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Nonlinear second-order delay differential equation

Nieliniowe równanie różniczkowe rzędu drugiego z opóźnieniem

Abstract

The aim of this paper is to prove the theorem on the existence and uniqueness of the classical solution of the initial-boundary value problem for a nonlinear second-order delay differential equation. For this purpose, we apply the Banach contraction principle and the Bielecki norm. The paper is based on publications [1–7] and is a generalisation of publication [6].

Keywords: second-order delay equation, initial-boundary value problem, Banach contraction principle, Bielecki norm

Streszczenie

W artykule udowodniono twierdzenie o istnieniu i jednoznaczności klasycznego rozwiązania zagadnienia początkowo-brzegowego dla nieliniowego równania różniczkowego rzędu drugiego z opóźnieniem. W tym celu stosowane jest twierdzenie Banacha o punkcie stałym i norma Bieleckiego. Artykuł bazuje na publikacjach [1–7] i jest uogólnieniem publikacji [6].

Słowa kluczowe: równanie rzędu drugiego z opóźnieniem, zagadnienie początkowo-brzegowe, twierdzenie Banacha o punkcie stałym, norma Bieleckiego.

1. Preliminaries

Is this paper, we study the following problem

$$x''(t) = f(t, x_t, x'(t)), \quad t \in [0,T],\ T > 0, \tag{1.1}$$

$$x_0 = \phi, \quad x'(T) = \beta x'(0), \quad \beta > 1, \tag{1.2}$$

where $f : [0,T] \times C([-\tau,0],\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ and $\phi \in C([-\tau,0],\mathbb{R})$, $\tau > 0$, are given functions.

Therefore, for any function $x : [-\tau, T] \to \mathbb{R}$ and any $t \in [0,T]$, we denote by x_t the function $x_t : [-\tau,0] \to \mathbb{R}$ defined by the formula $x_t(s) = x(t+s)$, $s \in [-\tau,0]$.

It is easy to see that the condition $x_0 = \phi$ means that $x(s) = \phi(s)$, $s \in [-\tau,0]$.

Moreover, for $\phi \in C([-\tau,0],\mathbb{R})$ we use the norm

$$\|\phi\|_0 = \sup_{-\tau \le s \le 0} |\phi(s)|.$$

2. Theorem on the existence and uniqueness of the classical solution

Let $C^* := C([-\tau,T],\mathbb{R}) \cap C^2([0,T],\mathbb{R})$.

Definition 2.1. The function $x \in C^*$ is said to be a solution of problem (1.1) – (1.2) if x satisfies equation (1.1) and conditions (1.2).

Now, we will prove the following lemma:

Lemma 2.1. Function $x \in C^*$ is a solution of problem (1.1)–(1.2), where $f \in C([0,T] \times C([-\tau,0],\mathbb{R}) \times \mathbb{R}, \mathbb{R})$ if and only if x is a solution of the following integral equation:

$$x(t) = \begin{cases} \phi(t), & t \in [-\tau,0], \\ \phi(0) + \dfrac{t}{\beta - 1} \displaystyle\int_0^T f(s, x_s, x'(s))ds + \int_0^t (t-s) f(s, x_s, x'(s))ds, & t \in [0,T]. \end{cases}$$

Proof. If $x \in C^*$ is a solution of (1.1) – (1.2) then we have

$$x''(t) = f(t, x_t, x'(t)), \quad t \in [0,T]. \tag{2.1}$$

Integration by parts gives

$$x(t)=x(0)+tx'(0)+\int_0^t (t-s)x''(s)ds. \tag{2.2}$$

Differentiating (2.2), we get

$$x'(t)=x'(0)+\int_0^t x''(s)ds.$$

Thus,

$$x'(T)=x'(0)+\int_0^T x''(s)ds.$$

Applying the boundary condition we obtain

$$x'(0)+\int_0^T x''(s)ds=\beta x'(0).$$

Thus,

$$x'(0)=\frac{1}{\beta-1}\int_0^T x''(s)ds. \tag{2.3}$$

Equation (2.2), together with (2.1) and (2.3), imply

$$x(t)=\phi(0)+\frac{t}{\beta-1}\int_0^T f(s,x_s,x'(s))ds+\int_0^t (t-s)f(s,x_s,x'(s))ds. \tag{2.4}$$

Conversely, if x is a solution of equation (2.4) then direct differentiation of (2.4) gives

$$x'(t)=\frac{1}{\beta-1}\int_0^T f(s,x_s,x'(s))ds+\int_0^t f(s,x_s,x'(s))ds,$$

$$x''(t)=f(t,x_t,x'(t)), \quad t\in[0,T].$$

Thus,

$$x'(0)=\frac{1}{\beta-1}\int_0^T f(s,x_s,x'(s))ds,$$

$$x'(T) = \frac{1}{\beta - 1}\int_0^T f(s, x_s, x'(s))ds + \int_0^T f(s, x_s, x'(s))ds = \frac{\beta}{\beta - 1}\int_0^T f(s, x_s, x'(s))ds,$$

which gives

$$x'(T) = \beta x'(0).$$

The proof of Lemma 2.1 is complete.

Now, using Lemma 2.1 and the Banach contraction theorem, we shall prove the existence and uniqueness of the solution for problem (1.1) – (1.2).

Theorem 2.1. Assume that $f \in C([0,T] \times C([-\tau,0],\mathbb{R}) \times \mathbb{R},\mathbb{R})$ and there exists $m \in L^1([0,T],\mathbb{R}_+)$ such that

$$|f(t,u,z) - f(t,\tilde{u},\tilde{z})| \le m(t)(\|u - \tilde{u}\|_0 + |z - \tilde{z}|) \tag{2.5}$$

for all $t \in [0,T]$, $u, \tilde{u} \in C([-\tau,0],\mathbb{R})$, $z, \tilde{z} \in \mathbb{R}$ and

$$M(T) < \frac{\ln\beta}{T+1}, \tag{2.6}$$

where $M(t) := \int_0^t m(r)dr$.

Then problem (1.1) – (1.2) has a unique solution $x \in C^*$.

Proof: for $x \in C^1([0,T],\mathbb{R})$ let

$$\|x\|_1 := \max_{s \in [0,T]}\left\{e^{-\gamma M(s)}(\max_{r \in [0,s]}|x(r)| + |x'(s)|)\right\},$$

where

$$T + 1 < \gamma < \frac{\ln\beta}{M(T)}. \tag{2.7}$$

Define an operator

$$F : C^1([0,T],\mathbb{R}) \to C^1([0,T],\mathbb{R})$$

by the formula

$$(Fx)(t)=\phi(0)+\frac{t}{\beta-1}\int_0^T f(s,x_s,x'(s))ds+\int_0^t (t-s)f(s,x_s,x'(s))ds,$$

where $x_s(r)=x(s+r)=\phi(s+r)$ for $s+r\le 0$.

For any $x,y\in C^1([0,T],\mathbb{R})$ and $t\in[0,T]$, by (2.5), we have

$$|(Fx)(t)-(Fy)(t)|\le\frac{t}{\beta-1}\int_0^T |f(s,x_s,x'(s))-f(s,y_s,y'(s))|ds$$

$$+\int_0^t (t-s)|f(s,x_s,x'(s))-f(s,y_s,y'(s))|ds$$

$$\le\frac{t}{\beta-1}\int_0^T m(s)\left(\|x_s-y_s\|_0+|x'(s)-y'(s)|\right)ds$$

$$+\int_0^t (t-s)m(s)\left(\|x_s-y_s\|_0+|x'(s)-y'(s)|\right)ds$$

$$\le\frac{T}{\beta-1}\int_0^T m(s)\left(\|x_s-y_s\|_0+|x'(s)-y'(s)|\right)ds$$

$$+T\int_0^t m(s)\left(\|x_s-y_s\|_0+|x'(s)-y'(s)|\right)ds.$$

Observe that (see[6])

$$\|x_s-y_s\|_0=\max\{|x(r)-y(r)|,\ r\in[0,s]\}\ \text{ if } s\in[0,\tau]$$

and

$$\|x_s-y_s\|_0\le\max\{|x(r)-y(r)|,\ r\in[0,s]\}\ \text{ if } s\in(\tau,T].$$

Therefore,

$$\left|(Fx)(t)-(Fy)(t)\right|$$

$$\leq \frac{T}{\beta-1}\int_0^T m(s)e^{\gamma M(s)}e^{-\gamma M(s)}\left(\max_{r\in[0,s]}\left|x(r)-y(r)\right|+\left|x'(s)-y'(s)\right|\right)ds$$

$$+T\int_0^t m(s)\ e^{\gamma M(s)}e^{-\gamma M(s)}\left(\max_{r\in[0,s]}\left|x(r)-y(r)\right|+\left|x'(s)-y'(s)\right|\right)ds$$

$$\leq \frac{T}{\beta-1}\left\|x-y\right\|_1\int_0^T m(s)e^{\gamma M(s)}ds+T\left\|x-y\right\|_1\int_0^t e^{\gamma M(s)}ds$$

$$=\frac{T}{\beta-1}\left\|x-y\right\|_1\frac{1}{\gamma}e^{\gamma M(s)}\Big|_0^T+T\left\|x-y\right\|_1\frac{1}{\gamma}e^{\gamma M(s)}\Big|_0^t$$

$$=\frac{T}{\gamma}\left\|x-y\right\|_1\left(\frac{e^{\gamma M(T)}-\beta}{\beta-1}+e^{\gamma M(t)}\right).$$

It follows from (2.7) that

$$e^{\gamma M(T)}-\beta<0.$$

Consequently,

$$\left|(Fx)(t)-(Fy)(t)\right|\leq\frac{T}{\gamma}\left\|x-y\right\|_1 e^{\gamma M(t)}. \tag{2.8}$$

Observe that

$$(Fx)'(t)=\frac{1}{\beta-1}\int_0^T f(s,x_s,x'(s))ds+\int_0^t f(s,x_s,x'(s))ds.$$

Thus,

$$\left|(Fx)'(t)-(Fy)'(t)\right|\leq\frac{1}{\gamma}\left\|x-y\right\|_1\ e^{\gamma M(t)}. \tag{2.9}$$

From (2.8), (2.9) and from the definition of the norm $\|\cdot\|_1$, we have

$$\|Fx - Fy\|_1 \leq \frac{T+1}{\gamma}\|x - y\|_1 . \tag{2.10}$$

By (2.10) and (2.7), F is a contractive operator. Consequently, by the Banach fixed point theorem, the proof of Theorem 2.1 is complete.

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