shaft, and it is thus seen that the resiliences of shafts of similar cross-sections are proportional to their volumes. Hence the resilience of a shaft under torsion is governed by laws similar to those of a beam under flexure (Art. 96).

The formula here established is only valid when the greatest unit-stress S_s does not surpass the elastic limit for shearing. When S_s corresponds to the elastic limit, the quantity $\frac{1}{2}\frac{S_s^2}{E_s}$ may be called the modulus of resilience for torsion or shearing, in analogy to the modulus of resilience for tension or compression (Art. 94).

As an example, let it be required to find the work necessary to strain a steel shaft 12 inches in diameter and 30 feet long up to its elastic limit, supposed to be 30 000 pounds per square inch. Here $S_s = 30\,000$ and $E_s = 11\,200\,000$ pounds per square inch (Art. 121); also, $c = 6$ inches, $A = 113.1$ square inches, $J = \frac{1}{32}\pi d^4 = \frac{1}{8}Ad^2 = 2036$ inches^4, $l = 360$ inches. Inserting all values, K is found to be 818 000 inch-pounds or 68 200 foot-pounds. Thus to produce this stress in the shaft in one minute more than 2 horse-powers are required.

Prob. 171. Compare the resilience of a square shaft and a round shaft, the cross-sections and lengths being equal.

ART. 123. HOLLOW AND SOLID SHAFTS.

It was mentioned in Art. 69 that a hollow shaft is stronger than a solid shaft of the same sectional area. A general comparison will now be made with respect to strength, stiffness, and resilience. Let A be the area of the cross-section in both cases, let D be the outer and d the inner diameter of the hollow shaft; then $A = \frac{1}{4}\pi(D^2 - d^2)$, and the diameter of the solid shaft is $d_1 = \sqrt{D^2 - d^2}$.

The strength of a shaft under torsion is measured by the twisting moment it can carry under a given unit-stress, and by

Art. 64 this is seen to vary as J/c, or as its polar moment of inertia divided by its radius. Hence for the hollow shaft

$$\frac{J}{c}=\frac{\pi(D^4-d^4)}{16D}=\frac{A(D^2+d^2)}{4D},$$

and for the solid shaft

$$\frac{J}{c}=\frac{\pi d_1^4}{16d_1}=\frac{A\sqrt{D^2-d^2}}{4}.$$

Therefore, dividing the first of these by the second, and letting k denote the value of D/d, there results

$$\frac{\text{hollow}}{\text{solid}}=\frac{k^2+1}{k\sqrt{k^2-1}},$$

which is the ratio of the strength of a hollow shaft to a solid one of the same sectional area.

The stiffness of a shaft under torsion is measured by the twisting moment it can carry with a given angle of torsion. As seen in Art. 66, this angle is

$$\theta=\frac{S_sl}{E_sc}=\frac{Ppl}{E_sJ},$$

and hence the stiffness varies directly as the polar moment of inertia. For the hollow shaft

$$J=\tfrac{1}{32}\pi(D^4-d^4)=\tfrac{1}{8}A(D^2+d^2),$$

and for the solid shaft

$$J=\tfrac{1}{32}\pi d_1^4=\tfrac{1}{8}A(D^2-d^2).$$

Therefore, dividing the first by the second, and designating the quantity D/d by k,

$$\frac{\text{hollow}}{\text{solid}}=\frac{k^2+1}{k^2-1},$$

which is the ratio of the stiffness of a hollow shaft to a solid one of the same sectional area.

The resilience of a shaft is measured by the work required to produce a given unit-stress. From (26) of the last Article this is seen to vary with J/c^2. Thus for the hollow shaft

$$\frac{J}{c^2} = \frac{\pi(D^4 - d^4)}{8D^2} = \frac{A(D^2 + d^2)}{2D^2},$$

and for the solid shaft

$$\frac{J}{c^2} = \frac{\pi d_1^4}{8d_1^2} = \tfrac{1}{2}A.$$

Dividing the first by the second, or putting D/d equal to k,

$$\frac{\text{hollow}}{\text{solid}} = \frac{k^2 + 1}{k^2},$$

which is the ratio of the resilience of a hollow shaft to a solid one of the same sectional area.

In practice the outer diameter is often about twice the inner diameter. For this case $k = 2$, and the above formulas show that a hollow shaft has 1.44 times the strength, 1.67 times the stiffness, and 1.25 times the resilience of a solid shaft of the same sectional area. These conclusions are valid only when the maximum stress is within the elastic limit of the material; for higher stresses a theoretic comparison cannot be satisfactorily made.

Shafts of the same material and length are of the same strength when their values of J/c are equal, of the same stiffness when their values of J are equal, and of the same resilience when their values of J/c^2 are equal. It is thus easy to show that the percentage of weight saved by using a hollow shaft instead of a solid one is

For equal strength, $$100\left(1 - \sqrt[3]{\frac{k^2(k^2 - 1)}{(k^2 + 1)^2}}\right),$$

For equal stiffness, $$100\left(1 - \sqrt{\frac{k^2 - 1}{k^2 + 1}}\right),$$

For equal resilience, $$\frac{100}{k^2 + 1},$$

in which k is the ratio of the outer to the inner diameter of the hollow shaft. See a paper by R. W. DAVENPORT in the Transactions of the Society of Naval Architects and Marine Engineers for 1893 (reprinted in *Engineering News*, Nov. 23, 1893) for a discussion of the practical advantages of hollow shafts made of steel having a high elastic limit.

Prob. 172. Compare the strength of a solid shaft 13 inches in diameter with that of a hollow shaft with outer diameter 17 inches and inner diameter 11 inches, the elastic strengths being 30 000 and 50 000 pounds per square inch respectively.

ART. 124. SHAFT COUPLINGS.

At *A* and *B* in Fig. 70 are shown the end and side views of a flange coupling for a shaft, the flanges being connected by bolts. These bolts in transmitting the torsion from one flange to another are subject to shearing stress, and they must be

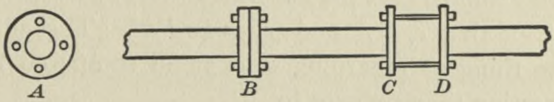

FIG. 70.

of sufficient strength to safely carry it. This shear differs from that in the main body of the shaft only in intensity, and it is the greatest upon the side of the bolt most remote from the axis.

Let J be the polar moment of inertia of the cross-section of a solid shaft, c its radius, and S_s the shearing unit-stress on the outer surface. Let J_1 be the polar moment of inertia of the cross-section of the bolts, c_1 the distance from the axis of the shaft to the side of the bolts farthest from the axis, and let the shearing unit-stress on that side be the same as that on the outer surface of the shaft. Then in order that the bolts may be equal in strength to the shaft it is necessary that J/c should equal J_1/c_1. The polar moment of inertia of the cross-section of one bolt with respect to the axis of the shaft is equal to its polar moment with respect to its own axis plus the area of the cross-section into the square of the distance between the two axes.

Let D be the diameter of the shaft, d the diameter of each of the bolts, h the distance of the center of a bolt from the axis of the shaft, and n the number of bolts; then

$$\frac{J}{c} = \tfrac{1}{16}\pi D^3, \qquad \frac{J_1}{c_1} = n\frac{\tfrac{1}{32}\pi d^4 + \tfrac{1}{4}\pi d^2 . h^2}{\tfrac{1}{2}d + h}$$

and equating these values there is found

$$D^3(d + 2h) = nd^2(d^2 + 8h^2),$$

which is the necessary relation between the quantities in order that the bolts may be equal in strength to the shaft, provided the material be the same.

This formula is an awkward one for determining d, and hence it is often assumed that the shear is uniformly distributed over the bolts, or that $c_1 = h$ and $J_1 = \frac{1}{4}\pi d^2 h^2$. This amounts to the same thing as regarding d as small compared to h, and the expression then reduces to

$$D^3 = 4nd^2h, \qquad \text{or} \qquad d = \tfrac{1}{2}\sqrt{\frac{D^3}{nh}}.$$

In practice the bolts are often made a little larger in diameter than this formula requires.

The above supposes the shaft to be solid. If it be hollow with outer diameter D and inner diameter d_1, the D^3 in the above expressions is to be replaced by $D^3 - d_1^4/D$, if d_1 be large enough to have any influence.

The case shown at CD in Fig. 70 is one that would not occur in practice, but it is here introduced in order to indicate that the bolts would be subject to a flexural as well as a shearing stress. It is clear that the flexural stress will increase with the length of the bolts, and that they should be greater in diameter than for the case of pure shearing. The flexural stress will also depend upon the work transmitted by the shaft. This case will be investigated in Art. 126 in connection with the discussion of the pin of a crank shaft.

Prob. 173. A solid shaft 6 inches in diameter is coupled by bolts $1\frac{1}{4}$ inches in diameter with their centers 5 inches from the axis. How many bolts are necessary?

Prob. 174. A hollow shaft 17 inches in outer and 11 inches in inner diameter is to be coupled by 12 bolts placed with their centers 20 inches from the axis. What should be the diameter of the bolts?

ART. 125. A CRANK PIN AND SHAFT.

A crank pin, *CD* in Fig. 71, is subject to a pressure *W* from the connecting rod which is uniformly distributed over nearly its entire length. This pressure varies at different positions in the stroke of the engine, but for ordinary computations may be taken at from 10 to 20 per cent greater than the total mean pressure on the steam piston; to this may be added the weight of the connecting and piston rods in case these should be vertical in position.

This maximum pressure *W* causes a cross-shear in the crank pin at the section *C*, and it also causes a flexural stress at *C* due to a uniform load over the cantilever *CD*. These may be computed by the methods of Chapter III, and their combined influence can be determined by formula (13) of Art. 75. The compressive or bearing stress on the surface of the pin is usually also to be considered, this being estimated per square unit of diametral area.

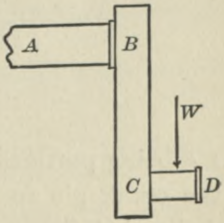

FIG. 71.

Owing to the constant alternation of these stresses as the crank arm revolves, the allowable working unit-stresses should be taken low in designing the pin, arm, and journal bearing. The crank arm is under flexure as a cantilever loaded at the end, while the part of the shaft *AB* which rests in the journal is subject to combined shearing, flexure, and torsion. The methods of Chapters VI and VII give all required to thoroughly discuss these cases. See UNWIN'S Elements of Machine Design for the special formulas generally used in practice.

Prob. 175. The crank *CD* in Fig. 71 is 8 inches long and 4 inches in diameter, the maximum pressure *W* being 60 000 pounds. Compute the bearing unit-stresses, the shearing unit-stress and the flexural unit-stress. Compute the maximum unit-stress due to combined shear and flexure.

ART. 126. A TRIPLE-CRANK PIN.

Double and triple cranks are used when several engines are to be attached to the same shaft, as is usual in ocean steamers. With the triple arrangement the cranks are set at angles of 120 degrees with each other, thus securing a uniform action upon the shaft. Fig. 72 shows one of these cranks, *AB* and

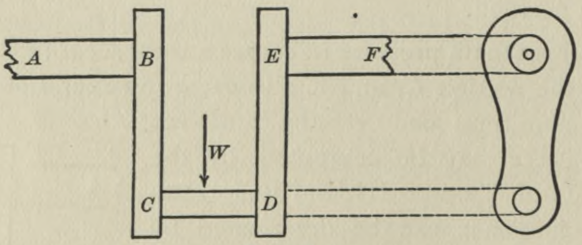

FIG. 72.

EF being portions of the shaft resting in journal bearings, *CD* the crank pin to which the connecting rod is attached, while *BC* and *DE* are the crank arms or webs which are usually shrunk upon the shaft and pins.

The complete investigation of the maximum stresses in such a crank shaft and pin is one of much difficulty. A brief abstract of such an investigation will, however, here be given for the crank pin. There are three cranks, and the one to be considered is the nearest to the propeller, so that the torsion from the other two cranks is transmitted through the pin *CD*. This steel crank pin is hollow, 18 inches in outer diameter and 6 inches in inner diameter, its length between webs being 24 inches, the thickness of each web 12 inches, and the distance from the axis of the shaft to the center of the pin being 30 inches. The three engines transmit 7200 horse-power to the shaft *EF*, of which 4800 horse-power is transmitted through the shaft *AB* and through the crank pin *CD*. The maximum pressure *W* brought by the connecting rod upon the crank pin is 156 000 pounds. It is required to determine the stresses when the crank makes 80 revolutions per minute.

The pressure *W* is distributed over about 17 inches of the length of the pin, so that the bearing compressive stress on the diametral area is

$$S_1 = \frac{156\,000}{17(18-6)} = 820 \text{ pounds per square inch,}$$

which is a low and safe value.

The shearing stress due to *W*, taken as uniformly distributed over the cross-section of the pin, is

$$S_2 = \frac{78\,000}{0.7854(18^2 - 6^2)} = 620 \text{ pounds per square inch,}$$

which is low, but will be much increased by the other stresses acting on the end section.

The shearing stress due to the horse-power transmitted through *BC* has its greatest value on the side of the pin farthest from the axis. The twisting moment *Pp* due this 4800 horse-power is found, from the first equation on page 141 of Art. 67, to be

$$Pp = \frac{198\,000 \times 4800}{3.1416 \times 80} = 3\,782\,000 \text{ pound-inches,}$$

and this is equal to the resisting moment of the crank pin, or to SJ_1/c_1, in which J_1 is the polar moment of inertia of the cross-section with respect to the axis of the shaft and c_1 is the distance from that axis to the side of the pin where the stress S is to be found. Now, $c = 30 + 9 = 39$ inches, and then $J_1 = 0.7854(18^4 - 6^4) + 0.7854(18^2 - 6^2) \times 39^2$. Hence, from formula (11) of Art. 64,

$$S_3 = \frac{Pp_1c_1}{J_1} = \frac{3\,782\,000 \times 39}{213\,500} = 690 \text{ pounds per square inch,}$$

which is the maximum shearing stress due to the transmitted power.

The flexure of the pin due to the torsion carried through it falls under a case not heretofore considered, except in the brief mention in Art. 124. The twisting moment Pp is equivalent to a force P acting at a distance of 30 inches from the shaft and normal to the crank arms; the value of P is

$$P = \frac{3\,782\,000}{30} = 126\,100 \text{ pounds,}$$

and this produces a bending moment in the pin which may be taken as a beam fixed at both ends while P acts in opposite directions at those ends. Hence there is a bending moment M' at each end, opposite in sign but equal in value, and the moment at any section is $M = M' + Px$; but when $x = l$ the value of M is $-M'$ and therefore $M' = \pm \frac{1}{2}Pl$, which is the maximum bending moment. Thus from (4) of Art. 21

$$S_4 = \frac{M'c}{I} = \frac{63\,000 \times 18 \times 9}{\frac{1}{64}\pi(18^4 - 6^4)} = 2060,$$

which is the flexural stress in pounds per square inch.

All of these stresses are light, but the pin is necessarily made heavier than they would require on account of the additional stresses due to the shrinking of the web upon the pin. The data here given are not sufficient to determine these with exactness, but there is a radial compressive unit-stress S_5 brought by the web upon the pin of probably 3000 pounds per square inch, and this is accompanied by a tangential compressive unit-stress S_6 of about 4000 pounds per square inch (Art. 142). These take effect in the fillet of the pin on the inside of the web at *D*, where also all of the other stresses concentrate except S_1. In Art. 134 it will be shown how these several values may be combined in order to obtain the maximum tensile, compressive, and shearing stresses.

Prob. 176. Draw the shear and moment diagram for the lateral flexure of the pin due to the transmitted torsion.

CHAPTER XIII.

APPARENT STRESSES AND TRUE STRESSES.

ART. 127. THE MATHEMATICAL THEORY OF ELASTICITY.

In Art. 5 are stated several laws, derived from experiment, which are the foundation of the science of Mechanics of Materials. Of these (A) and (B) relate to elasticity and are the basis of all discussions concerning stresses that do not surpass the elastic strength of the material, the latter being usually referred to as HOOKE'S law (Art. 81). All the theoretic formulas of the preceding pages are derived by the help of this law, and these hence constitute a part of the mathematical theory of elasticity.

This theory is one of vast extent and far-reaching consequences, and its full development would require volumes. It includes not only the complete investigation of the stresses and deformations produced in every part of a body by gradually applied exterior forces, but also those arising under conditions of impact. It deals not only with elastic solids, but with fluids, gases, and the æther of space. The discussion of stresses and deformations, both in homogeneous and crystalline bodies, leads to the investigation of wave propagations, the time and velocity of elastic oscillations, and numerous other phenomena of physics. In this Chapter will be presented a few of the fundamental principles with reference to homogeneous materials only.

Statics proper is concerned only with rigid bodies, while the theory of elasticity deals with bodies deformed under the action of exterior forces and which recover their original shape on the removal of these forces. All the principles and methods of statics apply in the discussion of elastic bodies, but in addition new principles based upon HOOKE'S law arise. The amount of deformation being small within the elastic limit for common materials, it is allowable to neglect the squares and higher powers of a unit-elongation in comparison with the elongation itself. Thus if l be the length of a bar, which under the action of stress is increased to the length $l(1 + s)$, the square of this new length may be taken as $l^2(1 + 2s)$ and the cube as $l^3(1 + 3s)$. For a substance like india rubber, where this assumption does not apply, some of the conclusions of the theory of elasticity are not necessarily valid.

Another axiom derived from experience is the following: Under tensile forces the volume of a body is increased and under compressive forces it is diminished. This is the case for the common materials, although bodies may exist for which it is not true. Thus if a cube be compressed upon two opposite faces the edges parallel to the forces are decreased while those at right angles to the forces are increased in length ; on the whole, however, the volume of the cube is slightly decreased.

The student should consult the article on Elasticity by KELVIN in the Encyclopædia Britannica, as also the History of TODHUNTER and PIERSON. The works of CLEBSCH (Elasticität fester Körper, 1862), WINKLER (Elasticität und Festigkeit, 1867), GRASHOF (Theorie der Elasticität und Festigkeit, 1878), and FLAMANT (Résistance des Matériaux, 1886) may be mentioned as treating the subject both from the theoretical and the engineering point of view.

Prob. 177. Consult TODHUNTER and PIERSON'S History of the Theory of Elasticity and of the Strength of Materials, and ascertain something about the investigations of SAINT VENANT.

ART. 128. LATERAL DEFORMATION.

It has already been noted, in Arts. 71 and 121, that a bar under tension not only elongates in the direction of its length, but is subject to a lateral contraction. So if a bar be under compression there occurs a longitudinal contraction and a lateral elongation. If P be the applied force and A the area of the cross-section the unit-stress is $P/A = S$; if l be the length and λ the longitudinal change of length, then $\lambda/l = s$ is the unit-elongation or unit-contraction. In the case of tension each unit-length of the bar is increased by s, but each unit at right angles to the length is decreased by the amount ϵs, where ϵ is an abstract number less than unity, called the factor of lateral contraction. In the case of compression ϵs is the lateral unit-elongation.

Let a cube in the interior of a bar under tension have each of its sides unity before the application of the stress; under this tension the cube is deformed so that its sides are $1 + s$, $1 - \epsilon s$, and $1 - \epsilon s$. The volume of the deformed body is, if the squares and cubes of s be neglected in comparison with the first power,

$$(1 + s)(1 - \epsilon s)^2 = 1 + (1 - 2\epsilon)s.$$

Now in order that the volume may be increased, $(1 - 2\epsilon)s$ must be positive, or ϵ must be less than $\frac{1}{2}$. When ϵ is 0 no lateral contraction occurs, when $\epsilon = \frac{1}{2}$ the lateral contraction is a maximum. Thus for all bodies whose volume is increased under tension the factor of lateral contraction must lie be tween 0 and $\frac{1}{2}$.

The same reasoning applies in the case of compression, for which as far as known ϵ has the same value as for tension. By precise measurements of bars under stresses within the elastic limit it has been found that ϵ generally lies between 0.2 and 0.4 for wrought iron and steel, a mean value extensively used being $\frac{1}{3}$. For cast iron ϵ is about $\frac{1}{4}$ or a little less.

Prob. 178. A bar of steel 2×2 inches and 6 feet long is pulled by a force of 50 000 pounds. Compute the percentage of increase of length, and the percentage of increase of volume.

ART. 129. TRUE TENSILE AND COMPRESSIVE STRESSES.

If a parallelopiped be subject to a unit-stress S in the direction of its length this is the true stress on all planes normal to its length. In directions at right angles to the length there exists, however, a lateral contraction which implies an internal stress of compression. If s be the unit-elongation due to S, the lateral unit-contraction is ϵs, and this is the same as would be produced by a lateral compressive unit-stress ϵS. Thus it is clear that the deformation at right angles to S is the same as that produced by an actual unit-stress ϵS. In a similar manner if forces act upon all the sides of the parallelopiped the true internal stresses are different from the apparent ones.

All the stresses computed thus far in this volume are apparent stresses, for the influence of lateral deformation has not been taken into account. In this Chapter the letter S will denote the apparent unit-stresses, while the true unit-stresses corresponding to the actual deformations will be designated by T. The true resistance of a body depends upon the actual deformations produced, and these are measured by the true internal stresses.

Let a homogeneous parallelopiped be subject to normal forces of tension or compression upon its six faces, those upon opposite faces being equal. Let the edges of the parallelopiped be designated by 01, 02, 03, as in Fig. 73. Let S_1 be the normal unit-stress upon the two faces perpendicular to the edge 01, and S_2 and S_3 those upon the faces normal to 02

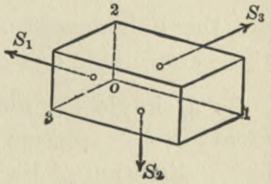

FIG. 73.

and 03; thus the directions of S_1, S_2, S_3 are parallel to 01, 02, 03, respectively. Then, supposing that these stresses are all tensile, and that the factor of lateral contraction ϵ is the same in all directions, the true internal unit-stresses in the three directions are,

$$(27)\qquad \begin{aligned} T_1 &= S_1 - \epsilon S_2 - \epsilon S_3, \\ T_2 &= S_2 - \epsilon S_3 - \epsilon S_1, \\ T_3 &= S_3 - \epsilon S_1 - \epsilon S_2, \end{aligned}$$

in which ϵ lies between 0 and $\frac{1}{2}$, as shown in Art. 128. If any apparent stress S be compressive, it is to be taken as negative in the formulas, and then the true stresses T_1, T_2, T_3 are tensile or compressive, according as their numerical values are positive or negative.

For example, let a cube be stressed upon all sides by the apparent compression S; then the true internal unit-stress T is $S(1 - 2\epsilon)$, or about $\frac{1}{3}S$, and its linear deformation is only about one-third of that due to a compressive stress S applied upon two opposite faces. Again, if a bar have a tension S_1 in the direction of its length, and no apparent stresses upon its sides, then $T_1 = S_1$ while $T_2 = T_3 = -\epsilon S_1$.

The true deformations corresponding to the true internal stresses will be denoted by t_1, t_2, t_3. If the coefficient of elasticity in all directions be E, then

$$t_1 = \frac{T_1}{E}, \quad t_2 = \frac{T_2}{E}, \quad t_3 = \frac{T_3}{E}$$

are the unit-elongations or unit-contractions parallel to the three coördinate axes.

As a simple example, let a steel bar 2 feet long and 3 × 2 inches in cross-section be subject to a tension of 60 000 pounds in the direction of its length and to a compression of 432 000 pounds upon the two opposite flat sides. Here $S_1 = 60\,000/6 = 10\,000$ pounds per square inch, $S_2 = -432\,000/72 = -6000$

pounds per square inch, and $S_3 = 0$. Then from (27), taking ϵ as $\frac{1}{3}$, the true internal stresses are

$$T_1 = +12\,000, \qquad T_2 = -9330, \qquad T_3 = -1330;$$

and it is thus seen that the true tensile unit-stress is 20 per cent greater than the apparent, while the true compressive unit-stress is more than 50 per cent greater than the apparent.

If the parallelopiped in Fig. 73 be subject to the action of oblique stresses R_1, R_2, R_3, each may be resolved into a stress normal to the face and into shearing stresses parallel to the edges. In such a case the true unit-stresses T_1, T_2, T_3 cannot be directly found, but it will be shown in the following articles that three planes can be determined upon which the apparent stresses are wholly normal, and that these are the maximum apparent tensile and compressive stresses due to R_1, R_2, R_3; these being found, formulas (27) are directly applicable.

Prob. 179. In Fig. 73 let a plane be passed through the edge 02 and through the edge diagonally opposite to it. Let the edges 01, 02, 03 be equal in length. Show that the apparent shearing unit-stress on this plane is $\frac{1}{2}(S_1 - S_3)$.

ART. 130. NORMAL AND TANGENTIAL STRESSES.

The general case of internal stress is that of an elementary parallelopiped held in equilibrium by apparent stresses applied to its forces in directions not normal. Here each oblique stress may be decomposed into three components parallel respectively to three coördinate axes, OX, OY, OZ. Upon each of the faces perpendicular to OX the normal component of the oblique unit-stress is designated by S_x and the two tangential components by S_{xy} and S_{xz}. A similar notation applies to each of the other faces. An S having but one subscript denotes a tensile or compressive stress, and its direction is parallel to the axis corresponding to that subscript. An S having two subscripts denotes a shearing stress, the first subscript designating the axis to which the face is perpendicular and the second designating the axis to which the stress is parallel; thus S_{zx} is

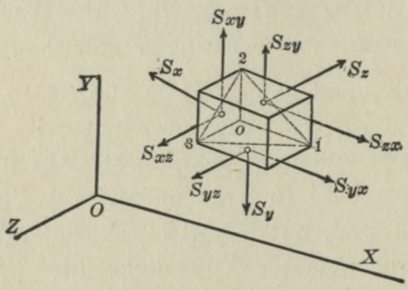

FIG. 74.

on the face perpendicular to OZ and its direction is parallel to OX. In Fig. 74 the six components for three sides of the parallelopiped are shown. Neglecting the weight of the parallelopiped the components upon the three opposite sides must be of equal intensity in order that equilibrium may obtain.

An elementary parallelopiped in the interior of a body is thus held in equilibrium under the action of six normal and twelve tangential stresses acting upon its faces. The normal stresses upon any two opposite faces must be equal in intensity and opposite in direction. The tangential stresses upon any two opposite faces must also be equal in intensity and opposite in direction.

A certain relation must also exist between the six shearing stresses shown in Fig. 74 in order that equilibrium may obtain. Let the parallelopiped be a cube with each edge equal to unity; then if no tendency to rotation exists with respect to an axis through the center of the cube and parallel to OX it is necessary that S_{yz} should equal S_{zy}. A similar condition obtains for each of the other rectangular axes, and hence

(28) $$S_{xy} = S_{yx}, \qquad S_{yz} = S_{zy}, \qquad S_{xz} = S_{zx},$$

that is, those shearing unit-stresses are equal which are upon any two adjacent faces and normal to their common edge.

The apparent unit-stresses designated by S are computed by the methods of the preceding Chapters; it is rare, however, that more than three or four of them exist, even under the action of complex forces. The general problem is then to find a parallelopiped such that the resultant stresses upon it are wholly normal. These resultant normal stresses will be S_1, S_2, S_3, from which by (27) the true normal stresses T_1, T_2, T_3 can be found. It will later be shown that these stresses S_1, S_2, S_3 are the maximum apparent stresses of tension or compression resulting from the given normal and tangential stresses.

Prob. 180. Let a, b, c be the angles which a line makes with the axes OX, OY, OZ, respectively. Show that the sum of the squares of the cosines of these angles is equal to unity.

ART. 131. RESULTANT STRESS.

The resultant unit-stress upon any face of the parallelopiped in Fig. 74 is the resultant of the three rectangular unit-stresses acting upon that face. Thus for the face normal to OZ the resultant unit-stress is given by

$$R_3^2 = S_z^2 + S^2_{zx} + S^2_{zy},$$

and the total resultant stress upon that face is the product of its area and R_3.

The resultant unit-stress R upon any elementary plane having any position can be determined when the normal and tangential stresses in the directions parallel to the coördinate axes are known. Let a plane be passed through the corners 1, 2, 3, of the parallelopiped in Fig. 74, and let a, b, c be the angles that its normal makes with the axes OX, OY, OZ, respectively. Let α, β, γ be the angles which the resultant unit-stress R makes with the same axes. Let A be the area of the triangle

1 2 3; then the total resultant stress upon that area is AR, and its components parallel to the three axes are $AR \cos \alpha$, $AR \cos \beta$, $AR \cos \gamma$. The triangle whose area is A, together with the three triangles 0 1 2, 0 2 3, 0 3 1, form a pyramid which is in equilibrium under the action of R and the stresses upon the three triangles. The areas of these triangles are $A \cos a$, $A \cos b$, $A \cos c$, and the stresses upon them are the products of the areas by the several unit-stresses. Now the components of these four stresses with respect to each rectangular axis must vanish as a necessary condition of equilibrium. Hence, cancelling out A, which occurs in all terms, there results

$$
\begin{aligned}
R \cos \alpha &= S_x \cos a + S_{yx} \cos b + S_{zx} \cos c, \\
(29) \quad R \cos \beta &= S_{xy} \cos a + S_y \cos b + S_{zy} \cos c, \\
R \cos \gamma &= S_{xz} \cos a + S_{yz} \cos b + S_z \cos c,
\end{aligned}
$$

in which the second members are all known quantities.

From these equations the values of $R \cos \alpha$, $R \cos \beta$, $R \cos \lambda$ can be computed; then the sum of the squares of these is R^2 since $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$. The value of $\cos \alpha$ is found by dividing that of $R \cos \alpha$ by R, and similarly for $\cos \beta$ and $\cos \gamma$. Now the angle θ between the directions of R and the normal to the plane is given by

$$\cos \theta = \cos a \cos \alpha + \cos b \cos \beta + \cos c \cos \gamma,$$

and then the tensile or compressive unit-stress normal to the given plane is $R \cos \theta$, while the resultant shearing unit-stress is $R \sin \theta$. This shearing stress may be resolved into two components in any two directions on the plane.

As a simple numerical example, let a bolt be subject to a tension of 12 000 pounds per square inch and also to a cross-shear of 8 000 pounds per square inch. It is required to find the apparent unit-stresses on a plane making an angle of 60 degrees with the axis of the bolt. Take OX parallel to the tensile force and OY parallel to the cross-shear. Then $S_x =$

$+ 12\,000$, $S_{xy} = 8000$, $S_{yx} = 8000$, and the other stresses are zero; also $a = 30°$, $b = 60°$, and $c = 90°$. Then from (29)

$R \cos \alpha = +14\,390$, $R \cos \beta = +6930$, $R \cos \gamma = 0$,

and the resultant stress in the plane is,

$R = \sqrt{14\,390^2 + 6930^2} = 15\,970$ pounds per square inch.

The direction made by R with the axis is,

$$\cos \alpha = \frac{1439}{1597} = 0.901, \qquad \alpha = 64\tfrac{1}{4}°,$$

$$\cos \beta = \frac{693}{1597} = 0.434, \qquad \beta = 25\tfrac{3}{4}°,$$

and the angle between the resultant R and the normal to the plane is given by

$$\cos \theta = 0.866 \times 0.901 + 0.5 \times 0.434 = 0.997.$$

Lastly, the normal tensile stress on the plane is found to be $R \cos \theta = 15\,920$ pounds per square inch, while the shearing stress on the plane is $R \sin \theta = 1200$ pounds per square inch.

Prob. 181. Find for the above example the position of a plane upon which there is no shearing stress.

ART. 132. THE ELLIPSOID OF STRESS.

At any point within a body let lines be drawn in every direction proportional to the resultant unit-stresses in those directions. It is now to be shown that a surface passing through the ends of those lines is the surface of an ellipsoid.

Let R_1, R_2, R_3 be the resultant unit-stresses upon the three faces of the parallelopiped in Fig. 74, and let θ_1, θ_2, θ_3 be the angles which they make with the coordinate axis OX; then

$$\cos \theta_1 = \frac{S_x}{R_1}, \qquad \cos \theta_2 = \frac{S_{yx}}{R_2}, \qquad \cos \theta_3 = \frac{S_{zx}}{R_3},$$

determine the directions of R_1, R_2, R_3. Now let these directions be taken as those of a new system of oblique coordinate axes, let R be the resultant unit-stress in any direction, and let

R_x, R_y, R_z be its components parallel to these new axes. Then $R \cos \alpha$ is the component of R parallel to OX, and

$$R \cos \alpha = R_x \cos \theta_1 + R_y \cos \theta_2 + R_z \cos \theta_3,$$

or, inserting for the cosines their values,

$$R \cos \alpha = S_x \frac{R_x}{R_1} + S_{yx} \frac{R_y}{R_2} + S_{zx} \frac{R_z}{R_3}.$$

Comparing this with the first equation in (30), it is seen that,

$$\cos a = \frac{R_x}{R_1}, \qquad \cos b = \frac{R_y}{R_2}, \qquad \cos c = \frac{R_z}{R_3}.$$

But the sum of the squares of these cosines is equal to unity; therefore,

$$\frac{R_x^2}{R_1^2} + \frac{R_y^2}{R_2^2} + \frac{R_z^2}{R_3^2} = 1,$$

in which the numerators are variable coordinates and the denominators are given quantities. This is hence the equation of the surface of an ellipsoid with respect to three coordinate axes having the directions of R_1, R_2, R_3.

The ellipsoid of stress is hence a figure whose radius vectors represent the intensity of the resultant stresses upon planes normal to their respective directions. If the forces be entirely confined to one plane, as in Art. 75, the stresses are represented by an ellipse.

If there be three planes at right angles to each other which are subject only to normal stresses, as in Fig. 75, the normal unit-stresses S_x, S_y, S_z correspond to R_1, R_2, R_3 in the above equation of the ellipsoid. In this case S_x, S_y, S_z are the three axes of the ellipsoid. If now shearing-stresses be applied to the faces the ellipsoid will be deformed, and the three axes will take other positions corresponding to three planes upon which no shear-

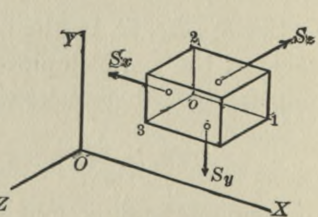

FIG. 75.

ing stresses act. The stresses corresponding to the axes of the ellipsoid are called principal stresses.

Prob. 182. If $S_y = S_z$ in Fig. 75, show that the ellipsoid becomes either a prolate spheroid or an oblate spheroid.

ART. 133. THE THREE PRINCIPAL STRESSES.

In general the resultant unit-stress R upon a given plane makes an angle θ with the normal to that plane, and hence can be resolved into a normal stress of tension or compression and into two tangential shearing stresses (Art. 131). It is evident, however, that planes may exist upon which only normal stresses act, so that θ is zero and R is pure tension or compression. In order to find these planes and the stresses upon them the angles α, β, γ in the equations (29) are to be made equal to a, b, c, respectively. Also replacing R by S, they become,

$$(S - S_x) \cos a = S_{yx} \cos b + S_{zx} \cos c,$$
$$(S - S_y) \cos b = S_{xy} \cos a + S_{zy} \cos c,$$
$$(S - S_z) \cos c = S_{xz} \cos a + S_{yz} \cos b,$$

in which S, $\cos a$, $\cos b$, $\cos c$ are quantities to be determined.

The three angles, however, are connected by the necessary relation,

$$\cos^2 a + \cos^2 b + \cos^2 c = 1,$$

and hence four equations exist between four unknowns.

Remembering the relation between the shearing stresses expressed in (28), the solution of the equations leads to a cubic equation for S, which is of the form,

$$(30) \qquad S^3 - AS^2 + BS - C = 0,$$

in which the values of the coefficients are,

$$A = S_x + S_y + S_z,$$
$$B = S_xS_y + S_yS_z + S_zS_x - S^2_{xy} - S^2_{yz} - S_{zx}^2,$$
$$C = S_xS_yS_z + 2S_{xy}S_{yz}S_{zx} - S_xS^2_{yz} - S_yS^2_{zx} - S_zS^2_{xy};$$

and the three roots of this cubic are the three normal stresses of tension or compression, often called the three principal stresses.

The directions of these principal stresses with respect to the axes OX, OY, OZ, are given by the values of cos a, cos b, cos c, which are found to be,

$$\cos a = \sqrt{\frac{m_1}{m}}, \qquad \cos b = \sqrt{\frac{m_2}{m}}, \qquad \cos c = \sqrt{\frac{m_3}{m}},$$

in which the values of the m's are,

$$\begin{aligned} m_1 &= (S_y - S)(S_z - S) - S^2_{yz}, \\ m_2 &= (S_z - S)(S_x - S) - S^2_{zx}, \\ m_3 &= (S_x - S)(S_y - S) - S_{xy}, \\ m &= m_1 + m_2 + m_3 ; \end{aligned}$$

and it will now be shown that each principal stress is perpendicular to the plane of the other two.

Let S_1, S_2, S_3 be the three roots of the cubic equation (30). Let a_1, b_1, c_1 be the angles which S_1 makes with the three coordinate axes OX, OY, OZ, and let a_2, b_2, c_2 be the angles which S_2 makes with the same axes. The angle between the directions of S_1 and S_2 is then given by

$$\cos \phi = \cos a_1 \cos a_2 + \cos b_1 \cos b_2 + \cos c_1 \cos c_2 .$$

Now in the first set of formulas of this article let S be made S_1 and a, b, c be changed to a_1, b_1, c_1; let the first equation be multiplied by cos a_2, the second by cos b_2, and the third by cos c_2; and let the three equations be added; then

$$S_1(\cos a_1 \cos a_2 + \cos b_1 \cos b_2 + \cos c_1 \cos c_2),$$

is one term in this sum. Again, let T be made T_2, and a, b, c, be changed to a_2, b_2, c_2; let the equations be multiplied by cos a_1, cos b_1, cos c_1, respectively, and added; then,

$$S_2(\cos a_1 \cos a_2 + \cos b_1 \cos b_2 + \cos c_1 \cos c_2),$$

is one term in the sum, while all the other terms are the same as before. Hence if S_1 and S_2 are unequal, the factor in the parenthesis, which is cos ϕ, must vanish ; ϕ is therefore a right angle or S_1 and S_2 are perpendicular. In the same manner it may be shown that S_3 is perpendicular to both S_1 and S_2.

The three principal stresses are hence perpendicular to each other, and as the only diameters of the ellipsoid which have this property are its axes, it follows that the directions of the principal stresses S_1, S_2, S_3 are those of the axes of the ellipsoid of stress. These principal stresses thus give the maximum normal stresses of tension or compression.

An interesting property of the three rectangular stresses S_x, S_y, S_z, is that their sum is constant, whatever may be the position of the coordinate axes. For the sum of the three principal stresses S_1, S_2, S_3 is equal to the coefficient A in the cubic of (30), and hence,

$$S_x + S_y + S_z = S_1 + S_2 + S_3;$$

that is, the sum of the normal unit-stresses in any three rectangular directions is constant.

Prob. 183. When two principal stressses are equal, show that the value of each is $(AB - 9C)/(2A^2 - 6B)$, where A, B, C are the coefficients in (30).

ART. 134. A NUMERICAL CASE.

To apply the preceding principles to a particular case a crank pin similar to that investigated in Art. 126 may be taken. The axis OX is assumed parallel to the axis of the pin, OY parallel to the crank arm, and OZ perpendicular to both. On one side of the crank pin near its junction with the arm there were found the following apparent stresses: A cross shear from the pressure of the connecting rod giving $S_{xz} = 300$ pounds per square inch, a shear due to the transmitted torsion giving $S_{xz} = 900$ pounds per square inch, a flexural stress due to the connecting rod giving $S_x = + 800$ pounds per square inch, a flexural stress due to the transmitted torsion giving $S_x = + 1\,600$ pounds per square inch, and two compressions due to shrinkage giving $S_y = - 4\,000$ and $S_z = - 2\,000$ pounds per square inch.

The two shears having the same direction add together, as also the two tensions, and the data then are,

$$S_{xz} = 1\,200, \quad S_x = + 2\,400, \quad S_y = - 4\,000, \quad S_z = - 2\,000.$$

Inserting these in (30) it becomes,

$$S^3 + 3\,600S^2 - 7\,840\,000S - 27\,200\,000\,000 = 0,$$

and the three roots of this are the three principal stresses. To solve this put $S = x - 1\,200$, and it reduces to

$$x^3 - 12\,160\,000x - 11\,096\,000\,000 = 0.$$

As this cubic has three real roots, it is to be solved by the help of a table of cosines; thus let

$$3r^2 = 12\,160\,000, \qquad 2r^3 \cos 3\phi = 11\,096\,000\,000,$$

from which $r = 2\,013$ and $\cos 3\phi = 0.6801$. Then from the table $3\phi = 47°\,09'$ whence $\phi = 15°\,43'$. The roots now are

$$x_1 = 2r \cos \phi \qquad\qquad = + 3\,880,$$
$$x_2 = 2r \cos (\phi + 120°) = - 2\,890,$$
$$x_3 = 2r \cos (\phi + 240°) = - \quad 990;$$

and finally the three principal stresses are,

$$S_1 = x_1 - 1\,200 = + 2\,680,$$
$$S_2 = x_2 - 1\,200 = - 4\,090,$$
$$S_3 = x_3 - 1\,200 = - 2\,190,$$

of which S_1 is the maximum tension and S_2 the maximum compression.

These are the apparent stresses. Taking $\epsilon = \frac{1}{3}$ for steel, the true principal stresses are now found by (27) to be,

$$T_1 = + 4\,770, \qquad T_2 = - 4\,250, \qquad T_3 = - 1\,720,$$

which shows that the maximum true tensile stress is nearly double the apparent, while the maximum true compressive stress is 6 per cent greater than the apparent.

In any case the method of procedure is similar. The apparent stresses being first computed for the point under consideration, their values establish the cubic equation (30) whose solution gives the three principal apparent stresses S_1, S_2, S_3; then from (27) the true maximum stresses of tension or compression are found. It frequently happens that one of the principal apparent stresses is zero; in this event the last term of the cubic vanishes and the ellipsoid becomes an ellipse. For instance let a bar be under a tension S_x and a single cross-shear S_{xy}. Then from (30) the principal stresses are

$$S_1 = \tfrac{1}{2}S_x + \sqrt{S^2_{xy} + \tfrac{1}{4}S_x^2}, \qquad S_2 = \tfrac{1}{2}S_x - \sqrt{S^2_{xy} + \tfrac{1}{4}S_x^2},$$

the first of which agrees with the result given in a different notation in equation (13) of Art. 75.

Prob. 184. Find the maximum true unit-stresses of tension or compression for the data of Prob. 121.

ART. 135. MAXIMUM SHEARING STRESSES.

As there are certain planes upon which the tensile and compressive unit-stresses are a maximum, so there are certain other planes upon which the shearing unit-stresses have their maximum values. In order to determine these it is well to take the axes of the ellipsoid as the coordinate axes, and upon the planes normal to these there are no shearing stresses. The stresses S_1, S_2, S_3 will give apparent shearing stresses on other planes, while T_1, T_2, T_3 will give the true shearing stresses.

Let 1 2 3 in Fig. 76 be any plane whose normal makes the angles a, b, c with the coordinate axes. Let R be the resultant unit-stress upon this plane, and α, β, γ be the angles which it makes with the same axes. The angle between R and the normal is expressed by,

$$\cos\theta = \cos a \cos\alpha + \cos b \cos\beta + \cos c \cos\gamma,$$

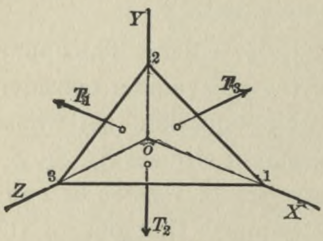

FIG. 76.

and the resultant shearing unit-stress on the plane is

$$R \sin\theta = R\sqrt{1 - \cos^2\theta}.$$

If R be apparent stress, this is the apparent shearing stress; if R be true stress, this is the true shearing stress.

The value of R, as a true stress, is given by

$$R^2 = (T_1 \cos a)^2 + (T_2 \cos b)^2 + (T_3 \cos c)^2.$$

Now, since both $R \cos\alpha$ and $T_1 \cos a$ are components of R in the direction OX, they are equal, and hence

$$\cos\alpha = \frac{T_1}{R}\cos a, \quad \cos\beta = \frac{T_2}{R}\cos b, \quad \cos\gamma = \frac{T_3}{R}\cos c.$$

Substituting these in the value of $\cos\theta$, the resulting true shearing unit-stress is expressed by

$$T^2 = (T_1 \cos a)^2 + (T_2 \cos b)^2 + (T_3 \cos c)^2 - (T_1 \cos^2 a + T_2 \cos^2 b + T_3 \cos^2 c)^2,$$

by the discussion of which the values of a, b, c, which render T a maximum, are deduced. Bearing in mind that the sum of the squares of the three cosines is unity, the discussion gives

$$\begin{aligned} &c = 90^\circ, \quad a = b = \pm 45^\circ, \quad T = \pm \tfrac{1}{2}(T_1 - T_2), \\ (31) \qquad &a = 90^\circ, \quad b = c = \pm 45^\circ, \quad T = \pm \tfrac{1}{2}(T_2 - T_3), \\ &b = 90^\circ, \quad c = a = \pm 45^\circ, \quad T = \pm \tfrac{1}{2}(T_3 - T_1); \end{aligned}$$

and therefore there are six planes of maximum shearing stress, each of which is parallel to one of the principal stresses and bisects the angle between the other two. On each of these planes the shearing unit-stress is one half the difference of the principal unit-stresses whose direction is bisected.

The same investigation applies equally well to the apparent shearing unit-stresses, whose maximum values are

(31)′ $S = \pm \frac{1}{2}(S_1 - S_2),\quad S = \pm \frac{1}{2}(S_2 - S_3),\quad S = \pm \frac{1}{2}(S_3 - S_1),$

and whose directions are the same as those of the maximum true shearing unit-stresses. The sign $\pm$ indicates that the shears have opposite directions on opposite sides of the plane, but in numerical work it is always convenient to take them as positive, or rather, as signless quantities.

As an example, let a bar be subject to a tension of 3 000 pounds per square inch in the direction of its length and to a compression of 6 000 pounds per square inch upon two opposite sides. Here $S_1 = +3\,000$, $S_2 = -6\,000$, $S_3 = 0$; then from (31)′ the maximum apparent shearing stresses are,

$$S = 4\,500, \qquad S_2 = 3\,000, \qquad S_3 = 1\,500.$$

But from (27), taking ϵ as $\frac{1}{3}$, the true tensile and compressive unit-stresses are $T_1 = +5\,000$, $T_2 = -7\,000$, $T_3 = +1\,000$, and then from (31) the maximum true shearing stresses are,

$$T = 6\,000, \qquad T = 4\,000, \qquad T = 2\,000,$$

which are 33 per cent greater than the apparent ones.

Prob. 185. Compute the maximum apparent and true shearing unit-stresses for the data given in Prob. 121.

ART. 136. THE ELLIPSE OF STRESS.

The ellipse of stress is that particular case where one of the principal stresses is zero, in which event the last term of (30) vanishes. An instance of this is where $S_z = 0$, $S_{yz} = 0$, $S_{xz} = 0$, which is that of a body subject to the normal stresses S_x, S_y, and to a cross-shear S_{xy}. The cubic equation then reduces to

$$S^2 - (S_x + S_y)S + S_{xy}S - S^2_{xy} = 0;$$

and the two roots of this are the two principal apparent stresses whose directions correspond to the two axes of the ellipse.

Let S_1 and S_2 be these two roots, and in Fig. 77 let OA and OB be laid off at right angles to represent their values. Let a quadrant of an ellipse be drawn upon these two semi-axes,

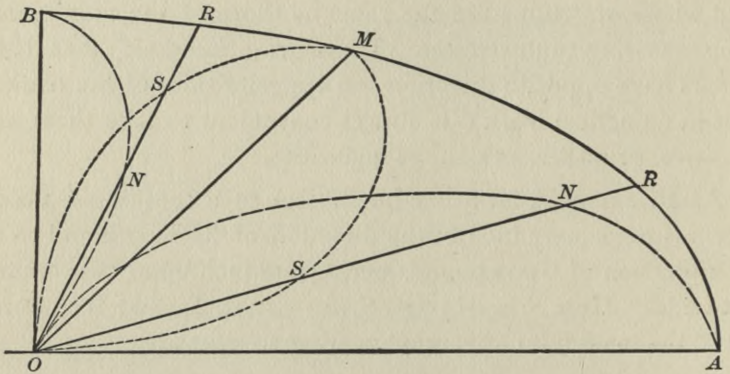

FIG. 77.

and let ϕ be the angle ROA which any line RO makes with OA. Upon a plane perpendicular to OR the normal unit-stress of tension or compression is

$$ON = S_1 \cos^2 \phi - S_2 \sin^2 \phi,$$

while the shearing unit-stress is

$$OS = (S_1 - S_2) \sin \phi \cos \phi,$$

and the resultant unit-stress is

$$OR = \sqrt{S_1^2 \cos^4 \phi + S_2^2 \sin^4 \phi}.$$

From these expressions the curves in Fig. 77 may be constructed. The values of ON are indicated by the broken lines, those of OS by the dotted curve, and those of OR by the quadrant of the ellipse. The dotted curve is tangent to the axes OA and OB, and also to the ellipse at M, the line OM bisecting the right angle AOB. The broken curves are not tangent to OM, but to a line making an angle of more than 45 degrees with OA. The figure thus gives a representation of the distribution of apparent stresses on planes through O. The curve

ONA may represent either tensions or compressions, as also may the curve *ONB*.

As a numerical illustration take the case of a bolt subject to a tension of 2000 pounds per square inch, and to a cross-shear of 3 000 pounds per square inch. Here $S_x = +2000$, $S_{xy} = 3\,000$, and $S_y = 0$; the above quadratic equation then gives $S_1 = +4\,160$ and $S_2 = -2160$ as the two principal stresses. The direction made by S_1 with the axis of the bolt as found by the value of cos a in Art. 133 is about $54\frac{1}{4}°$. The maximum tension is the value of S_1, the maximum compression is the value of S_2. From (31)′ the maximum shear is 3 160 pounds per square inch. On a plane making an angle of 60° with S_1 or an angle of $5\frac{3}{4}°$ with the axis of the bolt the normal stress *ON* is 2 660 pounds per square inch, tension; the shearing stress *OS* is 1 600 pounds per square inch, and their resultant *OR* is 3 100 pounds per square inch. All these are apparent stresses.

To find the true maximum stresses, formulas (27) give, taking $\frac{1}{3}$ as the factor of lateral contraction, $T_1 = +4\,880$, $T_2 = -3\,550$, $T_3 = -670$, as the principal tensions and compressions; then from (31) the greatest shearing stress is $T = 4\,220$ pounds per square inch. Here the maximum true tensile stress is 17 per cent greater than the apparent, the true compressive stress is 64 per cent greater, and the true shearing stress is 33 per cent greater. The true stresses cannot be represented by an ellipse, but an ellipsoid of internal stress exists, of which Fig. 77 may perhaps be regarded as a typical section.

Prob. 186. Given a body acted upon by two tensions at right angles to each other, one being 400 and the other 200 pounds per square inch. Construct the ellipse of apparent stresses, and find the position of a plane on which there is no tension or compression.

ART. 137. RECAPITULATION.

It has been shown in this chapter that the true internal stresses are different from the apparent ones on account of the deformations resulting from the elasticity of the material. It has also been shown how the true stresses may be obtained, and that often they are materially greater than the apparent ones. Combined stresses should be avoided when possible, so that the computed apparent stresses may be relied upon as closely agreeing with the true ones. Nevertheless in many important cases they cannot be avoided, and here an estimate of the true stresses should be made or the factors of safety be taken very high.

In Art. 119 it was shown that an external shear S on the web of a plate-girder produces an apparent tension S and an apparent compression S in directions at right angles to each other. It is now seen from (27) that the true tensile stress is $S - \epsilon(-S) = (1 + \epsilon)S$, and that the true compressive stress is $-(1 + \epsilon)S$. Then from (31) it follows that the true shearing stress is also $(1 + \epsilon)S$, or about $\frac{4}{3}S$. Thus the true internal stresses are about 33 per cent greater than their apparent values. If these conclusions be accepted, it follows that the real factors of safety in beams and plate girders are much lower than generally supposed.

A body under shear sometimes ruptures by tension, a crack forming in a direction making an angle of 45 degrees with that of the shear. The apparent shearing stress being S, the true tensile stress is $(1 + \epsilon)S$ and thus the ratio of the shearing to the tensile strength is $\frac{1}{1 + \epsilon}$. If $\epsilon = \frac{1}{3}$ this ratio is $\frac{3}{4}$. This has been taken by some authors as a kind of proof that the shearing strength of wrought iron is about three-fourths of its tensile strength; it is, however, not a valid proof, and it is not at all true for cast iron where $\epsilon = \frac{1}{4}$.

The investigations of Arts. 131–133 apply equally well to true internal stresses if S be replaced by T. Around any point in a body let a sphere of radius unity be described ; then if a single tensile force be applied the radius in that direction is elongated to $1 + s$, and those in directions at right angles are compressed to $1 - \epsilon s$. If three normal forces be applied in three rectangular directions, the sphere becomes an ellipsoid; if shearing forces be applied the ellipsoid is deformed so that its axes are oblique to the normal forces. Thus, even in simple cases, the ellipsoid and not the ellipse represents the true internal stresses.

Cases can, however, be imagined in which one of the true principal stresses is zero. Thus if S_1, S_2, S_3 are the apparent normal stresses in three rectangular directions, and if $S_3 - \epsilon S_1 - \epsilon S_2$ is zero, then the principal stress T_3 is also zero, as seen from (27). For instance, if a cube be under compression by three normal stresses of 30, 24, and 18 pounds per square inch, and if the material be such that $\epsilon = \frac{1}{3}$, then $T_1 = 16$, $T_2 = 8$, and $T_3 = 0$. Here the ellipse of stress has its correct application, as the true stresses are the same in all planes parallel to the plane of T_1 and T_2.

Prob. 187. Solve Prob. 129, deducing the true internal stresses by the methods of this chapter.

CHAPTER XIV.

STRESSES IN GUNS.

ART. 138. LAMÉ'S FORMULA.

Let a thick hollow cylinder, shown in longitudinal and cross-section in Fig. 78, be subject to a pressure p_1 on each square unit of the inner surface and to a pressure p_2 on each square unit of the outer surface. The inner pressure may be

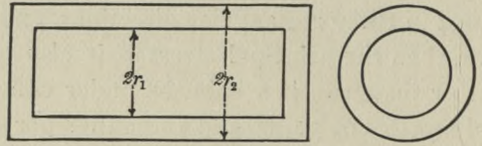

FIG. 78.

produced by the expansion of a gas and the outer pressure by the atmosphere or by other causes. It is required to determine the internal stresses produced by these pressures at any point in the cylindrical annulus.

The outer pressure on the end of the closed cylinder is $\pi r_2^2 p_2$ and the inner pressure on the end is $\pi r_1^2 p_1$. If the inner be greater than the outer pressure, as is often the case, the difference of these, or $\pi(r_1^2 p_1 - r_2^2 p_2)$ is the longitudinal tension on the annulus. For any part of the cylinder, not very near the end, this must be uniformly distributed over the cross-section. The longitudinal unit-stress on the annulus is hence a constant for all points, and its value is found by dividing the total stress by the area of the cross-section, or

$$q = \frac{r_1^2 p_1 - r_2^2 p_2}{r_2^2 - r_1^2},$$

which may be either tension or compression according as the numerator is positive or negative.

This longitudinal stress, together with the radial pressures, causes a longitudinal elongation of the cylinder, which also is to be regarded uniform for all parts of the annulus.

Let x be the distance from the center to any point within the annulus. Any elementary particle is here held in equilibrium by the longitudinal unit-stress q, a tangential unit-stress S, and a radial unit-stress p. The value of p is evidently intermediate between p_1 and p_2; in Fig. 79 it is, like S, drawn as if a tensile stress. Now from Art. 129 the effective longitudinal unit-elongation of the cylinder due to these three stresses is,

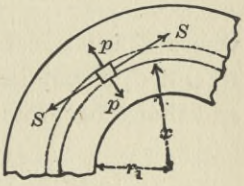

FIG. 79.

$$t = \frac{T}{E} = \frac{1}{E}(q - \epsilon S - \epsilon p),$$

in which ϵ is the factor of lateral contraction whose mean value is about $\frac{1}{3}$. But, as above noted, both q and t are constant for all parts of the annulus, and it hence follows that $S + p$ is also a constant, or

$$S + p = 2C_1,$$

which is one equation between S and p.

Let an elementary annulus of thickness dx be drawn; its inner radius is x and its outer radius is $x + dx$. The pressure for one unit of length in a direction perpendicular to any diameter is px upon the inner surface and $(p + dp)(x + dx)$ upon the outer surface of this elementary annulus. Thus, exactly as in the case of a thin pipe (Art. 9), the equation of equilibrium is,

$$(p + dp)(x + dx) - px = Sdx,$$

and, neglecting the product $dp \, . \, dx$, this reduces to

$$xdp + pdx = Sdx,$$

which is a second equation between S and p.

The solution of these two equations gives for S and p the values

$$S = C_1 + \frac{C_2}{x^2}, \qquad p = C_1 - \frac{C_2}{x^2},$$

where C_2 is a constant of integration. To find the values of C_1 and C_2 it may be noted that p becomes $-p_1$ when $x = r_1$, and that p becomes $-p_2$ when $x = r_2$: thus,

$$C_1 = \frac{p_1 r_1^2 - p_2 r_2^2}{r_2^2 - r_1^2}, \qquad C_2 = \frac{r_1^2 r_2^2 (p_1 - p_2)}{r_2^2 - r_1^2};$$

and these being inserted give

$$(32) \qquad S = \frac{r_1^2 p_1 - r_2^2 p_2}{r_2^2 - r_1^2} + \frac{r_1^2 r_2^2}{r_2^2 - r_1^2} \cdot \frac{p_1 - p_2}{x^2}, \qquad p = \frac{r_1^2 p_1 - r_2^2 p_2}{r_2^2 - r_1^2} - \frac{r_1^2 r_2^2}{r_2^2 - r_1^2} \cdot \frac{p_1 - p_2}{x^2},$$

which are LAMÉ'S formulas for the stresses in hollow cylinders under inner and outer pressures. In deriving these both S and p have been supposed to be tension. This will be the case if the formulas give positive values; if, however, either S or p has a negative value the stress is compression instead of tension.

The tangential unit-stress S is usually greater than the radial stress p, and is the controlling factor in the design of guns. It is seen to increase as x decreases, and hence it is the greatest at the inner surface of the cylinder. It may be either tension or compression, depending upon the relative values of the radii and pressures. The radial pressure p is always compression, its value ranging from p_1 at the inner to p_2 at the outer surface.

As a numerical example let a cylinder be one foot in inner and two feet in outer diameter, the inner pressure being 600 and the outer 15 pounds per square inch. Here $r_1 = 6$ inches, $r_2 = 12$ inches, $p_1 = 600$, $p_2 = 15$, and the formulas become,

$$S = 180 + \frac{28\,080}{x^2}, \qquad p = 180 - \frac{28\,080}{x^2}.$$

For the inner surface $x = 6$ inches, whence $S = +960$ and $p = -600$ pounds per square inch; at the outer surface $x = 12$ inches, whence $S = +375$ and $p = -15$ pounds per square inch, + denoting tension and − denoting compression.

Prob. 188. A solid cylinder is subject to a uniform outer pressure of p_2 pounds per square inch. Prove that the radial compression is uniform throughout. Prove that the tangential stress S is compressive and equal to the radial compression at all parts of the cylinder.

ART. 139. A SOLID GUN.

A gun tube without hoops is a solid hollow cylinder, subject upon the outer surface to atmospheric pressure and upon the inner surface to the pressure of the gas arising from the explosion of the powder. The outer pressure p_2 is so small compared to the inner pressure p_1 that it may be neglected. Then making x equal to r_1 in (32) they become,

$$S_1 = \frac{r_2^2 + r_1^2}{r_2^2 - r_1^2} p_1, \qquad p = -p_1,$$

which give the greatest tensile and compressive unit-stresses caused by the interior pressure. The tangential tension and the radial compression decrease as x increases, and at the outer surface where $x = r_2$ their values are

$$S_2 = \frac{2r_1^2}{r_2^2 - r_1^2} p_1, \qquad p = 0,$$

which are the least tensile and compressive unit-stresses caused by the inner pressure.

As a special case let the outer radius be twice the inner radius or $r_2 = 2r_1$. Then the tension for the inner surface becomes $S_1 = \frac{5}{3}p_1$, and for the outer surfaces it is $S_2 = \frac{2}{3}p_1$.

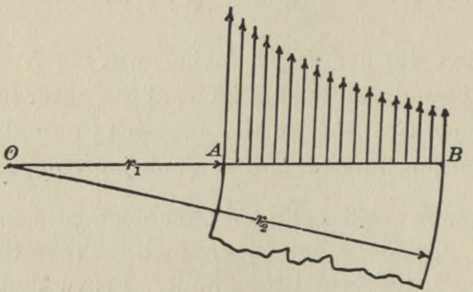

FIG. 80.

Thus the different parts of the annulus are quite unequally stressed, and hence the material is not utilized to the best advantage. The shaded area in Fig. 80 shows the manner in which the tangential tension varies in this case throughout the annulus.

The maximum stress in a solid gun is hence the tangential tension at the inner surface which is given by the formula,

$$S_1 = \frac{r_2^2 + r_1^2}{r_2^2 - r_1^2} p_1,$$

and this is the expression frequently used in cases of investigation and design.

As an example of investigation let a cast-steel gun have an inner diameter of 7.5 inches at the powder chamber and the thickness of the tube be 1.75 inches. What is the greatest tension produced when the inner pressure arising from the explosion is 10000 pounds per square inch? Here $r_1 = 3.75$ inches, $r_2 = 5.50$ inches, $p_1 = 10\,000$ pounds per square inch. Then from the formula S_1 is found to be 27 300 pounds per square inch, which is less than the elastic limit of steel, and hence not too large.

As an example of design let the inner diameter be 3.25 inches, the inner pressure caused by the explosion 15 000 pounds per square inch, and the allowable working strength in tension be 30 000 pounds per square inch. What should be the outer diameter? Here $r_1 = 1.625$ inches, $p_1 = 15\,000$ and $S_1 = 30\,000$ pounds per square inch. Then solving the last formula for r_2 there results

$$r_2 = r_1\sqrt{\frac{S_1 + p_1}{S_1 - p_1}},$$

from which the outer radius r_2 is found to be 2.815 inches; thus the thickness of the tube is 1.19 inches, and its outer diameter is 5.63 inches.

Prob. 189. A solid gun tube is 6 inches in diameter and 3 inches thick. What is the inner pressure that will produce a maximum tangential tension of 30 000 pounds per square inch?

ART. 140. A COMPOUND CYLINDER.

In a solid gun the maximum tension occurs at the inner surface during the explosion, rising suddenly from 0 up to its greatest value S_1. If now the metal near the bore can be brought into compression, this must be overcome before the tension can take effect, and thus the capacity to resist the inner pressure is increased. One method of producing this compression is by means of a hoop, or jacket, shrunk upon a tube so as to produce an outer pressure p_2 over the surface of the tube. This arrangement may be called a hollow compound cylinder.

In its normal state of rest the inner cylinder, or tube, has no pressure on its inner surface and p_2 on its outer surface. Making $p_1 = 0$ in (32), and also $x = r_1$ and $x = r_2$ in succession, there are found

$$S = -\frac{2r_2^2}{r_2^2 - r_1^2}p_2, \qquad S = -\frac{r_2^2 + r_1^2}{r_2^2 - r_1^2}p_2,$$

which are the tangential unit-stresses at the inner and outer surfaces of the tube due to the external pressure p_2. Both of these are compression, but the former is numerically greater than the latter, since $2r_2^2$ is greater than $r_2^2 + r_1^2$.

If the hoop is shrunk on so as to produce a compressive unit-stress S_c at the inner surface of the tube, the pressure p_2 upon the outer surface must be,

$$p_2 = \frac{r_2^2 - r_1^2}{2r_2^2}S_c,$$

and, if S_c be assumed, the shrinkage may be so regulated as to produce this pressure p_2 in the normal state of rest. Then the tangential stresses throughout the tube are all compression, while the radial pressures range from p_2 on the outer surface to 0 on the inner surface.

As an example, let $r_1 = 2$ inches, $r_2 = 3$ inches, and let it be required to find the outer pressure which will cause a tangential compressive unit-stress of 18 000 pounds per square inch at the inner surface of the tube. Here,

$$p_2 = \frac{9-4}{2 \times 9}18\,000 = 5\,000.$$

and hence the hoop must be shrunk on so as to produce this outer pressure on the hoop.

When the gun is fired the explosion of the powder causes an internal tangential tension S given by (32), whose greatest value is at the inner surface of the tube. Making $x = r_1$, this tensile unit-stress is,

$$(32)' \qquad S_1 = \frac{(r_1^2 + r_2^2)p_1 - 2r_2^2 p_2}{r_2^2 - r_1^2}$$

which is LAMÉ'S formula for the investigation of the tube of a compound gun. This tension first overcomes the initial com pression S_c due to shrinkage, so that the effective tension at the bore during firing is $S_1 - S_c$.

For example, let a tube whose inner and outer diameters are 4 inches and 6 inches be hooped so that a tangential compression of 18 000 pounds per square inch is caused at the bore, while the inner pressure caused by the explosion is 25 000 pounds per square inch. It is required to find the tangential stress at the bore during the explosion. Here $r_1 = 2$ and $r_2 = 3$ inches, $p_1 = 25\ 000$, and $p_2 = 5\ 000$, as seen above. Then $S_1 = 47\ 000$ pounds per square inch is the tension due to the explosion, but before this can take effect the initial compression of 18 000 pounds must be overcome. Hence the resultant tension at the bore during the firing is $47\ 000 - 18\ 000 = 29\ 000$ pounds per square inch.

If this tube be without a hoop the tension at the inner sur face, found by the method of the last article, is 57 200 pounds per square inch. The very great advantage of the hoop in diminishing the internal stresses during the firing is hence apparent.

Prob. 190. A gun-tube 3 inches in diameter and 1.5 inches thick is hooped so that the tangential compression on the inner surface is 30 000 pounds per square inch. What inner pressure p_1 will produce a resultant tangential tension on the inner surface of 30 000 pounds per square inch?

ART. 141. CLAVARINO'S FORMULAS.

The preceding method of investigating the strength of gun tubes is defective in that the two stresses S and p of formula (32) are only apparently the internal stresses. It was shown in Art. 71 and also in Art. 129 that the true internal stresses are those corresponding to the deformations produced. These deformations are influenced by the factor of lateral contraction of the material, which for steel and gun metal is usually taken as $\frac{1}{3}$ (Art. 128).

At any point in the annulus (Fig. 79) the apparent tangential, radial, and longitudinal unit-stress are S, p, and q, respectively. Let T be the true tangential unit-stress; then from (27) of Art. 129 its value is,

$$T = S - \tfrac{1}{3}p - \tfrac{1}{3}q.$$

Inserting in this the values of S, p, and q from Art. 138, it reduces to,

$$(33) \qquad T = \frac{r_1^2p_1 - r_2^2p_2}{3(r_2^2 - r_1^2)} + \frac{4r_1^2r_2^2}{3(r_2^2 - r_1^2)} \cdot \frac{p_1 - p_2}{x^2},$$

which is CLAVARINO'S principal formula for the investigation and design of guns.

This formula shows, as before, that the tangential stress is greatest at the inner surface of the cylinder. Making $x = r_1$, this maximum tension is

$$(33)' \qquad T_1 = \frac{(r_1^2 + 4r_2^2)p_1 - 5r_2^2p_2}{3(r_2^2 - r_1^2)},$$

which is the practical formula for the discussion of the most common cases. T_1 may be either tension or compression, depending upon the relative values of the pressures and radii.

CLAVARINO'S formulas are now frequently used in the investigation and design of guns, instead of those of LAMÉ. In order to compare them, the particular case where the outer diameter is double the inner diameter may be considered. Here $r_2 = 2r_1$, and (32)′ and (33)′ reduce to

$$S_1 = \frac{5p_1 - 8p_2}{3}, \qquad T_1 = \frac{17p_1 - 20p_2}{9}.$$

Now if $p_2 = 0$, the first formula gives a smaller stress than the second; if $p_2 = p_1$, the first gives a stress three times as large as the second; if $p_1 = 0$, the first gives a little larger stress than the second. Thus, since the second formula gives undoubtedly a better representation of the true stress than the first, it follows that LAMÉ'S method errs on the side of danger in a solid gun and on the side of safety in a hooped tube.

The value $\frac{1}{3}$ here used as the coefficient of lateral contraction is that employed in the United States by both the Army and the Navy in ordnance formulas, and also generally in Europe. In France, however, the value $\epsilon = \frac{1}{4}$ is adopted.

Prob. 191. Solve Problem 189 by the formulas of this article, and compare the two results.

ART. 142. BIRNIE'S FORMULAS.

The preceding articles present an outline of the methods of investigating stresses in guns by the formulas of LAMÉ and CLAVARINO. The formulas of LAMÉ refer to apparent stresses only; those of CLAVARINO, although referring to true stresses, contain an error which renders them not strictly correct for hooped guns. This error lies in taking for the longitudinal unit-stress q the value given in first equation of Art. 138. That value of q is correct for a hollow cylinder subject to pressure upon its ends as well as upon its curved surfaces. For a gun, however, q has a different value. When the gun is at rest q is zero, for then no exterior longitudinal forces act upon it.

For the investigation of a gun at rest the true tangential stress T should be deduced by making $q = 0$. Thus, in Art. 140 the correct value of T is $S - \frac{1}{3}p$, or,

$$(34) \qquad T = \frac{2r_1^2p_1 - 2r_2^2p_2}{3(r_2^2 - r_1^2)} + \frac{4r_1^2r_2^2}{3(r_2^2 - r_1^2)} \cdot \frac{p_1 - p_2}{x^2},$$

which is BIRNIE'S formula for the investigation and design of hooped guns. Making $x = r_1$, this becomes,

$$(34)' \qquad T_1 = \frac{(2r_1^2 + 4r_2^2)p_1 - 6r_2^2p_2}{3(r_2^2 - r_1^2)},$$

which is the tangential unit-stress at the inner surface of a hoop whose radii are r_1 and r_3. These formulas are used in the Ordnance Bureau of the United States army, not only for hoops, but for jackets and tubes, both at rest and during the firing.

Strictly speaking, BIRNIE'S formulas apply only to hoops and tubes upon which the exterior longitudinal stress q is zero. For a solid gun, or for a tube attached to the breech block, a more correct formula may be found by considering the actual value of q due to the inner pressure. Here the longitudinal pressure is $\pi r_1^2 p_1$, and this produces longitudinal tension upon the area $\pi(r_2^2 - r_1^2)$, so that

$$q = \frac{r_1^2 p_1}{r_2^2 - r_1^2}$$

is the apparent longitudinal unit-stress. Then the true tangential stress T is $S - \frac{1}{3}p - \frac{1}{3}q$, or,

$$T = \frac{r_1^2 p_1 - 2r_2^2 p_2}{3(r_2^2 - r_1^2)} + \frac{4r_1^2 r_2^2}{3(r_2^2 - r_1^2)} \cdot \frac{p_1 - p_2}{x^2},$$

and making in this $x = r_1$, it becomes

$$T_1 = \frac{(r_1^2 + 4r_2^2)p_1 - 6r_2^2 p_2}{3(r_2^2 - r_1^2)},$$

which is the true tangential unit-stress upon the inner surface of the bore.

To compare the formulas of CLAVARINO and BIRNIE the particular case where $r_2 = 2r_1$ may be considered. Then (33)′ and (34)′ reduce to

$$T_1 = \frac{17p_1 - 20p_2}{9}, \qquad T_1 = \frac{18p_1 - 24p_2}{9}.$$

Now, if $p_2 = 0$, as for a solid gun during firing, the second formula gives a stress 6 per cent larger than the first; if $p_1 = 0$, as for a hooped gun at rest, the second gives a stress 25 per cent greater than the first. Thus for this case, at least, CLAVARINO'S formulas give the stresses somewhat too low.

Prob. 192. Solve Problem 189 by the formulas of this article and compare the results with those of Problem 191.

ART. 143. HOOP SHRINKAGE.

Let λ be the elongation or contraction of any radius x due to inner or outer pressure, then $2\pi\lambda$ is the elongation or contraction of any circumference $2\pi x$. Now $2\pi\lambda/2\pi x$ is the change in the circumference per unit of length due to the unit-stress T; hence $\lambda/x = T/E$, or

$$\lambda = \frac{T}{E}x,$$

is the change in length of any radius to the circle where the tangential unit-stress is T. If $x = r_1$ the deformation λ_1 is that of the radius of the bore due to the unit-stress T_1: if $x = r_2$ the change λ_2 is that of the outer radius of the tube where the unit-stress is T_2.

Suppose a compound cylinder to be formed by shrinking a hoop upon a tube. The inner radius of the tube is r_1 and its outer radius r_2; the inner radius of the hoop is r_2 and its outer radius r_3. In consequence of the shrinkage the radial pressure p_2 is produced between the two surfaces; this causes the tube to be under tangential compression and the hoop to be under tangential tension. It is required to find these stresses when the original inner radius of the hoop is less than that of the outer radius of the tube by the amount λ.

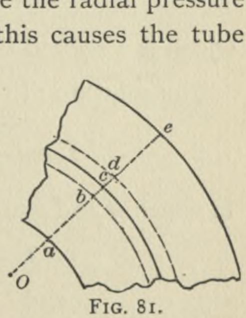

FIG. 81.

Let λ_2 be the decrease in the outer radius of the tube and λ_2' the increase in the inner radius of the hoop; then $\lambda = \lambda_2 + \lambda_2'$. In Fig. 81, which is much exaggerated, cd represents λ_2 and bc represents λ_2'. Also, let T_2 be the tangential compression at the outer surface of the tube due to the shortening λ_2, and let T_2' be the tangential tension at the inner surface of the hoop due to the elongation λ_2'. Then

$$\lambda = \frac{T_2}{E}r_2 + \frac{T_2'}{E}r_2 = (T_2 + T_2')\frac{r_2}{E}$$

or

$$T_2 + T_2' = \frac{E\lambda}{r_2},$$

which gives one equation between T_2 and T_2'.

Formula (34) is applied to the tube by making $p_1 = 0$ and $x = r_2$; thus the tangential compression is,

$$T_2 = \frac{4r_1^2 + 2r_2^2}{3(r_2^2 - r_1^2)}p_2 = ap_2.$$

Formula (34) is applied to the hoop by replacing p_1 by p_2, p_2 by 0, r_1 by r_2, and r_2 by r_3; then making $x = r_2$, there results,

$$T_2' = \frac{2r_2^2 + 4r_3^2}{3(r_3^2 - r_2^2)}p_2 = bp_2,$$

which is the tangential tension. Dividing the first of these by the second, gives

$$\frac{T_2}{T_2'} = \frac{a}{b} \quad \text{or} \quad T_2 b = T_2' a,$$

which is a second equation between T_2 and T_2'.

The solution of these equations gives the values

$$T_2 = \frac{E\lambda}{r_2} \cdot \frac{a}{a+b}, \qquad T_2' = \frac{E\lambda}{r_2} \cdot \frac{b}{a+b},$$

in which a and b are the known quantities,

$$a = \frac{2}{3} \cdot \frac{2r_1^2 + r_2^2}{r_2^2 - r_1^2}, \qquad b = \frac{2}{3} \cdot \frac{r_2^2 + 2r_3^2}{r_3^2 - r_1^2},$$

and thus the tangential compression at the outer surface of the tube and the tangential tension at the inner surface of the hoop may be computed. The tangential compression at the bore is, however, greater than T_2, and it may be found from (34)′ by substituting the value of p_2, now known; thus,

$$T_1 = \frac{3r_2^2}{2r_1^2 + r_2^2} T_2,$$

which is the greatest compression in the tube.

As a numerical example let a compound cylinder be formed of a steel tube whose inner radius is 3 inches and outer radius 5 inches, with a steel hoop whose thickness is 2 inches. It is required to find the stresses produced when the original difference between the outer radius of the tube and the inner radius of the hoop is 0.004 inches. First, the sum of the two tangential stresses at the surface of contact is,

$$\frac{E\lambda}{r_2} = \frac{30\,000\,000 \times 0.004}{5} = 24\,000.$$

Second, since $r_1 = 3$, $r_2 = 5$, and $r_3 = 7$ inches,

$$a = \frac{2}{3}\frac{18 + 25}{25 - 9} = \frac{43}{24}, \qquad b = \frac{2}{3}\frac{25 + 98}{49 - 25} = \frac{82}{24}.$$

Third, the tangential compression at the outer surface of the tube is,

$$T_2 = \frac{43}{43 + 82} \times 24\,000 = 8\,260 \text{ pounds per square inch.}$$

Fourth, the tangential tension at the inner surface of the hoop is,

$$T_2' = \frac{82}{43 + 82} \times 24\,000 = 15\,740 \text{ pounds per square inch.}$$

Thus it is seen that the hoop tension is nearly double the compression on the outer surface of the tube. At the bore of the tube, however, the tangential compression is found to be 14 400 pounds per square inch.

The decrease in the outer radius of the tube is next computed; thus,

$$\lambda_2 = \frac{T_2' r_2}{E} = 0.00138 \text{ inches,}$$

and the increase in the inner radius of the hoop is,

$$\lambda_2' = \frac{T_2' r_2}{E} = 0.00262 \text{ inches.}$$

Hence if the radius of the common surface of contact is to be exactly 5 inches after the shrinkage, the tube should be turned to an outer radius of 5.0014 inches, and the hoop to an inner radius of 4.9974 inches. The radius of the bore, however, will then be less than 3 inches by the quantity,

$$\lambda_1 = \frac{T_1 r_1}{E} = 0.00144 \text{ inches,}$$

and hence if its final radius is to be exactly 2 inches, it must be turned to a radius of 3.0014 inches.

If this example be solved by using the formulas of CLAVARINO instead of those of BIRNIE, the following values will be found: $T_2 = 7\,500$, $T_2' = 16\,500$, and $T_1 = 15\,400$ pounds per square inch; $\lambda_2 = 0.00125$, $\lambda_2' = 0.00275$, and $\lambda_1 = 0.00154$ inches. The shrinkages thus agree within 0.0002, which is as close as measurements can be relied upon.

Prob. 193. A solid steel shaft, 6 inches in radius, is to be hooped so that the greatest tensile stress in the hoop and the greatest compressive stress in the shaft shall be 15 000 pounds per square inch. Find the thickness of the hoop and the radius to which it should be turned.

ART. 144. HOOPED GUNS.

A hooped gun should be so constructed that neither the stresses due to hoop shrinkage nor those developed during the firing shall exceed the elastic limit of the material. The simple case of a tube with one hoop can here only be considered. If the radii be given, as also the inner pressure p_1 due to the explosion, it may be desired to find the shrinkages so that this requirement will be fulfilled. As p_1 is very large, it is desirable that the given dimensions should be such as to require the least amount of materiai.

The condition of minimum amount of material will be in general fulfilled when the stresses during the explosion are as great as allowable and as nearly equal as possible. The diagram in Fig. 82 represents the distribution of the internal stresses under this supposition. O is the center of the gun,

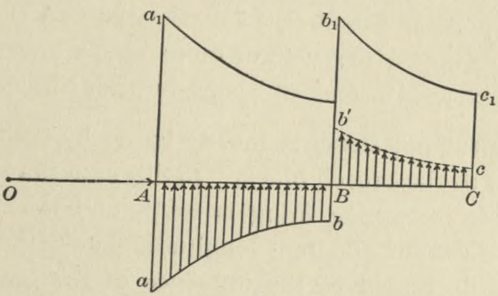

FIG. 82.

OA the inner radius r_1, while AB is the thickness of the tube and BC that of the hoop. The shaded areas show the stresses due to the hoop shrinkage, Aa and Bb being the tangential compressions T_1 and T_2 of the last article, while Bb' is the tangential tension T_2', and Cc is the tangential tension at the outer surface of the hoop. When the explosion occurs the two cylinders are thrown into tangential tension, Aa_1 and Bb_1 being those at the inner surfaces of the tube and hoop. The above principle indicates that both Aa_1 and Bb_1 should be equal to the maximum allowable unit-stress T_e, which for guns is often taken nearly as high as the elastic limit of the material.

In designing a gun the radius of the bore and the thickness of the tube may be assumed, and it may be required to find the thickness and shrinkage of the hoop so that the stresses Aa, Aa_1, and Bb_1 in Fig. 82 are each equal to the elastic limit of the material. Or, given the radius of the bore and the outer radius of the hoop, it may be required to find the intermediate radius under the same conditions. These problems can be solved as well as more complex ones relating to guns with several hoops.

The stresses in guns are greatest near the powder chamber, since the gas expands and its pressure decreases as the projectile moves outward. Hence the hoops increase in number toward the powder chamber, each being shorter than the one beneath it. Guns with seven or more hoops have been built. Wire has been used instead of hoops, but not with good results.

The longitudinal stress is mostly borne by the tube if that be attached to the breech block. In this case the longitudinal internal pressure during firing is $\pi r_1^2 p_1$ and this divided by $\pi(r_2^2 - r_1^2)$ gives the apparent longitudinal unit-stress q. This, however, is decreased by the influence of the tangential and radial pressures. Thus, from Arts. 129 and 138,

$$T_q = q - \tfrac{1}{3}S - \tfrac{1}{3}p = \frac{r_1^2 p_1 + 2r_2^2 p_2}{3(r_2^2 - r_1^2},$$

which is the true longitudinal unit-stress on the tube. When the jacket is attached to the breech block, which is more often the case, it is subject to the inner pressure p_2 from the tube, to the outer pressure p_3 from the hoop, and it carries the entire longitudinal stress $\pi r_1^2 p_1$; thus

$$T_q = \frac{3r_1^2 p_1 - 2(r_2^2 p_2 - r_3^2 p_3)}{3(r_3^2 - r_2^2)},$$

is the longitudinal unit-stress on the jacket.

Reference is made to the excellent work of MEIGS and INGERSOLL, The Elastic Strength of Guns (Baltimore, 1885), for a detailed presentation of the formulas and methods used in the United States Navy for the design of guns. The formulas used by the Ordnance Bureau of the Army will be found set forth in a thorough manner in STORY'S Elements of the Elastic Strength of Guns (Fort Monroe, 1894).

Prob. 194. Prove that a gun tube with one hoop is most advantageously designed when the common radius of tube and hoop is a mean proportional between the other two radii. (To solve this problem derive an expression for p_1 in terms of r_1, r_2, r_3, and T_e. Then the most advantageous value of r_2 is that which renders p_1 a maximum.)

CHAPTER XV.

PLATES, SPHERES, AND COLUMNS.

ART. 145. CIRCULAR PLATES.

Let a circular plate of radius r and uniform thickness d be subject on one side to a pressure p on each square unit of area and be supported or fixed around the circumference. The head of a cylinder under the pressure of water or steam is a circular plate in such a condition. The maximum stress caused by the flexure will evidently be at the middle, and this is required to be determined.

As the simplest case let the plate be merely supported around the circumference. The total load on the plate being $\pi r^2 p$, the total reaction of the support is also $\pi r^2 p$, or the reaction per linear unit is $\frac{1}{2}rp$. Now let a strip having the small width b be imagined to be cut out of the plate, so that its central line coincides with a diameter. The reaction at each end of this strip is $b.\frac{1}{2}rp$, and the load on the strip is $b.2r.p$. The sum of the two reactions being only one half the load, an upward shearing force equal to $b.r.p$ must act along the sides of the strip to maintain the equilibrium. At the center of the circle there can be no shearing stress and the most probable assumption regarding its distribution on the sides of the strip is to take it as varying uniformly from the center to the circumference, as shown by the dotted lines in Fig. 83.

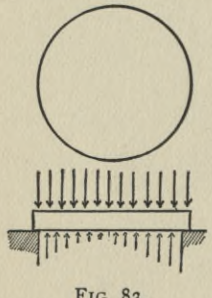

FIG. 83.

The strip whose width is b and length $2r$ is thus a beam acted upon by two vertical reactions, each equal to $\frac{1}{2}brp$, a downward load $2brp$, and two vertical shears on the sides each equal to $\frac{1}{2}brp$. The bending moment at the middle of this imaginary beam hence is,

$$M = \tfrac{1}{2}brp \,.\, r + \tfrac{1}{2}brp \,.\, \tfrac{2}{3}r - brp \,.\, \tfrac{1}{2}r = \tfrac{1}{3}br^2p,$$

and the maximum unit-stress on the upper or lower fiber at the middle of the strip is,

$$S_1 = \frac{Mc}{I} = \frac{6M}{bd^2} = 2p\frac{r^2}{d^2}.$$

This value of S_1 is not the real horizontal fiber stress at the center of the circle, but only the apparent stress due to considering the elementary strip. At the center the fiber stresses are acting in all directions. If a second strip be passed in Fig. 83 at right angles to the first, a stress S_2 equal in value but normal in direction to the first will be found. The true fiber-stress T will be determined from the principle of Art. 129, taking into account the factor of lateral contraction ϵ; thus on the upper side of the plate,

$$T = (S_1 - \epsilon S_2 + \epsilon p)$$

and on the lower side of the plate,

$$T = (S_1 - \epsilon S_2) = 2(1 - \epsilon)p\frac{r^2}{d^2},$$

which is the value to be used, since rupture will generally begin on the tensile side.

For iron and steel the mean value of the factor of lateral contraction ϵ is $\frac{1}{3}$. Hence

$$T = \frac{4}{3}\frac{r^2}{d^2}p$$

is the practical formula for the discussion of iron and steel plates under a uniform pressure when simply supported at the circumference. The unit-pressure that such a plate can carry with a given unit-stress hence varies directly as the square of its thickness and inversely as the square of its radius.

The more common case of a circular plate fixed around its circumference cannot be solved without determining the elastic curve into which a diameter deflects. The investigation is too difficult and lengthy for this elementary book, but it can be said that the general conclusion is that the true effective unit-stress T is about three-fourths of that for the supported plate. Using this result the coefficient in the last formula will be unity, or

$$T = \frac{r^2}{d^2}p$$

is the practical formula for iron and steel plates under uniform pressure when fixed around the circumference.

Assuming a safe working stress T, the proper thicknesses of plates under uniform pressure then are

$$d = r\sqrt{\frac{4p}{3T}}, \quad \text{and} \quad d = r\sqrt{\frac{p}{T}},$$

the first being for supported and the second for fixed circumference. For example, let a fixed cast-iron cylinder head of 3 feet diameter be required to sustain a uniform pressure of 250 pounds per square inch, so that the maximum tensile unit-stress T may be 3 600 pounds per square inch; then

$$d = 18\sqrt{\frac{250}{3\,600}} = 4\tfrac{3}{4} \text{ inches},$$

which is the required thickness.

The formulas derived by GRASHOF from a more extended theoretical analysis, are,

$$d = r\sqrt{\frac{5p}{6T}} \quad \text{and} \quad d = r\sqrt{\frac{2p}{3T}}$$

for the thickness of supported and fixed plates respectively.

The second of these, applied to the above numerical example, gives $d = 3\frac{7}{8}$ inches.

Prob. 195. Deduce a formula for a circular plate supported around the circumference and carrying a single load P at the center.

ART. 146. ELLIPTICAL PLATES.

Elliptical plates are commonly used for the covers of manholes in boilers and stand-pipes. Let p be the uniform unit-pressure on the plate, a the major axis and b the minor axis of the ellipse. It is required to find the maximum unit-stress S on the tensile side of the plate.

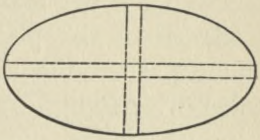

FIG. 84.

Taking the case where the plate is simply supported around the circumference, let two elementary strips be drawn as in Fig. 84, one along the major axis and the other along the minor axis. Let W_1 and W_2 be the loads on these strips, and R_1 and R_2 the reactions at their ends. At the center they have a common deflection, which by Art. 36 is

$$\Delta = \frac{W_1 a^3}{mEI} = \frac{W_2 b^3}{mEI};$$

and hence $W_1 a^3 = W_2 b^3$; or, since the reactions are proportional to the loads,

$$R_1 a^3 = R_2 b^3 \quad \text{or} \quad R_1/R_2 = b^3/a^3;$$

that is, the reactions at the ends of the axes are inversely as the cubes of the lengths of the axes. Hence the total weight $\frac{1}{4}\pi abp$ is not uniformly distributed on the support around the circumference, but the greatest reaction per linear unit will be found at the ends of the minor axis and the least at the ends of the major axis. It should hence be expected that the fiber stresses near the center are the greatest in directions parallel to the minor axis, and that in case of rupture a crack would begin at the center and run along the major axis; this is verified by tests.

A satisfactory solution of this very difficult problem has not been made. From the discussion given by BACH (Elasticität und Festigkeit, 1894) the following approximate formula results, which may advantageously be used for elliptical plates in the absence of other rules:

$$S = \frac{Ca^2b^2p}{(a^2 + b^2)d^2},$$

in which S is the maximum unit-stress at the center, d is the thickness of the plate, and C is a number which is probably about $\frac{2}{3}$ when the circumference is supported and about $\frac{1}{2}$ when it is fixed.

Prob. 196. A common proportion for manhole covers is to make $a = 1.5b$. If the thickness be $\frac{3}{4}$ inches, the major axis 16 inches, and the material wrought iron, find the safe pressure per square inch.

ART. 147. RECTANGULAR PLATES.

A rectangular plate of length a and breadth b, subject to a uniform pressure p per square unit, distributes that pressure over the supports in a similar manner to the elliptical plate. The reaction per linear unit is less on the ends than on the sides, and is greater at the middle of the ends and sides than near the corners. Rupture tends to occur near the center and parallel to the longer side. The approximate formula derived by BACH for such plates is,

$$S = \frac{Ca^2b^2p}{(a^2 + b^2)d^2},$$

in which S is the maximum unit-stress at the middle, and d is the thickness of the plate. The value of C as found by BACH, by experiments on square plates, ranged from $\frac{3}{4}$ to $\frac{3}{2}$, according as the condition of the edges approached that of a mere support or a state of fixedness.

For a square plate whose side is a this formula gives

$$S = \frac{9a^2p}{8d^2} \qquad \text{and} \qquad S = \frac{3a^2p}{4d^2}$$

for free and fixed supports respectively. A theoretic analysis by GRASHOF for square plates fixed at the middle of each side gives the true unit-stress as

$$T = (1 - \epsilon^2)\frac{a^2p}{d^2} = \frac{8a^2p}{9d^2}$$

if the factor of lateral contraction ϵ be taken at $\frac{1}{3}$, as usual for iron plates (Art. 129).

While the numerical coefficients, as deduced by different authors, vary somewhat, it is well established that the unit-stress at the middle of the plate varies directly as its area and the unit-pressure p, and inversely as the thickness of the plate. The strength of a plate as measured by the pressure p that it can carry varies directly as the square of the thickness and inversely as the area.

Prob. 197. Prove that the maximum unit-stress caused by a given uniform load W is independent of the size of the plate.

ART. 148. HOLLOW SPHERES.

Hollow spheres are used in certain forms of boilers under inner steam-pressure. The ends of steam and water cylinders are sometimes made hemispherical instead of plane, in order to avoid flexure (Art. 145). If the thickness of the sphere is small compared to its radius, the investigation is simple. Let r be the radius and t the thickness. Let p be the inner pressure per square unit, and S the tensile unit-stress on the annulus. Then on any great circle the total pressure is $\pi r^2 p$, and this is resisted by the tension $2\pi rtS$ in the section of the annulus. Equating these gives

$$rp = 2tS, \qquad \text{or} \qquad S = \frac{rp}{2t},$$

which is the formula generally used for thin spheres under inner pressure. But in strictness S is the apparent stress, while another equal in intensity acts at right angles to it. Thus from Art. 129 the true stress on the inner surface is

$$T = S - \tfrac{1}{3}S + \tfrac{1}{3}p = \frac{rp}{3t} + \frac{p}{3},$$

which gives the maximum true unit-stress, since that on the outer surface is $\frac{2}{3}S$. The usual formula thus errs on the side of safety.

The investigation of a thick hollow sphere under inner and outer pressure will be similar to that of the thick cylinder in Art. 138. Let r_1 be the inner and r_2 the outer radius, p_1 and p_2 being the corresponding pressures per square unit of surface. Fig. 79 of Art. 138 may represent a partial section of the sphere, x being the radius of any elementary annulus where the radial unit-stress is p and the tangential unit-stress is S. From the symmetry of the sphere it is seen that another stress S acts at right angles to the one shown in the figure. Thus an elementary particle at any position in the annulus is held in equilibrium by three principal stresses p, S, and S. The sum of these is regarded as constant throughout the annulus, or

$$2S + p = 3C_1$$

is one equation between S and p.

Now the inner pressure on a great circle whose radius is x is $\pi x^2 p_1$, and the outer pressure on a great circle whose radius is $x + dx$ is $\pi(x + dx)^2(p + dp)$. The difference of these is equal to the resistance of the elementary annulus, which is $2\pi x dx . S$, or

$$(x + dx)^2(p - dp) - x^2 p = 2Sxdx;$$

and this, omitting quantities of the second order, reduces to

$$xdp + 2pdx = 2Sdx,$$

which is a second equation between S and p.

Substituting in the second equation the value of S from the first, and integrating, the value of p in terms of x is found, and thus S also; the results are

$$S = C_1 + \frac{C_2}{x^3}, \qquad p = C_1 - \frac{C_2}{x^3},$$

in which C_2 is a constant of integration. To find the values of C_1 and C_2, the value of p becomes $-p_1$ when $x = r_1$, and $-p_2$ when $x = r_2$; thus

$$C_1 = \frac{r_1^3 p_1 - r_2^3 p_2}{r_2^3 - r_1^2}, \qquad C_2 = \frac{r_1^3 r_2^3 (p_1 - p_2)}{r_2^3 - r_1^3};$$

and these are the formulas for thick hollow spheres deduced by LAMÉ. It is seen that they are analogous to those for thick cylinders, the radii being cubed instead of squared. It is also seen that the formula for S is the important one and that its greatest value occurs at the inner surface of the sphere.

For the common case of inner pressure, let $p_2 = 0$, and make $x = r_1$, then the greatest tangential stress is

$$S_1 = \frac{r_1^3 + r_2^3}{r_2^3 - r_1^3} p_1,$$

which is the common formula for hollow spheres. This gives the apparent tensile unit-stress at the inner surface. The true unit-stress at the inner surface is, by Art. 129,

$$T = S_1 - \tfrac{1}{3}S_1 + \tfrac{1}{3}p_1 = \frac{r_1^3 + 3r_2^3}{r_2^3 - r_1^3} \cdot \frac{p_1}{3},$$

which will always be found to be less than S_1.

As an example, let a cylinder 4 inches in inner and 8 inches in outer radius have a hemispherical end with the same radii, and be subject to an inner water-pressure of 4 000 pounds per square inch. Then the apparent tensile stress on the inner surface of the hemisphere is

$$S_1 = \frac{64 + 512}{512 - 64} \times 4\,000 = 5\,140 \text{ pounds per square inch.}$$

while the true tensile stress is

$$T_1 = \frac{64 + 1\,536}{512 - 64} \times \frac{4\,000}{3} = 4\,800 \text{ pounds per square inch,}$$

which shows that the true value is about 6 per cent less than the apparent.

For the cylinder the apparent and true tensile unit-stresses at the inner surface are, from Arts. 138 and 140,

$$S_1 = \frac{r_1^2 + r_2^2}{r_2^3 - r_1^3} p_1, \qquad T_1 = \frac{r_1^2 + 4r_2^2}{r_2^2 - r_1^2} \frac{p_1}{3},$$

which give $S_1 = 6\,700$ and $T_1 = 7\,600$ pounds per square inch, so that the true stress is 13 per cent greater than the apparent.

If the end of the cylinder be a flat plate of the same thickness as the cylinder, or 4 inches, and be fixed around the circumference, the true stress on the outer side is

$$T = \tfrac{64}{16} \times 4\,000 = 16\,000 \text{ pounds per square inch,}$$

which is $3\frac{1}{3}$ times as great as for the hemisphere, and more than double the greatest stress in the cylinder. The advantage of hemispherical ends in reducing the stresses is thus seen to be very great. It may be remarked, in conclusion, that the theory of internal stress in cylinders and spheres is not perfect, for it fails to give the same results for the common surface of junction of a cylinder and hemisphere.

Prob. 198. A hollow sphere is to be subject to a steam-pressure of 600 pounds per square inch, its inner radius being 8 inches. Find its thickness so that the greatest stress may be 1 000 pounds per square inch.

Prob. 199. Investigate the discrepancy between the formulas for hollow cylinders and hollow spheres for the following numerical case. A hollow cylinder with hollow hemispherical ends, the inner diameters being 8 inches and the outer diameters 12 inches, is subject to an inner water pressure of 2400 pounds per square inch. Compute, by Art. 141 and by this article, the true maximum unit-stress T for the common plane of junction of cylinder and hemisphere.

ART. 149. EXPERIMENTS ON COLUMNS.

It is impossible to present here even a summary of the many experiments that have been made to determine the laws of resistance of columns. The interesting tests made by CHRISTIE in 1883 for the Pencoyd Iron Works will however be briefly described on account of their great value and completeness as regards wrought iron struts, embracing angle, tee, beam, and channel sections. See Transactions of the American Society of Civil Engineers, April, 1884.

The ends of the struts were arranged in different methods; first flat ends between parallel plates to which the specimen was in no way connected; second, fixed ends, or ends rigidly clamped; third, hinged ends, or ends fitted to hemispherical balls and sockets or cylindrical pins; fourth, round ends, or ends fitted to balls resting on flat plates.

The number of experiments was about 300, of which about one-third were upon angles, and one-third upon tees. The quality of the wrought iron was about as follows: elastic limit 32 000 pounds per square inch. Ultimate tensile strength 49 600 pounds per square inch, ultimate elongation 18 per cent in 8 inches. The length of the specimens varied from 6 inches up to 16 feet, and the ratio of length to least radius of gyration varied from 20 to 480. Each specimen was placed in a Fairbanks' testing machine of 50 000 pounds capacity and the power applied by hand through a system of gearing to two rigidly parallel plates between which the specimen was placed in a vertical position. The pressure or load was measured on an ordinary scale beam, pivoted on knife edges and carrying a moving weight which registered the pressure automatically. At each increment of 5 000 pounds, the lateral deflection of the column was measured. The load was increased until failure occurred.

The following are the combined average results of these care-

Length divided by Least Radius of Gyration.	Flat Ends.	Fixed Ends.	Hinged Ends.	Round Ends.
20	46 000	46 000	46 000	44 000
40	40 000	40 000	40 000	36 500
60	36 000	36 000	36 000	30 500
80	32 000	32 000	31 500	25 000
100	29 800	30 000	28 000	20 500
120	26 300	28 000	24 300	16 500
140	23 500	25 500	21 000	12 800
160	20 000	23 000	16 500	9 500
180	16 800	20 000	12 800	7 500
200	14 500	17 500	10 800	6 000
220	12 700	15 000	8 800	5 000
240	11 200	13 000	7 500	4 300
260	9 800	11 000	6 500	3 800
280	8 500	10 000	5 700	3 200
300	7 200	9 000	5 000	2 800
320	6 000	8 000	4 500	2 500
340	5 100	7 000	4 000	2 100
360	4 300	6 500	3 500	1 900
380	3 500	5 800	3 000	1 700
400	3 000	5 200	2 500	1 500
420	2 500	4 800	2 300	1 300
440	2 200	4 300	2 100	
460	2 000	3 800	1 900	
480	1 900		1 800	

fully conducted experiments. The first column gives the values of l/r and the other columns the values of P/A, the latter being the ultimate load in pounds per square inch. From the results it will be seen that there is little practical difference between the strength of the four classes when the strut is short. The strength of the long columns with round ends appears to be about $3\frac{1}{2}$ times that of those with round ends.

EULER'S formula fairly represents the results of the tests on the long round-ended columns. Taking $E = 25\,000\,000$ pounds per square inch for wrought iron, and $\pi^2 = 10$, Art. 53 gives,

$$\frac{P}{A} = 250\,000\,000\frac{r^2}{l^2},$$

from which are computed,

$$\text{for } l/r = 300, \quad 340, \quad 380, \quad 400,$$

$$P/A = 2800, \quad 2200, \quad 1700, \quad 1400,$$

while the experiments give the values

$$P/A = 2800, \quad 2100, \quad 1700, \quad 1300.$$

Since EULER'S formula is deduced under the laws of elasticity, it must be concluded that the elastic limit was not exceeded when these long columns failed by lateral flexure.

Prob. 200. Plot the above experiments on round-ended columns, taking P/A as abscissa and l/r as ordinate. Also plot on the same sheet EULER'S curve and the straight line given by T. H. JOHNSON'S formula.

ART. 150. EULER'S MODIFIED FORMULA.

EULER'S formula for columns expresses the condition of indifferent equilibrium or that state which borders between stable and unstable equilibrium. In all cases of indifferent equilibrium slight causes produce marked effects, and hence it seems important to inquire whether EULER'S formula, as given in Art. 53, represents the exact relation between P/A and l/r. It will be apparent on reflection that, while the formula contains but one length l, there are really three different lengths that should be taken into consideration. Let l represent the length of the straight column in its unstrained state, l_1 the length of the straight column after compression by the unit-stress P/A, and l_2 the chord of the deflected curve after lateral bending.

Referring now to formula (5) of Art. 33, it is seen that this does not include the effect of the longitudinal compression P/A. To introduce this let S be the total unit-stress produced by both flexure and compression; then in the demonstration the first four formulas will be thus modified:

$$\lambda = (S - P/A)\frac{dl}{E}, \quad \frac{\text{R}}{dl_1} = \frac{c}{\lambda}, \quad \frac{S - P/A}{c} = \frac{E}{\text{R}}, \quad \frac{S - P/A}{c} = \frac{M}{I},$$

and from these there results,

$$\frac{d^2y}{dx^2} = \frac{M}{EI}\frac{dl}{dl_1} = \frac{M}{EI}\frac{l}{l_1},$$

which is the correct differential equation of the elastic curve for a body under combined flexure and compression.

Passing now to EULER'S deduction in Art. 53, M is replaced by $-Py$, and the equation of the elastic curve for a round-ended column is

$$y = \Delta \sin x\left(\frac{P}{EI}\frac{l}{l_1}\right)^{\frac{1}{2}}.$$

Here $y = 0$ when $x = l_2$; hence by the same reasoning as before,

$$P = \pi^2 EI\frac{l_1}{ll_2^2} \quad \text{or} \quad \frac{P}{A} = \pi^2 Er^2\frac{l_1}{ll_2^2}$$

is the exact condition for the state of indifferent equilibrium.

To complete the investigation l_1 and l_2 are to be expressed in terms of l. Now $l - l_1$ is the deformation due to the longitudinal compression P/A, and hence from the fundamental definition of the coefficient of elasticity (Art. 4),

$$l_1 = l\left(1 - \frac{P}{AE}\right),$$

which gives the length of the straight column after longitudinal compression.

To find l_2 it is plain that $l_1 - l_2$ is the fall of the end of the column due to the lateral flexure and that $P(l_1 - l_2)$ is the work done in this fall. This external work must equal the internal work of the flexural stresses. From Art. 109,

$$dK = \frac{M^2 dx}{2EI} = \frac{P^2 y^2 dx}{2EI} = \frac{P^2 \Delta^2}{2EI} \sin^2 \frac{\pi x}{l_2} \,.\, dx,$$

and integrating this between the limits $x = 0$ and $x = l_2$, the internal work K is found to be $P^2 \Delta^2 l_2 / 4EI$. Thus, equating the two works,

$$l_1 - l_2 = \frac{P \Delta^2 l_2}{4EI},$$

and from this, in connection with the above value of l_1,

$$l_2 = \frac{1 - P/AE}{1 + P\Delta^2/4Ar^2} l,$$

which gives the length of the chord of the deflected curve.

The quantities P/AE and $P\Delta^2/4Ar^2$ are small in comparison with unity, and hence their squares and their product may be neglected; also the reciprocal of $1 - P/AE$ may be taken as $1 + P/AE$. Introducing the values of l_1 and l_2 into the above expression for P it reduces to

$$\frac{P}{A} = \pi^2 E \frac{r^2}{l^2}\left(1 + \frac{P}{AE} + \frac{P}{AE}\frac{\Delta^2}{2r^2}\right),$$

and since $\pi^2 E r^2 / l^2$ is a close approximation to the value of P/A it may replace the latter in the parenthesis, giving finally

$$\frac{P}{A} = \pi^2 E \frac{r^2}{l^2}\left(1 + \pi^2 \frac{r^2}{l^2} + \pi^2 \frac{\Delta^2}{2l^2}\right),$$

which is EULER'S modified formula for round-ended columns. By writing $2\frac{1}{4}\pi^2$ instead of π^2 it applies to a column with one end round and the other fixed, and by writing $4\pi^2$ instead of π^2 it applies to a column with both ends fixed.

A modification of EULER'S formula by use of the l_1 and l was given by WINCKLER in 1867, by GRASHOF in 1878, and by PRICHARD in 1897 (see Engineering News, April 1897). The formula as given by them can be obtained from the above by making $l_2 = l_1$ or by making $\Delta = 0$. Since the quantities $\pi^2 r^2/l^2$ and $\pi^2 \Delta/l^2$ are small compared with unity, EULER'S modified formula gives numerical results but little larger than the usual ones. It shows, however, that P really increases slightly as Δ increases, and that hence Δ is in strictness not wholly indeterminate, as the common theory teaches.

Prob. 201. Show that the fall of the load P, due to direct compression of the column, equals the fall due to the lateral flexure when $\Delta = 2r$.

ART. 151. THE DEFLECTION OF COLUMNS.

An ideal column always remains straight under the action of an axial load W, provided that this load is less than the value of P given by EULER'S formula. If a slight lateral force be applied it will deflect, recovering its straight condition, however, as soon as this force is removed. Under the action of such a slight lateral force there is a definite relation between the deflection Δ and the maximum unit-stress S on the concave side. This relation, as given by Art. 55, is

$$\Delta = \left(\frac{S}{W/A} - 1\right)\frac{r^2}{c},$$

and it shows that S increases with Δ.

Now if W becomes so great that the column does not spring back, but remains in indifferent equilibrium, its value is the P given by EULER'S formula, and

$$\Delta = \frac{Sl^2}{m\pi^2 Ec} - \frac{r^2}{c}.$$

Here S is indeterminate, and the column may remain in indifferent equilibrium with many different values of $\varDelta$. However, if $\varDelta$ be so great that S becomes equal to the elastic limit S_e, failure at once follows, and hence the greatest possible deflection is found by using S_e for S in the last equation.

When a steadily increasing load W is applied to an actual column it may, on account of being non-homogeneous or not perfectly straight, begin to bend without the application of any lateral force long before W reaches the value P given by EULER'S formula. Suppose, for instance, that the column is slightly crooked so that it has an initial deflection δ. The actual deflection $\varDelta$ will now be measured, not from the axis of ordinates, but from the original position of the axis of the column. Then for an ideal column with the deflection $\varDelta$ the bending moment is $P\varDelta$, while for the actual column the bending moment is $W(\delta + \varDelta)$, and equating these there is found,

$$\varDelta = \frac{W}{P - W}\delta.$$

Thus here the actual deflection increases with W, and no indeterminateness occurs. This case in fact is closely analogous with that of a column when the load is eccentric with respect to the axis, the deflection $\varDelta$ being always perfectly determinate (Art. 62). Even if δ be very small $\varDelta$ may become a considerable quantity, and as it increases the unit-stress S also increases, failure being practically complete as soon as S reaches the elastic limit.

It is seen from these considerations that it is not possible to establish a formula for the deflection of a column under an increasing load W, unless an initial eccentricity of load or an initial crookedness of the column be assumed. Reference is made to Chapter X of FIDLER'S Practical Treatise on Bridge Construction (London, 1887), for an excellent treatment of columns based on the above considerations.

APPENDIX.

Vibrations of a Beam.

In Art. 111 the case of deflection of a beam under impact was discussed and a formula for the momentary maximum deflection was deduced. It was also mentioned that a series of vibrations ensued, until finally the beam came to rest with a deflection due to the static load. After Art. 111 was in type, there appeared in *The Railroad Gazette* of May 31, 1895, an article by P. H. Dudley, in which an interesting experiment on deflection is given. Fig. 85 and the following description is taken from that article.

A railroad rail 30 feet long and weighing 80 pounds per yard was placed at its extreme ends upon rigid supports, having a scale-pan of 210 pounds weight suspended from the middle. Secured to the center of the rail was an attachment carrying a pencil which recorded upon moving paper the deflections and vibrations of the rail. A load of 100 pounds being suddenly applied the maximum deflection was 0.240 inches; but when the rail ceased vibrating the deflection was only 0.120 inches, or one-half that due to the sudden load. The weight of 100 pounds was next dropped from the height of 12 inches, and the maximum deflection was 0.910 inches. To produce the same deflection by a static load, 650 pounds was required to be added to the scale-pan, which with the 100 pounds already in the pan made a total of 750 pounds. After the weight was dropped it required about 42 seconds for the rail to cease vibrating, the total number of vibrations being about 120. Fig. 85 shows in a beautiful manner the record made by the pencil on the moving paper.

The formula applicable to this case is (22)″ of Art. 111, where $\varDelta = 0.120$ inches, the deflection due to the static load; $h = 12$ inches, the height of fall; $W = 800$ pounds, the weight of the beam; $P = 100$ pounds, the weight of the load. Thus $k = 8$, and $1 + qk = 4.886$. Then the momentary maximum deflection δ is, by the formula, 0.897 inches, whereas the experiment gave 0.910 inches. The static load Q required to produce the deflection δ is, since the deflections are proportional to the loads, $Q = P\delta/\varDelta = 747$ pounds, whereas the experiment gave 750 pounds. It hence appears that the theoretic formulas well represent observed facts when, as in this case, the elastic limit is not exceeded.

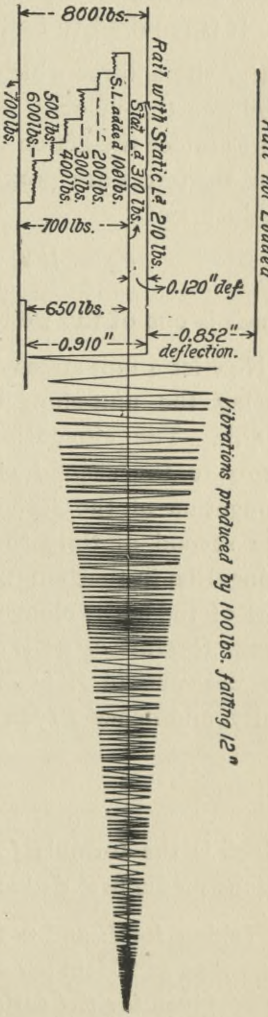

FIG. 85.

Prob. 202. Compute the maximum unit-stress for the above beam due to a load of 750 pounds at the middle, taking $I = 29.2$ inches^4, and $c = 2.6$ inches.

Velocity of Stress.

When an exterior force is applied to a body the stresses in the latter are not instantaneously generated, but are produced by a wave-like propagation through the mass. Especially is this true when the exterior force is applied suddenly or with impact. Thus there is a velocity of transmission of stress which will be shown to depend upon the stiffness and density of the material.

Let v be the velocity with which stress is transmitted in a body whose coefficient of elasticity is E, and whose weight per cubic unit is w at a place where the acceleration of gravity is g. It is required to find v in terms of E, w, and g.

If F be a force which acting continuously for one second produces the velocity u, and if a body of the weight W acquires under the action of gravity the velocity g in one second, then the forces are proportional to the accelerations that they produce, or

$$Fg = Wu, \qquad \text{whence} \qquad F = W\frac{u}{g},$$

which is one of the well-known formulas of mechanics.

Now let a unit-stress S be applied to the end of a bar, producing the unit-elongation s upon the first element of its length. The elongation of the first element transmits the stress to the second element, and this in turn produces an elongation of the second element, and so on. At the end of one second of time the length v is elongated, and the total elongation in that length will be sv. Thus in one second the end of the bar is elongated by the amount sv. Now referring to the formula $Fg = Wu$, the value of F is S, which is equal to Es, the value of W is wv, since the cross-section considered is unity, and hence $Esg = wv \,.\, sv$, whence

$$v = \sqrt{\frac{Eg}{w}},$$

which is the formula for the velocity of wave propagation in elastic media first deduced by NEWTON.

Taking for E and w their mean values given in Art. 80, and also g as 32.16 feet per second per second, since the unit-weights w are given for the surface of the earth, the mean values of the velocity of transmission of stress for different materials are found to be as follows:

For timber,	$v = 13\,200$ feet per second,
For stone,	$v = 13\,200$ feet per second,
For cast iron,	$v = 12\,400$ feet per second,
For wrought iron,	$v = 15\,500$ feet per second,
For steel,	$v = 17\,200$ feet per second.

In making this computation E must be taken in pounds per square foot, since both w and g are expressed in terms of feet.

The velocity of sound, light, and all wave propagations in elastic media is given by the above formula for v. The ratio of w to g is a constant for the same material at any point in space, and it expresses the density, while E is an index of stiffness. The æther that transmits waves of light must be lighter than air and stiffer than steel, in order that v may be the high value found by observation.

Prob. 203. What time is required for sound to travel a distance of 5 miles in water, the linear unit-compression for water being 0.00005?

ADVANCED PROBLEMS.

Many questions relating to flexure and torsion have not been treated in the preceding pages, as their discussion properly belongs to special works on special branches of applied mechanics. A few of these are here noted as advanced exercises that may be assigned by teachers as prize problems.

Prob. 204. Prove that the maximum bending moment caused by two equal loads rolling over a simple beam occurs at the section distant $\frac{1}{4}a$ from the middle, a being the distance between the two loads.

Prob. 205. Prove that the maximum bending moment at a given section in a simple beam, due to a given system of moving loads, occurs when the sum of those on the left of the section divided by the distance of the section from the left end equals the total load divided by the length of the span.

Prob. 206. Prove that the maximum maximorum bending moment in a simple beam, due to a given system of moving loads, occurs at a section so located that the distance between it and the center of gravity of the loads is bisected by the middle of the span.

Prob. 207. Let a helical spring consist of round wire, let r be the radius of the coil, d the diameter of the wire, and P the tensile and compressive load upon the spring. Show that

$$S = \frac{16Pr}{\pi d^3}$$

is the shearing or torsional unit-stress in the wire.

Prob. 208. The data being the same as in the last problem, and n being the number of coils, show that

$$\lambda = \frac{64nPr^3}{E_s d^4},$$

is the elongation or compression of the spring.

Prob. 209. A simple beam of given rectangular cross-section carries a load of w pounds per linear unit in addition to its own weight. If S be the allowable working stress show that the greatest possible length of the beam is

$$l = 2d\sqrt{\frac{Sbd^2}{3(w + bdu)}},$$

where u is the weight of a cubic unit of the material, b is the breadth and d the depth of the beam.

Prob. 210. If the flanges of an I beam be considered to carry all the bending moment and the web all the vertical shear, show that the most advantageous proportions are such that the cross-section of the flanges equals the cross-section of the web.

Prob. 211. If y be the elongation of a spring or bar under longitudinal impact, prove that

$$t = \pi\sqrt{y/g}$$

is the time of one oscillation.

Prob. 212. Show that the percentage of weight saved by using a hollow instead of a solid shaft is $\frac{100}{n^2+1}$ if they are made of equal resilience, n being the ratio of the outer to the inner diameter of the hollow section.

Prob. 213. Show, for a shaft of square cross-section, that the formulas for investigation and design are

$$S_s = 267\,500 \frac{H}{nd^3}, \qquad d = 64.4\sqrt[3]{\frac{H}{nS_s}},$$

in which the letters have the same meaning as in Art. 68.

Prob. 214. A load P is supported by three strings of equal size lying in the same vertical plane. The middle string is vertical and each of the others makes an angle θ with it. If S be the stress on the middle string and S_1 the stress on each of the others, show that

$$S = \frac{P}{1 + 2\cos^3\theta}, \qquad S_1 = \frac{P\cos^2\theta}{1 + 2\cos^3\theta}.$$

(Note: To solve this problem the condition must be introduced that the internal work of all the stresses is a minimum.)

Prob. 215. A load P is supported by three strings of equal size lying in the same plane. The middle string is vertical, one string makes with it the angle θ on one side, and the second string makes with it the angle ϕ on the other side. Find the stresses in the strings.

Prob. 216. A circular ring of radius R is pulled in the direction of a diameter by two tensile forces each equal to P. Show that the maximum bending moment is at the section where P is applied, and that its value is

$$M = \frac{PR^3}{\pi(R^2 + r^2)},$$

in which r is the radius of gyration of the cross-section of the ring. Deduce also an expression for the maximum negative bending moment.

Prob. 217. A continuous beam of five equal spans has a load

P on the second span at a distance kl from the second support. Show that the reaction of the first support is

$$R_1 = \frac{P}{209}(-97k + 168k^2 - 71k^3),$$

and that the reaction of the second support is

$$R_2 = \frac{P}{209}(209 - 45k - 381k^2 + 217k^3).$$

Prob. 218. Discuss the formula $\tan 2\phi = p/2v$ in Art. 75 and show that it represents two sets of shear curves, each being at right angles to the other. Draw the two sets of curves for the case of a simple beam of rectangular cross-section.

Prob. 219. A beam of two equal spans has a joint at the middle of the first span so that the moment there is always zero. Find the reactions due to a load P on the first span; (a) when k is less than $\frac{1}{2}$; (b) when k is greater than $\frac{1}{2}$. Draw for each case the diagrams of shears and moments.

Prob. 220. A cantilever bridge has three spans, the length of each end span being l and that of the middle span m. The middle span has two joints, the distance of each from the nearest support being n. Find the reactions at the supports; (a) when the load P is on an end span; (b) when it is on the middle span between a pier and the nearest joint; (c) when it is on the middle span between the two joints.

Prob. 221. A beam is fixed at the ends A and C, and loaded at B with a load P; the end C, however, being free to deflect, while B and C are kept on the same level. Show that the reactions at A and C are

$$R_a = P\frac{4k - 3k^2}{3 - 2k}, \qquad R_c = P\frac{3(1 - k)^2}{3 - 2k},$$

in which k represents the ratio of BC to AC.

Prob. 222. A continuous beam of three spans has each end span of length l and the middle span of length nl. Find the reactions due to a load P in an end span.

Answers to Problems.

Below will be found the answers to about nine-tenths of the problems stated in the preceding pages, the number of the problem being in parenthesis and the answer immediately following. It has been thought well that some answers should be omitted in order that the student may struggle with them to ascertain the truth, according to his best knowledge of the subject, rather than to make his numerical results agree with given figures. However satisfactory it may be to the student to know the result of an exercise he is to solve, let him remember that after commencement day the answers to problems will never be given.

The unit-stresses to be employed in solutions will be, unless otherwise stated in the problem, uniformly taken from the tables given in the text and in Art. 80. Considering the great variation in these data it has not been thought best to carry the numerical answers to more than three significant figures, but in making the solution four significant figures should be retained through the work in order that the third may be correct in the final result.

Chapter I. (1) 7.2, 7.06, and 86.4 square inches. (2) 173, 34.7, and 4.44 pounds. (3) 55 000 pounds per square inch. (4) 27 500 pounds. (5) 165 000 pounds. (6) 0.15 inches. (7) 26 250 000 pounds per square inch. (8) 0.004 inches. (9) About $3\frac{1}{2}$ inches in diameter. (11) 2 880 and 5 400 feet. (12) 0.00153 inches. (13) 52 900 pounds per square inch. (14) 849 pounds per square inch. (15) About $1\frac{3}{4}$ inches in diameter. (16) 9 for *AB* and 23 for *BC*.

Chapter II. (17) 0.88 inches if $f = 15$. (18) 1 170 pounds per square inch. (19) 2 500 pounds per square inch. (20) 1 620 pounds per square inch. (22) $2\frac{1}{4}$ inches. (23) 57 per cent; $f = 7.7$. (24) 3.28 inches; about 0.73. (25) 0.0032 inches.

Chapter III. (27) $2\frac{1}{8}$ inches. (29) 998 and 742 pounds. (30) + 800, + 160, and − 180 at 1, 3, and 5 feet from left end. (31) − 10, − 40, − 90, − 40, − 10 pound-feet. (33) $Y = 140$ and $X = 242$ pounds. (34) 2 700 pounds. (35) $X = -Z = 375$ pounds. (36) About 27. (37) 4.20 inches. (38) $c = 1.714$ inches, $I = 7.39$ inches^4. (39) $\frac{1}{12}bd^3$ and $\frac{1}{4}bd^3$. (41) At 5.37 feet from left end; $M = 689$ pound-feet. (42) No. (43) The bar will break. (44) 294 pounds per linear foot. (45) About 6 000 pounds. (47) 8.87 inches. (48) 0.0178 inches. (49) About 615 pounds. (51) Ultimate strengths about as 4 to 1, while working strengths for a steady load are about as 1.8 to 1. (52) 3.7 and 1.8. (53) 7 feet, 8 inches. (54) The beam will break. (56) $S = 5\,610$ and $S' = 3\,170$ pounds per square inch. (57) 209 inches, 418 inches, and ∞. (58) 0.622 inches. (59) 14 500 000 pounds per square inch. (62) As 8 to 3; as 64 to 9. (63) $7\frac{1}{4}$ inches. (64) 0.243 inches. (65) $x = 6\,000\,\frac{bd^2}{P}$ for the first case; the shear at supports is independent of x. (68) 0.72 inches.

Chapter IV. (69) The diagrams should always be drawn on cross-section paper. (70) $R_1 = 290$. (72) $k = 0.366$, and $k = 0.577$. (73) $l = 2.828\ m$. (74) Max. positive $M = 810$ pound-feet. (76) A heavy 15-inch beam; a light 15-inch beam. (78) 0.0269 inches. (80) $R_1 = R_4 = \frac{4}{10}wl$; $R_2 = R_3 = \frac{11}{10}wl$. (81) $-0° 25' 47''$. (83) $\frac{I}{c} = 7.2$ which requires the light 6-inch beam. (84) $-\frac{1}{20}wl$. (86) $n = 0.6095$.

Chapter V. (88) 9.15 inches. (89) 2 inches (90) 205 000 pounds. (92) 69.7 tons. (93) 5.05 inches. (95) $r = 0.48$ inches. (96) $1\frac{3}{8}$. (97) 2.35 and 24. (98) 250 000 pounds. (99) 23 100 pounds. (100) $13\frac{1}{2}$ and $16\frac{1}{2}$ inches square. (104) Draw Fig. 48 so as to make $bq = 0$; then state the equation of moments and reduce it by the relation between the similar triangles.

Chapter VI. (106) 30 pounds. (107) 105 degrees. (108) 720, 270, and 290 pound-inches. (109) 1 876 pounds per square inch. (110) $J = 0.0361d^4$ and $c = 0.577d$. (111) $J = 26.5$ and $c = 3.41$. (112) 1 680 pounds. (113) 9 380 000 pounds per square inch. (114) 64 horse-power. (115) 9.7. (116) 2.65 and 3.58 inches. (118) 6 500. (119) As $\sqrt{2\pi}$ to 3. (120) As 100 to 260.

Chapter VII. (122) 3 720 pounds per square inch. (123) 4 690 pounds per square inch. (124) The light 9-inch beam. (125) Nearly 8 inches. (126) 9 inches. (127) $t = 9\,420$, $\phi = 54°\ 20'$; $S = 7\,160$, $\phi = 9°\ 20'$. (129) 5.4. (130) $2\frac{1}{2}$ inches. (131) $S = 5\,660$ and $S_s = 202$ pounds per square inch. (132) At 3 inches from neutral surface $S = 2\,000$, $S_h = 250$, and $t = 2\,030$ pounds per square inch.

Chapter IX. (139) Theoretic stress is 3.3 per cent and theoretic elongation is 3.5 per cent greater than the observed values. (140) 1.34 horse-power. (141) 122 foot-pounds. (142) For the second case $K = S^2Al/16E$.

Chapter X. (144) 7 500 feet and 3.75 feet. (145) $\log x = \log b + 0.4343\frac{wbl}{P}y$. (146) About 33 feet per second. (148) Nearly 28 000 pounds per square inch. (149) 2 935 rollers 2 inches in diameter, or 82 rollers 1 foot in diameter. (150) 10. (151) 11 500 pounds per square inch.

Chapter XI. (152) For uniform load $M = -\frac{1}{2}wx^2$ and $K = W^2l^3/40EI$. (153) Deflection $= \frac{9}{16}$th of that for load at middle. (154) $\Delta = 0.174$ inches, $S = 4030$ pounds per square inch. (156) $S = 2\,250$ and $T = 21\,400$ pounds per square inch. (157) 88 000 pounds. (158) About 1.4 inches. (160) Over 600 miles per hour. (161) $W^2l/24E_sA$. (166) See 'Higher Mathematics,' Chap. IV, Art. 36.

Chapter XII. (167) $\frac{1}{2}(6\,000) - \frac{1}{2}(-2\,000) = 4\,000$. (168) About 12 horse-power. (170) $E_s = 9\,380\,000$, then find ϵ from formula (25). (171) As 2 to 3. (172) As 100 to 307. (173) 8 bolts. (174) $2\frac{1}{8}$ inches. (175) Bearing unit-stress = 1 880 pounds per square inch. (176) Shear is uniform throughout, while moment is zero at middle.

Chapter XIII. (178) 0.042 and 0.014 per cent. (181) 56°19′ with axis of bolt. (183) See Theory of Equations in Algebra. (184) Greatest tension = 36.3, compression = 87.1 pounds per square inch. (185) Apparent = 53.7, true = 61.7 pounds per square inch. (186) 54° 44′ with greater and 35° 46′ with lesser stress. (187) Maximum true compression = 12 900 pounds per square inch, or 27 per cent more than the apparent.

Chapter XIV. (188) Make $r_1 = 0$. (189) 18 000 pounds per square inch. (190) 54 000 pounds per square inch. (191) 15 900 pounds per square inch. (192) 15 000 pounds per square inch. (194) Deduce an expression for p_1 in terms of the given radii and S_e; then find the value of r_2 which renders p_1 a maximum.

Chapter XV. (195) See the books mentioned in Art. 127. (196) About 40 pounds per square inch with factor of 10. (198) Thickness = 1.6 inches. (201) See the demonstrations in Arts. 55 and 108.

Appendix. (202) See Railroad Gazette, June 7, 1895. (205) See Roofs and Bridges, Part I. (217) See London Philosophical Magazine, September, 1875. (218) See Engineering News, August 1, 1895. (221) See article by J. L. Greenleaf in Journal of Franklin Institute for July, 1895.

Description of Tables.

Tables I, II, III, and IV, in Art. 80 (pages 163 and 164) give mean constants of the elasticity and strength of the principal materials used in engineering. Other tables, not numbered, are noted in the Table of Contents (page ix).

Table V, on the next two pages, gives four-place logarithms of numbers which will be found very useful and sufficiently accurate for all computations in the mechanics of materials.

Table VI gives four-place squares of numbers from 1.00 to 9.99, the arrangement being the same as that of the logarithmic table. By properly moving the decimal point four-place squares of other numbers may also be taken out. For example, the square of 0.874 is 0.7639, that of 87.4 is 7 639, and that of 874 is 763 900, correct to four significant figures.

Table VII gives four-place areas of circles for diameters ranging from 1.00 to 9.99, arranged in the same manner. By properly moving the decimal point four-place areas for all circles may be found. For instance, if the diameter is 4.175 inches. the area is 13.69 square inches; if the diameter is 0.535 inches the area is 0.2248 square inches; if the diameter is 12.3 feet, the area is 116.9 square feet, all correct to four significant figures.

Table VIII gives weights per linear foot of wrought-iron bars both square and round, the side of the square or the diameter of the circle ranging from $\frac{1}{8}$ to $10\frac{3}{4}$ inches. Approximate weights of bars of other materials may be derived from this table by the following rules:

For timber, divide by 12;
For brick, divide by 4;
For stone, divide by 3;
For cast iron, subtract 6 per cent;
For steel, add 2 per cent.

For example, a cast-iron bar $6\frac{7}{8}$ inches square and 8 feet long weighs $8(157.6 - 0.06 \times 157.6) = 1\ 185$ pounds. In like manner a steel bar $2\frac{3}{16}$ inches in diameter and 4 feet 9 inches in length weighs $4\frac{3}{4}(12.53 + 0.02 \times 12.53) = 57.9$ pounds.

TABLE V. COMMON LOGARITHMS.

n	0	1	2	3	4	5	6	7	8	9	Diff.
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	42
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	38
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	35
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	32
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	30
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	28
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	27
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	25
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	24
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	22
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	18
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	18
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	17
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	15
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	14
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	11
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	11
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	9
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8
n	0	1	2	3	4	5	6	7	8	9	Diff.

TABLE V. COMMON LOGARITHMS.

n	0	1	2	3	4	5	6	7	8	9	Diff.
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	7
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	5
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	
n	0	1	2	3	4	5	6	7	8	9	Diff.

TABLE VI. SQUARES OF NUMBERS.

n.	0	1	2	3	4	5	6	7	8	9	Diff.
1.0	1.000	1.020	1.040	1.061	1.082	1.103	1.124	1.145	1.166	1.188	22
1.1	1.210	1.232	1.254	1.277	1.300	1.323	1.346	1.369	1.392	1.416	24
1.2	1.440	1.464	1.488	1.513	1.538	1.563	1.588	1.613	1.638	1.664	26
1.3	1.690	1.716	1.742	1.769	1.796	1.823	1.850	1.877	1.904	1.932	28
1.4	1.960	1.988	2.016	2.045	2.074	2.103	2.132	2.161	2.190	2.220	30
1.5	2.250	2.280	2.310	2.341	2.372	2.403	2.434	2.465	2.496	2.528	32
1.6	2.560	2.592	2.624	2.657	2.690	2.723	2.756	2.789	2.822	2.856	34
1.7	2.890	2.924	2.958	2.993	3.028	3.063	3.098	3.133	3.168	3.204	36
1.8	3.240	3.276	3.312	3.349	3.386	3.423	3.460	3.497	3.534	3.572	38
1.9	3.610	3.648	3.686	3.725	3.764	3.803	3.842	3.881	3.920	3.960	40
2.0	4.000	4.040	4.080	4.121	4.162	4.203	4.244	4.285	4.326	4.368	42
2.1	4.410	4.452	4.494	4.537	4.580	4.623	4.666	4.709	4.752	4.796	44
2.2	4.840	4.884	4.928	4.973	5.018	5.063	5.108	5.153	5.198	5.244	46
2.3	5.290	5.336	5.382	5.429	5.476	5.523	5.570	5.617	5.664	5.712	48
2.4	5.760	5.808	5.856	5.905	5.954	6.003	6.052	6.101	6.150	6.200	50
2.5	6.250	6.300	6.350	6.401	6.452	6.503	6.554	6.605	6.656	6.708	52
2.6	6.760	6.812	6.864	6.917	6.970	7.023	7.076	7.129	7.182	7.236	54
2.7	7.290	7.344	7.398	7.453	7.508	7.563	7.618	7.673	7.728	7.784	56
2.8	7.840	7.896	7.952	8.009	8.066	8.123	8.180	8.237	8.294	8.352	58
2.9	8.410	8.468	8.526	8.585	8.644	8.703	8.762	8.821	8.880	8.940	60
3.0	9.000	9.060	9.120	9.181	9.242	9.303	9.364	9.425	9.486	9.548	62
3.1	9.610	9.672	9.734	9.797	9.860	9.923	9.986	10.05	10.11	10.18	6
3.2	10.24	10.30	10.37	10.43	10.50	10.56	10.63	10.69	10.76	10.82	7
3.3	10.89	10.96	11.02	11.09	11.16	11.22	11.29	11.36	11.42	11.49	7
3.4	11.56	11.63	11.70	11.76	11.83	11.90	11.97	12.04	12.11	12.18	7
3.5	12.25	12.32	12.39	12.46	12.53	12.60	12.67	12.74	12.82	12.89	7
3.6	12.96	13.03	13.10	13.18	13.25	13.32	13.40	13.47	13.54	13.62	7
3.7	13.69	13.76	13.84	13.91	13.99	14.06	14.14	14.21	14.29	14.36	8
3.8	14.44	14.52	14.59	14.67	14.75	14.82	14.90	14.98	15.05	15.13	8
3.9	15.21	15.29	15.37	15.44	15.52	15.60	15.68	15.76	15.84	15.92	8
4.0	16.00	16.08	16.16	16.24	16.32	16.40	16.48	16.56	16.65	16.73	8
4.1	16.81	16.89	16.97	17.06	17.14	17.22	17.31	17.39	17.47	17.56	8
4.2	17.64	17.72	17.81	17.89	17.98	18.06	18.15	18.23	18.32	18.40	9
4.3	18.49	18.58	18.66	18.75	18.84	18.92	19.01	19.10	19.18	19.27	9
4.4	19.36	19.45	19.54	19.62	19.71	19.80	19.89	19.98	20.07	20.16	9
4.5	20.25	20.34	20.43	20.52	20.61	20.70	20.79	20.88	20.98	21.07	9
4.6	21.16	21.25	21.34	21.44	21.53	21.62	21.72	21.81	21.90	22.00	9
4.7	22.09	22.18	22.28	22.37	22.47	22.56	22.66	22.75	22.85	22.94	10
4.8	23.04	23.14	23.23	23.33	23.43	23.52	23.62	23.72	23.81	23.91	10
4.9	24.01	24.11	24.21	24.30	24.40	24.50	24.60	24.70	24.80	24.90	10
5.0	25.00	25.10	25.20	25.30	25.40	25.50	25.60	25.70	25.81	25.91	10
5.1	26.01	26.11	26.21	26.32	26.42	26.52	26.63	26.73	26.83	26.94	10
5.2	27.04	27.14	27.25	27.35	27.46	27.56	27.67	27.77	27.88	27.98	11
5.3	28.09	28.20	28.30	28.41	28.52	28.62	28.73	28.84	28.94	29.05	11
5.4	29.16	29.27	29.38	29.48	29.59	29.70	29.81	29.92	30.03	30.14	11
n.	0	1	2	3	4	5	6	7	8	9	Diff.

TABLE VI. SQUARES OF NUMBERS.

n.	0	1	2	3	4	5	6	7	8	9	Diff.
5.5	30.25	30.36	30.47	30.58	30.69	30.80	30.91	31.02	31.14	31.25	11
5.6	31.36	31.47	31.58	31.70	31.81	31.92	32.04	32.15	32.26	32.38	11
5.7	32.49	32.60	32.72	32.83	32.95	33.06	33.18	33.29	33.41	33.52	12
5.8	33.64	33.76	33.87	33.99	34.11	34.22	34.34	34.46	34.57	34.69	12
5.9	34.81	34.93	35.05	35.16	35.28	35.40	35.52	35.64	35.76	35.88	12
6.0	36.00	36.12	36.24	36.36	36.48	36.60	36.72	36.84	36.97	37.09	12
6.1	37.21	37.33	37.45	37.58	37.70	37.82	37.95	38.07	38.19	38.32	12
6.2	38.44	38.56	38.69	38.81	38.94	39.06	39.19	39.31	39.44	39.56	13
6.3	39.69	39.82	39.94	40.07	40.20	40.32	40.45	40.58	40.70	40.83	13
6.4	40.96	41.09	41.22	41.34	41.47	41.60	41.73	41.86	41.99	42.12	13
6.5	42.25	42.38	42.51	42.64	42.77	42.90	43.03	43.16	43.30	43.43	13
6.6	43.56	43.69	43.82	43.96	44.09	44.22	44.36	44.49	44.62	44.76	13
6.7	44.89	45.02	45.16	45.29	45.43	45.56	45.70	45.83	45.97	46.10	14
6.8	46.24	46.38	46.51	46.65	46.79	46.92	47.06	47.20	47.33	47.47	14
6.9	47.61	47.75	47.89	48.02	48.16	48.30	48.44	48.58	48.72	48.86	14
7.0	49.00	49.14	49.28	49.42	49.56	49.70	49.84	49.98	50.13	50.27	14
7.1	50.41	50.55	50.69	50.84	50.98	51.12	51.27	51.41	51.55	51.70	14
7.2	51.84	51.98	52.13	52.27	52.42	52.56	52.71	52.85	53.00	53.14	15
7.3	53.29	53.44	53.58	53.73	53.88	54.02	54.17	54.32	54.46	54.61	15
7.4	54.76	54.91	55.06	55.20	55.35	55.50	55.65	55.80	55.95	56.10	15
7.5	56.25	56.40	56.55	56.70	56.85	57.00	57.15	57.30	57.46	57.61	15
7.6	57.76	57.91	58.06	58.22	58.37	58.52	58.68	58.83	58.98	59.14	15
7.7	59.29	59.44	59.60	59.75	59.91	60.06	60.22	60.37	60.53	60.68	16
7.8	60.84	61.00	61.15	61.31	61.47	61.62	61.78	61.94	62.09	62.25	16
7.9	62.41	62.57	62.73	62.88	63.04	63.20	63.36	63.52	63.68	63.84	16
8.0	64.00	64.16	64.32	64.48	64.64	64.80	64.96	65.12	65.29	65.45	16
8.1	65.61	65.77	65.93	66.10	66.26	66.42	66.59	66.75	66.91	67.08	16
8.2	67.24	67.40	67.57	67.73	67.90	68.06	68.23	68.39	68.56	68.72	17
8.3	68.89	69.06	69.22	69.39	69.56	69.72	69.89	70.06	70.22	70.39	17
8.4	70.56	70.73	70.90	71.06	71.23	71.40	71.57	71.74	71.91	72.08	17
8.5	72.25	72.42	72.59	72.76	72.93	73.10	73.27	73.44	73.62	73.79	17
8.6	73.96	74.13	74.30	74.48	74.65	74.82	75.00	75.17	75.34	75.52	17
8.7	75.69	75.86	76.04	76.21	76.39	76.56	76.74	76.91	77.09	77.26	18
8.8	77.44	77.62	77.79	77.97	78.15	78.32	78.50	78.68	78.85	79.03	18
8.9	79.21	79.39	79.57	79.74	79.92	80.10	80.28	80.46	80.64	80.82	18
9.0	81.00	81.18	81.36	81.54	81.72	81.90	82.08	82.26	82.45	82.63	18
9.1	82.81	82.99	83.17	83.36	83.54	83.72	83.91	84.09	84.27	84.46	18
9.2	84.64	84.82	85.01	85.19	85.38	85.56	85.75	85.93	86.12	86.30	19
9.3	86.49	86.68	86.86	87.05	87.24	87.42	87.61	87.80	87.98	88.17	19
9.4	88.36	88.55	88.74	88.92	89.11	89.30	89.49	89.68	89.87	90.06	19
9.5	90.25	90.44	90.63	90.82	91.01	91.20	91.39	91.58	91.78	91.97	19
9.6	92.16	92.35	92.54	92.74	92.93	93.12	93.32	93.51	93.70	93.90	19
9.7	94.09	94.28	94.48	94.67	94.87	95.06	95.26	95.45	95.65	95.84	20
9.8	96.04	96.24	96.43	96.63	96.83	97.02	97.22	97.42	97.61	97.81	20
9.9	98.01	98.21	98.41	98.60	98.80	99.00	99.20	99.40	99.60	99.80	20
n.	0	1	2	3	4	5	6	7	8	9	Diff.

TABLE VII. AREAS OF CIRCLES.

d	0	1	2	3	4	5	6	7	8	9	Diff.
1.0	.7854	.8012	.8171	.8332	.8495	.8659	.8825	.8992	.9161	.9331	
1.1	.9503	.9677	.9852	1.003	1.021	1.039	1.057	1.075	1.094	1.112	
1.2	1.131	1.150	1.169	1.188	1.208	1.227	1.247	1.267	1.287	1.307	19
1.3	1.327	1.348	1.368	1.389	1.410	1.431	1.453	1.474	1.496	1.517	21
1.4	1.539	1.561	1.584	1.606	1.629	1.651	1.674	1.697	1.720	1.744	22
1.5	1.767	1.791	1.815	1.839	1.863	1.887	1.911	1.936	1.961	1.986	24
1.6	2.011	2.036	2.061	2.087	2.112	2.138	2.164	2.190	2.217	2.243	26
1.7	2.270	2.297	2.324	2.351	2.378	2.405	2.433	2.461	2.488	2.516	27
1.8	2.545	2.573	2.602	2.630	2.659	2.688	2.717	2.746	2.776	2.806	29
1.9	2.835	2.865	2.895	2.926	2.956	2.986	3.017	3.048	3.079	3.110	30
2.0	3.142	3.173	3.205	3.237	3.269	3.301	3.333	3.365	3.398	3.431	32
2.1	3.464	3.497	3.530	3.563	3.597	3.631	3.664	3.698	3.733	3.767	34
2.2	3.801	3.836	3.871	3.906	3.941	3.976	4.012	4.047	4.083	4.119	35
2.3	4.155	4.191	4.227	4.264	4.301	4.337	4.374	4.412	4.449	4.486	36
2.4	4.524	4.562	4.600	4.638	4.676	4.714	4.753	4.792	4.831	4.870	38
2.5	4.909	4.948	4.988	5.027	5.067	5.107	5.147	5.187	5.228	5.269	40
2.6	5.309	5.350	5.391	5.433	5.474	5.515	5.557	5.599	5.641	5.683	41
2.7	5.726	5.768	5.811	5.853	5.896	5.940	5.983	6.026	6.070	6.114	43
2.8	6.158	6.202	6.246	6.290	6.335	6.379	6.424	6.469	6.514	6.560	44
2.9	6.605	6.651	6.697	6.743	6.789	6.835	6.881	6.928	6.975	7.022	46
3.0	7.069	7.116	7.163	7.211	7.258	7.306	7.354	7.402	7.451	7.499	48
3.1	7.548	7.596	7.645	7.694	7.744	7.793	7.843	7.892	7.942	7.992	49
3.2	8.042	8.093	8.143	8.194	8.245	8.296	8.347	8.398	8.450	8.501	51
3.3	8.553	8.605	8.657	8.709	8.762	8.814	8.867	8.920	8.973	9.026	52
3.4	9.079	9.133	9.186	9.240	9.294	9.348	9.402	9.457	9.511	9.566	54
3.5	9.621	9.676	9.731	9.787	9.842	9.898	9.954	10.01	10.07	10.12	56
3.6	10.18	10.24	10.29	10.35	10.41	10.46	10.52	10.58	10.64	10.69	6
3.7	10.75	10.81	10.87	10.93	10.99	11.04	11.10	11.16	11.22	11.28	6
3.8	11.34	11.40	11.46	11.52	11.58	11.64	11.70	11.76	11.82	11.88	6
3.9	11.95	12.01	12.07	12.13	12.19	12.25	12.32	12.38	12.44	12.50	6
4.0	12.57	12.63	12.69	12.76	12.82	12.88	12.95	13.01	13.07	13.14	7
4.1	13.20	13.27	13.33	13.40	13.46	13.53	13.59	13.66	13.72	13.79	7
4.2	13.85	13.92	13.99	14.05	14.12	14.19	14.25	14.32	14.39	14.45	7
4.3	14.52	14.59	14.66	14.73	14.79	14.86	14.93	15.00	15.07	15.14	7
4.4	15.21	15.27	15.34	15.41	15.48	15.55	15.62	15.69	15.76	15.83	7
4.5	15.90	15.98	16.05	16.12	16.19	16.26	16.33	16.40	16.47	16.55	7
4.6	16.62	16.69	16.76	16.84	16.91	16.98	17.06	17.13	17.20	17.28	7
4.7	17.35	17.42	17.50	17.57	17.65	17.72	17.80	17.87	17.95	18.02	8
4.8	18.10	18.17	18.25	18.32	18.40	18.47	18.55	18.63	18.70	18.78	8
4.9	18.86	18.93	19.01	19.09	19.17	19.24	19.32	19.40	19.48	19.56	8
5.0	19.63	19.71	19.79	19.87	19.95	20.03	20.11	20.19	20.27	20.35	8
5.1	20.43	20.51	20.59	20.67	20.75	20.83	20.91	20.99	21.07	21.16	8
5.2	21.24	21.32	21.40	21.48	21.57	21.65	21.73	21.81	21.90	21.98	8
5.3	22.06	22.15	22.23	22.31	22.40	22.48	22.56	22.65	22.73	22.82	8
5.4	22.90	22.99	23.07	23.16	23.24	23.33	23.41	23.50	23.59	23.67	9
d	0	1	2	3	4	5	6	7	8	9	Diff.

TABLE VII. AREAS OF CIRCLES.

d	0	1	2	3	4	5	6	7	8	9	Diff.
5.5	23.76	23.84	23.93	24.02	24.11	24.19	24.28	24.37	24.45	24.54	9
5.6	24.63	24.72	24.81	24.89	24.98	25.07	25.16	25.25	25.34	25.43	9
5.7	25.52	25.61	25.70	25.79	25.88	25.97	26.06	26.15	26.24	26.33	9
5.8	26.42	26.51	26.60	26.69	26.79	26.88	26.97	27.06	27.15	27.25	9
5.9	27.34	27.43	27.53	27.62	27.71	27.81	27.90	27.99	28.09	27.18	9
6.0	28.27	28.37	28.46	28.56	28.65	28.75	28.84	28.94	29.03	29.13	9
6.1	29.22	29.32	29.42	29.51	29.61	29.71	29.80	29.90	30.00	30.09	10
6.2	30.19	30.29	30.39	30.48	30.58	30.68	30.78	30.88	30.97	31.07	10
6.3	31.17	31.27	31.37	31.47	31.57	31.67	31.77	31.87	31.97	32.07	10
6.4	32.17	32.27	32.37	32.47	32.57	32.67	32.78	32.88	32.98	33.08	10
6.5	33.18	33.29	33.39	33.49	33.59	33.70	33.80	33.90	34.00	34.11	10
6.6	34.21	34.32	34.42	34.52	34.63	34.73	34.84	34.94	35.05	35.15	10
6.7	35.26	35.36	35.47	35.57	35.68	35.78	35.89	36.00	36.10	36.21	10
6.8	36.32	36.42	36.53	36.64	36.75	36.85	36.96	37.07	37.18	37.28	11
6.9	37.39	37.50	37.61	37.72	37.83	37.94	38.05	38.16	38.26	38.37	11
7.0	38.48	38.59	38.70	38.82	38.93	39.04	39.15	39.26	39.37	39.48	11
7.1	39.59	39.70	39.82	39.93	40.04	40.15	40.26	40.38	40.49	40.60	11
7.2	40.72	40.83	40.94	41.06	41.17	41.28	41.40	41.51	41.62	41.74	11
7.3	41.85	41.97	42.08	42.20	42.31	42.43	42.54	42.66	42.78	42.89	11
7.4	43.01	43.12	43.24	43.36	43.47	43.59	43.71	43.83	43.94	44.06	12
7.5	44.18	44.30	44.41	44.53	44.65	44.77	44.89	45.01	45.13	45.25	12
7.6	45.36	45.48	45.60	45.72	45.84	45.96	46.08	46.20	46.32	46.45	12
7.7	46.57	46.69	46.81	46.93	47.05	47.17	47.29	47.42	47.54	47.66	12
7.8	47.78	47.91	48.03	48.15	48.27	48.40	48.52	48.65	48.77	48.89	12
7.9	49.02	49.14	49.27	49.39	49.51	49.64	49.76	49.89	50.01	50.14	12
8.0	50.27	50.39	50.52	50.64	50.77	50.90	51.02	51.15	51.28	51.40	13
8.1	51.53	51.66	51.78	51.91	52.04	52.17	52.30	52.42	52.55	52.68	13
8.2	52.81	52.94	53.07	53.20	53.33	53.46	53.59	53.72	53.85	53.98	13
8.3	54.11	54.24	54.37	54.50	54.63	54.76	54.89	55.02	55.15	55.29	13
8.4	55.42	55.55	55.68	55.81	55.95	56.08	56.21	56.35	56.48	56.61	13
8.5	56.75	56.88	57.01	57.15	57.28	57.41	57.55	57.68	57.82	57.95	13
8.6	58.09	58.22	58.36	58.49	58.63	58.77	58.90	59.04	59.17	59.31	14
8.7	59.45	59.58	59.72	59.86	59.99	60.13	60.27	60.41	60.55	60.68	14
8.8	60.82	60.96	61.10	61.24	61.38	61.51	61.65	61.79	61.93	62.07	14
8.9	62.21	62.35	62.49	62.63	62.77	62.91	63.05	63.19	63.33	63.48	14
9.0	63.62	63.76	63.90	64.04	64.18	64.33	64.47	64.61	64.75	64.90	14
9.1	65.04	65.18	65.33	65.47	65.61	65.76	65.90	66.04	66.19	66.33	14
9.2	66.48	66.62	66.77	66.91	67.06	67.20	67.35	67.49	67.64	67.78	15
9.3	67.93	68.08	68.22	68.37	68.51	68.66	68.81	68.96	69.10	69.25	15
9.4	69.40	69.55	69.69	69.84	69.99	70.14	70.29	70.44	70.58	70.73	15
9.5	70.88	71.03	71.18	71.33	71.48	71.63	71.78	71.93	72.08	72.23	15
9.6	72.38	72.53	72.68	72.84	72.99	73.14	73.29	73.44	73.59	73.75	15
9.7	73.90	74.05	74.20	74.36	74.51	74.66	74.82	74.97	75.12	75.28	15
9.8	75.43	75.58	75.74	75.89	76.05	76.20	76.36	76.51	76.67	76.82	16
9.9	76.98	77.13	77.29	77.44	77.60	77.76	77.91	78.07	78.23	78.38	16
d	0	1	2	3	4	5	6	7	8	9	Diff.

TABLE VIII.—WEIGHT OF WROUGHT-IRON BARS.

Side or Diameter. Inches.	Pounds per Linear Foot.		Side or Diameter. Inches	Pounds per Linear Foot.		Side or Diameter. Inches.	Pounds per Linear Foot.	
	Square Bars.	Round Bars.		Square Bars.	Round Bars.		Square Bars.	Round Bars.
0			2	13.33	10.47	5	83.33	65.45
1/16	0.013	0.010	1/16	14.18	11.14	1/8	87.55	68.76
1/8	0.052	0.041	1/8	15.05	11.82	1/4	91.88	72.16
3/16	0.117	0.092	3/16	15.95	12.53	3/8	96.30	75.64
1/4	0.208	0.164	1/4	16.88	13.25	1/2	100.8	79.19
5/16	0.326	0.256	5/16	17.83	14.00	5/8	105.5	82.83
3/8	0.469	0.368	3/8	18.80	14.77	3/4	110.2	86.56
7/16	0.638	0.501	7/16	19.80	15.55	7/8	115.1	90.36
1/2	0.833	0.654	1/2	20.83	16.36	6	120.0	94.25
9/16	1.055	0.828	9/16	21.89	17.19	1/8	125.1	98.22
5/8	1.302	1.023	5/8	22.97	18.04	1/4	130.2	102.3
11/16	1.576	1.237	11/16	24.08	18.91	3/8	135.5	106.4
3/4	1.875	1.473	3/4	25.21	19.80	1/2	140.8	110.6
13/16	2.201	1.728	13/16	26.37	20.71	5/8	146.3	114.9
7/8	2.552	2.004	7/8	27.55	21.64	3/4	151.9	119.3
15/16	2.930	2.301	15/16	28.76	22.59	7/8	157.6	123.7
1	3.333	2.618	3	30.00	23.56	7	166.3	128.3
1/16	3.763	2.955	1/8	32.55	25.57	1/4	175.2	137.6
1/8	4.219	3.313	1/4	35.21	27.65	1/2	187.5	147.3
3/16	4.701	3.692	3/8	37.97	29.82	3/4	200.2	157.2
1/4	5.208	4.091	1/2	40.83	32.07	8	213.3	167.6
5/16	5.742	4.510	5/8	43.80	34.40	1/4	226.9	178.2
3/8	6.302	4.950	3/4	46.88	36.82	1/2	240.8	189.2
7/16	6.888	5.410	7/8	50.05	39.31	3/4	255.2	200.4
1/2	7.500	5.890	4	53.33	41.89	9	270.0	212.1
9/16	8.138	6.392	1/8	56.72	44.55	1/4	285.2	224.0
5/8	8.802	6.913	1/4	60.21	47.29	1/2	300.8	236.3
11/16	9.492	7.455	3/8	63.80	50.11	3/4	316.9	248.9
3/4	10.21	8.018	1/2	67.50	53.01	10	333.3	261.8
13/16	10.95	8.601	5/8	71.30	56.00	1/4	350.2	275.1
7/8	11.72	9.204	3/4	75.21	59.07	1/2	367.5	288.6
15/16	12.51	9.828	7/8	79.22	62.22	3/4	385.2	302.5

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